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Nonlinear Vibration Systems with Two Parallel Random Excitations

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In memory to Prof. Dr. P. Günther

Abstract. Systems of nonlinear vibration differential equations are investigated where the non-linearities are given by polynomials of any degree. The random excitations are induced by two parallel processes. These random excitations of an often applied type are expressed by linear functionals of weakly correlated processes with correlation length ε. The moments of the solutions and their first and second derivatives are expanded with respect to ε where all terms up to order ε^2 are included. Approximations of the correlation functions are given explicitely. Only the quadratic and cubic non-linearities have an influence on the correlation functions in this approximation order.

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1. Problem

Discrete mechanical models of n degrees of freedom are described by systems of n ordinary differential equations of second order. Defining the vector

$$z = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n;\, x_1, x_2, \ldots, x_n)^T$$

where $x_1, x_2, \ldots, x_n$ are the deviations of the masses the mathematical model leads to a system of $2n$ differential equations of first order

$$\left.\begin{aligned} M\dot{z} + Nz + \eta \sum_{k=2}^{m} B_k(z) &= F(t,\omega) \\ z(t_0) &= z_0 \end{aligned}\right\} \tag{1}$$

where M and N are $(2n \times 2n)$-matrices and η is a small parameter. In the following let the matrices M and N be regular. Furthermore, the matrix $M^{-1}N$ is assumed to have eigenvalues with positive real parts only.

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The nonlinear terms are approximated by polynomials, i.e. the coordinates of the vectors B_k are defined by

$$B_{k,p}(z) = \sum_{i_1,\dots,i_k=1}^{2n} b_{pi_1\dots i_k} z_{i_1} \cdots z_{i_k}$$

where the coefficients $b_{pi_1\dots i_k}$ are symmetrical with respect to the indices $i_1, \dots, i_k$ and $b_{pi_1\dots i_k} = 0$ for $n < p \leq 2n$. Furthermore,

$$\begin{aligned} F(t,\omega) &= P_{0L}\bar{f}_L(t,\omega) + P_{1L}\dot{\bar{f}}_L(t,\omega) + P_{2L}\ddot{\bar{f}}_L(t,\omega) \\ &\quad + P_{0R}\bar{f}_R(t,\omega) + P_{1R}\dot{\bar{f}}_R(t,\omega) + P_{2R}\ddot{\bar{f}}_R(t,\omega) \end{aligned} \tag{3}$$

with $(2n \times 2n)$-matrices P_{jL} and P_{jR} and

$$\begin{aligned} \bar{f}_{Lk}(t,\omega) &= \begin{cases} f_L(t+v_k,\omega) & \text{if } k = 1,\dots,r \\ 0 & \text{if } k = r+1,\dots,2n. \end{cases} \\ \bar{f}_{Rk}(t,\omega) &= \begin{cases} f_R(t+v_k,\omega) & \text{if } k = 1,\dots,r \\ 0 & \text{if } k = r+1,\dots,2n. \end{cases} \end{aligned} \qquad (r \leq n) \tag{4}$$

are the coordinates of the vector processes $\bar{f}_L$ and $\bar{f}_R$, respectively. That means, the time-shifted excitations $\bar{f}_{Lk}$ and $\bar{f}_{Rk}$ are induced by two centred random processes f_L and f_R, respectively.

A vehicle considered as multibody vibration system (cf. [3, 4, 6]) is an example for such models. The discrete masses are coupled by springs and dampers whose characteristics are nonlinear functions approximated by polynomials. The model is excited by two parallel tracks f_L and f_R of random road surfaces, $\bar{f}_{Lk}$ and $\bar{f}_{Rk}$ are the time-shifted random excitations at the left and right wheels, respectively.

The excitations are random processes. Hence, the deviations x_k of masses and subsequently their velocities $\dot{x}_k$ and accelerations $\ddot{x}_k$ are random processes, too. The aim is to obtain characteristics with respect to their stochastic behaviour in form of expectations and correlation functions which are the basis for further characteristics, e.g. spectral densities and expected numbers of threshold crossings.

2. Remarks on weakly correlated processes

A wide-sense stationary process $f_\varepsilon = f_\varepsilon(t,\omega)$ with expectation $\langle f_\varepsilon(t)\rangle \equiv 0$ is called *weakly correlated* if the influence of the process does not reach far, i.e. the values of this process at two points t_1 and t_2 do not correlate if their distance $t_2 - t_1$ exceeds a certain quantity $\varepsilon > 0$. The *correlation length* ε is always assumed to be sufficiently small. Hence, weakly correlated processes can also be characterized as processes without "distant effect" or as processes of "noise-natured character". In particular, the correlation function has the property

$$R_{f_\varepsilon f_\varepsilon}(t_2 - t_1) := \langle f_\varepsilon(t_1) f_\varepsilon(t_2)\rangle = 0 \qquad \text{for } |t_2 - t_1| > \varepsilon.$$

The precise definition includes a decomposition property with respect to all higher moments $\langle f_\varepsilon(t_1)f_\varepsilon(t_2)\cdots f_\varepsilon(t_m)\rangle$ $(m \geq 2)$ (cf. [5: p. 23 ff]). A characteristic quantity of a weakly correlated process is the intensity a defined by

$$a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{+\varepsilon} R_{f_\varepsilon f_\varepsilon}(t)\, dt. \tag{5}$$

Let $f_\varepsilon = f_\varepsilon(t,\omega)$ be a weakly correlated and weakly stationary process with almost surely sample functions and $\langle |f_\varepsilon(t)|^j\rangle \leq c_j < \infty$ for all $t \in \mathcal{I}$, $\mathcal{I} \subset \mathbb{R}$ some intervall, and $j \geq 1$. Let ϕ_i $(i = 1,\ldots,m)$ be bounded deterministic functions on subintervals $\mathcal{I}_i \subseteq \mathcal{I}$ with $\phi_i \in L_1(\mathcal{I}_i) \cap L_2(\mathcal{I}_i)$. Then all moments of the linear functional

$$\Phi_i(\omega) = \int_{\mathcal{I}_i} \phi_i(t) f_\varepsilon(t,\omega)\, dt$$

can be expanded with respect to the correlation length ε. The approximation order of the k-th moments is given by

$$\begin{aligned} \langle \Phi_i \rangle &= 0 \\ \langle \Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_k}\rangle &= \begin{cases} O(\varepsilon^{\frac{k}{2}}) & \text{if } k \geq 2 \text{ is even} \\ O(\varepsilon^{\frac{k+1}{2}}) & \text{if } k \geq 3 \text{ is odd} \end{cases} \end{aligned} \tag{6}$$

Especially, the second moments are

$$\langle \Phi_i \Phi_j \rangle = \varepsilon a \int_{\mathcal{I}_i \cap \mathcal{I}_j} \phi_i(t)\phi_j(t)\, dt + O(\varepsilon^2). \tag{7}$$

A detailed theoretical concept of these random processes and proofs of the above properties can be found in [5: p. 25 ff], [7: p. 152 ff] and [8].

3. Model of excitations

In many applications weakly stationary excitations (e.g. random road profiles) are expressed as processes with a correlation function

$$R(t) = \sigma^2 e^{-\gamma|t|} \qquad (\gamma > 0) \tag{8}$$

and corresponding spectral density

$$S(\alpha) = \frac{\sigma^2}{\pi} \frac{\gamma}{\gamma^2 + \alpha^2}$$

(cf. [3: p. 27 ff] and [4: p. 206 ff]). The correlation function (8) is not differentiable at $t = 0$, and therefore these processes need not be differentiable. Hence, processes of this type are not suitable as excitations in system (1) if derivatives of the excitations are included.

To obtain differentiable processes of a similar type the process f is expressed by a linear functional of a weakly correlated and weakly stationary process f_ε in the form

$$f(t,\omega) = \int_{-\infty}^{t} Q(t-s) f_\varepsilon(s,\omega)\, ds \tag{9}$$

where Q is a twice continuously differentiable function with

$$Q(0) = 0, \qquad Q'(0) = 0, \qquad Q''(0) = 0 \tag{10}$$

(cf. [1], [6: p. 58 ff] and [9]). Hence, the process f is twice continuously differentiable with a uniform presentation of the process and its derivatives, i.e.

$$f^{(k)}(t,\omega) = \int_{-\infty}^{t} Q^{(k)}(t-s) f_\varepsilon(s,\omega)\, ds \qquad (k = 0,1,2). \tag{11}$$

Applying approximation theorems for linear functionals of weakly correlated processes (cf. (6) and (7)) the correlation functions are calculated as

$$R_{f^{(k)}f^{(l)}}(t_1,t_2) = \langle f^{(k)}(t_1) f^{(l)}(t_2)\rangle = R^1_{f^{(k)}f^{(l)}}(t_1,t_2) + O(\varepsilon^2) \quad (k,l = 0,1,2)$$

where

$$R^1_{f^{(k)}f^{(l)}}(t_1,t_2) = \varepsilon a \int_{-\infty}^{\min(t_1,t_2)} Q^{(k)}(t_1 - s) Q^{(l)}(t_2 - s)\, ds.$$

Linear transformations lead to

$$R^1_{f^{(k)}f^{(l)}}(t_1,t_2) = R^1_{f^{(k)}f^{(l)}}(t_2 - t_1)$$

$$R^1_{f^{(k)}f^{(l)}}(t) = \begin{cases} \varepsilon a \int_0^\infty Q^{(k)}(s) Q^{(l)}(t+s)\, ds & \text{if } t \geq 0 \\ \varepsilon a \int_0^\infty Q^{(k)}(s-t) Q^{(l)}(s)\, ds & \text{if } t \leq 0. \end{cases}$$

Choosing $Q(t) = Q_0(t) e^{-\gamma t}$ $(\gamma, \delta > 0)$ where

$$Q_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ 6\left(\frac{t}{\delta}\right)^5 - 15\left(\frac{t}{\delta}\right)^4 + 10\left(\frac{t}{\delta}\right)^3 & \text{if } 0 \leq t \leq \delta \\ 1 & \text{if } t > \delta \end{cases}$$

the first approximations of the correlation functions are calculated as

$$R^1_{f^{(k)}f^{(l)}}(t) = \frac{\varepsilon a}{2} (-1)^{k+l} \gamma^{k+l-1} e^{-\gamma(|t|+2\delta)}$$

$$+ \varepsilon a \begin{cases} \int_0^\delta Q^{(k)}(s) Q^{(l)}(t+s)\, ds & \text{if } t \geq 0 \\ \int_0^\delta Q^{(k)}(s-t) Q^{(l)}(s)\, ds & \text{if } t \leq 0 \end{cases} \qquad (k,l = 0,1,2).$$

Setting $l = k = 0$ the Lebesgue theorem on dominated convergence leads to

$$\lim_{\delta \to 0} R^1_{ff}(t) = \frac{\varepsilon a}{2\gamma} e^{-\gamma|t|}.$$

Therefore, this method can also be interpreted as smoothing of the correlation function (8) in a δ-neighbourhood of $t = 0$. The parameters γ and δ of Q as well as εa can be determined by comparison with given correlation functions (8) or with measurement results (cf. [6: p. 66 ff]).

To model the parallel excitations f_L and f_R orthotropic behaviour is assumed, i.e.

$$\left.\begin{aligned} R_{f_L f_L}(t) = R_{f_R f_R}(t) &= \sigma^2 e^{-\gamma|t|} \\ R_{f_L f_R}(t) &= \sigma^2 e^{-\gamma(b+|t|)} \quad (b > 0). \end{aligned}\right\}$$

Defining a mean profile m and a difference profile d by

$$m(t,\omega) = \frac{1}{2}\big(f_L(t,\omega) + f_R(t,\omega)\big) \qquad \text{and} \qquad d(t,\omega) = \frac{1}{2}\big(f_L(t,\omega) - f_R(t,\omega)\big)$$

the correlation functions

$$\left.\begin{aligned} R_{md}(t) &= 0 \\ R_{mm}(t) &= \sigma_m^2 e^{-\gamma|t|} \quad \text{with } \sigma_m^2 = \frac{1}{2}\sigma^2(1 + e^{-\gamma b}) \\ R_{dd}(t) &= \sigma_d^2 e^{-\gamma|t|} \quad \text{with } \sigma_d^2 = \frac{1}{2}\sigma^2(1 - e^{-\gamma b}) \end{aligned}\right\}$$

are obtained. Hence, as above the presentations

$$m(t,\omega) = \int_{-\infty}^{t} Q(t-s) f_{1\varepsilon}(s,\omega)\, ds \quad \text{and} \quad d(t,\omega) = \int_{-\infty}^{t} Q(t-s) f_{2\varepsilon}(s,\omega)\, ds$$

are used where $f_{1\varepsilon}$ and $f_{2\varepsilon}$ are independent, weakly correlated and weakly stationary processes with correlation length ε and intensities a_1 and a_2, respectively. It follows

$$\left.\begin{aligned} f_L^{(k)}(t,\omega) &= \int_{-\infty}^{t} Q^{(k)}(t-s)\big(f_{1\varepsilon}(s,\omega) + f_{2\varepsilon}(s,\omega)\big)\, ds \\ f_R^{(k)}(t,\omega) &= \int_{-\infty}^{t} Q^{(k)}(t-s)\big(f_{1\varepsilon}(s,\omega) - f_{2\varepsilon}(s,\omega)\big)\, ds \end{aligned}\right. \qquad (k = 0,1,2). \tag{12}$$

4. Linear systems

First, the linear system (1) with $\eta = 0$ is considered. The solution z of (1) has the form

$$z(t,\omega) = G(t-t_0)Mz_0 + \int_{t_0}^{t} G(t-s)F(s,\omega)\,ds$$

where the matrix function G is defined by

$$G(t) = \exp(-M^{-1}Nt)M^{-1}.$$

The vector process

$$\bar{z}(t,\omega) = \int_{-\infty}^{t} G(t-s)F(s,\omega)\,ds$$

is a solution of the differential equation (1), but $\bar{z}$ does not satisfy the initial condition. Because the matrix $M^{-1}N$ has eigenvalues with positive real parts only the difference $\bar{z} - z$ converges exponentially to 0 as t tends to infinity. Hence, the vector process $\bar{z}$ can be regarded as a solution of the system (1) after a sufficiently large transient time. Using (3), (4) and (12) the coordinates of $\bar{z}$ can be expressed by

$$\begin{aligned}\bar{z}_i(t,\omega) = &\sum_{j=1}^{r} \int_{-\infty}^{t+v_j} \widehat{G}_{ij}^{+}(t+v_j-u)f_{1\varepsilon}(u,\omega)\,du \\ &+\sum_{j=1}^{r} \int_{-\infty}^{t+v_j} \widehat{G}_{ij}^{-}(t+v_j-u)f_{2\varepsilon}(u,\omega)\,du \qquad (i=1,\ldots,2n)\end{aligned} \tag{13}$$

with matrix functions

$$\widehat{G}^{\pm}(t) = \sum_{l=0}^{2} \int_{0}^{t} G(t-v)P_l^{\pm}Q^{(l)}(v)\,dv \quad \text{where} \quad P_l^{\pm} = P_{lL} \pm P_{lR}. \tag{14}$$

To investigate the accelerations of the deviations the first derivatives of $\bar{z}_i$ are determined. Using equations (13) and (14) a uniform representation of the processes and their derivatives can be obtained, i.e.

$$\begin{aligned}\bar{z}_i^{(k)}(t,\omega) = &\sum_{j=1}^{r} \int_{-\infty}^{t+v_j} \widehat{G}_{ij}^{+(k)}(t+v_j-u)f_{1\varepsilon}(u,\omega)\,du \\ &+\sum_{j=1}^{r} \int_{-\infty}^{t+v_j} \widehat{G}_{ij}^{-(k)}(t+v_j-u)f_{2\varepsilon}(u,\omega)\,du \qquad (k=0,1).\end{aligned}$$

The equation

$$\widehat{G}^{\pm(k)}(t) = \sum_{l=0}^{2} \int_0^t G(t-v) P_l^{\pm} Q^{(l+k)}(v)\, dv \qquad (k = 0, 1)$$

follows from partial integration and conditions (10). Now, approximation theorems for linear functionals of weakly correlated processes can be used to calculate moments of the deviations and their first and second derivatives. Especially, (7) leads to the approximations

$$\begin{aligned} R_{\bar{z}_i^{(k)}\bar{z}_j^{(l)}}(t_1, t_2) &= \langle \bar{z}_i^{(k)}(t_1) \bar{z}_j^{(l)}(t_2) \rangle \\ &= \varepsilon a_1 \sum_{p,q=1}^{r} \int_{-\infty}^{\min(t_1+v_p, t_2+v_q)} \widehat{G}_{ip}^{+(k)}(t_1 + v_p - s) \widehat{G}_{jq}^{+(l)}(t_2 + v_q - s)\, ds \\ &+ \varepsilon a_2 \sum_{p,q=1}^{r} \int_{-\infty}^{\min(t_1+v_p, t_2+v_q)} \widehat{G}_{ip}^{-(k)}(t_1 + v_p - s) \widehat{G}_{jq}^{-(l)}(t_2 + v_q - s)\, ds \\ &+ O(\varepsilon^2) \end{aligned}$$

of the correlation functions. Some transformations yield

$$R_{\bar{z}_i^{(k)}\bar{z}_j^{(l)}}(t_1, t_2) = \varepsilon D_{ij}^{kl}(t_2 - t_1) + O(\varepsilon^2) \qquad (k, l = 0, 1) \tag{15}$$

where

$$D_{ij}^{kl}(t) = a_1 \sum_{p,q=1}^{r} T_{ijpq}^{+kl}(t + v_q - v_p) + a_2 \sum_{p,q=1}^{r} T_{ijpq}^{-kl}(t + v_q - v_p)$$

and

$$T_{ijpq}^{\pm kl}(s) = \begin{cases} \int_0^\infty \widehat{G}_{ip}^{\pm(k)}(u-s) \widehat{G}_{jq}^{\pm(l)}(u)\, du & \text{if } s \le 0 \\ \int_0^\infty \widehat{G}_{ip}^{\pm(k)}(u) \widehat{G}_{jq}^{\pm(l)}(u+s)\, du & \text{if } s \ge 0. \end{cases}$$

These approximations of the correlation functions depend only on the difference $t_2 - t_1$, i.e. the corresponding processes are weakly stationary.

5. Nonlinear systems

The solutions of nonlinear systems (1) can be deduced by perturbation methods. Substituting the series

$$z(t,\omega) = \sum_{l=0}^{\infty} {}^{l}z(t,\omega)\,\eta^{l} \tag{16}$$

into the nonlinear terms (2) it follows

$$\left.\begin{aligned} B_{k,p}(z) &= \sum_{l=0}^{\infty} {}^{l}B_{k,p}(z)\,\eta^{l} \\ {}^{l}B_{k,p}(z) &= \sum_{i_1,\dots,i_k=1}^{2n} b_{pi_1\dots i_k} \sum_{l_1+\dots+l_k=l} {}^{l_1}z_{i_1}\cdots{}^{l_k}z_{i_k}. \end{aligned}\right\}$$

Now, a comparison of coefficients in system (1) leads to a system of linear systems of differential equations

$$\left.\begin{aligned} M\,{}^{0}\dot{z} + N\,{}^{0}z &= F(t,\omega) \\ M\,{}^{l}\dot{z} + N\,{}^{l}z &= -\sum_{k=2}^{m} {}^{l-1}B_k(z) \quad (l \geq 1) \end{aligned}\right\} \tag{17}$$

which can be solved recursively. As in the linear case the vector processes

$$\left.\begin{aligned} {}^{0}\bar{z}(t,\omega) &= \int_{-\infty}^{t} G(t-s)F(s,\omega)\,ds \\ {}^{l}\bar{z}(t,\omega) &= -\sum_{k=2}^{m} \int_{-\infty}^{t} G(t-s)\,{}^{l-1}B_k(\bar{z}(s,\omega))\,ds \quad (l \geq 1) \end{aligned}\right\} \tag{18}$$

can be regarded as solutions of system (17) after a transient time. The vector process ${}^{0}\bar{z}$ is the solution of the corresponding linear system and was investigated in the previous section. Because of the recursive presentation (18) all coordinates ${}^{l}\bar{z}_k$ of ${}^{l}\bar{z}$ $(l \geq 1)$ can be expressed as sums of integrals in dependence of the coordinates of ${}^{0}\bar{z}$ which are linear functionals of weakly correlated processes (cf. (13)). The integrands of ${}^{l}\bar{z}_k$ include products of processes ${}^{0}\bar{z}_1, \dots, {}^{0}\bar{z}_{2n}$. It can be shown inductively that the minimal number of factors is $l+1$. Subsequently, it follows from (6) that

$$\langle {}^{l}\bar{z}_i(t) \rangle = \begin{cases} O(\varepsilon^{\frac{l+1}{2}}) & \text{if } l \text{ odd} \\ O(\varepsilon^{\frac{l+2}{2}}) & \text{if } l \text{ even} \end{cases}$$

$$\langle {}^{\mu}\bar{z}_i(t_1)\,{}^{\nu}\bar{z}_j(t_2) \rangle = \begin{cases} O(\varepsilon^{\frac{l+2}{2}}) & \text{if } l \text{ even} \\ O(\varepsilon^{\frac{l+3}{2}}) & \text{if } l \text{ odd} \end{cases} \qquad (l = \mu + \nu).$$

Especially,

$$\begin{aligned}
\langle {}^0\bar{z}_i(t)\rangle &= 0 \\
\langle {}^1\bar{z}_i(t)\rangle &= O(\varepsilon) \\
\langle {}^2\bar{z}_i(t)\rangle &= O(\varepsilon^2) \\
\langle {}^3\bar{z}_i(t)\rangle &= O(\varepsilon^2)
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
\langle {}^0\bar{z}_i(t_1){}^0\bar{z}_j(t_2)\rangle &= O(\varepsilon) \\
\langle {}^1\bar{z}_i(t_1){}^0\bar{z}_j(t_2)\rangle &= O(\varepsilon^2) \\
\langle {}^1\bar{z}_i(t_1){}^1\bar{z}_j(t_2)\rangle &= O(\varepsilon^2) \\
\langle {}^0\bar{z}_i(t_1){}^2\bar{z}_j(t_2)\rangle &= O(\varepsilon^2)
\end{aligned}$$

while all the other first and second moments are at least of order $O(\varepsilon^3)$. Hence, the expansion (16) leads to the mean values

$$\langle \bar{z}_i(t)\rangle = \eta\langle {}^1\bar{z}_i(t)\rangle + \eta^2\langle {}^2\bar{z}_i(t)\rangle + \eta^3\langle {}^3\bar{z}_i(t)\rangle + O(\varepsilon^3)$$

and finally to the correlation functions

$$\begin{aligned}
R_{\bar{z}_i\bar{z}_j}(t_1,t_2) &= \Big\langle \big(\bar{z}_i(t_1) - \langle \bar{z}_i(t_1)\rangle\big)\big(\bar{z}_j(t_2) - \langle \bar{z}_j(t_2)\rangle\big)\Big\rangle \\
&= \big\langle {}^0\bar{z}_i(t_1){}^0\bar{z}_j(t_2)\big\rangle \\
&\quad + \eta\left\{\big\langle {}^0\bar{z}_i(t_1){}^1\bar{z}_j(t_2)\big\rangle + \big\langle {}^1\bar{z}_i(t_1){}^0\bar{z}_j(t_2)\big\rangle\right\} \\
&\quad + \eta^2\left\{\big\langle {}^1\bar{z}_i(t_1){}^1\bar{z}_j(t_2)\big\rangle - \big\langle {}^1\bar{z}_i(t_1)\big\rangle\big\langle {}^1\bar{z}_j(t_2)\big\rangle\right. \\
&\quad \left. + \big\langle {}^0\bar{z}_i(t_1){}^2\bar{z}_j(t_2)\big\rangle + \big\langle {}^2\bar{z}_i(t_1){}^0\bar{z}_j(t_2)\big\rangle\right\} \\
&\quad + O(\varepsilon^3).
\end{aligned} \tag{19}$$

These investigations show that it is necessary to consider all terms up to order ε^2 to obtain differences to linear models.

Now, all the first and second moments of (19) are expanded with respect to the correlation length ε where only the terms of order ε and ε^2 have to be calculated explicitely. Using approximation theorems for linear functionals of weakly correlated processes all these terms can be expressed analogously to (7) (cf. [5: p. 25 ff] and [8]).

Here, for simplification it is assumed that $f_{1\varepsilon}$ and $f_{2\varepsilon}$ are weakly correlated processes with

$$\int_{-\varepsilon}^{\varepsilon} R_{f_{1\varepsilon}f_{1\varepsilon}}(t)\,dt = \varepsilon a_1 + O(\varepsilon^3) \qquad \text{and} \qquad \int_{-\varepsilon}^{\varepsilon} R_{f_{2\varepsilon}f_{2\varepsilon}}(t)\,dt = \varepsilon a_2 + O(\varepsilon^3)$$

(cf. (5) and [5: p. 88 ff], [6: p. 44 ff] and [8]). Then all terms of order ε^2 vanish in the expansions (15). Additionally, it is supposed that $f_{1\varepsilon}$ and $f_{2\varepsilon}$ are Gaussian processes. Hence, all even moments of these processes can be expressed by second moments and all odd moments vanish (cf. [2. p. 149]). These properties can be transferred to the moments of the coordinates of ${}^0\bar{z}$. In this case some straight forward calculations and approximation theorems lead to the following expressions:

$$\langle {}^0\bar{z}_i(t_1)\,{}^0\bar{z}_j(t_2)\rangle = \varepsilon D_{ij}^{00}(t_2-t_1) + O(\varepsilon^3) \quad \text{(cf. (15))}$$

$$\langle {}^1\bar{z}_i(t_1)\,{}^0\bar{z}_j(t_2)\rangle = -3\varepsilon^2 \sum_{p=1}^{n} \sum_{i_1,i_2,i_3=1}^{2n} b_{pi_1i_2i_3} D_{i_1i_2}^{00}(0) \times \int_0^\infty G_{ip}(s) D_{i_3j}^{00}(t_2-t_1+s)\,ds + O(\varepsilon^3)$$

$$\langle {}^1\bar{z}_i(t_1)\,{}^1\bar{z}_j(t_2)\rangle = \varepsilon^2 \Bigg\{ \sum_{p_1,p_2=1}^{n} \sum_{i_1,\ldots,i_4=1}^{2n} N_{ip_1}^{-1} N_{jp_2}^{-1} b_{p_1i_1i_2} b_{p_2i_3i_4} D_{i_1i_2}^{00}(0) D_{i_3i_4}^{00}(0) + 2 \sum_{p_1,p_2=1}^{n} \sum_{i_1,\ldots,i_4=1}^{2n} b_{p_1i_1i_2} b_{p_2i_3i_4} \int_0^\infty\int_0^\infty G_{ip_1}(u) G_{jp_2}(v) \times D_{i_1i_3}^{00}(t_2-t_1+u-v) D_{i_2i_4}^{00}(t_2-t_1+u-v)\,dv\,du \Bigg\} + O(\varepsilon^3)$$

$$\langle {}^2\bar{z}_i(t_1)\,{}^0\bar{z}_j(t_2)\rangle = \varepsilon^2 \Bigg\{ 2 \sum_{p_1,p_2=1}^{n} \sum_{i_1,\ldots,i_4=1}^{2n} N_{i_1p_2}^{-1} b_{p_1i_1i_2} b_{p_2i_3i_4} D_{i_3i_4}^{00}(0) \times \int_0^\infty G_{ip_1}(s) D_{i_2j}^{00}(t_2-t_1+s)\,ds + 4 \sum_{p_1,p_2=1}^{n} \sum_{i_1,\ldots,i_4=1}^{2n} b_{p_1i_1i_2} b_{p_2i_3i_4} \times \int_0^\infty\int_0^\infty G_{ip_1}(s) G_{i_1p_2}(u) D_{i_3i_2}^{00}(u) D_{i_4j}^{00}(t_2-t_1+u+s)\,du\,ds \Bigg\} + O(\varepsilon^3)$$

$$\langle {}^1\bar{z}_i(t)\rangle = -\sum_{p=1}^{n} N_{ip}^{-1} \Bigg\{ \varepsilon \sum_{i_1,i_2=1}^{2n} b_{pi_1i_2} D_{i_1i_2}^{00}(0) + 3\varepsilon^2 \sum_{i_1,\ldots,i_4=1}^{2n} b_{pi_1i_2i_3i_4} D_{i_1i_2}^{00}(0) D_{i_3i_4}^{00}(0) \Bigg\} + O(\varepsilon^3).$$

All these moments only depend on the difference $t_2 - t_1$ and therefore the correlation functions (19) only depend on the difference $t_2 - t_1$ up to terms of second order, too, i.e. the corresponding processes are weakly stationary.

The correlation functions of the derivatives of the solutions are calculated on the base of

$$\begin{aligned}
R_{\bar{z}_i^{(k)}\bar{z}_j^{(l)}}(t) &= \left\langle \left(\bar{z}_i^{(k)}(t_1) - \langle \bar{z}_i^{(k)}(t_1)\rangle\right)\left(\bar{z}_j^{(l)}(t_2) - \langle \bar{z}_j^{(l)}(t_2)\rangle\right)\right\rangle \\
&= \frac{\partial^{k+l}}{\partial t_1^k \partial t_2^l}\left\langle \left(\bar{z}_i(t_1) - \langle \bar{z}_i(t_1)\rangle\right)\left(\bar{z}_j(t_2) - \langle \bar{z}_j(t_2)\rangle\right)\right\rangle \\
&= \frac{\partial^{k+l}}{\partial t_1^k \partial t_2^l} R_{\bar{z}_i\bar{z}_j}(t_2 - t_1) \\
&= (-1)^k R_{\bar{z}_i\bar{z}_j}^{(k+l)}(t) \qquad (k, l = 0, 1;\ t = t_2 - t_1).
\end{aligned}$$

The equations

$$\left.\begin{aligned}
(-1)^k \frac{d^{k+l}}{dt^{k+l}} D_{ij}^{00}(t+s) &= D_{ij}^{kl}(t+s) \\
(-1)^k \frac{d^{k+l}}{dt^{k+l}} D_{i_1i_2}^{00}(t+s) D_{i_3i_4}^{00}(t+s) &= \sum_{\mu=0}^{k}\sum_{\nu=0}^{l} D_{i_1i_2}^{\mu\nu}(t+s) D_{i_3i_4}^{k-\mu\ l-\nu}(t+s) \\
&\qquad (k, l = 0, 1)
\end{aligned}\right\}$$

derived from (15) lead to the following final result for the correlation functions of the vibration deviations and their velocities and accelerations:

$$\begin{aligned}
R_{\bar{z}_i^{(k)}\bar{z}_j^{(l)}}(t) &= (-1)^k \frac{d^{k+l}}{dt^{k+l}} R_{\bar{z}_i\bar{z}_j}(t) \\
&= \varepsilon D_{ij}^{kl}(t) + \varepsilon^2 \Bigg\{ -3\eta \sum_{p=1}^{n} \sum_{i_1,i_2,i_3=1}^{2n} b_{pi_1i_2i_3} D_{i_1i_2}^{00}(0) \\
&\times \left[\int_0^\infty G_{ip}(s) D_{i_3j}^{kl}(t+s)\,ds + (-1)^{k+l} \int_0^\infty G_{jp}(s) D_{i_3i}^{kl}(-t+s)\,ds \right] \\
&+ 2\eta^2 \sum_{p_1,p_2=1}^{n} \sum_{i_1,\ldots,i_4=1}^{2n} b_{p_1i_1i_2} b_{p_2i_3i_4} \Bigg\{ \sum_{\mu=0}^{k}\sum_{\nu=0}^{l} \int_0^\infty \int_0^\infty G_{ip_1}(s) G_{jp_2}(u) \\
&\times D_{i_1i_3}^{\mu\nu}(t+s-u) D_{i_2i_4}^{k-\mu\ l-\nu}(t+s-u)\,du\,ds \\
&+ N_{i_1p_2}^{-1} D_{i_3i_4}^{00}(0) \\
&\times \left[\int_0^\infty G_{ip_1}(s) D_{i_2j}^{kl}(t+s)\,ds + (-1)^{k+l} \int_0^\infty G_{jp_1}(s) D_{i_2i}^{kl}(-t+s)\,ds \right] \\
&+ 2 \int_0^\infty \int_0^\infty G_{ip_1}(s) G_{i_1p_2}(u) D_{i_3i_2}^{00}(u) D_{i_4j}^{kl}(t+u+s)\,du\,ds \\
&+ 2(-1)^{k+l} \int_0^\infty \int_0^\infty G_{jp_1}(s) G_{i_1p_2}(u) D_{i_3i_2}^{00}(u) D_{i_4i}^{kl}(-t+u+s)\,du\,ds \Bigg\} \Bigg\} \\
&+ O(\varepsilon^3) \qquad (k, l = 0, 1).
\end{aligned}$$

It is obvious that only the quadratic and cubic non-linearities have an influence to these correlation functions in this approximation order.

The method presented allows also the approximation of higher moments in a similar way. Furthermore, it is possible to approximate the distribution functions. In the first approximation Gaussian distributions are obtained (cf. [5: p. 36 ff] and [8]).

References

[1] Drescher, G., Fellenberg, B. and J. vom Scheidt: *Simulation of random functions for applications to dynamic systems.* Dyn. Systems Appl. 2 (1993), 275 - 289.

[2] Gichman, I. I. and A. W. Skorochod: *The Theory of Stochastic Processes*, Vol. I. Berlin: Springer-Verlag 1974.

[3] Mitschke, M.: *Dynamik der Kraftfahrzeuge.* Vol. B: *Schwingungen.* Berlin et al.: Springer-Verlag 1984.

[4] Popp, P. and W. O. Schiehlen: *Fahrzeugdynamik.* Stuttgart: B. G. Teubner 1993.

[5] Scheidt, J. vom: *Stochastic Equations of Mathematical Physics.* Berlin: Akademie-Verlag 1990.

[6] Scheidt, J. vom, Fellenberg, B. and U. Wöhrl: *Analyse und Simulation stochastischer Schwingungssysteme.* Stuttgart: B. G. Teubner 1994.

[7] Scheidt, J. vom and W. Purkert: *Random Eigenvalue Problems.* Berlin: Akademie-Verlag 1983, und New York et al.: North-Holland 1983.

[8] Scheidt, J. vom and U. Wöhrl: *Approximation der Verteilung linearer Funktionale schwach korreliert verbundener Vektorprozesse - Anwendung auf Schwingungsdifferentialgleichungen.* Z. Angew. Math. Mech. (ZAMM) 67 (1987), 607 - 615.

[9] Scheidt, J. vom, Wöhrl, U. and S. Wulff: *A method for the numerical treatment of non-linear stochastic vibrations of vehicles.* Dyn. Systems Appl. 1 (1992), 187 - 204.

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