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On the Integral Giving the Degree of a Map and a Rouché Type Theorem

T. Hatziafratis and A. Tsarpalias

Abstract. An analytic approach to the degree of a map $f : \partial D \longrightarrow \mathbb{R}^n \setminus \{0\}$ is given (where $D \subset \mathbb{R}^n$ is a bounded domain with smooth boundary) and a Rouché type theorem is proved.

Keywords: *Degree of a map, differential forms, fixed points*

AMS subject classification: Primary 58 A 10, secondary 58 C 30

1. Introduction

In this paper we give a simple analytic proof that the integral

$$\frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{\partial D} \frac{1}{|f|^n} \sum_{j=1}^{n} (-1)^{j-1} f_j \, df_1 \wedge \cdots (j) \cdots \wedge df_n$$

is an integer for a continuously differentiable map

$$f : \partial D \longrightarrow \mathbb{R}^n \setminus \{0\},$$

where D is a bounded domain in $\mathbb{R}^n$ with smooth boundary; this integral gives the degree of f.

The basic idea of this proof is the following: if we call $\eta(f)$ the integrand in the above integral then, although $\eta(f)$ is not, in general, d-exact on the $(n-1)$-dimensional manifold ∂D, its derivative

$$\frac{\partial \eta(f(\,\cdot\,,t))}{\partial t}$$

is d-exact, when we let f depend on a parameter t; and this is proved by constructing explicitly a d-primitive (see Lemma 2). As for the parametrization of f, it is done with a perturbation argument, based on Sard's theorem. For another analytic proof see Heinz [2]. For the history of the above integral as well as its connections with polynomial equations and the Gauss-Bonnet theorem see Siegberg [3]. We also prove a version of Rouché's principle which gives a proof of Brouwer's fixed point theorem. A special case of it is contained in [5].

T. Hatziafratis: University of Athens, Dept. Math., Panepistimiopolis, 157 84 Athens, Greece
A. Tsarpalias: University of Athens, Dept. Math., Panepistimiopolis, 157 84 Athens, Greece

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2. Preliminaries

Let us recall some basic facts about differential forms in $\mathbb{R}^n$. A p-form in an open set G in $\mathbb{R}^n$ is a function ω, symbolically represented by the sum

$$\omega = \sum_{1 \le i_1, \dots, i_p \le n} f_{i_1,\dots,i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

which assigns to each smooth p-surface $\mathcal{X}$ in G a number $\omega(\mathcal{X}) = \int_{\mathcal{X}} \omega$ according to the rule

$$\int_{\mathcal{X}} \omega = \sum \int_{\Omega} f_{i_1,\dots,i_p}(\mathcal{X}(t)) \frac{\partial(x_{i_1}, \dots, x_{i_p})}{\partial(t_1, \dots, t_p)}\, dt_1 \cdots dt_p,$$

where Ω is the parameter domain of $\mathcal{X}$, and $f_{i_1,\dots,i_p}$ are real C^1-functions on G. The number $\int_{\mathcal{X}} \omega$ is called also the integral of ω on $\mathcal{X}$. The algebra of differential forms obeys the laws of exterior algebra. We recall that a basic rule of exterior algebra is the general anticommutative law:

$$dx_{j_1} \wedge \cdots \wedge dx_{j_p} = \operatorname{sign}\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix} dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $(j_1 \cdots j_p)$ is a permutation of $(i_1 \cdots i_p)$ and $\operatorname{sign}\binom{i_1 \cdots i_p}{j_1 \cdots j_p}$ is the sign of the permutation. Also the differentiation of differential forms is done according to the rule

$$d\omega = \sum df_{i_1,\dots,i_p}(x) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

With the above terminology and notation Stokes' theorem takes the following form.

Stokes' Theorem. *If $D \subset \mathbb{R}^n$ is an open bounded set with smooth boundary ∂D, and ω is an $(n-1)$-form in a neighborhood of $\overline{D}$, then*

$$\int_D d\omega = \int_{\partial D} \omega.$$

We will use determinants with entries differential forms: if a_{ij} are differential forms, then

$$\det \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} = \det\left[a_{1j}, \dots, a_{nj}\right] =: \sum_{\sigma} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \wedge \cdots \wedge a_{n\sigma(n)},$$

where the summation is extended over all permutations σ of $\{1, \dots, n\}$. (The elements of each column are assumed to be differential forms of the same degree; this degree may change from column to column.) Thus when we write $\det\left[a_{1j}, \dots, a_{nj}\right]$, we mean that j runs from 1 to n forming the n rows of the determinant. The value of such a determinant does not change if we add to a row a multiple of another row (we mean multiplied by a function). In some determinants a column may be repeated, and we put an index to indicate how many times this column is repeated. For example, in the determinant

$$\det_{n-2}\begin{bmatrix} a_{11} & a_{12} & b_1 & \cdots & b_1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & b_n & \cdots & b_n \end{bmatrix}$$

the column $[b_1, \ldots, b_n]^T$ is repeated $n-2$ times. A determinant of the form

$$\det_{n-1}\begin{bmatrix} a_1 & b_1 & \cdots & b_1 \\ \vdots & \vdots & & \vdots \\ a_n & b_n & \cdots & b_n \end{bmatrix}$$

is denoted by $\det_{1,n-1}[a_j, b_j]$. Now if the a_j are functions and the b_j are 1-forms then, expanding the above determinant, we obtain

$$\det_{1,n-1}[a_j, b_j] = (n-1)! \sum_{j=1}^{n} (-1)^{j-1} a_j \cdot b_1 \wedge \cdots (j) \cdots \wedge b_n.$$

If furthermore $db_j = 0$, then

$$d(\det_{1,n-1}[a_j, b_j]) = \det_{1,n-1}[da_j, b_j].$$

All these properties follow from the corresponding properties of the usual determinants if we take into consideration the anticommutative law for differential forms:

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega,$$

where

$$p = \deg \omega \qquad \text{and} \qquad q = \deg \eta.$$

For more properties of such determinants see [1: p. 8] and for the calculus of differential forms, that we are using, see [4: Chapter 4].

Definition. Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. For a C^1-map $f : \partial D \longrightarrow \mathbb{R}^n \setminus \{0\}$ we define

$$\eta(f) = \frac{c}{|f|^n} \sum_{j=1}^{n} (-1)^{j-1} f_j \, df_1 \wedge \cdots (j) \cdots \wedge df_n$$

where $|f|^2 = \sum_{j=1}^{n} f_j^2$ and $c = \frac{\Gamma(n/2)}{2\pi^{n/2}}$. The *degree* $\delta(f)$ of f is defined by the integral

$$\delta(f) = \int_{\partial D} \eta(f).$$

3. The main results

We start be proving some lemmas from which the main results, Theorems 1 and 2, will follow; Lemma 2 is the main step.

Lemma 1. *Let $D \subset \mathbb{R}^n$ be a domain and $f : \partial D \longrightarrow \mathbb{R}^n \setminus \{0\}$ a C^2-map. Then $d\eta(f) = 0$ in D, and for a C^2-function $\phi : D \longrightarrow (0, \infty)$ we have $\eta(\phi \cdot f) = \eta(f)$.*

Proof. Let us notice that

$$\eta(f) = \frac{c'}{|f|^n} \det_{1,n-1}[f_j, df_j]$$

where $c' = \frac{c}{(n-1)!}$. Therefore,

$$\begin{aligned}
\eta(\phi \cdot f) &= \frac{c'}{\phi^n\,|f|^n} \det_{1,n-1}[\phi\, f_j, \phi\, df_j + f_j\, d\phi] \\
&= \frac{c'}{\phi^{n-1}\,|f|^n} \det_{1,n-1}[f_j, \phi\, df_j + f_j\, d\phi] \\
&= \frac{c'}{\phi^{n-1}\,|f|^n} \det_{1,n-1}[f_j, \phi\, df_j] \\
&= \frac{c'}{\phi^n\,|f|^n} \det_{1,n-1}[f_j, df_j] \\
&= \eta(f)
\end{aligned}$$

which proves the second assertion of the lemma; for the first assertion apply what we have just proved with $\phi = |f|^{-1}$ in order to obtain

$$\eta(f) = \eta(g) = c' \det_{1,n-1}[g_j, dg_j]$$

where $g = |f|^{-1} f$. Then

$$d\eta(f) = d\eta(g) = c' \det_{1,n-1}[dg_j, dg_j] = c'\, n!\, dg_1 \wedge \cdots \wedge dg_n.$$

But $\sum_{j=1}^n g_j^2 = 1$ and therefore $\sum_{j=1}^n g_j dg_j = 0$ which implies $dg_1 \wedge \cdots \wedge dg_n = 0$, i.e. $d\eta(f) = 0$ ∎

Lemma 2. *If $f = f(x,t) : (\partial D) \times [0,1] \longrightarrow \mathbb{R}^n \setminus \{0\}$ is a C^2-map, then the differential form $\frac{\partial}{\partial t}\eta(f(\cdot, t))$ is d-exact; more precisely,*

$$\frac{\partial}{\partial t}\eta(f(\cdot, t)) = d_x\theta \tag{1}$$

where

$$\theta = \frac{(n-1)\cdot c'}{|f|^n} \det_{n-2} \begin{bmatrix} f_1 & f_{1,t} & df_1 & \cdots & df_1 \\ \vdots & \vdots & \vdots & & \vdots \\ f_n & f_{n,t} & df_n & \cdots & df_n \end{bmatrix}$$

with $f_{j,t} = \frac{\partial f_j}{\partial t}$.

Proof. Exactly as in the case of η (see Lemma 1), θ remains the same if f is multiplied by a C^2-function

$$\phi(x,t) : (\partial D) \times [0,1] \longrightarrow (0,\infty).$$

Hence we may assume, without loss of generality, that

$$\sum_{j=1}^{n} f_j^2(x,t) = 1. \tag{2}$$

We may also assume that, near the point at which we want to prove (1), $f_1 \neq 0$. Then

$$f_1\,\eta(f(\,\cdot\,,t)) = c'\,\det{}_{n-1}\begin{bmatrix} f_1^2 & f_1\,df_1 & \cdots & f_1\,df_1 \\ f_2 & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ f_n & df_n & \cdots & df_n \end{bmatrix}.$$

(Throughout this proof $d = d_x$.) Now multiplying each j-th row of this determinant by f_j $(2 \le j \le n)$ and adding them to the first row we obtain, in view of (2),

$$f_1\,\eta(f(\,\cdot\,,t)) = c'\,\det{}_{n-1}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ f_2 & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ f_n & df_n & \cdots & df_n \end{bmatrix} = c\,df_2 \wedge \cdots \wedge df_n.$$

Hence

$$\eta(f(\,\cdot\,,t)) = \frac{c}{f_1}\,df_2 \wedge \cdots \wedge df_n$$

and therefore

$$\frac{\partial}{\partial t}\eta(f(\,\cdot\,,t)) = -c\,\frac{f_{1,t}}{f_1^2}\,df_2 \wedge \cdots \wedge df_n + \frac{c}{f_1}\sum_{j=2}^{n} df_2 \wedge \cdots \wedge df_{j,t} \wedge \cdots \wedge df_n. \tag{3}$$

On the other hand, as a similar computation shows,

$$\begin{aligned} f_1 \cdot \theta &= (n-1)\,c'\,\det{}_{n-2}\begin{bmatrix} f_1^2 & f_1 f_{1,t} & f_1\,df_1 & \cdots & f_1\,df_1 \\ f_2 & f_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & \vdots & & \vdots \\ f_n & f_{n,t} & df_n & \cdots & df_n \end{bmatrix} \\ &= (n-1)\,c'\,\det{}_{n-2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ f_2 & f_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & \vdots & & \vdots \\ f_n & f_{n,t} & df_n & \cdots & df_n \end{bmatrix} \\ &= (n-1)\,c'\,\det{}_{n-2}\begin{bmatrix} f_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ f_{n,t} & df_n & \cdots & df_n \end{bmatrix} \end{aligned}$$

Hence

$$\theta = \frac{(n-1)\,c'}{f_1}\,\mathrm{det}_{n-2}\begin{bmatrix} f_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ f_{n,t} & df_n & \cdots & df_n \end{bmatrix}$$

and therefore

$$\begin{aligned} d\theta = &-\frac{(n-1)\,c'}{f_1^2}\,df_1 \wedge \mathrm{det}_{n-2}\begin{bmatrix} f_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ f_{n,t} & df_n & \cdots & df_n \end{bmatrix} \\ &+\frac{(n-1)\,c'}{f_1}\,\mathrm{det}_{n-2}\begin{bmatrix} df_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ df_{n,t} & df_n & \cdots & df_n \end{bmatrix}. \end{aligned} \tag{4}$$

Now (2) gives

$$f_1\,df_1 = -\sum_{j=2}^{n} f_j\,df_j \qquad \text{and} \qquad f_1\,f_{1,t} = -\sum_{j=2}^{n} f_j\,f_{j,t}.$$

Hence

$$\begin{aligned} &f_1\,df_1 \wedge \mathrm{det}_{n-2}\begin{bmatrix} f_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ f_{n,t} & df_n & \cdots & df_n \end{bmatrix} \\ &\quad = \left(-\sum_{j=2}^{n} f_j\,df_j\right) \wedge \mathrm{det}_{n-2}\begin{bmatrix} f_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ f_{n,t} & df_n & \cdots & df_n \end{bmatrix} \\ &\quad = -(n-2)!\left(\sum_{j=2}^{n} f_j\,df_j\right) \wedge \left(\sum_{j=2}^{n}(-1)^j f_{j,t}\,df_2 \wedge \cdots (j) \cdots \wedge df_n\right) \\ &\quad = -(n-2)!\left(\sum_{j=2}^{n} f_j\,f_{j,t}\right) df_2 \wedge \cdots \wedge df_n \\ &\quad = (n-2)!\,f_1\,f_{1,t}\,df_2 \wedge \cdots \wedge df_n. \end{aligned} \tag{5}$$

Also,

$$\mathrm{det}_{n-2}\begin{bmatrix} df_{2,t} & df_2 & \cdots & df_2 \\ \vdots & \vdots & & \vdots \\ df_{n,t} & df_n & \cdots & df_n \end{bmatrix} = (n-2)\sum_{j=2}^{n} df_2 \wedge \cdots \wedge df_{j,t} \wedge \cdots \wedge df_n. \tag{6}$$

Substituting (5) and (6) into (4) and comparing with (3), we obtain (1). The proof is complete ∎

Lemma 3. *If $f = f(x,t) : (\partial D) \times [0,1] \longrightarrow \mathbb{R}^n \setminus \{0\}$ is a C^2-map, then the integral*

$$\delta(f(\cdot\,,t)) = \int_{\partial D} \eta(f(\cdot\,,t))$$

is independent of t.

Proof. In view of Lemma 2 and Stokes' theorem, we obtain

$$\frac{\partial}{\partial t}\delta(f(\cdot\,,t)) = \int_{\partial D} \frac{\partial}{\partial t}\eta(f(\cdot\,,t)) = \int_{\partial D} d_x\theta = \int_{\partial(\partial D)} \theta = 0$$

and the assertion of the lemma follows ∎

Lemma 4. *Let $f : \overline{D} \longrightarrow \mathbb{R}^n$ be a C^1-map with $f(x) \neq 0$ for all $x \in \partial D$. Then for $\varepsilon > 0$ there is an $y \in \mathbb{R}^n$ with $|y| < \varepsilon$ so that the equation $f(x) = y$ has at most a finite number of solutions, say $x^1, \ldots, x^p \in D$, at which $J_f(x^i) \neq 0$ for $i = 1, \ldots, p$ (here J_f denotes the Jacobian of f) and such that $f(x) \neq y$ for all $x \in \partial D$.*

Proof. Let $C = \{x \in D : J_f(x) = 0\}$. Then, by Sard's theorem, the set $f(C)$, the image of C under f, has measure zero. Hence there is an $y \in \mathbb{R}^n$ with $|y| < \varepsilon$ so that $y \notin f(C)$. Since $f(x) \neq 0$ for all $x \in \partial D$, we may also choose y sufficiently close to 0 so that $f(x) \neq y$ for all $x \in \partial D$.

We claim that the equation $f(x) = y$ has at most a finite number of solutions; for otherwise there would exist a sequence $\{x^\nu\}_{\nu \geq 1} \subset D$ of distinct points with

$$x^\nu \to x \in \overline{D} \text{ as } \nu \to \infty \qquad \text{and} \qquad f(x^\nu) = y \text{ for all } \nu \geq 1. \tag{7}$$

But then $f(x) = y$ whence $x \in D$ and $J_f(x) \neq 0$. Therefore, by the inverse function theorem, f is one-to-one on a neighborhood of x, which contradicts (7) and the proof is complete ∎

Lemma 5. *Suppose that $f : U \longrightarrow \mathbb{R}^n$ is a C^2-map on an open set $U \subset \mathbb{R}^n$ such that f is a diffeomorphism on a neighborhood of a point $a \in U$ to a neighborhood of $b = f(a)$. Then*

$$\int_{\partial B(a,\varepsilon)} \eta(f-b) = 1 \qquad \text{or} \qquad \int_{\partial B(a,\varepsilon)} \eta(f-b) = -1$$

for sufficiently small $\varepsilon > 0$, where $B(x,\varepsilon) = \{y \in \mathbb{R}^n : |y - x| < \varepsilon\}$.

Proof. We have

$$\int_{\partial B(a,\varepsilon)} \eta(f-b) = c \int_{f(B(a,\varepsilon))} \frac{1}{|y-b|^n} \sum_{j=1}^{n} (-1)^{j-1}(y_j - b_j)\, dy_1 \wedge \cdots (j) \cdots \wedge y_n.$$

But, by Lemma 1 and Stokes' theorem, the above integral becomes

$$\delta(f) = c \int_{\partial B(0,1)} \sum_{j=1}^{n} (-1)^{j-1} y_j\, dy_1 \wedge \cdots (j) \cdots \wedge dy_n$$

(here $\partial B(0,1)$ could have either orientation depending upon f). Therefore, by Stokes' theorem again,

$$\delta(f) = \pm c \int_{B(0,1)} n\, dy_1 \wedge \cdots \wedge dy_n = \pm c\mathrm{Vol}(B(0,1)) = \pm 1$$

and the proof is complete ∎

Theorem 1. *Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $f : \partial D \longrightarrow \mathbb{R}^n \setminus \{0\}$ a C^1-map. Then $\delta(f)$ is an integer.*

Proof. Let us assume first that f is a C^2-map and let $\tilde{f} : \overline{D} \longrightarrow \mathbb{R}^n$ be a C^2-extension of f. By Lemma 4 there is an $y \in \mathbb{R}^n$, sufficiently close to zero so that the equation $\tilde{f}(x) = y$ has finitely many solutions $x^1, \ldots, x^p \in D$ with $J_{\tilde{f}}(x^i) \neq 0$ and $f(x) \neq u$ for $x \in \partial D$ and $|u| \leq |y|$.

But, by Lemma 3, $\delta(f) = \delta(f - y)$. Also, for small $\varepsilon > 0$,

$$\delta(f - y) = \int_{\partial D} \eta(f - y) = \sum_{i=1}^{p} \int_{\partial B(x^i, \varepsilon)} \eta(\tilde{f} - y)$$

where $B(x^i, \varepsilon) = \{x \in \mathbb{R}^n : |x - x^i| \leq \varepsilon\}$; the last equation follows from Stokes' theorem applied to $\eta(\tilde{f} - y)$ and the domain

$$\Omega = D \setminus \bigcup_{i=1}^{p} B(x^i, \varepsilon),$$

since $d\eta(\tilde{f} - y) = 0$ by Lemma 1.

But $\tilde{f} - y$ is a C^2-diffeomorphism on a neighbothood of x^i and therefore, if ε is sufficiently small,

$$\int_{\partial B(x^i, \varepsilon)} \eta(\tilde{f} - y) = \pm 1,$$

by Lemma 5. Hence $\delta(f - y)$ is an integer and so is $\delta(f)$. This completes the proof in the case f is of class C^2. If f is only of class C^1, then we can approximate f by C^2-functions, uniformly on ∂D and prove that, in this case too, $\delta(f)$ is an integer. This completes the proof ∎

Theorem 2. *Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that $f : \overline{D} \longrightarrow \mathbb{R}^n$ is a continuous map and $g : \overline{D} \longrightarrow \mathbb{R}^n$ is a C^1-map with $g(x) \neq 0$ for all $x \in \partial D$ and $\delta(g) \neq 0$. If $|f(x)| \leq |g(x)|$ for every $x \in \partial D$, then the equation $f(x) + g(x) = 0$ has at least one solution x in $\overline{D}$.*

Proof. Let us prove the statement first in the case f is of class C^1 and $|f(x)| < |g(x)|$ for every $x \in \partial D$. Then for $t \in [0, 1]$ and $x \in \partial D$ we have

$$|g(x) + tf(x)| \geq |g(x)| - t|f(x)| > 0.$$

Therefore $\delta(g + tf)$ is independent of t, since it is continuous in t and integer-valued (by Theorem 1). Hence $\delta(0) = \delta(1)$, i.e. $\delta(g) = \delta(f + g)$ and therefore $\delta(f + g) \neq 0$. But if $f(x) + g(x) \neq 0$ for all $x \in \overline{D}$, then

$$\delta(f + g) = \int_{\partial D} \eta(f + g) = 0,$$

by Lemma 1 and Stokes' theorem, since $d[\eta(f + g)] = 0$ in $\overline{D}$. This is a contradiction which completes the proof in this case.

Now we consider the general case in which f is only continuous and $|f(x)| \leq |g(x)|$ for $x \in \partial D$. Then, by the Stone-Weierstrass theorem, for each $\lambda \in (0,1)$ there is a sequence of polynomials p_ν so that

$$p_\nu \to \lambda f \qquad \text{uniformly on } \overline{D} \text{ and } |p_\nu| < |g| \text{ on } \overline{D}.$$

Thus, by the first case considered, there is $x^\nu \in \overline{D}$ so that $p_\nu(x^\nu) + g(x^\nu) = 0$. Passing to a subsequence, we may assume that $x^\nu \to x_\lambda \in \overline{D}$. Therefore

$$\lambda f(x_\lambda) + g(x_\lambda) = 0.$$

Finally, choosing $\lambda_j \to 1$ so that $x_{\lambda_j} \to x \in \overline{D}$, we obtain that $f(x) + g(x) = 0$ and the proof is complete ∎

The following immediate consequence of Theorem 2 is Brouwer's fixed point theorem. It follows from Theorem 2 by applying it with $\overline{D}$ the closed unit ball of $\mathbb{R}^n$ and $g(x) = -x$.

Corollary. *Let* $\mathbf{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$. *Every continuous map* $f : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *has a fixed point.*

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