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On Some Interrelations between the Generalized Nehari Problem for the Carathéodory Class and Prediction Theory

B. Fritzsche and B. Kirstein

Dedicated to the memory of Prof. Paul Günther

Abstract. This paper is a continuation of the authors' former studies on generalized Nehari problems (see [40, 44, 47]). We indicate the stochastic background of the generalized Nehari problem for the Carathéodory class. Moreover, we discuss some intimately related questions of prediction theory for stationarily cross-correlated stationary sequences.

Keywords: *Matrix-valued Carathéodory functions, generalized Nehari problem, prediction theory*

AMS subject classification: 30 E 05, 47 A 57, 60 G 10, 60 G 25

0. Introduction

Throughout this paper, let p and q be positive integers, and let H be an infinite-dimensional complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Further, let $\mathbb{N}_0$ and $\mathbb{C}$ be the sets of all non-negative integers and complex numbers, respectively. We will use the notation $\mathbb{C}^{p\times q}$ to denote the set of all $(p\times q)$-matrices all entries of which are complex numbers. A kernel $\mathcal{K}:\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{C}^{q\times q}$ is said to be ***non-negative definite*** if, for all $j\in\mathbb{N}_0$, the block matrix $(\mathcal{K}(m,n))_{m,n=0}^{j}$ is non-negative Hermitian. It is known that, for several interpolation problems as the classical interpolation problems of Schur, Carathéodory, Nevanlinna-Pick and Nehari (and their matricial generalizations), the solvability of the problem can be described by the fact that some kernel appropriately constructed from the given interpolation data is non-negative definite (see, e.g., [33]). A famous result due to Kolmogorov [53] shows that if $\mathcal{K}:\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{C}^{q\times q}$ is a given kernel, then $\mathcal{K}$ is non-negative definite if and only if there is a sequence $(h_n)_{n=0}^{\infty}$ from H^q such that $\mathcal{K}(m,n)$ is exactly the Gramian of h_m and h_n for all non-negative integers m and n. In particular, one can choose H as the subspace of all equivalence classes of square-integrable complex-valued random variables on a probability space $(\Omega,\mathfrak{A},P)$ with finite variance and zero expectation. For this reason, one can expect that mathematical objects describing non-negative definite kernels have a clear probabilistic meaning. In [37, 42], the authors stated how the matricial versions of the interpolation problems of Carathéodory and Schur are associated with multivariate stationary sequences. A

B. Fritzsche: Mathematisches Institut der Universität, Augustuspl. 10, D – 04109 Leipzig
B. Kirstein: Mathematisches Institut der Universität, Augustuspl. 10, D – 04109 Leipzig

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correlation-theoretical interpretation of Schur analysis of non-negative Hermitian block matrices is given in [43].

The main goal of the present paper is to work out the stochastic meaning of the Nehari-Type Problem for matrix-valued Carathéodory functions defined on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. A matrix-valued function $\Phi : \mathbb{D} \to \mathbb{C}^{q \times q}$ is called $(q \times q)$-*Carathéodory function* (on $\mathbb{D}$) if it has the following two properties:

(i) Φ is holomorphic in $\mathbb{D}$.

(ii) For each $z \in \mathbb{D}$, the real part $\operatorname{Re}\Phi(z) := \frac{1}{2}(\Phi(z) + [\Phi(z)]^*)$ of the matrix $\Phi(z)$ is non-negative Hermitian.

The Nehari-Type Problem for matrix-valued Carathéodory Functions can then be formulated as follows.

Problem (NTPCF). Let $\alpha : \mathbb{D} \to \mathbb{C}^{p \times p}$, $\beta : \mathbb{D} \to \mathbb{C}^{p \times q}$ and $\delta : \mathbb{D} \to \mathbb{C}^{q \times q}$ be given matrix-valued functions holomorphic in $\mathbb{D}$.

(a) Describe the set $\mathcal{N}_0[\alpha, \beta, \delta]$ of all matrix-valued functions $\xi : \mathbb{D} \to \mathbb{C}^{q \times p}$ such that

$$\Omega := \begin{pmatrix} \alpha & \beta \\ \xi & \delta \end{pmatrix} \tag{1}$$

is a $((p+q) \times (p+q))$-Carathéodory function (on $\mathbb{D}$) satisfying $[\Omega(0)]^* = \Omega(0)$.

(b) If k is a non-negative integer and if $(\gamma_j)_{j=0}^k$ is a given sequence of complex $(q \times p)$-matrices, then describe the set $\mathcal{N}_0[\alpha, \beta, \delta; (\gamma_j)_{j=0}^k]$ of all $\xi \in \mathbb{N}_0[\alpha, \beta, \delta]$ satisfying $\xi(0) = \gamma_0$ and, if $k \in \mathbb{N}$,

$$\frac{1}{j!} \frac{d^j \xi}{dz^j}(0) = 2\gamma_j$$

for all integers j such that $1 \leq j \leq k$.

Part (a) of problem (NTPCF) was posed by Katsnelson [51, 52]. It turns out to be a reformulation of a generalization of a classical problem studied by Nehari [57]. Fundamental results associated with this classical problem and its matricial version were obtained by Adamjan, Arov and Krein in their famous papers [1 - 4]. The solution of problem (NTPCF) leads to the study of certain non-negative definite kernels of mixed Toeplitz-Hankel type. Those kernels originate in a series of papers by Arocena, Cotlar, Leon and Sadosky and were intensively studied in [7 - 19, 25 - 28] (see also Alegria [5]). In [40] the authors gave a necessary and sufficient condition for the case that problem (NTPCF) has a solution. Moreover, all solutions were described by their Taylor coefficients. This description shows that the set of all solutions contains an element which is distinguished for geometrical reasons, namely the so-called *central* solution. We will indicate that the central solution can be also characterized as the unique solution of an appropriate approximation problem in the framework of prediction theory.

1. The Nehari-type completion problem for matrix-valued Carathéodory functions

In this section, we will state an answer to problem (NTPCF). For this reason, we will give some notation. Thoughout this paper, let $\mathbb{Z}$ be the set of all integers. If $r \in \mathbb{Z} \cup \{-\infty\}$ and $s \in \mathbb{Z} \cup \{+\infty\}$ satisfy $r \leq s$, then let $\mathbb{Z}_{r,s}$ be the set of all integers k which fulfill $r \leq k \leq s$. If A is a complex $(p \times q)$-matrix, then A^+ designates the Moore-Penrose inverse of A. We will use the Löwner semi-ordering for Hermitian matrices. If $A \in \mathbb{C}^{q\times q}$ and $B \in \mathbb{C}^{q\times q}$ are Hermitian, then $A \geq B$ (or $B \leq A$) means that $A - B$ is non-negative Hermitian. Let I_p be the identity matrix which belongs to $\mathbb{C}^{p\times p}$, and let $\mathbb{K}_{p\times q}$ be the set of all $A \in \mathbb{C}^{p\times q}$ which are contractive, i.e., which satisfy the inequality $I_p \geq AA^*$. If $M \in \mathbb{C}^{p\times q}$, $A \in \mathbb{C}^{p\times p}$ and $B \in \mathbb{C}^{q\times q}$ are given, then the set

$$\mathfrak{K}(M; A, B) := \{M + AKB : K \in \mathbb{K}_{p\times q}\}$$

is called the *closed matrix ball* with center M, left semi-radius A and right radius B. For a detailed study of matrix (and operator) balls, we refer to the paper [58] of Smuljan.

Now we assume that $(\alpha_j)_{j=-\infty}^{\infty}$ and $(\delta_j)_{j=-\infty}^{\infty}$ are sequences of complex $(p \times p)$- and $(q \times q)$-matrices, respectively. Then, for all $m \in \mathbb{N}_0$, let A_m and Δ_m be the block Toeplitz matrices defined by

$$A_m := (\alpha_{r-s})_{r,s=0}^{m} \qquad \text{and} \qquad \Delta_m := (\delta_{r-s})_{r,s=0}^{m}.$$

If $k \in \mathbb{N}_0$, and if a sequence $(\beta_j)_{j=-k}^{\infty}$ of complex $(q \times q)$-matrices is given, then let

$$B_{l,n} := (\beta_{r-s+l})_{r,s=0}^{n} \qquad (n \in \mathbb{N}_0,\ l \in \mathbb{Z}_{n-k,\infty})$$

$$\mathcal{S}_{m,n} := \begin{pmatrix} A_{m+n} & B_{n,m+n} \\ B_{n,m+n}^* & \Delta_{m+n} \end{pmatrix} \qquad (m \in \mathbb{Z}_{-2,k},\ n \in \mathbb{N}_0 \cap \mathbb{Z}_{-m,\infty}) \tag{2}$$

and, for all $n \in \mathbb{N}_0$ and all $m \in \mathbb{N}_0$ which fulfill $1 - n \leq m \leq k + 1$,

$$d_{m,n} := \left(\beta_{-(m-1)}^*, \beta_{-(m-2)}^*, \dots, \beta_n^*, \delta_{m+n}, \delta_{m+n-1}, \dots, \delta_1\right),$$
$$e_{m,n} := \left(\alpha_{-1}, \alpha_{-2}, \dots, \alpha_{-(m+n)}, \beta_n, \beta_{n-1}, \dots, \beta_{-(m-1)}\right)$$

and

$$L_{m,n} := \delta_0 - d_{m,n}\mathcal{S}_{m-2,n+1}^+ d_{m,n}^*,$$
$$R_{m,n} := \alpha_0 - e_{m,n}\mathcal{S}_{m-2,n+1}^+ e_{m,n}^*,$$
$$M_{m,n} := d_{m,n}\mathcal{S}_{m-2,n+1}^+ e_{m,n}^*.$$

Furthermore, let $L_{00} := \delta_0, R_{00} := \alpha_0$ and $M_{00} := 0_{q\times p}$ where $0_{q\times p}$ stands for the null matrix that belongs to $\mathbb{C}^{q\times p}$.

Comparing parts (a) and (b) of problem (NTPCF), we immediately see the following.

Remark 1. If Ω is a $((p+q)\times(p+q))$-Carathéodory function (on $\mathbb{D}$) which has the block partition (1), then $[\Omega(0)]^* = \Omega(0)$ implies $\xi(0) = [\beta(0)]^*$. Hence,

$$\mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^0] = \begin{cases} \mathcal{N}_0[\alpha,\beta,\delta] & \text{if } \gamma_0 = [\beta(0)]^* \\ \emptyset & \text{if } \gamma_0 \neq [\beta(0)]^*. \end{cases}$$

In view of Remark 1, we can focus our attention to part (b) of problem (NTPCF) where $\gamma_0 = [\beta(0)]^*$. The following theorem gives an answer to this problem.

Theorem 1. *Suppose that $\alpha : \mathbb{D} \to \mathbb{C}^{p\times p}$, $\beta : \mathbb{D} \to \mathbb{C}^{p\times q}$ and $\delta : \mathbb{D} \to \mathbb{C}^{q\times q}$ are holomorphic matrix-valued functions with Taylor series representations*

$$\alpha(z) = \alpha_0 + 2\sum_{j=1}^{\infty} \alpha_j z^j \quad \textit{and} \quad \beta(z) = \beta_0 + 2\sum_{j=1}^{\infty} \beta_j z^j \qquad (z \in \mathbb{D}) \tag{3}$$

and

$$\delta(z) = \delta_0 + 2\sum_{j=1}^{\infty} \delta_j z^j \qquad (z \in \mathbb{D}). \tag{4}$$

Let $k \in \mathbb{N}_0$, and let $(\gamma_j)_{j=0}^k$ be a sequence of complex $(q \times p)$-matrices where $\gamma_0 = \beta_0^*$. Then:

(a) *The following statements are equivalent:*

(i) *The set $\mathcal{N}_0\left[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k\right]$ is non-empty.*

(ii) *For all $n \in \mathbb{N}_0$, the matrix $S_{k,n}$ is non-negative Hermitian where*

$$\beta_{-j} := \gamma_j^* \qquad (j \in \mathbb{Z}_{1,k}). \tag{5}$$

(iii) *The kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ given by (5),*

$$\alpha_{-j} := \alpha_j^* \quad \textit{and} \quad \delta_{-j} := \delta_j^* \qquad (j \in \mathbb{N}) \tag{6}$$

and

$$\mathcal{K}_k(m,n) := \begin{pmatrix} \alpha_{m-n} & \beta_{m+n-k} \\ \beta_{m+n-k}^* & \delta_{n-m} \end{pmatrix} \tag{7}$$

is non-negative definite, i.e., for all $j \in \mathbb{N}_0$, the matrix $(\mathcal{K}_k(m,n))_{m,n=0}^j$ is non-negative Hermitian.

(b) *Suppose that $\mathcal{N}_0\left[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k\right]$ is non-empty. Then the following procedure yields all functions $\xi \in \mathcal{N}_0\left[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k\right]$ by their Taylor series representations*

$$\xi(z) = \xi_0 + 2\sum_{j=1}^{\infty} \xi_j z^j \qquad (z \in \mathbb{D}) :$$

Step (I) *Set $\xi_j := \gamma_j$ for all $j \in \mathbb{Z}_{0,k}$.*

Step (II) *Assume that* $m \in \mathbb{Z}_{k+1,\infty}$ *and that, in the case* $m > k+1$, *the coefficients* $\xi_{k+1}, \xi_{k+2}, \dots, \xi_{m-1}$ *are already determined. If* $\beta_{-j} := \xi_j^*$ *for all* $j \in \mathbb{Z}_{1,m-1}$, *then there exist the limits*

$$M_{m,*} := \lim_{n\to\infty} M_{m,n}, \qquad L_{m,*} := \lim_{n\to\infty} L_{m,n}, \qquad R_{m,*} := \lim_{n\to\infty} R_{m,n}$$

where $L_{m,*} \geq 0$ *and* $R_{m,*} \geq 0$. *Choose*

$$\xi_m \in \mathfrak{K}\left(M_{m,*}; \sqrt{L_{m,*}}, \sqrt{R_{m,*}}\right).$$

A proof of Theorem 1 was given in [40: Theorems 1 and 2]. There the formulation of the theorem was stated only in the case $k = 0$. However, a closer analysis of the proof given in [40] shows that it goes also through for the more general situation considered here.

Remark 2. If $k \in \mathbb{N}$ and if the kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ is given by (7), then it is readily checked that $\mathcal{K}_k$ is non-negative definite if and only if all the kernels $\mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_k$ are non-negative definite.

If the set $\mathcal{N}_0\left[\alpha, \beta, \delta; (\gamma_j)_{j=0}^k\right]$ is non-empty, then we see from Theorem 1 that it contains an element which is distinguished for geometrical reasons, namely the function ξ the Taylor coefficients of which are successively chosen as the centers of the matrix balls in question (see Theorem 1/(b)):

$$\xi_0, \xi_1, \dots, \xi_k, \xi_{k+1} = M_{k+1,*}, \qquad \xi_{k+2} = M_{k+2,*}, \qquad \xi_{k+3} = M_{k+3,*}, \ \dots \ .$$

This function ξ will be called *the central element* of $\mathcal{N}_0\left[\alpha, \beta, \delta; (\gamma_j)_{j=0}^k\right]$. Later we will see that it can be also characterized by a certain extremality property in the context of prediction theory.

2. Some facts on multivariate stationary sequences

In this section, we will summarize some facts later on stationary sequences in Hilbert space. For a comprehensive survey on this topic, we refer the reader to Masani's paper [56].

In the following, we again suppose that H is a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. Further, let H^q be the Cartesian product of H with itself q times, i.e., the set of all column vectors $g = \operatorname{col}(g^{(1)}, g^{(2)}, \dots, g^{(q)})$ with $g^{(k)} \in H$ for each $k \in \mathbb{Z}_{1,q}$ ($k = 1, \dots, q$). Obviously, if $g \in H^q$ and if $A = (a_{jk})_{j=1}^{p}{}_{k=1}^{q} \in \mathbb{C}^{p\times q}$, then the vector

$$Ag = \operatorname{col}\left(\sum_{k=1}^{q} a_{1k}\, g^{(k)}, \sum_{k=1}^{q} a_{2k}\, g^{(k)}, \dots, \sum_{k=1}^{q} a_{pk}\, g^{(k)}\right)$$

belongs to H^p. If $f \in H^p$ and $g \in H^q$, then the Gramian (f,g) of the ordered pair $[f,g]$ is defined by the matrix

$$(f,g) := \left(\langle f^{(j)}, g^{(k)}\rangle\right)_{j=1}^{p}{}_{k=1}^{q}.$$

If $\operatorname{tr} A$ denotes the trace of a complex $(q \times q)$-matrix A, then H^q turns out to be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle_{H^q}$ and associated norm $\| \cdot \|_{H^q}$ given by

$$\langle f, g \rangle_{H^q} := \operatorname{tr}(f, g) \qquad \text{and} \qquad \|g\|_{H^q} := \sqrt{\operatorname{tr}(g, g)}$$

for every choice of f and g in H^q. Let U be a closed linear subspace of H, and let $g \in H^q$. Then U^q is a closed matrix-linear subspace of H^q, and there is a unique vector $\widehat{g} \in U^q$ such that $(g - \widehat{g}, h) = 0_{q \times q}$ for all $h \in U^q$. One can also characterize $\widehat{g}$ as the unique vector $g_{\blacksquare}$ which belongs to U^q and which satisfies

$$\|g - g_{\blacksquare}\|_{H^q} \leq \|g - h\|_{H^q}$$

for each $h \in U^q$. We will write $(g|U^q)$ for this so-called *Gramian orthogonal projection* $\widehat{g}$ of g onto U^q. Observe that this projection admits the representation

$$(g|U^q) = \operatorname{col}((g^{(1)}|U), (g^{(2)}|U), ..., (g^{(q)}|U)).$$

Remark 3. Wiener and Masani (see [59: Lemma 5.8] and [60: Lemma 1.17]) proved that $(g|U^q)$ can also be characterized as the unique vector $g_{\square}$ which belongs to U^q and which fulfills the matrix inequality

$$(g - g_{\square}, g - g_{\square}) \leq (g - h, g - h)$$

for all $h \in U^q$.

Wiener and Masani (see [59, 60]) also observed that if the linear subspace U of H is finite-dimensional, then Gramian orthogonal projections onto U^q admit useful representations:

Remark 4. Let $f \in H^p$, and let $\operatorname{sp}[f^{(1)}, f^{(2)}, ..., f^{(p)}]$ denote the linear span of the components $f^{(1)}, f^{(2)}, ..., f^{(p)}$ of f. Then for each $g \in H^q$ the vector

$$\widehat{g} = (g | (\operatorname{sp}[f^{(1)}, f^{(2)}, ..., f^{(p)}])^q)$$

admits the representation

$$\widehat{g} = (g, f)(f, f)^+ f$$

and satisfies

$$(g - \widehat{g}, g - \widehat{g}) = (g, g) - (g, f)(f, f)^+ (f, g).$$

In the following, we will continue to use the notation $\operatorname{sp}[f^{(1)}, f^{(2)}, ..., f^{(p)}]$ introduced in Remark 4. It will be advantageous to give some further properties of Gramian orthogonal projections.

Remark 5. Let U be a closed linear subspace of the complex Hilbert space H. Let $g \in H^q$, and let $\widehat{g} := (g|U^q)$. Then it is readily checked that the vector

$$\widetilde{g} = \sqrt{(g - \widehat{g}, g - \widehat{g})}^{\,+} (g - \widehat{g})$$

satisfies $(\widetilde{g},\widetilde{g})^2 = (\widetilde{g},\widetilde{g})$ and

$$g - \widehat{g} = \sqrt{(g-\widehat{g}, g-\widehat{g})}\,\widetilde{g}.$$

If G is the smallest closed linear subspace $\mathcal{L}$ of H with $U \subseteq \mathcal{L}$ and $g^{(j)} \in \mathcal{L}$ for all $j \in \mathbb{Z}_{1,q}$, then

$$G \ominus U = \operatorname{sp}\left[\widetilde{g}^{(1)}, \widetilde{g}^{(2)}, ..., \widetilde{g}^{(q)}\right]$$

where $\widetilde{g}^{(1)}, \widetilde{g}^{(2)}, ..., \widetilde{g}^{(q)}$ are the components of $\widetilde{g}$.

Lemma 1. *Let $f \in H^p$ and $g \in H^q$. Further, let U be a closed linear subspace of H, let*

$$\widehat{f} := (f|U^p) \qquad \text{and} \qquad \widehat{g} := (g|U^q), \tag{8}$$

and let

$$\widetilde{f} := \sqrt{(f-\widehat{f}, f-\widehat{f})}^{\,+}(f-\widehat{f}) \qquad \text{and} \qquad \widetilde{g} := \sqrt{(g-\widehat{g}, g-\widehat{g})}^{\,+}(g-\widehat{g}). \tag{9}$$

Then $(\widetilde{g},\widetilde{f})$ is a contractive matrix:

$$(\widetilde{g},\widetilde{f})(\widetilde{g},\widetilde{f})^* \leq I_q. \tag{10}$$

Proof. Because of Remark 5, both matrices $(\widetilde{f},\widetilde{f})$ and $(\widetilde{g},\widetilde{g})$ are idempotent and Hermitian. Therefore, $I_p \geq (\widetilde{f},\widetilde{f})$ and $I_q \geq (\widetilde{g},\widetilde{g})$. Thus we see that

$$\begin{aligned}\begin{pmatrix} I_p & (\widetilde{f},\widetilde{g}) \\ (\widetilde{g},\widetilde{f}) & I_q \end{pmatrix} &= \begin{pmatrix} (\widetilde{f},\widetilde{f}) & (\widetilde{f},\widetilde{g}) \\ (\widetilde{g},\widetilde{f}) & (\widetilde{g},\widetilde{g}) \end{pmatrix} + \begin{pmatrix} I_p - (\widetilde{f},\widetilde{f}) & 0 \\ 0 & I_q - (\widetilde{g},\widetilde{g}) \end{pmatrix} \\ &\geq \begin{pmatrix} (\widetilde{f},\widetilde{f}) & (\widetilde{f},\widetilde{g}) \\ (\widetilde{g},\widetilde{f}) & (\widetilde{g},\widetilde{g}) \end{pmatrix} = \left(\begin{pmatrix} \widetilde{f} \\ \widetilde{g} \end{pmatrix}, \begin{pmatrix} \widetilde{f} \\ \widetilde{g} \end{pmatrix}\right).\end{aligned} \tag{11}$$

As a Gramian the matrix stated on the right-hand side of (11) is non-negative Hermitian. Therefore, the matrix on the left-hand side of (11) has this property as well. Consequently, applying a lemma which characterizes non-negative Hermitian block matrices (see, e.g., [33: Lemmas 1.1.9 and 1.1.12]) we obtain (10) ∎

Lemma 2. *Suppose that U is a closed linear subspace of the complex Hilbert space H. Let $f \in H^p$ and $g \in H^q$, and let $\widehat{f}, \widehat{g}, \widetilde{f}$ and $\widetilde{g}$ be given by (8) and (9). Further, let $M := (\widehat{g},\widehat{f})$, $L := (g-\widehat{g}, g-\widehat{g})$ and $R := (f-\widehat{f}, f-\widehat{f})$. Then there is a unique matrix $K \in \mathbb{C}^{q\times p}$ such that*

$$(g,f) = M + \sqrt{L}K\sqrt{R}, \tag{12}$$

$$LL^+K = K \qquad \text{and} \qquad KR^+R = K. \tag{13}$$

This matrix K is contractive and admits the representations

$$K = (\tilde{g},\tilde{f}) \qquad \text{and} \qquad K = \sqrt{L}^{\,+}[(g,f) - M]\sqrt{R}^{\,+}.$$

In particular, the matrix (g,f) belongs to the matrix ball $\mathfrak{K}(M;\sqrt{L},\sqrt{R})$. Moreover, $(g,f)=M$ if and only if $(\tilde{g},\tilde{f})=O_{q\times p}$.

Proof. If there is a matrix $K\in\mathbb{C}^{q\times p}$ satisfying the three identities stated in (12) and (13), then, in view of $L^+L=LL^+=\sqrt{L}^{\,+}\sqrt{L}=\sqrt{L}\sqrt{L}^{\,+}$,

$$K=\sqrt{L}^{\,+}\sqrt{L}K=\sqrt{L}^{\,+}\sqrt{L}K\sqrt{R}\sqrt{R}^{\,+}=\sqrt{L}^{\,+}[(g,f)-M]\sqrt{R}^{\,+}. \tag{14}$$

In particular, there is at most one matrix $K\in\mathbb{C}^{q\times p}$ satisfying the equations given in (12) and (13). Because of $(\widehat{g},f-\widehat{f})=0_{q\times q}$ and $(g-\widehat{g},\widehat{f})=0_{q\times q}$, we have

$$(g,f)=(\widehat{g}+g-\widehat{g},\widehat{f}+f-\widehat{f})=M+(g-\widehat{g},f-\widehat{f}).$$

In view of Remark 5, then

$$(g,f)=M+\sqrt{L}(\widetilde{g},\widetilde{f})\sqrt{R} \tag{15}$$

and

$$\sqrt{L}^{\,+}[(g,f)-M]\sqrt{R}^{\,+}=\sqrt{L}^{\,+}\sqrt{L}(\widetilde{g},\widetilde{f})\sqrt{R}\sqrt{R}^{\,+}=(\widetilde{g},\widetilde{f}). \tag{16}$$

Thus, if there is a matrix $K\in\mathbb{C}^{q\times p}$ satisfying the equations given in (12) and (13), then it follows from (14) and (16) that $K=(\widetilde{g},\widetilde{f})$. From (15) and (16) one can conversely see that the matrix $K=(\widetilde{g},\widetilde{f})$ really satisfies (12) and (13). Lemma 1 shows that the matrix $(\widetilde{g},\widetilde{f})$ is contractive. Hence, the identity (15) provides finally $(g,f)\in\mathfrak{K}(M;\sqrt{L},\sqrt{R})$ ∎

The following lemma due to Masani [56] will play a key role in our further considerations.

Lemma 3. *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of closed linear subspaces of the complex Hilbert space H which satisfy $U_n\subseteq U_{n+1}$ for each $n\in\mathbb{N}$. Let U denote the smallest closed linear subspace of H which fulfils $U_n\subseteq U$ for all $n\in\mathbb{N}$. Then, for each $f\in H^p$,*

$$(f|U^p)=\lim_{n\to\infty}(f|U_n^p)$$

with respect to the norm $\|\cdot\|_{H^p}$ of the Hilbert space H^p.

Assume that $r,\rho\in\mathbb{Z}\cup\{-\infty\}$ and $s,\sigma\in\mathbb{Z}\cup\{+\infty\}$ satisfy $r\le s$ and $\rho\le\sigma$, and that $f_t\in H^p$ $(t\in\mathbb{Z}_{r,s})$ and $g_\tau\in H^q$ $(\tau\in\mathbb{Z}_{\rho,\sigma})$ are given. Then we will use $\mathcal{L}_{r,s/\rho,\sigma}$ to denote the closed linear subspace of H generated by all $f_t^{(\mu)}$ $(\mu\in\mathbb{Z}_{1,p},t\in\mathbb{Z}_{r,s})$ and all $g_\tau^{(\nu)}$ $(\nu\in\mathbb{Z}_{1,q},\tau\in\mathbb{Z}_{\rho,\sigma})$. Now let $r,s,\rho,\sigma\in\mathbb{Z}$ be such that $r\le s$ and $\rho\le\sigma$. Then we set

$$F_{r,s}:=\begin{pmatrix}f_r\\f_{r+1}\\\vdots\\f_s\end{pmatrix}\qquad\text{and}\qquad G_{\rho,\sigma}:=\begin{pmatrix}g_\rho\\g_{\rho+1}\\\vdots\\g_\sigma\end{pmatrix}.$$

Let $l\in\mathbb{Z}$. If $f_l\in H^p$ and $g_l\in H^q$) are given, then, in view of Remark 4, the Gramian orthogonal projections

$$f_{l\bullet r,s/\rho,\sigma}:=(f_l|\mathcal{L}^p_{r,s/\rho,\sigma})$$

and

$$g_{l\bullet r,s/\rho,\sigma} := \left(g_l | \mathcal{L}^q_{r,s/\rho,\sigma}\right),$$

admit the representations

$$f_{l\bullet r,s/\rho,\sigma} = \left(f_l, \begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix}\right) \cdot \left(\begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix}, \begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix}\right)^+ \cdot \begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix} \tag{17}$$

and

$$g_{l\bullet r,s/\rho,\sigma} = \left(g_l, \begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix}\right) \cdot \left(\begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix}, \begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix}\right)^+ \cdot \begin{pmatrix} F_{r,s} \\ G_{\rho,\sigma} \end{pmatrix}, \tag{18}$$

respectively. Furthermore, we will work with the vectors

$$\widetilde{f}_{l\bullet r,s/\rho,\sigma} := \sqrt{(f_l - f_{l\bullet r,s/\rho,\sigma}, f_l - f_{l\bullet r,s/\rho,\sigma})}^{\,+} (f_l - f_{l\bullet r,s/\rho,\sigma})$$

and

$$\widetilde{g}_{l\bullet r,s/\rho,\sigma} := \sqrt{(g_l - g_{l\bullet r,s/\rho,\sigma}, g_l - g_{l\bullet r,s/\rho,\sigma})}^{\,+} (g_l - g_{l\bullet r,s/\rho,\sigma}).$$

Now we will turn our attention to stationary sequences in H^q. We again assume that $r, \rho \in \mathbb{Z} \cup \{-\infty\}$ and $s, \sigma \in \mathbb{Z} \cup \{+\infty\}$, where $r \leq s$ and $\rho \leq \sigma$. A sequence $(g_j)_{j \in \mathbb{Z}_{r,s}}$ in H^q is called *stationary* if $(g_{m+l}, g_{n+l}) = (g_m, g_n)$ for all $m, n \in \mathbb{Z}_{r,s}$ and $l \in \mathbb{Z}$ with $r \leq m + l \leq s$ and $r \leq n + l \leq s$. The sequences $(f_j)_{j \in \mathbb{Z}_{\rho,\sigma}}$ and $(g_j)_{j \in \mathbb{Z}_{r,s}}$ in H^p and H^q are said to be *stationarily cross-correlated* if $(f_{m+l}, g_{n+l}) = (f_m, g_n)$ for all $m \in \mathbb{Z}_{\rho,\sigma}$, $n \in \mathbb{Z}_{r,s}$ and $l \in \mathbb{Z}$ with $\rho \leq m + l \leq \sigma$ and $r \leq n + l \leq s$.

Theorem 2. *Suppose that H is an infinite-dimensional complex Hilbert space and that k is a non-negative integer. Let $(\alpha_j)_{j=0}^{\infty}$, $(\beta_m)_{m=-k}^{\infty}$ and $(\delta_j)_{j=0}^{\infty}$ be sequences from $\mathbb{C}^{p\times p}$, $\mathbb{C}^{q\times p}$ and $\mathbb{C}^{q\times q}$, respectively. Then the following two statements are equivalent:*

(i) *The kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ given by* (6) *and* (7) *is non-negative definite.*

(ii) *There are stationarily cross-correlated stationary sequences $(f_j)_{j=0}^{\infty}, (g_j)_{j=-\infty}^{k}$ in H^p and H^q, respectively, such that the identities*

$$(f_j, f_0) = \alpha_j \qquad \text{and} \qquad (g_0, g_{-j}) = \delta_j \tag{19}$$

and

$$(f_j, g_k) = \beta_{j-k} \tag{20}$$

are satisfied for all $j \in \mathbb{N}_0$.

If (ii) *holds true, then*

$$\left(\begin{pmatrix} f_m \\ g_{k-m} \end{pmatrix}, \begin{pmatrix} f_n \\ g_{k-n} \end{pmatrix}\right) = \mathcal{K}_k(m,n) \tag{21}$$

for all non-negative integers m and n, and

$$\left(\begin{pmatrix} f_j \\ g_j \end{pmatrix}, \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}\right) = \begin{pmatrix} \alpha_j & \beta_j \\ \beta^*_{-j} & \delta_j \end{pmatrix} \tag{22}$$

for all integers j satisfying $0 \leq j \leq k$.

Proof. (i) $\Rightarrow$ (ii): By virtue of a famous result due to Kolmogorov (see [53: Lemma 2]), there exists a sequence $(h_m)_{m=0}^{\infty}$ in H^{p+q} such that $(h_m, h_n) = \mathcal{K}_k(m,n)$ for all $m, n \in \mathbb{N}_0$. For $m \in \mathbb{N}_0$, let $f_m := (I_p, 0_{p\times q})h_m$ and $g_{k-m} := (0_{q\times p}, I_q)h_m$. Then

$$(f_{m+l}, f_{n+l}) = \alpha_{m-n}, \quad (g_{k-m-l}, g_{k-n-l}) = \delta_{n-m}, \quad (f_m, g_{k-n}) = \beta_{m+n-k}$$

for all non-negative integers m, n and l. Consequently, $(f_m)_{m=0}^{\infty}$ and $(g_m)_{m=-\infty}^{k}$ are stationary sequences which are stationarily cross-correlated. Moreover, we see that (19) and (20) are fulfilled for all $j \in \mathbb{N}_0$.

(ii) $\Rightarrow$ (i): Let $(h_m)_{m=0}^{\infty}$ be defined by $h_m = \binom{f_m}{g_{k-m}}$. Then, for all $m, n \in \mathbb{N}_0$ with $m \geq n$, in view of (6)

$$\begin{aligned}(h_m, h_n) &= \begin{pmatrix} (f_m, f_n) & (f_m, g_{k-n}) \\ (g_{k-m}, f_n) & (g_{k-m}, g_{k-n}) \end{pmatrix} \\ &= \begin{pmatrix} (f_{m-n}, f_0) & (f_{m+n}, g_k) \\ (f_{m+n}, g_k)^* & (g_0, g_{n-m})^* \end{pmatrix} = \mathcal{K}_k(m,n).\end{aligned}$$

In particular, for all $j \in \mathbb{N}_0$

$$\left(\mathcal{K}_k(m,n)\right)_{m,n=0}^{j} = \left((h_m, h_n)\right)_{m,n=0}^{j} \geq 0_{(p+q)\times(p+q)}.$$

This implies (i). The rest of the assertion follows by straightforward calculation ∎

Lemma 4. *Let $(\alpha_j)_{j=0}^{\infty}$, $(\beta_j)_{j=-\infty}^{\infty}$ and $(\delta_j)_{j=0}^{\infty}$ be sequences from $\mathbb{C}^{p\times p}$, $\mathbb{C}^{p\times q}$ and $\mathbb{C}^{q\times q}$, respectively, and let the sequence $(C_j)_{j=0}^{\infty}$ be given by*

$$C_j = \begin{pmatrix} \alpha_j & \beta_j \\ \beta_{-j}^* & \delta_j \end{pmatrix}. \tag{23}$$

Then the following statements are equivalent:

(i) *For all $k \in \mathbb{N}_0$, the kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ given by (6) and (7) is non-negative definite.*

(ii) *The sequence $(C_j)_{j=0}^{\infty}$ is non-negative definite.*

Proof. Assume that H is an infinite-dimensional complex Hilbert space.

(i) $\Rightarrow$ (ii): Let $k \in \mathbb{N}_0$. From Theorem 2 we see that there are stationarily cross-correlated stationary sequences $(f_j)_{j=0}^{\infty}$ and $(g_j)_{j=-\infty}^{k}$ in H^p and H^q, respectively, such that (22) holds for all $j \in \mathbb{Z}_{0,k}$. Hence,

$$\begin{pmatrix} C_0 & C_1^* & \cdots & C_k^* \\ C_1 & C_0 & \cdots & C_{k-1}^* \\ \vdots & \vdots & & \vdots \\ C_k & C_{k-1} & \cdots & C_0 \end{pmatrix} = \left(\left(\binom{f_r}{g_r}, \binom{f_s}{g_s}\right)\right)_{r,s=0}^{k}.$$

As a Gramian, the right-hand side of (24) is non-negative Hermitian. Since k is an arbitrarily chosen non-negative integer, statement (ii) follows.

(ii) $\Rightarrow$ (i): By virtue of Kolmogorov's result [53: Lemma 2] there is a stationary sequence $(h_j)_{j=0}^{\infty}$ in H^{p+q} such that $(h_j, h_0) = C_j$ for all $j \in \mathbb{N}_0$. Now let $k \in \mathbb{N}_0$. To verify (i) we consider an arbitrary non-negative integer r, and we will show that the matrix $(\mathcal{K}_k(m,n))_{m,n=0}^{r}$ is non-negative Hermitian. Setting

$$f_j := (I_p, 0_{p\times q})h_{r+j} \qquad \text{and} \qquad g_{k-j} := (0_{q\times p}, I_q)h_{r+k-j}$$

for all $j \in \mathbb{Z}_{0,r}$, we have

$$\begin{aligned}(f_m, f_n) &= (I_p, 0_{p\times q})(h_{r+m}, h_{r+n})(I_p, 0_{p\times q})^* = (I_p, 0_{p\times q})C_{m-n}(I_p, 0_{p\times q})^*,\\ (g_{k-m}, g_{k-n}) &= (0_{q\times p}, I_q)(h_{r+k-m}, h_{r+k-n})(0_{q\times p}, I_q)^* = \delta_{n-m},\\ (f_m, g_{k-n}) &= (I_p, 0_{p\times q})(h_{r+m}, h_{r+k-n})(0_{q\times p}, I_q)^* = \beta_{m+n-k}\end{aligned}$$

for every choice of m and n in $\mathbb{Z}_{0,r}$. Therefore

$$(\mathcal{K}_k(m,n))_{m,n=0}^{r} = \left(\left(\begin{pmatrix} f_m \\ g_{k-m}\end{pmatrix}, \begin{pmatrix} f_n \\ g_{k-n}\end{pmatrix}\right)\right)_{m,n=0}^{r} \geq 0_{(p+q)\times(p+q)}.$$

Consequently, statement (i) holds true ∎

For our further considerations, we need a modification of the already mentioned result due to Kolmogorov [53: Lemma 2].

Theorem 3. *Let $(C_j)_{j=-\infty}^{\infty}$ be a sequence of complex $(q\times q)$-matrices. Further, let H be an infinite-dimensional complex Hilbert space. Then the following statements are equivalent:*

(i) *There exists a sequence $(h_n)_{n=-\infty}^{\infty}$ from H^q such that*

$$(h_\rho, h_\sigma) = C_{\rho-\sigma} \tag{25}$$

for every choice of ρ and σ in $\mathbb{Z}$.

(ii) *The sequence $(C_n)_{n=-\infty}^{\infty}$ is non-negative definite, i.e., for each $r \in \mathbb{N}$ the block Toeplitz matrix $B_r = (C_{m-n})_{m,n=0}^{r}$ is non-negative Hermitian.*

For a proof of Theorem 3, we refer the reader to [6: Theorem 7].

Corollary 1. *Suppose that H is an infinite-dimensional complex Hilbert space. Let $(\alpha_j)_{j=0}^{\infty}$, $(\beta_j)_{j=-\infty}^{\infty}$ and $(\delta_j)_{j=0}^{\infty}$ be sequences from $\mathbb{C}^{p\times p}$, $\mathbb{C}^{p\times q}$ and $\mathbb{C}^{q\times q}$, respectively, and let $(\alpha_{-j})_{j=1}^{\infty}$ and $(\delta_{-j})_{j=1}^{\infty}$ be given by (6). Then the following two statements are equivalent:*

(i) *The sequence $(C_j)_{j=0}^{\infty}$ given by (23) is non-negative definite.*

(ii) *There are stationarily cross-correlated stationary sequences $(f_j)_{j=-\infty}^{\infty}$, $(g_j)_{j=-\infty}^{\infty}$ in H^p and H^q, respectively, such that (19) and (20) are satisfied for every choice of $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$.*

If (ii) *holds, then* (21) *is satisfied for all non-negative integers* k, m *and* n, *and* (22) *is valid for all integers* j.

Proof. If (ii) holds, then we see from (22) that $(h_n)_{n=-\infty}^{\infty}$ given by

$$h_n := \begin{pmatrix} f_n \\ g_n \end{pmatrix} \qquad (n \in \mathbb{Z}) \tag{26}$$

is a stationary sequence in H^{p+q} which satisfies (25) for all integers ρ and σ. Conversely, if a stationary sequence $(h_n)_{n=-\infty}^{\infty}$ in H^{p+q} is given which fulfills (25) for all integers ρ and σ, then $(f_n)_{n=-\infty}^{\infty}$ and $(g_n)_{n=-\infty}^{\infty}$ defined by

$$f_n := (I_p, 0_{p\times q})h_n \qquad \text{and} \qquad g_n := (0_{q\times p}, I_q)h_n \tag{27}$$

are stationarily cross-correlated stationary sequences in H^p and H^q, respectively, which satisfy (19) and (20) for all integers j and k. Obviously, $\alpha_{-j} = (f_0, f_j) = \alpha_j^*$ and $\delta_{-j} = (g_{-j}, g_0) = \delta_j^*$, and hence $C_{-j} = C_j^*$ for all $j \in \mathbb{N}_0$. Using Theorem 3, we then obtain the asserted equivalence. Thus, the rest of the assertion follows from Lemma 4 and Theorem 2 ∎

3. Prediction-theoretical interpretation of the parameters which describe the solution set of problem (NTPCF)

In Section 1, we stated that the solution set of an arbitrary Nehari-type problem for matrix-valued Carathéodory functions can be described by a sequence of matrix balls. This section is aimed to give a stochastic interpretation of the parameters of these matrix balls.

Lemma 5. *Let* $(\alpha_j)_{j=0}^{\infty}$ *and* $(\delta_j)_{j=0}^{\infty}$ *be sequences of complex* $(p \times p)$- *and* $(q \times q)$-*matrices, respectively. Let* $k \in \mathbb{N}_0$, *and let* $(\beta_m)_{m=-k}^{\infty}$ *be a sequence of complex* $(p \times q)$-*matrices. Suppose that the kernel* $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ *given by* (6) *and* (7) *is non-negative definite. Let* $(f_j)_{j=0}^{\infty}$ *and* $(g_j)_{j=-\infty}^{k}$ *be stationarily cross-correlated stationary sequences in* H^p *and* H^q, *respectively, such that* (19) *and* (20) *are satisfied for all* $j \in \mathbb{N}_0$. *Then:*

(a) *For all integers* m *and all non-negative integers* n *and* j *which satisfy* $m+n \geq 0$, $m \geq -2$ *and* $m + j \leq k$, *the matrix* $S_{m,n}$ *given in* (2) *admits the representation*

$$S_{m,n} = \left(\begin{pmatrix} F_{j,j+m+n} \\ G_{j-n,j+m} \end{pmatrix}, \begin{pmatrix} F_{j,j+m+n} \\ G_{j-n,j+m} \end{pmatrix} \right). \tag{28}$$

(b) *For all non-negative integers* m, n *and* j *which satisfy* $m + n \geq 1$ *and* $m + j \leq k + 1$, *we have*

$$d_{m,n} = \left(g_{k-j}, \begin{pmatrix} F_{k-m+1-j,k+n-j} \\ G_{k-m-n-j,k-1-j} \end{pmatrix} \right), \tag{29}$$

$$e_{m,n} = \left(f_j, \begin{pmatrix} F_{j+1,j+m+n} \\ G_{j-n,j+m-1} \end{pmatrix} \right), \tag{30}$$

$$L_{m,n} = \Big(g_{k-j} - g_{k-j\bullet k-m+1-j,k+n-j/k-m-n-j,k-1-j},$$
$$g_{k-j} - g_{k-j\bullet k-m+1-j,k+n-j/k-m-n-j,k-1-j}\Big), \tag{31}$$
$$R_{m,n} = \Big(f_j - f_{j\bullet j+1,j+m+n/j-n,j+m-1},\ f_j - f_{j\bullet j+1,j+m+n/j-n,j+m-1}\Big) \tag{32}$$

and, if the inequality $m + j \leq k$ additionally holds,

$$\begin{aligned} M_{m,n} &= \Big(g_{k-j\bullet k-m+1-j,k+n-j/k-m-n-j,k-1-j},\ f_{k-m-j}\Big) \\ &= \Big(g_{m+j},\ f_j - f_{j\bullet j+1,j+m+n/j-n,j+m-1}\Big) \\ &= \Big(g_{k-j\bullet k-m+1-j,k+n-j/k-m-n-j,k-1-j}, \\ &\qquad f_{k-m-j\bullet k-m+1-j,k+n-j/k-m-n-j,k-1-j}\Big) \end{aligned} \tag{33}$$

whereas $L_{00} = (g_k, g_k)$ and $R_{00} = (f_0, f_0)$.

Proof. The identities (28) - (30) follow by straightforward calculation. Let m, n and j be non-negative integers with $m + n \geq 1$ and $m + j \leq k + 1$. Then $\delta_0 = (g_{k-j}, g_{k-j})$, $\alpha_0 = (f_j, f_j)$ and

$$\mathcal{S}_{m-2,n+1} = \left(\begin{pmatrix} F_{k-m+1-j,k+n-j} \\ G_{k-m-n-j,k-1-j} \end{pmatrix}, \begin{pmatrix} F_{k-m+1-j,k+n-j} \\ G_{k-m-n-j,k-1-j} \end{pmatrix}\right),$$
$$\mathcal{S}_{m-2,n+1} = \left(\begin{pmatrix} F_{j+1,j+m+n} \\ G_{j-n,j+m-1} \end{pmatrix}, \begin{pmatrix} F_{j+1,j+m+n} \\ G_{j-n,j+m-1} \end{pmatrix}\right).$$

Applying Remark 4 we then get (31) and (32). If $m + j \leq k$, the equations stated in (33) can be verified analogously from

$$d_{m,n} = \left(g_{m+j}, \begin{pmatrix} F_{j+1,j+m+n} \\ G_{j-n,j+m-1} \end{pmatrix}\right)$$

and

$$e^*_{m,n} = \left(\begin{pmatrix} F_{k-m+1-j,k+n-j} \\ G_{k-m-n-j,k-1-j} \end{pmatrix}, f_{k-m-j}\right).$$

Thus the assertion is proved ∎

Proposition 1. *Let $(\alpha_j)_{j=0}^{\infty}$ and $(\delta_j)_{j=0}^{\infty}$ be sequences of complex $(p\times p)$- and $(q\times q)$-matrices. Let $k \in \mathbb{N}_0$, and let $(\beta_m)_{m=-k}^{\infty}$ be a sequence of complex $(p \times q)$-matrices. Suppose that the kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ defined by (6) and (7) is non-negative definite. Let $(f_j)_{j=0}^{\infty}$ and $(g_j)_{j=-\infty}^{k}$ be stationarily cross-correlated stationary sequences in H^p and H^q, respectively, such that (19) and (20) are satisfied for all $j \in \mathbb{N}_0$. Let $m \in \mathbb{Z}_{0,k}$, and let $n \in \mathbb{N}_0$. Then there is a unique complex $(q \times p)$-matrix $K_{m,n}$ such that*

$$(g_m, f_0) = M_{m,n} + \sqrt{L_{m,n}}K_{m,n}\sqrt{R_{m,n}}$$

and

$$L_{m,n}L_{m,n}{}^{+}K_{m,n} = K_{m,n} \qquad \text{and} \qquad K_{m,n}R_{m,n}{}^{+}R_{m,n} = K_{m,n}.$$

This matrix $K_{m,n}$ is contractive and admits the representations

$$K_{m,n} = \sqrt{L_{m,n}}^{+}(\beta^{*}_{-m} - M_{m,n})\sqrt{R_{m,n}}^{+}$$

and

$$K_{m,n} = \begin{cases} \left(\sqrt{(g_0, g_0)}^{+} g_0, \sqrt{(f_0, f_0)}^{+} f_0\right) & \text{if } m = n = 0 \\ \left(\widetilde{g}_{k \bullet k-m+1, k+n/k-m-n, k-1}, \widetilde{f}_{k-m \bullet k-m+1, k+n/k-m-n, k-1}\right) & \text{if } m+n \geq 1. \end{cases}$$

In particular, the matrix (g_m, f_0) belongs to the matrix ball $\mathfrak{K}(M_{m,n}; \sqrt{L_{m,n}}, \sqrt{R_{m,n}})$.

Proof. The case $m = n = 0$ is trivial. Assume that $m + n \geq 1$. If we set $U := \mathcal{L}_{k-m+1, k+m/k-m-n, k-1}$, then the application of Lemmas 2 and 5 yields the assertion ∎

Observe that Proposition 1 can be also obtained from Theorem 2 in [43].

In Theorem 1 we already stated that the matrix sequences $(M_{m,n})_{n=0}^{\infty}$, $(L_{m,n})_{n=0}^{\infty}$ and $(R_{m,n})_{n=0}^{\infty}$ converge. The proof of the existence of these limits given in [40] uses Smuljan's [58] results on sequences of nested matrix balls. Applying Lemmas 3 and 5, one can not only get an alternative proof, but also further statements on the existence of certain other limits which are of interest.

If complex $(p \times q)$-matrices $A_{m,n}$ are given for all non-negative integers m and n, then we will use the notation

$$\lim_{m,n \to \infty} A_{m,n}$$

if there is a matrix $A \in \mathbb{C}^{p \times q}$ such that for all positive real numbers ε there is a non-negative integer τ such that $|A_{m,n} - A|_E < \varepsilon$ for all $m, n \in \mathbb{Z}_{\tau,\infty}$ where $|\cdot|_E$ denotes the Euclidean matrix norm. In this case, we will write $\lim_{m,n\to\infty} A_{m,n}$ for this unique limit A.

Theorem 4. *Let $(\alpha_j)_{j=0}^{\infty}$ and $(\delta_j)_{j=0}^{\infty}$ be sequences of complex $(p \times p)$- and $(q \times q)$-matrices. Let $k \in \mathbb{N}_0$, and let $(\beta_m)_{m=-k}^{\infty}$ be a sequence of complex $(p \times q)$-matrices. Suppose that the kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ defined by (6) and (7) is non-negative definite. Then:*

(a) *For all $m \in \mathbb{Z}_{0,k+1}$, both sequences $(L_{m,n})_{n=0}^{\infty}$ and $(R_{m,n})_{n=0}^{\infty}$ are monotonously non-increasing (with respect to the Löwner semi-ordering of Hermitian matrices).*

(b) *For each $n \in \mathbb{N}_0$, both sequences $(L_{m,n})_{m=0}^{k+1}$ and $(R_{m,n})_{m=0}^{k+1}$ are monotonously non-increasing.*

(c) *For each $m \in \mathbb{Z}_{0,k+1}$, there exist the limits*

$$L_{m,*} := \lim_{n\to\infty} L_{m,n} \qquad \text{and} \qquad R_{m,*} := \lim_{n\to\infty} R_{m,n} \tag{34}$$

which satisfy

$$0 \leq L_{m,*} \leq L_{m,n} \qquad \text{and} \qquad 0 \leq R_{m,*} \leq R_{m,n}$$

for all $n \in \mathbb{N}_0$. *Moreover, both sequences* $(L_{m,*})_{m=0}^{k+1}$ *and* $(R_{m,*})_{m=0}^{k+1}$ *are monotonously non-increasing.*

(d) *For each* $m \in \mathbb{Z}_{0,k}$, *there exists the limit*

$$M_{m,*} := \lim_{n\to\infty} M_{m,n}. \tag{35}$$

(e) *Suppose that* $(f_j)_{j=0}^{\infty}$ *and* $(g_j)_{j=-\infty}^{k}$ *be stationarily cross-correlated stationary sequences in* H^p *and* H^q, *respectively, such that* (19) *and* (20) *are satisfied for all* $j \in \mathbb{N}_0$. *Then, for all non-negative integers* m *and* j *which satisfy* $m + j \le k+1$,

$$\begin{aligned} L_{m,*} = \Big(&g_{k-j} - g_{k-j\bullet k-m+1-j,\infty/-\infty,k-1-j},\\ &g_{k-j} - g_{k-j\bullet k-m+1-j,\infty/-\infty,k-1-j}\Big), \end{aligned} \tag{36}$$

$$R_{m,*} = \Big(f_j - f_{j\bullet j+1,\infty/-\infty,j+m-1},\ f_j - f_{j\bullet j+1,\infty/-\infty,j+m-1}\Big) \tag{37}$$

and, if $m + j \le k$,

$$\begin{aligned} M_{m,*} &= \Big(g_{k-j\bullet k-m+1-j,\infty/-\infty,k-1-j},\ f_{k-m-j}\Big)\\ &= \Big(g_{m+j}\ ,\ f_{j\bullet j+1,\infty/-\infty,j+m-1}\Big)\\ &= \Big(g_{k-j\bullet k-m+1-j,\infty/-\infty,k-1-j},\ f_{k-m-j\bullet k-m+1-j,\infty/-\infty,k-1-j}\Big). \end{aligned}$$

Proof. By virtue of Theorem 2, there are stationarily cross-correlated stationary sequences $(f_j)_{j=0}^{\infty}$ and $(g_j)_{j=-\infty}^{k}$ in H^p and H^q, respectively, such that (19) and (20) are satisfied for all $j \in \mathbb{N}_0$. Then Lemma 5 yields the representations (31) and (32) for all non-negative integers m, n and j with $m + n \ge 1$ and $m + j \le k+1$. Since

$$\mathcal{L}_{\rho,\sigma/\mu,\nu} \subseteq \mathcal{L}_{\rho,\sigma+r/\mu,\nu+s} \tag{39}$$

is satisfied for all $\rho, \sigma, s, r \in \mathbb{N}_0$ and all $\mu, \nu \in \mathbb{Z}$ with $\rho \le \sigma$ and $\mu \le \nu \le k - s$, we obtain from Remark 3 the assertion stated in parts (a) and (b). Lemma 3 shows that the limits stated in (34) exist and that the representations (36) and (37) are valid for all $m, j \in \mathbb{N}_0$ with $m + j \le k+1$. Similarly, we get from (33) that the limit (35) exists for all $m \in \mathbb{Z}_{0,k}$ and that the representations stated in (38) are fulfilled for all $m, j \in \mathbb{N}_0$ with $m + j \le k$. Remark 3 yields the rest of the assertion ∎

Theorem 5. *Let* $(\alpha_j)_{j=0}^{\infty}$, $(\delta_j)_{j=0}^{\infty}$ *and* $(\beta_j)_{j=-\infty}^{\infty}$ *be sequences from* $\mathbb{C}^{p\times p}$, $\mathbb{C}^{q\times q}$ *and* $\mathbb{C}^{p\times q}$, *respectively. Suppose that the sequence* $(C_j)_{j=0}^{\infty}$ *given by* (23) *is non-negative definite. Then:*

(a) *For each* $m \in \mathbb{Z}_{0,k+1}$, *there exist the limits*

$$L_{*,n} := \lim_{m\to\infty} L_{m,n} \qquad \text{and} \qquad R_{*,n} := \lim_{m\to\infty} R_{m,n} \tag{40}$$

which satisfy

$$0 \le L_{*,n} \le L_{m,n} \qquad \text{and} \qquad 0 \le R_{*,n} \le R_{m,n}$$

for all $n \in \mathbb{N}_0$. Moreover, both sequences $(L_{*,n})_{n=0}^{\infty}$ and $(R_{*,n})_{n=0}^{\infty}$ are monotonously non-increasing (with respect to the Löwner semi-ordering of Hermitian matrices).

(b) *There exist the limits*

$$L := \lim_{n\to\infty} L_{*,n} \quad \text{and} \quad R := \lim_{n\to\infty} R_{*,n} \tag{41}$$

and, furthermore, they admit the representations

$$L = \lim_{m\to\infty} L_{m,*} \quad \text{and} \quad R = \lim_{m\to\infty} R_{m,*} \tag{41}$$

and

$$L = \lim_{m,n\to\infty} L_{m,n} \quad \text{and} \quad R = \lim_{m,n\to\infty} R_{m,n}. \tag{43}$$

(c) *If $(f_j)_{j=-\infty}^{\infty}$ and $(g_j)_{j=-\infty}^{\infty}$ are stationarily cross-correlated stationary sequences in H^p and H^q, respectively, which satisfy* (19) *and* (20) *for all non-negative integers j and k, then*

$$L = \left(g_k - g_{k\bullet-\infty,\infty/-\infty,k-1},\, g_k - g_{k\bullet-\infty,\infty/-\infty,k-1}\right) \tag{44}$$

and

$$R = \left(f_k - f_{k\bullet k+1,\infty/-\infty,\infty},\, f_k - f_{k\bullet k+1,\infty/-\infty,\infty}\right) \tag{45}$$

for all $k \in \mathbb{Z}$. Morover,

$$L_{*,n} = \left(g_k - g_{k\bullet-\infty,k+n/-\infty,k-1},\, g_k - g_{k\bullet-\infty,k+n/-\infty,k-1}\right) \tag{46}$$

and

$$R_{*,n} = \left(f_k - f_{k\bullet k+1,\infty/k-n,\infty},\, f_k - f_{k\bullet k+1,\infty/k-n,\infty}\right) \tag{47}$$

for all non-negative integers n and all integers k.

Proof. We see from Lemma 4 that, for all $k \in \mathbb{N}_0$, the kernel $K_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ given by (6) and (7) is non-negative definite. Let H be an infinite-dimensional complex Hilbert space. According to Corollary 1, let $(f_j)_{j=-\infty}^{\infty}$ and $(g_j)_{j=-\infty}^{\infty}$ be stationarily cross-correlated stationary sequences in H^p and H^q, respectively, which satisfy (19) and (20) for all $j \in \mathbb{Z}$ and all $k \in \mathbb{Z}$. Then we see from Lemma 5 that

$$L_{m,n} = \left(g_m - g_{m\bullet 1,m+n/-n,m-1},\, g_m - g_{m\bullet 1,m+n/-n,m-1}\right) \tag{48}$$

and

$$R_{m,n} = \left(f_0 - f_{0\bullet 1,m+n/-n,m-1},\, f_0 - f_{0\bullet 1,m+n/-n,m-1}\right) \tag{49}$$

hold for all non-negative integers m and n with $m+n \geq 1$. By virtue of the stationarity properties of $(f_j)_{j=-\infty}^{\infty}$ and $(g_j)_{j=-\infty}^{\infty}$, we get from Remark 4 that

$$L_{m,n} = \left(g_k - g_{k\bullet k-m+1,k+n/k-m-n,k-1},\, g_k - g_{k\bullet k-m+1,k+n/k-m-n,k-1}\right)$$

and

$$R_{m,n} = \left(f_k - f_{k\bullet k+1,k+m+n/k-n,k+m-1},\, f_k - f_{k\bullet k+1,k+m+n/k-n,k+m-1}\right)$$

are satisfied for all $k \in \mathbb{Z}$ and all $m, n \in \mathbb{N}_0$ with $m+n \geq 1$. From Lemma 3 we obtain then the existence of the limits stated in (40) and the representations (46) and (47), which hold true for all $n \in \mathbb{N}_0$ and all $k \in \mathbb{Z}$. Then (39) and Lemma 3 provide that the limits stated in (41) exist and admit the representations (44) and (45) for all $k \in \mathbb{Z}$. In view of (39), one can analogously verify that (42) and (43) are satisfied ∎

Observe that, in general, the sequences $(M_{m,n})_{n=0}^{\infty}$ and $(K_{m,n})_{m=0}^{\infty}$ $(n \in \mathbb{N}_0)$, and $(K_{m,n})_{n=0}^{\infty}$ $(m \in \mathbb{N}_0)$ do not converge.

Theorem 6. *Let $(\alpha_j)_{j=0}^{\infty}$ and $(\delta_j)_{j=0}^{\infty}$ be sequences of complex $(p \times p)$- and $(q \times q)$-matrices, respectively. Let $k \in \mathbb{N}_0$, and let $(\beta_m)_{m=-k}^{\infty}$ be a sequence of complex $(p \times q)$-matrices. Suppose that the kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ defined by (6) and (7) is non-negative definite. For $m \in \mathbb{Z}_{0,k+1}$, let the matrices $L_{m,*}$ and $R_{m,*}$ be defined by (34), whereas the matrices $M_{m,*}$ $(m \in \mathbb{Z}_{0,k})$ are given by (35). Then:*

(a) *For all $m \in \mathbb{Z}_{0,k}$, there is a unique complex $(q \times p)$-matrix K_m such that*

$$\beta_{-m}^* = M_{m,*} + \sqrt{L_{m,*}} K_m \sqrt{R_{m,*}} \tag{50}$$

and

$$L_{m,*} L_{m,*}{}^+ K_m = K_m \qquad \text{and} \qquad K_m R_{m,*}{}^+ R_{m,*} = K_m,$$

namely

$$K_m = \sqrt{L_{m,*}}^+ (\beta_{-m} - M_{m,*}) \sqrt{R_{m,*}}^+.$$

This matrix K_m is contractive.

(b) *For all $m \in \mathbb{Z}_{0,k}$, the identities*

$$L_{m+1,*} = \sqrt{L_{m,*}} (I - K_m K_m^*) \sqrt{L_{m,*}}, \tag{52}$$

$$L_{m+1,*} = L_{m,*} - (\beta_{-m}^* - M_{m,*}) R_{m,*}{}^+ (\beta_{-m}^* - M_{m,*})^*, \tag{53}$$

$$R_{m+1,*} = \sqrt{R_{m,*}} (I - K_m^* K_m) \sqrt{R_{m,*}} \tag{54}$$

and

$$R_{m+1,*} = R_{m,*} - (\beta_{-m}^* - M_{m,*})^* L_{m,*}{}^+ (\beta_{-m}^* - M_{m,*}) \tag{54}$$

hold true.

(c) *Let $m \in \mathbb{Z}_{0,k}$. Then the following statements are equivalent:*

(i) $L_{m+1,*} = L_{m,*}$.

(ii) $R_{m+1,*} = R_{m,*}$.

(iii) $\beta_{-m}^* = M_{m,*}$.

(d) *If $(f_j)_{j=0}^{\infty}$ and $(g_j)_{j=-\infty}^{k}$ are stationarily cross-correlated stationary sequences in H^p and H^q, respectively, such that (19) and (20) hold true for all $j \in \mathbb{N}_0$, then, for all $m \in \mathbb{Z}_{0,k}$, the matrix K_m admits the representation*

$$K_m = \left(\widetilde{g}_{m\bullet 1,\infty/-\infty,m-1},\ \widetilde{f}_{0\bullet 1,\infty/-\infty,m-1} \right). \tag{56}$$

Proof. Let H be an infinite-dimensional complex Hilbert space. According to Theorem 2, let $(f_j)_{j=0}^{\infty}$ and $(g_j)_{j=-\infty}^{k}$ be stationarily cross-correlated stationary sequences in H^p and H^q, respectively, which satisfy (19) and (20) for all $j \in \mathbb{N}_0$. Let $m \in \mathbb{Z}_{0,k}$. Then we see from Lemma 2 and Theorem 4/(e) that the matrix given in (56) is the unique complex $(q \times p)$-matrix K_m which satisfies (50) and (51). Furthermore, we get that K_m is contractive. In order to prove the identity (52) we observe that

$$g_{m\bullet 0,\infty/-\infty,m-1} = g_{m\bullet 1,\infty/-\infty,m-1} + \widehat{g}_m$$

where

$$\widehat{g}_m := \left(g_m \middle| (\mathcal{L}_{0,\infty/-\infty,m-1} \ominus \mathcal{L}_{1,\infty/-\infty,m-1})^q\right)$$

fulfills $(g_{m\bullet 1,\infty/-\infty,m-1}, \widehat{g}_m) = 0$. Thus, we obtain from Theorem 4

$$\begin{aligned}
L_{m+1,*} &= \left(g_m - g_{m\bullet 0,\infty/-\infty,m-1},\ g_m - g_{m\bullet 0,\infty/-\infty,m-1}\right) \\
&= \left(g_m, g_m) - (g_{m\bullet 0,\infty/-\infty,m-1},\ g_{m\bullet 0,\infty/-\infty,m-1}\right) \\
&= \left(g_m, g_m) - (g_{m\bullet 1,\infty/-\infty,m-1},\ g_{m\bullet 1,\infty/-\infty,m-1}) - (\widehat{g}_m, \widehat{g}_m\right) \\
&= L_{m,*} - (\widehat{g}_m, \widehat{g}_m).
\end{aligned} \tag{57}$$

In view of Remark 5 we have

$$\begin{aligned}
&\mathcal{L}_{0,\infty/-\infty,m-1} \ominus \mathcal{L}_{1,\infty/-\infty,m-1} \\
&\quad = \mathrm{sp}\left(f_0^{(1)} - f_{0\bullet 1,\infty/-\infty,m-1}^{(1)}, f_0^{(2)} - f_{0\bullet 1,\infty/-\infty,m-1}^{(2)}, \dots, f_0^{(p)} - f_{0\bullet 1,\infty/-\infty,m-1}^{(p)}\right).
\end{aligned}$$

Hence, Remarks 4 yields

$$\begin{aligned}
\widehat{g}_m = &\left(g_m, f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right) \\
&\times \left(f_0 - f_{0\bullet 1,\infty/-\infty,m-1}, f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right)^+ \\
&\times \left(f_0 - f_{0\bullet 1,\infty/-\infty,m-1}, g_m\right).
\end{aligned}$$

This representation of $\widehat{g}_m$ implies

$$(\widehat{g}_m, \widehat{g}_m) = \left(g_m, f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right) R_{m,*}{}^+ \left(f_0 - f_{0\bullet 1,\infty/-\infty,m-1}, g_m\right). \tag{58}$$

Obviously, Theorem 4/(e) yields

$$\left(g_m, f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right) = (g_m, f_0) - M_{m,*}. \tag{59}$$

The identities (58) and (59) immediately imply (53). From

$$\left(g_{m\bullet 1,\infty/-\infty,m-1}\ ,\ f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right) = 0,$$

Remark 5 and Theorem 4/(e) we see

$$\begin{aligned}
\left(g_m, f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right) &= \left(g_m - g_{m\bullet 1,\infty/-\infty,m-1},\ f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right) \\
&= \sqrt{L_{m,*}}\left(\widetilde{g}_{m\bullet 1,\infty/-\infty,m-1},\ f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right).
\end{aligned}$$

Therefore, by virtue of (58) and (56), there follows

$$
\begin{aligned}
(\widehat{g}_m, \widehat{g}_m) &= \sqrt{L_{m,*}}\left(\widetilde{g}_{m\bullet 1,\infty/-\infty,m-1},\ f_0 - f_{0\bullet 1,\infty/-\infty,m-1}\right) R_{m,*}{}^{+} \\
&\quad \times \left(f_0 - f_{0\bullet 1,\infty/-\infty,m-1},\ \widetilde{g}_{m\bullet 1,\infty/-\infty,m-1}\right)\sqrt{L_{m,*}} \\
&= \sqrt{L_{m,*}} K_m K_m^* \sqrt{L_{m,*}}.
\end{aligned}
\tag{60}
$$

Now we can conclude from (57) that (52) holds true. The identities (54) and (55) can be verified analogously.

It remains to show part (c). If (i) holds, then (52) and the first identity in (51) imply $K_m = 0_{q\times p}$. Hence, in view of (54), (ii) follows. Similarly, condition (ii) implies $K_m = 0_{q\times p}$ and, by virtue of (50), then (iii) as well. If we suppose (iii), then (50) and (51) yield $K_m = 0_{q\times p}$ and, according to (52), condition stated in (i). The proof is complete ∎

Identities of the types stated in (52) and (54) are characteristic for one-step extension problems for contractive or non-negative Hermitian schemes (see, e.g., Dym and Gohberg [34], Constantinescu [22, 23] and the authors' papers [32], [33: Sections 3.2 and 3.3] and [36, 38, 41]). Slightly modified formulas occur in the papers of Kovalishina and Potapov [54] and Dubovoj (see [31] and [33: Theorem 5.5.6]). Taking into account the representations of the semi-radii as Gramian matrices given in Theorem 4 the extensions stated in Theorem 6 admit a clear prediction-theoretical interpretation. They describe the improvement of the accurracy of prediction error matrices in forward and backward prediction. In particular, the formulas (53) and (55) express explicit interrelations between certain forward and backward prediction error matrices.

4. A characterization of the central element of $\mathcal{N}_0\left[\alpha,\beta,\delta;(\gamma_j)_{j=0}^{k}\right]$

In [45] and [46] the authors investigated a pair of functions the values of which are non-negative Hermitian $(m\times m)$-matrices that is associated with a given non-degenerate $(m\times m)$-Carathéodory function via Weyl matrix balls. In this section, for a given matricial Carathéodory function Ω with prescribed block partition, we will consider appropriate matrix balls to construct a pair of non-negative Hermitian matrices which will be used for a characterization of central elements.

Let Ω be a $((p+q)\times(p+q))$-Carathéodory function (on $\mathbb{D}$) which satisfies $[\Omega(0)]^* = \Omega(0)$. If

$$\Omega(z) = C_0 + 2\sum_{j=1}^{\infty} C_j z^j \qquad (z\in\mathbb{D}) \tag{61}$$

is the Taylor series representation of Ω, then the combination of the matricial versions of the F. Riesz-Herglotz Theorem and the Herglotz-Bochner Theorem shows that the sequence $(C_j)_{j=0}^{\infty}$ is non-negative definite. Conversely, if an arbitrary non-negative sequence $(C_j)_{j=0}^{\infty}$ of complex $((p+q)\times(p+q))$-matrices is given, then the function $\Omega : \mathbb{D} \to \mathbb{C}^{(p+q)\times(p+q)}$ defined by

$$\Omega(z) := C_0 + 2\sum_{j=1}^{\infty} C_j z^j \qquad (z \in \mathbb{D}) \tag{62}$$

is a $\big((p+q)\times(p+q)\big)$-Carathéodory function which fulfils $[\Omega(0)]^* = \Omega(0)$ (see, e.g., [33: Theorems 2.2.1 and 2.2.2]). In this sense, if $(\alpha_j)_{j=0}^{\infty}$, $(\delta_j)_{j=0}^{\infty}$ and $(\beta_j)_{j=-\infty}^{\infty}$ are sequences of complex $(p\times p)$-, $(q\times q)$- and $(p\times q)$-matrices, respectively, and if the sequence $(C_j)_{j=0}^{\infty}$ is given by (23), then Lemma 4 shows that (62) defines an Ω which belongs to $\mathcal{C}_{p+q}(\mathbb{D})$ and which satisfies $[\Omega(0)]^* = \Omega(0)$ if and only if, for every non-negative integer k, the kernel $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ given by (6) and (7) is non-negative definite. In view of Theorem 1, we thus have the following situation if a function $\Omega \in \mathcal{C}_{(p+q)\times(p+q)}(\mathbb{D})$ with $[\Omega(0)]^* = \Omega(0)$ is given:

If (61) is the Taylor series representation of Ω around the origin and if, for each $j \in \mathbb{N}_0$, the matrix C_j is partitioned into blocks via

$$C_j = \begin{pmatrix} \alpha_j & \beta_j \\ \beta_{-j} & \delta_j \end{pmatrix}$$

with $(p\times p)$-block α_j, then the matrix sequences $(L_{m,*})_{m=1}^{\infty}$, $(R_{m,*})_{m=1}^{\infty}$ and $(M_{m,*})_{m=1}^{\infty}$ are well-defined. (It is readily checked that, for each $m \in \mathbb{N}$, all the matrices $L_{m,*}$, $R_{m,*}$ and $M_{m,*}$ only depend on the matrix sequences $(\alpha_j)_{j=0}^{\infty}$, $(\delta_j)_{j=0}^{\infty}$ and $(\beta_j)_{j=-(m-1)}^{\infty}$.) We set

$$L_{m,*}^{[\Omega]} = L_{m,*}, \qquad R_{m,*}^{[\Omega]} = R_{m,*}, \qquad M_{m,*}^{[\Omega]} = M_{m,*}$$

for all $m \in \mathbb{N}$. Moreover, we see from Theorem 5 that there exist the limits

$$L^{[\Omega]} := \lim_{m\to\infty} L_{m,*}^{[\Omega]} \qquad \text{and} \qquad R^{[\Omega]} := \lim_{m\to\infty} R_{m,*}^{[\Omega]}.$$

Remark 6. Let $\alpha : \mathbb{D} \to \mathbb{C}^{p\times p}$, $\beta : \mathbb{D} \to \mathbb{C}^{p\times q}$ and $\delta : \mathbb{D} \to \mathbb{C}^{q\times q}$ be holomorphic matrix-valued functions. Let $k \in \mathbb{N}_0$, and let $(\gamma_j)_{j=0}^{k}$ be a sequence of complex $(q\times p)$-matrices with $\gamma_0 = [\beta(0)]^*$. Suppose that the set $\mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^{k}]$ is non-empty. If $\tilde{\mathcal{C}}[\alpha,\beta,\delta;(\gamma_j)_{j=0}^{k}]$ denotes the set of all $\Delta \in \mathcal{C}_{p+q}(\mathbb{D})$ which admits the block representation

$$\Delta = \begin{pmatrix} \alpha & \beta \\ \xi & \delta \end{pmatrix}$$

with some function ξ belonging to $\mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^{k}]$, then in view of (5) and Theorem 1, it is readily checked that

$$L_{m,*}^{[\Omega]} = L_{m,*}^{[\Xi]}, \qquad R_{m,*}^{[\Omega]} = R_{m,*}^{[\Xi]} \qquad \text{and} \qquad M_{m,*}^{[\Omega]} = M_{m,*}^{[\Xi]}$$

for all $m \in \mathbb{Z}_{1,k+1}$ and every choice of Ω and Ξ in $\tilde{\mathcal{C}}[\alpha,\beta,\delta;(\gamma_j)_{j=0}^{k}]$.

Theorem 7. *Let* $\alpha : \mathbb{D} \to \mathbb{C}^{p\times p}$, $\beta : \mathbb{D} \to \mathbb{C}^{p\times q}$ *and* $\delta : \mathbb{D} \to \mathbb{C}^{q\times q}$ *be holomorphic matrix-valued functions. Let* $k \in \mathbb{N}_0$, *and let* $(\gamma_j)_{j=0}^k$ *be a sequence of complex* $(q \times p)$-*matrices where* $\gamma_0 = [\beta(0)]^*$. *Suppose that the set* $\mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k]$ *is non-empty. Further, let* $\xi \in \mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k]$, *and let* Ω *be given by* (1). *Then the following statements are equivalent:*

(i) *The function* ξ *is the central element of* $\mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k]$.

(ii) $L^{[\Omega]} = L_{k+1,*}^{[\Omega]}$.

(iii) $R^{[\Omega]} = R_{k+1,*}^{[\Omega]}$.

Proof. In view of the considerations above and Remark 6, Theorem 4/(c) shows that both sequences $(L_{m,*}^{[\Omega]})_{m=0}^\infty$ and $(R_{m,*}^{[\Omega]})_{m=0}^\infty$ are monotonously non-increasing. Thus, assertion (ii) is equivalent to $L_{m+1,*}^{[\Omega]} = L_{m,*}^{[\Omega]}$ for all $m \in \mathbb{Z}_{k+1,\infty}$, whereas assertion (iii) is satisfied if and only if $R_{m+1,*}^{[\Omega]} = R_{m,*}^{[\Omega]}$ for each $m \in \mathbb{Z}_{k+1,\infty}$. Hence, Theorem 6/(c) provides all the asserted equivalences ∎

Corollary 2. *Let* $\alpha : \mathbb{D} \to \mathbb{C}^{p\times p}$, $\beta : \mathbb{D} \to \mathbb{C}^{p\times q}$ *and* $\delta : \mathbb{D} \to \mathbb{C}^{q\times q}$ *be holomorphic matrix-valued functions. Let* $k \in \mathbb{N}_0$, *and let* $(\gamma_j)_{j=0}^k$ *be a sequence of complex* $(q \times p)$-*matrices where* $\gamma_0 = [\beta(0)]^*$ *such that the kernel* $\mathcal{K}_k : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{(p+q)\times(p+q)}$ *given by* $(5-7)$ *is non-negative definite. Suppose that* $L_{k+1,*} = 0_{p\times p}$ *or* $R_{k+1,*} = 0_{q\times q}$. *Then* $\mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k]$ *contains exactly one element, namely its central element. Moreover,* $L_{k+1+j,*} = L_{k+1,*}$ *and* $R_{k+1+j,*} = R_{k+1}$ *for all positive integers* j.

Proof. In view of Theorem 1, the set $\mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k]$ is non-empty. Let $\xi \in \mathcal{N}_0[\alpha,\beta,\delta;(\gamma_j)_{j=0}^k]$ be arbitrary, and let Ω be given by (1). Then Theorem 1/(c) and Remark 6 show that $L_{k+1,*} = 0_{p\times p}$ implies $L^{[\Omega]} = 0_{p\times p}$ and that $R^{[\Omega]} = 0_{q\times q}$ is necessary for $R_{k+1,*} = 0_{q\times q}$. Thus the application of Theorems 6 and 7 completes the proof ∎

The determination of elements of extremal entropy in the image of a linear fractional transformation of matrices the generating functions of which have a particular type can be traced back to the fundamental work of Arov and Krein [20, 21]. This was the beginning of a period of intensive studies of entropy optimization in the context of interpolation problems (see, e.g., Constantinescu [22, 23], Dewilde and Dym [29, 30], Dym and Gohberg [34, 35], Gohberg, Kaashoek and Woerdeman [48 - 50], Landau [55] and the authors' papers [37, 39, 41]). Constantinescu [24] indicated a maximum entropy principle for the set of contractive intertwining dilations. This work was extended by Arocena [11] who developed a prediction theory approach that characterizes a distinguished element in the set of all unitary extensions of a given isometric operator which can be considered as the most innovative one. Finally, it should be mentioned that Cotlar and Sadosky [26] discussed some interrelations between generalized Toeplitz kernels, stationarity and harmonizability.

References

[1] Adamjan, V. M., Arov, D. Z. and M. G. Krein: *On infinite Hankel matrices and generalized Carathéodory-Fejér and F. Riesz problems* (in Russian). Funk. Anal. i. Prilo. 2 (1968)1, 1 – 19; Engl. transl. in: Funct. Anal. Appl. 2 (1968), 1 – 18.

[2] Adamjan, V. M., Arov, D. Z. and M. G. Krein: *Infinite Hankel matrices and generalized Carathéodory-Fejér and Schur problems* (in Russian). Funk. Anal. i. Prilo. 2 (1968)4, 1 – 17; Engl. transl. in: Funct. Anal. Appl. 2 (1968), 269 – 281.

[3] Adamjan, V. M., Arov, D. Z. and M. G. Krein: *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem* (in Russian). Math. USSR - Sbornik 86 (1971), 34 – 75; Engl. transl. in: Math. USSR, Sbornik 15 (1971, 31 – 73.

[4] Adamjan, V. M., Arov, D. Z. and M. G. Krein: *Infinite block Hankel matrices and related extension problems* (in Russian). Izv. Akad. Nauk Armjan. SSR, Ser. Mat. 6 (1971), 87 – 112; Engl. transl. in: Amer. Math. Soc. Transl. 111 (1978), 133 – 156.

[5] Allegria, P.: *On the Adamjan-Arov-Krein and Arocena parametrizations: A constructive and generalized version.* Acta Cient. Venezolana 39 (1988), 107 – 116.

[6] Apitzsch, W., Fritzsche, B. and B. Kirstein: *A Schur analysis approach to maximum distance problems in Hilbert space and prediction.* Optimization 21 (1990), 879 – 904.

[7] Arocena, R.: *On the parametrization of Adamjan, Arov and Krein.* Publ. Math. d'Orsay 83-02 (1983), 7 – 23.

[8] Arocena, R.: *Generalized Toeplitz kernels and dilations of intertwining operators.* Int. Equ. Oper. Theory 6 (1983), 759 – 778.

[9] Arocena, R.: *On generalized Toeplitz kernels and their relation with a paper of Adamjan, Arov and Krein.* In: Functional Analysis, Holomorphy and Approximation Theory II (ed.: G. Zapata; North-Holland Math. Studies: Vol. 86). Amsterdam: North-Holland 1984, pp. 1 –22.

[10] Arocena, R.: *Generalized Toeplitz kernels and dilations of intertwining operators.* Part II: *The continuous case.* Acta Sci. Math. (Szeged) 53 (1989), 123 – 137.

[11] Arocena, R.: *Schur analysis of a class of translation invariant forms.* In: Analysis and Partial Differential Equations — A Collection of Papers dedicated to Mischa Cotlar (ed.: C. Sadosky; Lect. Notes Pure Appl. Math.: Vol. 122). New York – Basel: Marcel Dekker 1990, pp. 355 – 369.

[12] Arocena, R. and M. Cotlar: *Continuous generalized Toeplitz kernels in* $\mathbb{R}$. Portugalia Math. 39 (1980), 419 – 439.

[13] Arocena, R. and M. Cotlar: *A generalized Herglotz-Bochner theorem and* L^2*-weighted inequalities with finite measures.* In: Harmonic Analysis Confenence in Honour of A. Zygmund (ed.: W. Beckner). Wadsworth Intern. Math. Ser. 1982, pp. 258-269.

[14] Arocena, R. and M. Cotlar: *Dilation of generalized Toeplitz kernels and some vectorial moment and weighted problems.* Lect. Notes Math. 908 (1982), 189 – 188.

[15] Arocena, R. and M. Cotlar: *Generalized Toeplitz kernels and Adamjan-Arov-Krein moment problems.* In Toeplitz Centennial (ed.: I. Gohberg; Oper. Theory: Adv. Appl.: Vol 4). Basel: Birkhäuser Verlag 1982, pp. 37 – 55.

[16] Arocena, R. and M. Cotlar: *On a lifting theorem and its relation to some approximation problems.* In: Functional Analysis, Holomorphy and Approximation Theory (ed.: J. Barroso; North-Holland Math. Studies: Vol. 71). Amsterdam: North-Holland 1982, pp. 1 – 25.

[17] Arocena, R. and M. Cotlar: *Generalized Toeplitz kernels, Hankel forms and Sarason's commutation theorem.* Acta Cient. Venezolana 33 (1982), 89 - 98.

[18] Arocena, R., Cotlar, M. and J. Leon: *Toeplitz kernels, scattering structures and covariant systems.* In: Aspects of Mathematics and Its Applications. North-Holland Math. Library 34 (1986), pp. 77 - 95.

[19] Arocena, R., Cotlar, M. and C. Sadosky: *Weighted inequalities in L^2 and lifting properties.* In: Mathematical Analysis and Applications, Part A (ed.: L. Nachbin; Adv. Math. Suppl. Studies: Vol. 7A). New York: Academic Press 1981, pp. 95 - 128.

[20] Arov, D. Z. and M. G. Krein: *Problems of the search of the minimum of entropy in indeterminate extension problems* (in Russian), Funk. Anal. i Prilo. 15 (1981)2, 61 - 64; Engl. transl. in: Funct. Anal. Appl. 15 (1981), 123 - 126.

[21] Arov, D. Z. and M. G. Krein: *On computations of entropy functionals and their minimums in indeterminate extension problems* (in Russian). Acta Sci. Math. (Szeged) 45 (1983), 33 - 50.

[22] Constantinescu, T.: *On the structure of positive Toeplitz forms.* In: Dilation Theory, Toeplitz Operators and Other Topics (eds.: C. Apostol et al.; Operator Theory: Advances and Applications: Vol. 11). Basel-Boston-Stuttgart: Birkhäuser Verlag 1983, pp. 127 - 149.

[23] Constantinescu, T.: *An algorithm for the operatorial Carathéodory-Fejér problem.* In: Spectral Theory of Operators and Related Topics (eds.: H. Helson et al.; Operator Theory: Advances and Applications: Vol. 14). Basel - Boston - Stuttgart: Birkhäuser Verlag 1984, pp. 81 - 107.

[24] Constantinescu, T.: *A maximum entropy principle for contractive intertwining dilations.* In: Operators in Indefinite Metric Spaces, Scattering Theory and Other Topics (eds.: H. Helson et al.; Operator Theory: Advances and Applications: Vol. 24). Basel - Boston - Stuttgart: Birkhäuser Verlag 1987, pp. 69 - 85.

[25] Cotlar M. and C. Sadosky: *On the Helson-Szegö theorem and a related class of modified Toeplitz kernels.* In: Harmonic Analysis in Euclidean Spaces (eds.: G. Weiss and S. Wainger; Proc. Symp. Pure Math.: Vol. 35/Part 1). Providence (Rhode Island): Amer. Math. Soc. 1979, pp. 383 - 407.

[26] Cotlar M. and C. Sadosky: *Generalized Toeplitz kernels, stationarity and harmonizability.* J. d'Analyse Math. 44 (1984/85), 117 - 133.

[27] Cotlar M. and C. Sadosky: *A lifting theorem for subordinated invariant kernels.* J. Funct. Anal. 67 (1986), 345 - 359.

[28] Cotlar M. and C. Sadosky: *Lifting properties, Nehari Theorem and Paley Lacunary inequality.* Rev. Mat. Iberoamer. 2 (1986), 55 - 71.

[29] Dewilde P. and H. Dym: *Schur recursion error formulas and convergence of rational estimations for stationary processes.* IEEE Trans. Inf. Theory 27 (1981), 416 - 461.

[30] Dewilde, P. and H. Dym: *Lossless chain scattering matrices and optimum linear prediction: the vector case.* Intern. J. Circuit Theory Appl. 9 (1981), 135 - 175.

[31] Dubovoj, V. K.: *Indefinite metric in the interpolation problem of Schur for analytic matrix functions,* Part II (in Russian). Teor. Funkcii, Funkcional. Anal. i Prilozen. (Kharkov) 38 (1982), 32 - 39; Engl. transl. in: Amer. Math. Soc. Transl. (Series 2) 144 (1989), 61 - 70.

[32] Dubovoj, V. K., Fritzsche, B. and B. Kirstein: *On a class of matrix completion problems.* Math. Nachr. 143 (1989), 211 - 226.

[33] Dubovoj, V. K., Fritzsche, B. and B. Kirstein: *Matricial Version of the Classical Schur Problem* (Teubner-Texte zur Mathematik: Vol. 129). Stuttgart - Leipzig: B. G. Teubner1992.

[34] Dym, H. and I. Gohberg: *A maximum entropy principle for contractive interpolants.* J. Funct. Anal. 65 (1986), 83 - 125.

[35] Dym, H. and I. Gohberg: *A new class of contractive interpolants and maximum entropy principles.* In: Topics in Operator Theory and Interpolation (ed.: I. Gohberg; Operator Theory: Advances and Applications: Vol. 29). Basel - Boston - Berlin: Birkhäuser Verlag 1988, pp. 117 - 150.

[36] Fritzsche, B. and B. Kirstein: *An extension problem for non-negative Hermitian block Toeplitz matrices.* Math. Nachr. 130 (1987), 121 - 135.

[37] Fritzsche, B. and B. Kirstein: *An extension problem for non-negative Hermitian block Toeplitz matrices*, Part II. Math. Nachr. 131 (1987), 287 - 297.

[38] Fritzsche, B. and B. Kirstein: *A Schur type matrix extension problem.* Math. Nachr. 134 (1987), 257 - 271.

[39] Fritzsche, B. and B. Kirstein: *A Schur type matrix extension problem*, Part II. Math. Nachr. 138 (1988), 195 - 216.

[40] Fritzsche, B. and B. Kirstein: *On generalized Nehari problems.* Math. Nachr. 138 (1988), 217 - 237.

[41] Fritzsche, B. and B. Kirstein: *A matrix extension problem with entropy optimization.* Optimization 19 (1988), 85 - 99.

[42] Fritzsche, B. and B. Kirstein: *A Schur type matrix extension problem.* Part IV. Math. Nachr. 147 (1990, 235 - 258.

[43] Fritzsche, B. and B. Kirstein: *A correlation-theoretical interpretation of Schur analysis.* Statistics 21 (1990), 137 - 148.

[44] Fritzsche, B. and B. Kirstein: *A Nehari type problem for matricial Schur functions.* Z. Anal. Anw. 9 (1990), 37 - 84.

[45] Fritzsche, B. and B. Kirstein: *On the Weyl matrix balls associated with nondegenerate matrix-valued Carathéodory functions.* Z. Anal. Anw. 12 (1993), 239 - 261.

[46] Fritzsche, B. and B. Kirstein: *On the largest minorants associated with a matrix-valued Carathéodory function and spectral factorization.* Z. Anal. Anw. 12 (1993), 471 - 490.

[47] Fritzsche, B. and B. Kirstein: *On spectral factorization and generalized Nehari problems.* Z. Anal. Anw. 13 (1994), 97 - 109.

[48] Gohberg, I., Kaashoek, M. A. and H.-J. Woerdeman: *The band method for positive and contractive extension problems.* J. Oper. Theory 22 (1989), 109 - 155.

[49] Gohberg, I., Kaashoek, M. A. and H.-J. Woerdeman: *The band method for positive and contractive extensions: An alternate version and new applications.* Int. Equ. Oper. Theory 12 (1989), 343 - 382.

[50] Gohberg, I., Kaashoek, M. A. and H.-J. Woerdeman: *A maximum entropy principle in the general framework of the band method.* J. Funct. Anal. 95 (1991), 231 - 254.

[51] Katsnelson, V. E.: *Integral representations of Hermitian positive kernels of mixed type and the generalized Nehari problem* (in Russian). Teor. Funkcii, Funkcional. Anal. i Prilozen. (Kharkov) 43 (1985), 54 - 70; Engl. transl. in: J. Soviet Math. 48 (1990), 162 - 176.

[52] Katsnelson, V. E.: *Completion of an incomplete matrix function as a classical interpolation problem* (in Russian). Teor. Funkcii, Funkcional. Anal. i Prilozen. (Kharkov) 56 (1991), 105 - 122; Engl. transl. in: J. Math. Sci. 76 (1995), 2525 - 2537.

[53] Kolmogorov, A. N.: *Stationary sequences in Hilbert space* (in Russian). Bull. Math. Univ. Moscow 2 (1941), 1 - 40.

[54] Kovalishina, I. V. and V. P. Potapov: *The radii of a Weyl disc in the matrix Nevanlinna problem* (in Russian). In: Operator Theory in Function Spaces and Its Applications (ed.: V. A. Marcenko). Kiev: Naukova Dumka 1981, pp. 25 - 49; Engl. Transl. in: Amer. Math. Soc. Transl. (Series 2): Vol. 138, pp. 37 - 54.

[55] Landau, M. J.: *Maximum entropy and the moment problem.* Bull. Amer. Math. Soc. (New Series) 16 (1987), 47 - 77.

[56] Masani, P. R.: *Recent trends in multivariate prediction theory.* In: Multivariate Analysis (ed.: P. R. Krishnaiah). New York: Academic Press 1966, pp. 351 - 382.

[57] Nehari, Z.: *On bounded bilinear forms.* Ann. Math. 65 (1957), 153 - 162.

[58] Smuljan, J. L.: *Operator balls* (in Russian). Teor. Funkcii, Funkcional. Anal. i Prilozen. (Kharkov) 6 (1968), 68 - 81; Engl. Transl. in: Int. Equ. Oper. Theory 13 (1990), 864 - 882.

[59] Wiener, N. and P. R. Masani: *The prediction theory of multivariate stochastic processes.* Part I: *The regularity condition.* Acta Math. 98 (1957), 111 - 150.

[60] Wiener, N. and P. R. Masani: *The prediction theory of multivariate stochastic processes.* Part II: *The linear predictor.* Acta Math. 99 (1958), 93 - 137.

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