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Weighted Inequalities for the Fractional Maximal Operator in Lorentz Spaces via Atomic Decomposition of Tent Spaces

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Abstract. Consider the usual fractional maximal operator M_α with $0 \leq \alpha < n$. A characterization of $\mathbb{R}^n$ weight functions $u(\cdot)$ and $\sigma(\cdot)$ for which $M_\alpha d\sigma$ sends the (generalized) Lorentz space $\Lambda^s_\sigma(w_1)$ into $\Lambda^r_u(w_2)$ with $1 < s \leq r < \infty$ is obtained by using a suitable atomic decomposition of tent spaces.

Keywords: *Weighted inequalities, maximal operators, tent spaces, Lorentz spaces*

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1. Introduction

The Lorentz space $\Lambda^r_{d\nu}(w)$ is defined as the space of measurable functions $f(\cdot)$ on $\mathbb{R}^n$ satisfying

$$\|f(\cdot)\|^r_{\Lambda^r_{d\nu}(w)} = \int_0^\infty [f^*_\nu(t)]^r w(t)\,dt < \infty.$$

Here $0 < r < \infty$, $w(\cdot)$ is a weight function on $[0,\infty)$ (i.e. a non-negative locally integrable function), $d\nu(\cdot)$ is a locally finite positive Borel measure on $\mathbb{R}^n$ ($n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$), and $f^*_\nu(\cdot)$ is the decreasing rearrangement of $f(\cdot)$ defined on $[0,\infty)$ by

$$f^*_\nu(t) = \inf\left\{\lambda \geq 0 \;\middle|\; \int_{\{x\in\mathbb{R}^n:\,|f(x)|>\lambda\}} d\nu(x) = \big|\{\,|f(\cdot)| > \lambda\,\}\big|_\nu \leq t\right\}.$$

This function space is merely denoted as $\Lambda^r_u(w)$ when $d\nu(x) = u(x)dx$ with $u(\cdot)$ a weight function and dx the usual Lebesgue measure on $\mathbb{R}^n$. Many of usual spaces are particular cases of $\Lambda^r_{d\nu}(w)$. Indeed, the Lebesgue space $L^r(\mathbb{R}^n, d\nu(\cdot))$ is just $\Lambda^r_{d\nu}(1)$, and the classical Lorentz space $L^{qr}(\mathbb{R}^n, d\nu(\cdot))$ is obtained by putting $w(t) = t^{\frac{q}{r}-1}$. The space $L^{qr}[(\log L)^{\gamma q}](\mathbb{R}^n, d\nu(\cdot))$, useful in interpolation spaces, appears by taking $w(t) = t^{\frac{q}{r}-1}(1 + |\log t|)^{q\gamma}$.

The fractional maximal operator M_α $(0 \leq \alpha < n)$ is defined as

$$(M_\alpha f)(x) = \sup\left\{|Q|^{\frac{\alpha}{n}-1}\int_Q |f(y)|\,dy \;\middle|\; Q \text{ is a cube with } Q \ni x\right\}.$$

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All cubes Q considered have their sides parallel to the coordinate axis. So M_0 is the classical Hardy-Littlewood maximal operator.

The purpose of this paper is to characterize the weight functions $u(\cdot)$ and $\sigma(\cdot)$ for which there is a constant $C > 0$ so that

$$\|(M_\alpha f d\sigma)(\cdot)\|_{\Lambda_u^r(w_2)} \leq C \|f(\cdot)\|_{\Lambda_\sigma^s(w_1)} \qquad \text{for all } f(\cdot) \geq 0. \tag{1.1}$$

Here $1 < s \leq r < \infty$, and $w_1(\cdot)$ and $w_2(\cdot)$ are given weight functions on $[0, \infty)$. For convenience, inequality (1.1) will be denoted by

$$M_\alpha d\sigma : \Lambda_\sigma^s(w_1) \to \Lambda_u^r(w_2).$$

This embedding has an important link with $M_\alpha : \Lambda_v^s(w_1) \to \Lambda_u^r(w_2)$, i.e.

$$\|(M_\alpha g)(\cdot)\|_{\Lambda_u^r(w_2)} \leq C \|g(\cdot)\|_{\Lambda_v^s(w_1)} \qquad \text{for all } g(\cdot) \geq 0. \tag{1.2}$$

To the best of our knoweldge, a characterization of weights $u(\cdot)$ and $v(\cdot)$ for which (1.2) holds is an open problem. Indeed, only results for $M_0 : \Lambda_v^s(w_1) \to \Lambda_u^r(w_2)$ and with weights $u(\cdot)$ belonging to the Muckenhoupt class are available in the literature (see, for instance, [3, 4, 7]).

The first reason to deal with inequality (1.1) is that in many applications, for instance in trace inequality, the case of $d\sigma = dx$ is the most significant and interesting inequality under consideration. Next, inequality (1.1) yields a solution to inequality (1.2) when $w_1(\cdot) = 1$. As a third reason, problem (1.2) can be solved by using (1.1) when the weight functions $v(\cdot)$ belong to some Muckenhoupt class. However for the general case, the two embeddings $M_\alpha d\sigma : \Lambda_\sigma^s(w_1) \to \Lambda_u^r(w_2)$ and $M_\alpha : \Lambda_v^s(w_1) \to \Lambda_u^r(w_2)$ are completely different.

Our approach of (1.1) is based on atomic decomposition of some suitable tent space (see Section 5). The idea of using tent spaces to tackle maximal inequalities was already alluded by many authors (see, for instance, [11]). But the systematic development with various weights as presented here is not done. So we hope with the present work to fill this lack in the literature. The technique used here is inspired on the author's paper [8], where weighted inequalities for M_α on classical weighted Lebesgue spaces were considered.

So in Theorem 2.1 we obtain a characterization of the embedding $M_\alpha d\sigma : \Lambda_\sigma^s(w_1) \to \Lambda_u^r(w_2)$. As a consequence, the boundedness of $M_\alpha : L_v^p \to \Lambda_u^r(w)$ is stated in Proposition 2.2. And the embedding $M_\alpha : \Lambda_v^s(w_1) \to \Lambda_u^r(w_2)$ is also characterized (in Theorem 2.4) whenever the weight $v(\cdot)$ belongs to some Muckenhoupt class. As it is known [9] in the Lebesgue case, a characterizing condition is in general difficult to check, so this question is also examined in Corollary 2.3. Finally the statements of our results for the classical case $M_\alpha d\sigma : L_\sigma^{ps} \to L_u^{qr}$ or $M_\alpha : L_v^{ps} \to L_u^{qr}$ are also included in Corollaries 2.5, 2.6 and 2.7.

2. Main Results

To study the boundedness $M_\alpha d\sigma : \Lambda^s_\sigma(w_1) \to \Lambda^r_u(w_2)$ or

$$\|(M_\alpha f d\sigma)(\cdot)\|_{\Lambda^r_u(w_2)} \le C\,\|f(\cdot)\|_{\Lambda^s_\sigma(w_1)} \qquad \text{for all } f(\cdot) \ge 0, \tag{2.1}$$

some restrictions on α, s, r, $w_1(\cdot)$, $w_2(\cdot)$, $u(\cdot)$ and $\sigma(\cdot)$ are done. So it is always assumed that

$$0 \le \alpha < n \qquad \text{and} \qquad 1 < s \le r < \infty.$$

Further:

- $u(\cdot)$ and $\sigma(\cdot)$ are weight functions on $\mathbb{R}^n$ such that $u(\cdot) \notin L^1(\mathbb{R}^n, dx)$ and $\sigma(\cdot) > 0$ a.e.
- $w_1(\cdot)$ and $w_2(\cdot)$ are weight functions on $[0,\infty)$ for which the following growth conditions are satisfied:

$$w_1(\cdot) \in B_s, \qquad w_2(\cdot) \in B_r; \tag{2.2}$$

there is a real ε such that $s \le \varepsilon \le r$ and

$$\Big(\sum_j [W_1(t_j)]^{\frac{\varepsilon}{s}}\Big)^{\frac{s}{\varepsilon}} \le c W_1\Big(\sum_j t_j\Big) \qquad \text{for all } t_j > 0; \tag{2.3}$$

for $s = r$ it is assumed that $w_2(\cdot) \in B_{1\infty}$, else $w_2(\cdot) \in B_{\frac{r}{\varepsilon}}$. (2.4)

Here $c > 0$ is a fixed constant which only depends on $w_1(\cdot)$. And $W_1(\cdot)$ is defined as $W_1(R) = \int_0^R w_1(t)\,dt$. For $p > 1$, the condition $w(\cdot) \in B_p$ means

$$\int_R^\infty w(t)t^{-p}\,dt \le CR^{-p}\int_0^R w(t)\,dt \qquad \text{for all } R > 0.$$

And $w(\cdot) \in B_{1\infty}$ if there is $C > 0$ such that

$$R_2^{-1}W(R_2) \le C\,R_1^{-1}W(R_1) \qquad \text{for } 0 < R_1 \le R_2.$$

Condition $w(\cdot) \in B_p$ (resp. $w(\cdot) \in B_{1\infty}$) ensures that $\|\cdot\|_{\Lambda^p_u(w)}$ (resp. $\|\cdot\|_{\Lambda^1_u(w)}$) is equivalent to a norm (see [10] and [2]). Thus for a fixed constant $C > 0$

$$\Big\|\sum_j F_j(\cdot)\Big\|_{\Lambda^p_u(w)} \le C\sum_j \Big\|F_j(\cdot)\Big\|_{\Lambda^p_u(w)} \qquad \text{for all } F_j(\cdot) \ge 0. \tag{2.5}$$

A sort of converse of (2.5) is held under condition (2.3). Precisely,

$$\sum_j \Big\|G_j(\cdot)\Big\|^\varepsilon_{\Lambda^s_\sigma(w)} \le C\Big\|\sum_j G_j(\cdot)\Big\|^\varepsilon_{\Lambda^s_\sigma(w)} \quad \text{for } G_j(\cdot) \ge 0 \text{ with disjoint supports.} \tag{2.6}$$

The reason, why the growth conditions (2.2), (2.3) and (2.4) are introduced, can now roughly explained. Indeed, by using a suitable atomic decomposition of tent spaces (see Section 5), the left side in (2.1) is broken into pieces by applying the rule (2.5). Next the test condition (2.7) (see below) leads to do summations as displayed in (2.6) and in order to capture again the initial function $f(\cdot)$.

Now the first main result can be stated.

Theorem 2.1.

(a) *Suppose $M_\alpha d\sigma : \Lambda^s_\sigma(w_1) \to \Lambda^r_u(w_2)$. Then for a constant $A > 0$*

$$\left\|(M_\alpha 1\!\!1_Q d\sigma)(\cdot)1\!\!1_Q(\cdot)\right\|_{\Lambda^r_u(w_2)} \leq A\left\|1\!\!1_Q(\cdot)\right\|_{\Lambda^s_\sigma(w_1)} \qquad \textit{for all cubes } Q. \tag{2.7}$$

Here $1\!\!1_Q(\cdot)$ is the characteristic function of the cube Q.

(b) *For the converse, the growth conditions* (2.2), (2.3) *and* (2.4) *are assumed. So the test condition* (2.7) *implies $M_\alpha d\sigma : \Lambda^s_\sigma(w_1) \to \Lambda^r_u(w_2)$.*

Condition (2.7) is the analogue of the famous Sawyer's condition [9] for the Lebesgue spaces setting.

This result leads to the characterization of the weight functions $u(\cdot)$ and $v(\cdot)$ for which $M_\alpha : L^p_v \to \Lambda^r_u(w)$, where $L^p_v = L^p(\mathbb{R}^n, v(x)dx) = \Lambda^p_{v(x)dx}(1)$. So from now, the following is supposed:

- $1 < p \leq r < \infty$.
- $v(\cdot)$ and $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot)$ are weight functions on $\mathbb{R}^n$.
- $w(\cdot)$ is a weight function on $[0, \infty)$.

Proposition 2.2.

(a) *Suppose $M_\alpha : L^p_v \to \Lambda^r_u(w)$. Then for a constant $A > 0$*

$$\left\|(M_\alpha v^{-\frac{1}{p-1}} 1\!\!1_Q)(\cdot)1\!\!1_Q(\cdot)\right\|_{\Lambda^r_u(w)} \leq A\left\|v^{-\frac{1}{p-1}}(\cdot)1\!\!1_Q(\cdot)\right\|_{L^p_v} \qquad \textit{for all cubes } Q. \tag{2.8}$$

(b) *Conversely, condition* (2.8) *implies $M_\alpha : L^p_v \to \Lambda^r_u(w)$ whenever there is ε such that $p \leq \varepsilon \leq r$ and for which condition* (2.4) *with $s = p$ and $w_2(\cdot) = w(\cdot)$ is satisfied.*

Although Theorem 2.1 and Proposition 2.2 yield respectively characterizations of $M_\alpha d\sigma : \Lambda^s_\sigma(w_1) \to \Lambda^r_u(w_2)$ and $M_\alpha : L^p_v \to \Lambda^r_u(w)$, the conditions under consideration are in general difficult to check since they are expressed in term of the fractional maximal function M_α itself. However easily verifiable conditions can be derived under the reverse doubling condition RD_ρ with $\rho > 0$. Thus $w(\cdot) \in RD_\rho$ whenever there is $c > 0$ such that

$$\int_{Q_1} w(y)dy \leq c\left(\frac{|Q_1|}{|Q|}\right)^\rho \int_Q w(y)dy \qquad \text{for all cubes } Q_1 \text{ and } Q \text{ with } Q_1 \subset Q.$$

Many of usual weight functions have this property.

Corollary 2.3.

(a) *Suppose $\sigma(\cdot) \in RD_\rho$ with $1 - \frac{\alpha}{n} \leq \rho$. Then condition* (2.7) *in Theorem* 2.1 *can be replaced by*

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q \sigma(y)dy\right)\left[W_2\left(\int_Q u(x)dx\right)\right]^{\frac{1}{r}} \leq A\left[W_1\left(\int_Q \sigma(y)dy\right)\right]^{\frac{1}{s}} \tag{2.9}$$

for all cubes Q.

(b) *With the same hypothesis and $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot)$, then condition (2.8) in Proposition 2.1 can be replaced by*

$$|Q|^{\frac{\alpha}{n}-1}\left[W\Big(\int_Q u(x)dx\Big)\right]^{\frac{1}{r}}\Big(\int_Q v^{-\frac{1}{p-1}}(y)dy\Big)\Bigg]^{1-\frac{1}{p}} \leq A \qquad \textit{for all cubes } Q. \tag{2.10}$$

Theorem 2.1 yields a characterization of $u(\cdot)$ and $v(\cdot)$ for which $M_\alpha : \Lambda_v^s(w_1) \to \Lambda_u^r(w_2)$ and whenever $w_1(t) = 1$. To study this embedding for more general weights $w_1(\cdot)$, the standard Muckenhoupt conditions $v(\cdot) \in A_t$ $(t \geq 1)$ are needed. Remind that $v(\cdot) \in A_1$ if for a constant $c > 0$

$$|Q|^{-1}\int_Q v(y)\,dy \leq c \inf_{z\in Q} v(z) \qquad \text{for all cubes } Q,$$

and that $v(\cdot) \in A_t$ $(t > 1)$ if

$$\Big(|Q|^{-1}\int_Q v(y)\,dy\Big)^{\frac{1}{t}}\Big(|Q|^{-1}\int_Q v^{-\frac{1}{t-1}}(y)\,dy\Big)^{1-\frac{1}{t}} \leq c \qquad \text{for all cubes } Q.$$

The second main result for this paper can be stated as follows:

Theorem 2.4.

(a) *Suppose $M_\alpha : \Lambda_v^s(w_1) \to \Lambda_u^r(w_2)$. Then for a constant $A > 0$*

$$|Q|^{\frac{\alpha}{n}}\left[W_2\Big(\int_Q u(y)\,dy\Big)\right]^{\frac{1}{r}} \leq A\left[W_1\Big(\int_Q v(y)\,dy\Big)\right]^{\frac{1}{s}} \qquad \textit{for all cubes } Q. \tag{2.11}$$

(b) *For the converse, suppose $v(\cdot) \in A_t$ for some t with $1 \leq t < s$. Then condition (2.11) implies $M_\alpha : \Lambda_v^s(w_1) \to \Lambda_u^r(w_2)$ whenever $w_1(\cdot) \in B_{\frac{s}{t}}$ and there is ε such that*

$$\Big(\sum_j [W_1(t_j)]^{\frac{t\varepsilon}{s}}\Big)^{\frac{s}{t\varepsilon}} \leq cW_1\Big[\sum_j t_j\Big] \qquad \textit{for all } t_j > 0, \tag{2.12}$$

and $w_2(\cdot) \in B_{1\infty}$ if $s = r = t\varepsilon$, else $w_2(\cdot) \in B_{\frac{r}{t\varepsilon}}$ (and in this case $s \leq t\varepsilon < r$).

Finally, we end this section by stating the corresponding results for classical Lorentz spaces $L_{d\omega}^{ps}$, which can be seen as $\Lambda_{d\omega}^s(w)$ with $w(\tau) = \tau^{\frac{s}{p}-1}$. So from now it is assumed that

$$1 < p, s, q, r < \infty,$$

with roughly speaking $\max(p,s) \leq \min(q,r)$. Precisely the restriction done is described by one of the following inequalities:

$$\begin{array}{lll} p \leq s < q \leq r & p < s = q = r & p = s = q = r \\ p \leq s \leq r < q & s < p < q \leq r & s < p \leq r < q. \end{array} \tag{2.13}$$

Corollary 2.5.

(a) *Suppose* $M_\alpha d\sigma : L^{ps}_\sigma \to L^{qr}_u$. *Then for a constant* $A > 0$

$$\left\|(M_\alpha d\sigma 1\!\!1_Q)(\cdot)1\!\!1_Q(\cdot)\right\|_{L^{qr}_u} \leq A\left\|1\!\!1_Q(\cdot)\right\|_{L^{ps}_\sigma} \quad \textit{for all cubes } Q. \tag{2.14}$$

(b) *Conversely, condition* (2.14) *implies* $M_\alpha d\sigma : L^{ps}_{d\sigma} \to L^{qr}_u$ *whenever one of the inequalities in assumption* (2.13) *is satisfied.*

(c) *Inequality* (2.14) *in parts* (*a*) *and* (*b*) *can be replaced by*

$$|Q|^{\frac{\alpha}{n}-1}\Big(\int_Q \sigma(y)dy\Big)^{1-\frac{1}{p}}\Big(\int_Q u(x)dx\Big)^{\frac{1}{q}} \leq A \quad \textit{for all cubes } Q \tag{2.15}$$

whenever $\sigma(\cdot) \in RD_\rho$ *with* $1 - \frac{\alpha}{n} \leq \rho$.

Corollary 2.6.

(a) *Suppose* $M_\alpha : L^p_v \to L^{qr}_u$. *Then for a constant* $C > 0$

$$\left\|(M_\alpha v^{-\frac{1}{p-1}})(\cdot)1\!\!1_Q(\cdot)\right\|_{L^{qr}_u} \leq C\left\|v^{-\frac{1}{p-1}}(\cdot)1\!\!1_Q(\cdot)\right\|_{L^p_v} \quad \textit{for all cubes } Q. \tag{2.16}$$

(b) *Conversely, condition* (2.16) *implies* $M_\alpha : L^p_v \to L^{qr}_u$ *whenever one of the inequalities in assumption* (2.13) *is satisfied.*

(c) *Inequality* (2.16) *in parts* (*a*) *and* (*b*) *can be replaced by* (2.15) *whenever* $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot) \in RD_\rho$ *and* $1 - \frac{\alpha}{n} \leq \rho$.

Corollary 2.7.

(a) *Suppose* $M_\alpha : L^{ps}_v \to L^{qr}_u$. *Then for a constant* $A > 0$

$$|Q|^{\frac{\alpha}{n}}\Big(\int_Q u(y)\,dy\Big)^{\frac{1}{q}} \leq A\Big(\int_Q v(x)\,dx\Big)^{\frac{1}{p}} \quad \textit{for all cubes } Q. \tag{2.17}$$

(b) *For the converse, let* $v(\cdot) \in A_t$ *for some* t *with* $1 \leq t < s, p$. *Then condition* (2.17) *implies* $M_\alpha : L^{ps}_v \to L^{qr}_u$, *whenever one of the inequalities in assumptions* (2.13) *is satisfied.*

The results and method introduced in this paper may be easily generalized to the setting of homogeneous type spaces [5]. But for convenience, this generalization is not treated here.

Proofs of Proposition 2.2 and Corollaries 2.3, 2.5, 2.6 and 2.7 will be given in the next Section 3. With the help of a basic result (Theorem 4.1), Theorem 2.1 will be proved in Section 4. The proof of this basic result will be done in Section 5. And the last Section 6 will be devoted to the proof of Theorem 2.4.

3. Proofs of Proposition 2.2 and Corollaries 2.3 and 2.5 - 2.7

This section is devoted to the proofs of some consequences of our main results (Theorems 2.1 and 2.4).

Proof of Proposition 2.2. Assume that $M_\alpha : L^p_v \to \Lambda^r_u(w)$. Taking $f(\cdot) = v^{-\frac{1}{p-1}} 1\!\!1_Q(\cdot)$ in the corresponding inequality then condition (2.8) appears.

Next suppose (2.8) is satisfied. The extra condition (2.3) is trivially satisfied since $W_1(R) = R$ and $1 \leq \frac{\varepsilon}{p}$. Since $\|v^{-\frac{1}{p-1}}(\cdot)g(\cdot)\|_{L^p_v} = \|g(\cdot)\|_{L^p_\sigma} = \|g(\cdot)\|_{\Lambda^p_\sigma(w_1)}$ with $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot)$ and $w_1(\cdot) = 1$ $(\in B_p)$, then condition (2.8) is nothing else than (2.7). So, by Theorem 2.1, $M_\alpha d\sigma : \Lambda^p_\sigma(w_1) \to \Lambda^r_u(w)$ which is also equivalent to $M_\alpha : L^p_v \to \Lambda^r_u(w)$ ∎

Proof of Corollary 2.3. The proof can be restricted to part (a). To check condition (2.7), the point is just to use

$$(M_\alpha 1\!\!1_Q d\sigma)(x) 1\!\!1_Q(x) \leq C|Q|^{\frac{\alpha}{n}-1} \int_Q \sigma(y)dy \qquad \text{for all cubes } Q \tag{3.1}$$

where the constant $C > 0$ only depends on α, n and σ.

To prove (3.1), take a cube Q and $x \in Q$. It is sufficient to estimate $\mathcal{Q} = |Q'|^{\frac{\alpha}{n}-1} \int_{Q' \cap Q} \sigma(y)dy$ $Q' \ni x$, by the right member of (3.1). If Q' is a big cube, or presicely $\frac{1}{100}|Q| \leq |Q'|$, then clearly $\mathcal{Q} \leq C|Q|^{\frac{\alpha}{n}-1} \int_Q d\sigma(y)$. Next consider the case of a small cube, i.e. $0 < |Q'| < \frac{1}{100}|Q|$. One can find a cube $Q_1 \subset Q$ with $|Q_1| = |Q'|$ and $Q' \cap Q \subset Q_1$ such that $Q_1 = Q'$ if $Q' \subset Q$. Using $\sigma(\cdot) \in RD_\rho$ and $0 \leq \rho + \frac{\alpha}{n} - 1$, then

$$\begin{aligned} \mathcal{Q} &\leq |Q'|^{\frac{\alpha}{n}-1} \int_{Q_1} \sigma(y)dy \leq C|Q'|^{\frac{\alpha}{n}-1} \left(\frac{|Q_1|}{|Q|}\right)^\rho \int_Q \sigma(y)dy \\ &\leq C\left(\frac{|Q'|}{|Q|}\right)^{\rho+\frac{\alpha}{n}-1} |Q|^{\frac{\alpha}{n}-1} \int_Q \sigma(y)dy \leq C|Q|^{\frac{\alpha}{n}-1} \int_Q \sigma(y)dy \end{aligned}$$

and the assertion is proved ∎

Proof of Corollaries 2.5 and 2.6. Only parts (b) of these results need to be proved. Since $L^{ps}_\sigma = \Lambda^s_\sigma(w_1(\tau) = \tau^{\frac{s}{p}-1}$ and $L^{qr}_u = \Lambda^r_\sigma(w_2(\tau) = \tau^{\frac{r}{q}-1}$, then the conclusion is obtained by applying Theorem 2.1, Proposition 2.2 and Corollary 2.3, and the main problem is reduced to see that conditions (2.2), (2.3) and (2.4) are held under $1 < p, q, s, q, r < \infty$ and one of the inequalities in (2.13).

First $w_1(\cdot) \in B_s$ if $\frac{s}{p} < s$ or $p > 1$, and $w_2(\cdot) \in B_r$ if $\frac{r}{q} < r$ or $q > 1$. Thus condition (2.2) is satisfied. Next since $W_1(\tau) \approx \tau^{\frac{s}{p}}$, then clearly condition (2.3) is satisfied whenever

$$s \leq \varepsilon \leq r \qquad \text{and} \qquad p \leq \varepsilon. \tag{3.2}$$

On the other hand $w_2(\cdot) \in B_{\frac{r}{\varepsilon}}$ if $\frac{r}{q} < \frac{r}{\varepsilon}$ or $\varepsilon < q$ and $w_2(\cdot) \in B_{1\infty}$ if $\frac{r}{q} - 1 \leq 0$ or $r \leq q$. Consequently, to satisfy condition (2.4) it is needed that

$$\varepsilon < q \quad \text{for } s < r \qquad \text{and} \qquad r \leq q \quad \text{if } s = r. \tag{3.3}$$

The real ε, for which both conditions (3.2) and (3.3) are satisfied, exits under one of the retrictions (2.13) on s, p, q, and r ∎

Proof of Corollary 2.7. As above to prove part (b) of this result, it is sufficient to check the conditions needed to the conclusions in part (b) of Theorem 2.4. The details are similar to the above, except that instead directly of (2.13) the restriction used is on $\frac{p}{t}$, $\frac{s}{t}$, $\frac{q}{t}$ and $\frac{r}{t}$ ∎

4. Proof of Theorem 2.1

Theorem 2.1 is based on weighted inequalities for the dyadic version of the maximal operator M_α, which is defined as

$$(\mathcal{M}_\alpha g)(x) = \sup_{Q \ni x} \left\{ |Q|^{\frac{\alpha}{n}-1} \int_Q |g(y)|\, dy \Big| Q \text{ a closed dyadic cube} \right\}.$$

Remind that a closed dyadic cube is a product of n intevalls $[x_i, x_i + 2^k)$ where $x = (x_i)_i \in 2^k\mathbb{Z}^n$ for some $k \in \mathbb{Z}$. Assume that $d\omega(\cdot)$ and $d\sigma(\cdot)$ are locally finite positive measures which do not charge points of $\mathbb{R}^n$ with $0 < \int_Q d\sigma(x) < \infty$ for all cubes Q and $\int_{\mathbb{R}^n} d\omega(x) = \infty$. The main result on which Theorem 2.1 lies is the following

Theorem 4.1.

(a) *Suppose* $\mathcal{M}_\alpha d\sigma : \Lambda^s_{d\sigma}(w_1) \to \Lambda^r_{d\omega}(w_2)$. *Then for a constant* $A > 0$

$$\left\|(\mathcal{M}_\alpha 1\!\!1_Q d\sigma)(\cdot) 1\!\!1_Q(\cdot)\right\|_{\Lambda^r_{d\omega}(w_2)} \leq A \left\|1\!\!1_Q(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \qquad \text{for all dyadic cubes } Q. \tag{4.1}$$

(b) *For the converse, the growth conditions* (2.2), (2.3) *and* (2.4) *are assumed. So the test condition* (4.1) *implies* $\mathcal{M}_\alpha d\sigma : \Lambda^s_{d\sigma}(w_1) \to \Lambda^r_{d\omega}(w_2)$, *precisely:*

$$\left\|(\mathcal{M}_\alpha f d\sigma)(\cdot)\right\|_{\Lambda^r_{d\omega}(w_2)} \leq cA \left\|f(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \qquad \text{for all } f(\cdot) \geq 0 \tag{4.2}$$

with a constant $c > 0$ *which only depends on* n, r, s, $w_1(\cdot)$, *and* $w_2(\cdot)$ (*but not on* $d\omega(\cdot)$ *and* $d\sigma(\cdot)$).

This result will be proved in the next section, and for the moment we are proceeding to prove the embedding $M_\alpha d\sigma : \Lambda^s_\sigma(w_1) \to \Lambda^r_u(w_2)$. Due to the monotone convergence theorem, it is sufficient to find a constant $c > 0$ such that

$$\left\|(M_\alpha^{2^N} f d\sigma)(\cdot)\right\|_{\Lambda^r_u(w_2)} \leq cA \left\|f(\cdot)\right\|_{\Lambda^s_\sigma(w_1)} \qquad \text{for all } f(\cdot) \geq 0, \tag{4.3}$$

and all integers N. Here the truncated maximal function M_α^R is defined as usual by $(M_\alpha^R f)(x) = \sup_{Q \ni x}\{|Q|^{\frac{\alpha}{n}-1} \int_Q |f(y)|\, dy \big| |Q|^{\frac{1}{n}} \leq R\}$. As in [9], the first point to get (4.3) is

$$(M_\alpha^{2^N} g)(x) \leq c_1 \int_{[-2^{N+2}, 2^{N+2}]^n} ({}^z M_\alpha g)(x) \frac{dz}{2^{n(N+3)}}. \tag{4.4}$$

Here $c_1 > 0$ does not depend on x, $z \in \mathbb{R}^n$ and $N \in \mathbb{N}^*$, and ${}^zM_\alpha$ is defined as $({}^zM_\alpha f)(x) = \sup_{Q\ni x}\{|Q|^{\frac{\alpha}{n}-1}\int_Q |f(y)|\,dy \,|\, Q-z \text{ a closed dyadic cube}\}$. The second point for (4.3) is the existence of a constant $c_2 > 0$ for which

$$\left\|({}^zM_\alpha f d\sigma)(\cdot)\right\|_{\Lambda^r_u(w_2)} \le c_2 A\left\|f(\cdot)\right\|_{\Lambda^s_\sigma(w_1)} \qquad \text{for all } f(\cdot)\ge 0 \tag{4.5}$$

and all $z \in \mathbb{R}^n$. Indeed, using (4.4) and the fact that $\|\cdot\|_{\Lambda^r_u(w_2)}$ is equivalent to a norm (since $w_2(\cdot) \in B_r$) then (4.3) appears as follows:

$$\begin{aligned}
\left\|(M^{2^N}_\alpha f d\sigma)(\cdot)\right\|_{\Lambda^r_u(w_2)} &\le c_1 \int_{[-2^{N+2},2^{N+2}]^n} \left\|({}^zM_\alpha f d\sigma)(\cdot)\right\|_{\Lambda^r_u(w_2)} \frac{dz}{2^{n(N+3)}}\\
&\le c_1 \int_{[-2^{N+2},2^{N+2}]^n} c_2 A \left\|f(\cdot)\right\|_{\Lambda^s_\sigma(w_1)} \frac{dz}{2^{n(N+3)}} \quad \text{(by (4.5))}\\
&\le c_3 A\left\|f(\cdot)\right\|_{\Lambda^s_\sigma(w_1)} \quad \textit{(since } c_2 \textit{ does not depend on } z\textit{)}.
\end{aligned}$$

We end with the proof of Theorem 2.1 by proving inequality (4.5). It is for this purpose why Theorem 4.1 is crucial in the proof. First note that the test condition (2.7) implies (4.1) with the measures $d\sigma_z(\cdot) = \sigma(\cdot+z)dx$ and $d\omega_z(\cdot) = u(\cdot+z)dx$, and the constant $A > 0$ independent on z. Indeed, for each dyadic cubes Q then

$$\begin{aligned}
&\left\|(\mathcal{M}_\alpha d\sigma_z \mathbb{1}_Q)(\cdot)\mathbb{1}_Q(\cdot)\right\|_{\Lambda^r_{d\omega_z}(w_2)}\\
&\le \left\|(M_\alpha d\sigma_z \mathbb{1}_Q)(\cdot)\mathbb{1}_Q(\cdot)\right\|_{\Lambda^r_{d\omega_z}(w_2)} = \left\|(M_\alpha d\sigma \mathbb{1}_{Q+z})(\cdot)\mathbb{1}_{Q+z}(\cdot)\right\|_{\Lambda^r_u(w_2)}\\
&\le A\left\|\mathbb{1}_{Q+z}(\cdot)\right\|_{\Lambda^s_\sigma(w_1)} = A\left\|\mathbb{1}_Q(\cdot)\right\|_{\Lambda^s_{d\sigma_z}(w_1)}.
\end{aligned}$$

So, by Theorem 4.1, the embedding (4.2) with the measures $d\sigma_z(\cdot)$ and $d\omega_z(\cdot)$ can be assumed to hold with the constant cA where c does not depend on z.

Now, with some notations abuse, inequality (4.5) appears as follows:

$$\begin{aligned}
\left\|({}^zM_\alpha f d\sigma)(\cdot)\right\|_{\Lambda^r_u(w_2)} &= \left\|({}^zM_\alpha f d\sigma)(x)\right\|_{\Lambda^r_{u(x)dx}(w_2)}\\
&= \left\|(\mathcal{M}_\alpha[f(\cdot+z)d\sigma_z])(x)\right\|_{\Lambda^r_{d\omega_z(x)}(w_2)}\\
&\qquad \textit{(by using the definition of } {}^zM_\alpha\textit{)}\\
&\le cA\left\|f(x+z)\right\|_{\Lambda^s_{d\sigma_z(x)}(w_1)}\\
&= cA\left\|f(x+z)\right\|_{\Lambda^s_{\sigma(x+z)dx}(w_1)}\\
&= cA\left\|f(\cdot)\right\|_{\Lambda^s_\sigma(w_1)}.
\end{aligned}$$

5. Proof of Theorem 4.1

To prove Part (b) of Theorem 4.1 then, by translation and reflection, it is sufficient to find a constant $c > 0$ such that

$$\left\|(\mathcal{M}_\alpha^{Q[0,R]} f d\sigma)(\cdot)1\!\!1_{Q[0,R]}(\cdot)\right\|_{\Lambda^r_{d\omega}(w_2)} \leq cA\left\|f(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \quad \text{for all } f(\cdot) \geq 0 \tag{5.1}$$

and all $R > 0$. Here $Q[0,R] = (0,R)^n$ and

$$(\mathcal{M}_\alpha^{Q[0,R]} g)(x) = \sup_{Q \ni x}\left\{|Q|^{\frac{\alpha}{n}-1}\int_Q |g(y)|\,dy \,\middle|\, Q \text{ a closed dyadic cube with } Q \subset Q[0,R]\right\}.$$

Proposition 5.1. *Let $w_1(\cdot) \in B_s$. For all $\varepsilon > 0$ there is a constant $C > 0$ such that, for all $f(\cdot) \in \Lambda^s_{d\sigma}(w_1)$ and all $R > 0$, one can find $\lambda_j > 0$ and dyadic cubes Q_j satisfying*

$$(\mathcal{M}_\alpha^{Q[0,R]} f d\sigma)^\varepsilon(\cdot)1\!\!1_{Q[0,R]}(\cdot) \leq \sum_j \lambda_j^\varepsilon [W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}(\mathcal{M}_\alpha 1\!\!1_{Q_j} d\sigma)^\varepsilon(\cdot)1\!\!1_{Q_j}(\cdot) \tag{5.2}$$

and

$$\Big(\sum_j \lambda_j^\varepsilon\Big)^{\frac{1}{\varepsilon}} \leq C\left\|f(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \tag{5.3}$$

whenever $s \leq \varepsilon$ and condition (2.3) *is satisfied.*

Here $|E|_\sigma = \int_E d\sigma(x)$ for each set E.

Proposition 5.1 contains all of the philosophy of weighted inequalities (5.1). Indeed, (5.2) yields a sort of cut off $(\mathcal{M}_\alpha^{Q[0,R]} f d\sigma)(\cdot)$. Summation of the resulting pieces is ensured by (5.3). It is for this result that a suitable atomic decomposition of tent spaces associated to $\Lambda^s_{d\sigma}(w_1)$ is needed. The proof of Proposition 5.1 is postponed below, and for the moment we show how precisely inequality (5.1) can be obtained from the test condition (4.1).

The equivalence of $\|\cdot\|_{\Lambda^{\frac{r}{\varepsilon}}_{d\omega}(w_2)}$ with a norm denoted by $\|\cdot\|_{\Gamma^{\frac{r}{\varepsilon}}_{d\omega}(w_2)}$ is needed. Therefore inequality (5.1) appears since

$$\begin{aligned}
&\left\|(\mathcal{M}_\alpha^{Q[0,R]} f d\sigma)(\cdot)\amalg_{Q[0,R]}(\cdot)\right\|^{\varepsilon}_{\Lambda^r_{d\omega}(w_2)} \\
&= \left\|(\mathcal{M}_\alpha^{Q[0,R]} f d\sigma)^{\varepsilon}(\cdot)\amalg_{Q[0,R]}(\cdot)\right\|_{\Lambda^{\frac{r}{\varepsilon}}_{d\omega}(w_2)} \\
&\le \left\|\sum_j \lambda_j^{\varepsilon}[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}(\mathcal{M}_\alpha \amalg_{Q_j} d\sigma)^{\varepsilon}(\cdot)\amalg_{Q_j}(\cdot)\right\|_{\Lambda^{\frac{r}{\varepsilon}}_{d\omega}(w_2)} \quad \text{(by (5.2))} \\
&\le c_1\left\|\sum_j \lambda_j^{\varepsilon}[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}(\mathcal{M}_\alpha \amalg_{Q_j} d\sigma)^{\varepsilon}(\cdot)\amalg_{Q_j}(\cdot)\right\|_{\Gamma^{\frac{r}{\varepsilon}}_{d\omega}(w_2)} \\
&\le c_1\sum_j \lambda_j^{\varepsilon}[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}\left\|(\mathcal{M}_\alpha \amalg_{Q_j} d\sigma)^{\varepsilon}(\cdot)\amalg_{Q_j}(\cdot)\right\|_{\Gamma^{\frac{r}{\varepsilon}}_{d\omega}(w_2)} \\
&\le c_2\sum_j \lambda_j^{\varepsilon}\left\|\amalg_{Q_j}(\cdot)\right\|^{-\varepsilon}_{\Lambda^s_{d\sigma}(w_1)}\left\|(\mathcal{M}_\alpha \amalg_{Q_j} d\sigma)(\cdot)\amalg_{Q_j}(\cdot)\right\|^{\varepsilon}_{\Lambda^r_{d\omega}(w_2)} \\
&\le c_2 A^{\varepsilon}\sum_j \lambda_j^{\varepsilon} \quad \text{(by (4.1))} \\
&\le c_3 A^{\varepsilon}\left\|f(\cdot)\right\|^{\varepsilon}_{\Lambda^s_{d\sigma}(w_1)} \quad \text{(by (5.3))}.
\end{aligned}$$

To prove Proposition 5.1, we need a suitable atomic decomposition of dyadic tent spaces associated to $\Lambda^s_{d\sigma}(w_1)$, which is now introduced. Let X be the set $(0,\infty)^n$ minus the dyadic points $z = (z_i)_i \in 2^k\mathbb{Z}^n$, and let $\widetilde{X} = X \times 2^{\mathbf{Z}}$. For each $x \in X$, we write

$$(y, 2^k) \in \widetilde{\Gamma}(x) \qquad \text{if} \quad x \in Q[y, 2^k] \tag{5.4}$$

where $Q[y,2^k]$ is the unique dyadic cube which contains y and with length 2^k. Note that $Q[x,2^k] = Q[y,2^k]$. Also,

$$\widehat{\Omega} = \left(\cup\,\{\widetilde{\Gamma}(x) \mid x \in \Omega^c\}\right)^c \tag{5.5}$$

for each set $\Omega \subset X$. Thus

$$(y, 2^k) \in \widehat{\Omega} \qquad \Longleftrightarrow \qquad Q[y, 2^k] \subset \Omega. \tag{5.6}$$

The functional $\mathcal{A}_\infty$, acting on each measurable function $\widetilde{f}(\cdot,\cdot)$ of $\widetilde{X}$, is given by

$$(\mathcal{A}_\infty \widetilde{f})(x) = \sup\left\{\left|\widetilde{f}(y,2^k)\right| \, (y,2^k) \in \widetilde{\Gamma}(x)\right\}. \tag{5.7}$$

Finally, for each measurable function $\widetilde{f}(y,2^k)$ and $R > 0$, define that

$$\widetilde{f}(\cdot,\cdot) \in \mathcal{T}^{s,dya}_{d\sigma}(w_1)[Q[0,R]] \tag{5.8}$$

if $\widetilde{f}(y,2^k)$ is supported by $\widehat{(0,R)^n}$, and the set $\{(\mathcal{A}_\infty \widetilde{f})(\cdot) > \lambda\}$, $\lambda > 0$, is an union of dyadic cubes, and $\|(\mathcal{A}_\infty \widetilde{f})(\cdot)\|_{\Lambda^s_{d\sigma}(w_1)} < \infty$.

Now the atomic decomposition result for $\mathcal{T}^{s,dya}_{d\sigma}(w_1)[Q(0,R)]$ can be stated.

Lemma 5.2. *Let $w_1(\cdot) \in B_s$. There is a constant $C > 0$ such that, for all functions $\widetilde{f}(\cdot,\cdot) \in \mathcal{T}^{s,dya}_{d\sigma}(w_1)[Q[0,R]]$ $(R > 0)$ one can find $\lambda_j > 0$, dyadic cubes Q_j, and functions $\widetilde{a}_j(y,2^k)$ with disjoint supports such that*

$$|\widetilde{a}_j(y,2^k)| \leq [W_1(|Q_j|_\sigma)]^{-\frac{1}{s}} \widetilde{\mathrm{II}}_{\widehat{Q}_j}(y,2^k) \tag{5.9}$$

$$\widetilde{f}(y,2^k) = \sum_j \lambda_j \widetilde{a}_j(y,2^k) \quad a.e. \tag{5.10}$$

and

$$\sum_j \lambda_j^\varepsilon \leq C \|(\mathcal{A}_\infty \widetilde{f})(\cdot)\|^\varepsilon_{\Lambda^s_{d\sigma}(w_1)} \tag{5.11}$$

whenever $s \leq \varepsilon$ and the growth condition (2.3) is satisfied.

The proof of this result will be given below but for the moment we explain how to derive the

Proof of Proposition 5.1. Let $f(\cdot) \in \Lambda^s_{d\sigma}$ and $R > 0$. To obtain (5.2) observe that for $x \in Q[0,R]$ then

$$\begin{aligned}
&(\mathcal{M}^{Q[0,R]}_\alpha f d\sigma)(x) \\
&\approx \sup\left\{ |Q[y,2^k]|^{\frac{\alpha}{n}-1} \int_{Q[y,2^k]} |f(z)| d\sigma(z) \,\middle|\, x \in Q[y,2^k] \subset Q[0,R] \right\} \\
&= \sup\left\{ \widetilde{\Theta}(y,2^k)\widetilde{f}(y,2^k) \,\middle|\, x \in Q[y,2^k] \subset Q[0,R] \right\}
\end{aligned}$$

where

$$\widetilde{\Theta}(y,2^k) = |Q[y,2^k]|^{\frac{\alpha}{n}-1} \int_{Q[y,2^k]} d\sigma(z)$$

and

$$\widetilde{f}(y,2^k) = \begin{cases} |Q[y,2^k]|_\sigma^{-1} \int_{Q[y,2^k]} f(z) d\sigma(z) & \text{if } (y,2^k) \in \widehat{Q[0,R]} \\ 0 & \text{else.} \end{cases}$$

These expressions are well defined since by the hypothesis on $d\sigma(\cdot)$ then $0 < |Q[y,2^k]|_\sigma < \infty$.

To take profit from Lemma 5.2, we are going to prove that $\widetilde{f}(\cdot,\cdot) \in \mathcal{T}^{s,dya}_{d\sigma}(w_1)[Q[0,R]]$. Indeed, first the supports of $\widetilde{f}(\cdot,2^k)$ and $(\mathcal{A}_\infty \widetilde{f})(\cdot)$ are respectively contained in $\widehat{Q[0,R]}$ and $Q[0,R]$. Next, by (5.4) and (5.7), if $(\mathcal{A}_\infty \widetilde{f})(x) > \lambda > 0$, then there is $(y,2^k)$ such that $x \in Q[y,2^k]$ and $|Q[y,2^k]|_\sigma^{-1} \int_{Q[y,2^k]} f(z) d\sigma(z) > \lambda$. So the set $\{(\mathcal{A}_\infty \widetilde{f})(\cdot) > \lambda\}$ is a union of such dyadic cubes $Q[y,2^k]$. Finally,

$$(\mathcal{A}_\infty \widetilde{f})(\cdot) \leq (N_\sigma f)(\cdot)$$

with $(N_\sigma f)(x) = \sup_{Q[y,2^k] \ni x} \{|Q[y,2^k]|_\sigma^{-1} \int_{Q[y,2^k]} |f(z)| d\sigma(z)\}$. So the fact that $\|(\mathcal{A}_\infty \widetilde{f})(\cdot)\|_{\Lambda^s_{d\sigma}(w_1)} < \infty$ can be obtained from

Lemma 5.3. *Let $w_1(\cdot) \in B_s$. Then there is a constant $c > 0$ such that*

$$\left\|(\mathcal{A}_\infty \widetilde{f})(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \le \left\|(N_\sigma f)(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \le c\left\|f(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \quad \textit{for all } f(\cdot) \ge 0. \tag{5.12}$$

This Lemma will be proved below, but for the moment the sequel of the proof of Proposition 5.1 is performed.

Since $\widetilde{f}(\cdot,\cdot) \in \mathcal{T}^{s,dya}_{d\sigma}(w_1)[Q[0,R]]$ then, by Lemma 5.2, there are $\lambda_j > 0$ and dyadic cubes Q_j such that (5.9), (5.10) and (5.11) are satisfied.

Inequality (5.2). To estimate $(\mathcal{M}^{Q[0,R]}_\alpha f d\sigma)^\varepsilon(x)$ it is sufficient to control $\widetilde{\Theta}^\varepsilon(y,2^k)$ $\widetilde{f}^\varepsilon(y,2^k)$, where $x \in Q[y,2^k] \subset Q[0,R]$. So it follows that

$$\begin{aligned}
&\widetilde{\Theta}^\varepsilon(y,2^k)\widetilde{f}^\varepsilon(y,2^k)\\
&= \Theta^\varepsilon(y,2^k)\sum_j \lambda_j^\varepsilon|\widetilde{a}_j(y,2^k)|^\varepsilon\\
&\qquad \textit{(by (5.10) and since the supports of the } \widetilde{a}_j \textit{ are disjoints)}\\
&\le \sum_j \lambda_j^\varepsilon[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}\widetilde{\Theta}^\varepsilon(y,2^k)\widetilde{1\!\!1}_{\widehat{Q}_j}(y,2^k)\\
&\qquad \textit{(by (5.9))}\\
&= \sum_j \lambda_j^\varepsilon[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}\left[|Q[y,2^k]|^{\frac{\alpha}{n}-1}\int_{Q[y,2^k]} d\sigma(z)\right]^\varepsilon \widetilde{1\!\!1}_{\widehat{Q}_j}(y,2^k)\\
&\qquad \textit{(see the definition of } \widetilde{\Theta})\\
&= \sum_j \lambda_j^\varepsilon[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}\left[|Q[y,2^k]|^{\frac{\alpha}{n}-1}\int_{Q[y,2^k]\cap Q_j} d\sigma(z)\right]^\varepsilon \widetilde{1\!\!1}_{\widehat{Q}_j}(y,2^k)\\
&\qquad \textit{(note that by (5.6) } Q[y,2^k]\subset Q_j)\\
&\le \sum_j \lambda_j^\varepsilon[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}\left[|Q[y,2^k]|^{\frac{\alpha}{n}-1}\int_{Q[y,2^k]} 1\!\!1_{Q_j}(z)d\sigma(z)\right]^\varepsilon 1\!\!1_{Q_j}(x)\\
&\qquad \textit{(remind that } x \in Q[y,2^k] \subset Q_j)\\
&\le \sum_j \lambda_j^\varepsilon[W_1(|Q_j|_\sigma)]^{-\frac{\varepsilon}{s}}(\mathcal{M}_\alpha 1\!\!1_{Q_j} d\sigma)^\varepsilon(x)1\!\!1_{Q_j}(x).
\end{aligned}$$

Inequality (5.3). It is not difficult to obtain this inequality since by (5.11) and (5.12) then

$$\Big(\sum_j \lambda_j^\varepsilon\Big)^{\frac{1}{\varepsilon}} \le C\left\|(\mathcal{A}_\infty\widetilde{f})(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)} \le Cc\left\|f(\cdot)\right\|_{\Lambda^s_{d\sigma}(w_1)}.$$

Therefore the proof of Proposition 5.1 (and consequently of Theorem 4.1) will be completed, once we will finish to prove Lemmas 5.3 and 5.2.

Proof of Lemma 5.3. The second inequality in (5.12) is the same as

$$\int_0^\infty [(N_\sigma f)^*_\sigma(t)]^s w_1(t)dt \le c\int_0^\infty [f^*_\sigma(t)]^s w_1(t)dt \qquad \text{for all } f(\cdot) \ge 0.$$

The first point to obtain such an inequality is

$$(N_\sigma f)^*_\sigma(t) \le Ct^{-1}\int_0^t f^*_\sigma(\tau)\,d\tau = C(Hf^*_\sigma)(t) \tag{5.13}$$

and the second one is

$$\int_0^\infty (Hg)^s(t)w_1(t)dt \le c\int_0^\infty g^s(t)w_1(t)dt \qquad \text{for all } g(\cdot)\searrow.$$

But this last inequality is well-known to be equivalent to $w_1(\cdot)\in B_s$ (see [1]). The first one was proved by Herz [6] for the Lebesgue measure dx. Therefore (5.13) can be obtained by adapting the ideas of this author. For the convenience the complete proof is given.

The first key to get (5.13) is the fact that $N_\sigma : L^1(d\sigma)\to L^{1\infty}(d\sigma)$, which can be written as

$$t(N_\sigma f_1)^*_\sigma(t) \le C\int_{\mathbb{R}^n}|f_1(x)|\,d\sigma(x) \qquad \text{for all } f_1(\cdot). \tag{5.14}$$

Due to the special properties of dyadic cubes, this embedding is well-known to be true with a constant $C>0$ depending only on the dimension n. Without any inconvenience, in (5.14) it can be assumed that $C>2$. The second point to obtain (5.13) is the fact that $N_\sigma : L^\infty(d\sigma)\to L^\infty(d\sigma)$ or

$$\left\|(N_\sigma f_2)(\cdot)\right\|_{L^\infty(d\sigma)} = \limsup_{x\in\mathbb{R}^n}(N_\sigma f_2)(x) \le \left\|f_2(\cdot)\right\|_{L^\infty(d\sigma)} \tag{5.15}$$

for all functions $f_2(\cdot)$. Now to see (5.13), it can be assumed that $f(\cdot)\ge 0$ and $f(\cdot) = f_1(\cdot)+f_2(\cdot)$ with $f_1(\cdot) = [f(\cdot)-f^*_\sigma(t)]1\!\!1_{E_t}(\cdot)$, $f_2(\cdot) = f(\cdot)-f_1(\cdot)$ and $E_t = \{x \,|\, f(x) > f^*_\sigma(t)\}$. Observe that

(i) $|E_t|_\sigma = t$

(ii) $\|f_2(\cdot)\|_{L^\infty(d\sigma)} \le 2f^*_\sigma(t)$

(iii) $\int_{\mathbb{R}^n} f_1(x)\,d\sigma(x) \le \int_0^{|E_t|_\sigma} f^*_\sigma(\tau)d\tau - tf^*_\sigma(t) = t\big[(Hf^*_\sigma)(t) - f^*_\sigma(t)\big]$

(this last follows by the definition of $f_1(\cdot)$ and (i)). By (ii) and (5.15):

(iv) $(N_\sigma f)(\cdot) \le (N_\sigma f_1)(\cdot) + (N_\sigma f_2)(\cdot) \le (N_\sigma f_1)(\cdot) + 2f^*_\sigma(t)$.

Inequality (5.13) appears since

$$\begin{aligned}(N_\sigma f)^*_\sigma(t) &\le (N_\sigma f_1)^*_\sigma(t) + 2f^*_\sigma(t) \quad \text{(by (iv))}\\ &\le C\big[(Hf^*_\sigma)(t) - f^*_\sigma(t)\big] + 2f^*_\sigma(t) \quad \text{(by (5.14) and (iii))}\\ &\le C(Hf^*_\sigma)(t) \quad (\text{by the choice } C>2).\end{aligned}$$

Proof of Lemma 5.2 (Atomic decomposition of $T^{s,dya}_{d\sigma}(w_1)[Q[0,R]]$). The main key is the following

Lemma 5.4 *For each bounded open set $\Omega = \bigcup_{Q\in\mathcal{I}} Q \subset X$ (with Q being dyadic cubes) one can find a sequence of (maximal) dyadic cubes $(Q_i)_i$ $(Q_i \in \mathcal{I})$ with pairwise disjoint interiors and such that $\Omega = \bigcup_i Q_i$ and $\widehat{\Omega} = \bigcup_i \widehat{Q}_i$.*

This result is just an easy consequence of well-known dyadic cubes properties.

Now take a function $\widetilde{f}(\cdot,\cdot) \in \mathcal{T}^{s,dya}_{d\sigma}(w_1)[Q[0,R]]$. For any integer j, let $\Omega_j = \{x|\,(\mathcal{A}_\infty\widetilde{f})(x) > 2^j\}$. Then

$$c2^{js}W_1[|\Omega_j|_\sigma] \leq \|(\mathcal{A}_\infty\widetilde{f})(\cdot)\|^s_{\Lambda^s_{d\sigma}(w_1)} < \infty$$

for some $c > 0$. So it is clear that $W_1[|\Omega_j|_\sigma] < \infty$. Moreover,

$$\widehat{\Omega}_{j+1} \subset \widehat{\Omega}_j \tag{5.16}$$

$$|\widetilde{f}(y,2^k)| \leq 2^{j+1} \text{ on } \widehat{\Omega}^c_{j+1}. \tag{5.17}$$

Since $\Omega_j = \bigcup_{Q\in\mathcal{I}} Q \subset (0,R)^n$, then by the above Lemma 5.4

$$\Omega_j = \bigcup_i Q_{ij} \qquad \text{and} \qquad \sum_i \mathrm{1\!I}_{Q_{ij}}(\cdot) = \mathrm{1\!I}_{\Omega_j}(\cdot) \tag{5.18}$$

$$\widehat{\Omega}_j = \bigcup_i \widehat{Q}_{ij} \qquad \text{and} \qquad \sum_i \widetilde{\mathrm{1\!I}}_{\widehat{Q}_{ij}}(\cdot,\cdot) = \widetilde{\mathrm{1\!I}}_{\widehat{\Omega}_j}(\cdot,\cdot). \tag{5.19}$$

Define the scalars

$$\lambda_{ij} = 2^{j+1}\left[W_1(|Q_{ij}|_\sigma)\right]^{\frac{1}{s}}$$

and the functions

$$\widetilde{a}_{ij}(y,2^k) = 2^{-(j+1)}\left[W_1(|Q_{ij}|_\sigma)\right]^{-\frac{1}{s}}\widetilde{f}(y,2^k) \times \widetilde{\mathrm{1\!I}}_{\widehat{Q}_{ij}-\widehat{\Omega}_j}(y,2^k).$$

These quantities are well defined whenever $0 < W_1(|Q_{ij}|_\sigma) < \infty$. But this is the case since $W_1(|Q_{ij}|_\sigma) < W_1(|\Omega_j|_\sigma) < \infty$ and $0 < |Q_{ij}|_\sigma < \infty$ implies $0 < W_1(|Q_{ij}|_\sigma)$. This implication is true since by the condition $w_1(\cdot) \in B_s$, the fact that $W_1(\tau) = \int_0^\tau w_1(t)dt = 0$ for some $0 < \tau < \infty$ means that $w_1(\cdot) = 0$ a.e.

Clearly, by (5.19), the supports $\widetilde{E}_{ij} = \widehat{Q}_{ij} - \widehat{\Omega}_j$ of the $\widetilde{a}_{ij}$'s are almost disjoints. Inequality (5.17) and the definition of $\widetilde{a}_{ij}(y,2^k)$ yield

$$|\widetilde{a}_{ij}(y,2^k)| \leq \left[W_1(|Q_{ij}|_\sigma)\right]^{-\frac{1}{s}}\widetilde{\mathrm{1\!I}}_{\widehat{Q}_{ij}}(y,2^k)$$

which is the estimate (5.9). Again by (5.19) and the definition of λ_{ij} and $\widetilde{a}_{ij}(y,2^k)$ then

$$\widetilde{f}(y,2^k) = \sum_{j,i}\lambda_{ij}\widetilde{a}_{ij}(y,2^k) \qquad \text{a.e.}$$

which is the pointwise equality announced in (5.10).

Finally, to get inequality (5.11) let $s \leq \varepsilon$ with an ε for which (2.3) is satisfied. Then

$$\begin{aligned}
\sum_{j,i} \lambda_{ij}^{\varepsilon} &= \sum_j 2^{(j+1)\varepsilon} \sum_i [W_1(|Q_{ij}|_\sigma)]^{\frac{\varepsilon}{s}} \\
&\leq c \sum_j 2^{(j+1)\varepsilon} \Big(W_1 \Big[\sum_i |Q_{ij}|_\sigma \Big] \Big)^{\frac{\varepsilon}{s}} \quad \textit{(by condition (2.3))} \\
&= c \sum_j 2^{(j+1)\varepsilon} \Big(W_1 \Big[|\Omega_j|_\sigma \Big] \Big)^{\frac{\varepsilon}{s}} \quad \textit{(the cubes } Q_{ij} \textit{ have disjoint interiors)} \\
&\leq c \Big(\sum_j 2^{(j+1)s} W_1 [|\Omega_j|_\sigma] \Big)^{\frac{\varepsilon}{s}} \quad \textit{(remind that } \tfrac{\varepsilon}{s} \geq 1) \\
&= c \Big(\sum_j 2^{js} W_1 \big[|\{(\mathcal{A}_\infty \widetilde{f})(\cdot) > 2^j\}|_\sigma \big] \Big)^{\frac{\varepsilon}{s}} \quad \textit{(by the definition of } \Omega_j) \\
&\leq c' \big\| (\mathcal{A}_\infty \widetilde{f})(\cdot) \big\|_{\Lambda^s_{d\sigma}(w_1)}^{\varepsilon} \quad \textit{(by the discretisation of } \|\cdot\|^s_{\Lambda^s_{d\sigma}(w_1)} \textit{ as in [3]).}
\end{aligned}$$

6. Proof of Theorem 2.4

Since the proof is quite similar to that of Theorem 2.1, we essentially emphasize on the main points rather than on details. One of the points to get Part b) of Theorem 2.4 is the pointwise inequality

$$(M_\alpha f)(\cdot) \leq c(N_{\alpha t,v} f^t)^{\frac{1}{t}}(\cdot). \tag{6.1}$$

Here $c > 0$ depends only on the constant in the A_t-condition for $v(\cdot)$, and $N_{\lambda,v}$ is the maximal operator $(N_{\lambda,v} g)(x) = \sup\{ |Q|^{\frac{\lambda}{n}} \left(\int_Q v(z)\, dz \right)^{-1} \int_Q |g(y)| v(y) dy \,|\, Q \ni x \}$.

With (6.1), the proof of $M_\alpha : \Lambda^s_v(w_1) \to \Lambda^r_u(w_2)$ is reduced to

$$\Big\| (N_{\alpha t,v} g)(\cdot) \Big\|_{\Lambda_u^{\frac{r}{t}}(w_2)} \leq c A^t \|g(\cdot)\|_{\Lambda_v^{\frac{s}{t}}(w_1)} \qquad \text{for all } g(\cdot) \geq 0 \tag{6.2}$$

which is denoted by $N_{\alpha t,v} : \Lambda_v^{\frac{s}{t}}(w_1) \to \Lambda_u^{\frac{r}{t}}(w_2)$.

Inequality (6.1) is true for $t = 1$ and $v(\cdot) \in A_1$ since

$$\begin{aligned}
|Q|^{\frac{\alpha}{n}-1} \int_Q f(y)\, dy &= |Q|^{\frac{\alpha}{n}} \left(\int_Q v(z)\, dz \right)^{-1} \int_Q f(y) v(y) \left[\frac{1}{|Q|} \int_Q v(z)\, dz \right] \frac{1}{v(y)} dy \\
&\leq c |Q|^{\frac{\alpha}{n}} \left(\int_Q v(z)\, dz \right)^{-1} \int_Q f(y) v(y)\, dy.
\end{aligned}$$

Also (6.1) holds for $t > 1$ and $v(\cdot) \in A_t$ since, by applying the Hölder inequality,

$$|Q|^{\frac{\alpha}{n}-1} \int_Q f(y)\, dy \leq |Q|^{\frac{\alpha}{n}-1} \left(\int_Q f^t(y) v(y) dy \right)^{\frac{1}{t}} \left(\int_Q v^{-\frac{1}{t-1}}(y)\, dy \right)^{1-\frac{1}{t}}$$

$$= \left[|Q|^{\frac{\alpha t}{n}} \left(\int_Q v(z)\,dz\right)^{-1} \int_Q f^t(y)v(y)\,dy\right]^{\frac{1}{t}}$$

$$\times |Q|^{-1} \left(\int_Q v(y)\,dy\right)^{\frac{1}{t}} \left(\int_Q v^{-\frac{1}{t-1}}(y)\,dy\right)^{1-\frac{1}{t}}$$

$$\leq c\left[|Q|^{\frac{\alpha t}{n}} \left(\int_Q v(z)\,dz\right)^{-1} \int_Q f^t(y)v(y)\,dy\right]^{\frac{1}{t}}.$$

As in Theorem 2.1, (see Theorem 4.1) to obtain $N_{\alpha t,v} : \Lambda_v^{\frac{s}{t}}(w_1) \to \Lambda_u^{\frac{r}{t}}(w_2)$, the idea is also to prove the corresponding dyadic version $\mathcal{N}_{\alpha t,v} : \Lambda_v^{\frac{s}{t}}(w_1) \to \Lambda_u^{\frac{r}{t}}(w_2)$. This last embedding will be based on

$$(\mathcal{N}_{\alpha t,v}^{Q[0,R]} g)^\varepsilon(\cdot)1\!\!1_{Q[0,R]}(\cdot) \leq \sum_j \lambda_j^\varepsilon [W_1(|Q_j|_v)]^{-\frac{\varepsilon t}{s}} |Q_j|^{\frac{\varepsilon \alpha t}{n}} 1\!\!1_{Q_j}(\cdot) \qquad (\varepsilon > 0) \tag{6.3}$$

and

$$\left(\sum_j \lambda_j^\varepsilon\right)^{\frac{1}{\varepsilon}} \leq c\|g(\cdot)\|_{\Lambda_v^{\frac{s}{t}}(w_1)}. \tag{6.4}$$

Here $\lambda_j > 0$ and the Q_j's are dyadic cubes and $\mathcal{N}_{\lambda,v}^{Q[0,R]}$ is the maximal operator defined as $N_{\lambda,v}$ by means of dyadic cubes $Q \subset Q[0,R] = (0,R)^n$. Details on the obtention of (6.3) and (6.4) from atomic decomposition of a suitable tent space can be done as in the proof of Proposition 5.1.

The fact that (6.3) and (6.4) imply $\mathcal{N}_{\alpha t,v} : \Lambda_v^{\frac{s}{t}}(w_1) \to \Lambda_u^{\frac{r}{t}}(w_2)$, by assuming the test condition (2.11) (more exactly with dyadic cubes), can be obtained as follows:

$$\begin{aligned}
&\left\|(N_{\alpha,v}^{Q[0,R]} g)(\cdot)1\!\!1_{Q[0,R]}(\cdot)\right\|_{\Lambda_u^{\frac{r}{t}}(w_2)}^\varepsilon \\
&= \left\|(N_{\alpha,v}^{Q[0,R]} g)^\varepsilon(\cdot)1\!\!1_{Q[0,R]}(\cdot)\right\|_{\Lambda_u^{\frac{r}{\varepsilon t}}(w_2)} \\
&\leq \left\|\sum_j \lambda_j^\varepsilon [W_1(|Q_j|_v)]^{-\frac{\varepsilon t}{s}} |Q_j|^{\frac{\varepsilon \alpha t}{n}} 1\!\!1_{Q_j}(\cdot)\right\|_{\Lambda_u^{\frac{r}{\varepsilon t}}(w_2)} \qquad \text{(by using (6.3))} \\
&\leq c_1 \sum_j \lambda_j^\varepsilon [W_1(|Q_j|_v)]^{-\frac{\varepsilon t}{s}} |Q_j|^{\frac{\varepsilon \alpha t}{n}} \left\|1\!\!1_{Q_j}(\cdot)\right\|_{\Lambda_u^{\frac{r}{t}}(w_2)}^\varepsilon \\
&\qquad \left(\text{since } \|\cdot\|_{\Lambda_u^{\frac{r}{\varepsilon t}}(w_2)} \text{ is equivalent to a norm}\right) \\
&= c_1 \sum_j \lambda_j^\varepsilon \left([W_1(|Q_j|_v)]^{-\frac{1}{s}} |Q_j|^{\frac{\alpha}{n}} [W_2(|Q_j|_u)]^{\frac{1}{r}}\right)^{\varepsilon t} \\
&\leq c_1 A^{\varepsilon t} \sum_j \lambda_j^\varepsilon \qquad \text{(by condition (2.11))} \\
&\leq c_2 A^{\varepsilon t} \left\|g(\cdot)\right\|_{\Lambda_v^{\frac{s}{t}}(w_1)}^\varepsilon \qquad \text{(by using (6.4))}.
\end{aligned}$$

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