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Existence Results for the Quasistationary Motion of a Free Capillary Liquid Drop

M. Günther and G. Prokert

Abstract. We consider instationary creeping flow of a viscous liquid drop with free boundary driven by surface tension. This yields a nonlocal surface motion law involving the solution of the Stokes equations with Neumann boundary conditions given by the curvature of the boundary. The surface motion law is locally reformulated as a fully nonlinear parabolic (pseudodifferential) equation on a smooth manifold. Using analytic expansions, invariance properties, and a priori estimates we give, under suitable presumptions, a short-time existence and uniqueness proof for the solution of this equation in Sobolev spaces of sufficiently high order. Moreover, it is shown that if the initial shape of the drop is near the ball, then the evolution problem has a solution for all positive times which exponentially decays to the ball.

Keywords: *Stokes flows, quasisteady motions, surface tensions, nonlinear parabolic equations, surface motion laws*

AMS subject classification: Primary 35 R 35, secondary 35 Q 35, 76 D 07

1. Introduction

In fluid dynamics problems with very small Reynolds number the concept of "creeping flow" is used. This means that the inertial forces are neglected and, in the case of Newtonian flow, the Navier-Stokes equations simplify to the Stokes equations. When one uses such a simplification to describe liquid motions that are actually instationary, it could be called a quasistationary approximation. This idea is the basis for the following model of the motion of a viscous liquid drop under the influence of capillary forces which is successfully used in the description of the so-called viscous sintering process in glass production [21].

The liquid is assumed to be incompressible and to have constant viscosity, density, and (positive) surface tension coefficient. The only driving mechanism we consider is the force from surface tension. In dimensionless form this leads to the linear boundary

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value problem

$$\left.\begin{aligned} -\Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \Omega(t)$$
$$\mathcal{T}(u,p)n_t = \kappa_t n_t \quad \text{on } \Gamma(t) = \partial\Omega(t), \tag{1.1}$$

where $\Omega(t) \subset \mathbb{R}^N$ is the (bounded) domain occupied by the drop at time $t \geq 0$, u and p are the velocity and pressure fields in $\Omega(t)$ at this time,

$$\mathcal{T}(u,p) = \left((\nabla u) + (\nabla u)^T\right) - pI$$

denotes the stress tensor, κ_t and n_t denote the double mean curvature and the outer normal vector of $\Gamma(t)$. The sign of κ_t is taken such that it is negative if $\Omega(t)$ is convex.

As will be shown below, the equations (1.1) essentially determine u and p at time t. For the description of the motion of the drop the kinematic boundary condition

$$V_n(t) = u|_{\Gamma(t)} \cdot n_t \qquad \text{on } \Gamma(t) \tag{1.2}$$

has to be added where $V_n(t)$ denotes the normal velocity of $\Gamma(t)$. This condition is an equivalent expression for the demand that the set of particles that constitute the boundary of the drop does not change in time.

The problem (1.1), (1.2) is a moving boundary problem that can be considered as a problem of evolution of $\Gamma = \Gamma(t)$ by a nonlocal surface motion law, comparable, e.g., to Hele-Shaw flow driven by surface tension [7 - 9]. The problem (1.1), (1.2) and its counterpart concerning outer domains, which is a model for bubbles in a viscous liquid, have recently been investigated in the two-dimensional case. This has been done by methods from complex function theory, using, in particular, time-dependent conformal mappings and the solution of Hilbert problems [3, 4, 14, 15, 18]. For the numerical treatment of the problem we refer to [21] and the bibliography therein.

The aim of this paper is to provide an analysis of this problem in N dimensions (for the sake of simplicity sometimes restricted to $N = 3$) as far as this can be done by local methods. Accordingly, we prove, under suitable presumptions, a short-time existence and uniqueness result for general initial domains and global existence and exponential decay of the solution near the stable equilibrium solutions that are given by the balls.

Notation. All differentiations with exception of those with respect to the time variable t are to be understood in generalized sense. We will use the symbols C and c for "large" and "small" positive real constants, respectively. Sometimes an index is used to indicate their dependence on parameters. A function that is given on a (sufficiently regular) domain Ω and its restriction or trace at the boundary Γ of this domain are often denoted by the same symbol. The norms in the Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$ are denoted by $\|\cdot\|_s^\Omega$ and $\|\cdot\|_s^\Gamma$, respectively, and the same notation is used for the norms of the corresponding Sobolev spaces of vector-valued functions $(H^s(\Omega))^k$ and $(H^s(\Gamma))^k$. (These norms are specified later, at the moment it is sufficient to demand that they generate the usual topologies.)

For convenience we generalize some notions of vector algebra and analysis to $\mathbb{R}^N$. Let K be an arbitrary but fixed bijection from the set $\{(i,j)\,|\,1 \le i < j \le N\}$ to the set $\{1,\dots,\binom{N}{2}\}$. We define the bilinear mappings

$$\times : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^{\binom{N}{2}}$$
$$\otimes : \mathbb{R}^{\binom{N}{2}} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$$

by

$$(a \times b)_{K(i,j)} = a_i b_j - a_j b_i \qquad (1 \le i < j \le N)$$

and

$$(c \otimes a)_i = \sum_{j=1}^{i-1} c_{K(j,i)} a_j - \sum_{j=i+1}^{N} c_{K(i,j)} a_j \qquad (i = 1,\dots,N).$$

It is easy to check that

$$c \cdot (a \times b) = b \cdot (c \otimes a) \qquad \forall a, b \in \mathbb{R}^N, c \in \mathbb{R}^{\binom{N}{2}}. \tag{1.3}$$

We define, moreover, for any sufficiently smooth N-vector function v given on an open subset of $\mathbb{R}^N$, the $\binom{N}{2}$-vector-valued differential operator rot by

$$(\operatorname{rot} v)_{K(i,j)} = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \qquad (1 \le i < j \le N)$$

for which we have the integral theorem

$$\int_\Omega \operatorname{rot} v \, dx = \int_\Gamma n \times v \, d\Gamma. \tag{1.4}$$

Note that if $N = 3$, then the usual definitions of the outer product and the curl (rotation) of a vector field can be obtained, up to the sign of the second component, by choosing the suitable bijection K.

2. The boundary value problem on a fixed domain

We consider the boundary value problem

$$\left.\begin{aligned} -\Delta u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \Omega$$
$$\mathcal{T}(u,p)n = \kappa n \quad \text{on } \Gamma = \partial\Omega \tag{2.1}$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ that is taken fixed in this section. The quantities κ and n are defined analogously to κ_t and n_t in (1.1).

At first we establish the unique solvability of a generalized weak formulation of (2.1) with auxiliary conditions. We introduce the Hilbert spaces

$$X = (H^1(\Omega_0))^N \times L^2(\Omega_0) \times (\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}})$$
$$Y = ((H^1(\Omega_0))^N)' \times \left(L^2(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}\right)$$

and the (bi-)linear operators

$$\begin{aligned} L &: X \longrightarrow Y \\ A &: (H^1(\Omega))^N \longrightarrow ((H^1(\Omega))^N)' \\ B &: (H^1(\Omega))^N \longrightarrow L^2(\Omega_0) \times (\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}) \\ a &: (H^1(\Omega))^N \times (H^1(\Omega))^N \longrightarrow \mathbb{R} \\ \varphi_1 &: (H^1(\Omega))^N \longrightarrow \mathbb{R}^N \\ \varphi_2 &: (H^1(\Omega))^N \longrightarrow \mathbb{R}^{\binom{N}{2}} \end{aligned}$$

defined by

$$L \begin{bmatrix} u \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} Au + B' \begin{bmatrix} p \\ \lambda \end{bmatrix} \\ Bu \end{bmatrix} .$$

$$(Au)v = a(u,v)$$

$$Bu = \begin{bmatrix} -\operatorname{div} u \\ \varphi_1(u) \\ \varphi_2(u) \end{bmatrix}$$

$$a(u,v) = \frac{1}{2} \int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dx$$

$$\varphi_1(u) = \int_\Omega u \, dx$$

$$\varphi_2(u) = \int_\Omega \operatorname{rot} u \, dx$$

where $B' : L^2(\Omega) \times (\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}) \longrightarrow ((H^1(\Omega_0))^N)'$ is the dual of B.

Lemma 1 (Weak formulation).

(i) *The operator L is a homeomorphism between X and Y.*

(ii) *Suppose $L[u\, p\, \lambda]^T = [f\, 0]^T$ with*

$$f(v) = \int_\Gamma \kappa n \cdot v \, d\Gamma \qquad \forall v \in (H^1(\Omega))^N .$$

Then $\lambda = 0$ and (u,p) is a weak solution of (2.1).

Proof. Statement (i): The equation

$$L[u\,p\,\lambda]^T = F \tag{2.2}$$

is a variational problem with linear restrictions to which the usual existence results apply (see, e.g., [6]). In order to establish (i) it is therefore sufficient to show that a is elliptic on $(\ker B, \|\cdot\|_1^\Omega$ and B is surjective.

The first statement follows from Poincarés inequality [10]

$$\int_\Omega |\nabla w|^2\,dx + \left(\int_\Omega w\,dx\right)^2 \geq c\,\|w\|_1^{\Omega 2} \qquad \forall w \in H^1(\Omega)$$

and Korns second inequality [11]

$$a(v,v) \geq c \sum_{i,j=1}^N \int_\Omega \left(\frac{\partial v_i}{\partial x_j}\right)^2 dx \qquad \forall v \in \{v \in (H^1(\Omega))^N \,|\, \varphi_2(v) = 0\}.$$

Taking into account that φ_1 and φ_2 are surjective from $\{v \in (H^1(\Omega))^N \,|\, \operatorname{div} v = 0\}$ to $\mathbb{R}^N$ and $\mathbb{R}^{\binom{N}{2}}$, respectively, it remains to show that the equation $-\operatorname{div} v = q$ in Ω is solvable in $(H^1(\Omega))^N$ for all $q \in L^2(\Omega_0)$. This can be done by considering a solution $\Phi \in H^2(\Omega)$ of $-\Delta\Phi = q$ in Ω and setting $v = \nabla\Phi$.

Statement (ii): Consider the space

$$V_0 = \left\{ v \in (H^1(\Omega))^N \;\middle|\; v_i(x) = \sum_{j=1}^N s_{ij}x_j + c_i \;\; (s_{ij}, c_i \in \mathbb{R},\; s_{ij} = -s_{ji}) \right\}$$

$(i = 1, \ldots, N)$ and note that $a(\cdot, v)$, $a(u, \cdot)$, and div vanish on V_0. The same holds for f because the Green formula for closed surfaces yields

$$\int_\Gamma \kappa n \cdot v\,d\Gamma = \int_\Gamma \Delta_\Gamma x \cdot v\,d\Gamma = -\sum_{i=1}^N \int_\Gamma \nabla_\Gamma x_i \cdot \nabla_\Gamma v_i\,d\Gamma$$

where x_i and v_i are the coordinates of x and v in a fixed Cartesian basis of $\mathbb{R}^N$, and ∇_Γ and Δ_Γ are the generalized gradient and the Laplace-Beltrami operator on Γ, respectively. Hence $\lambda = 0$ because φ_1 and φ_2 are surjective from V_0 to $\mathbb{R}^N$ and $\mathbb{R}^{\binom{N}{2}}$, respectively. The fact that in this case (2.2) is a weak formulation of (2.1) follows from the integral identity

$$\begin{aligned} \frac{1}{2}\sum_{i,j=1}^N \int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx - \int_\Omega p \operatorname{div} v\,dx \\ = \int_\Omega (-\Delta u + \nabla p)\cdot v\,dx - \int_\Omega \nabla(\operatorname{div} u)\cdot v\,dx + \int_\Gamma \mathcal{T}(u,p)n \cdot v\,d\Gamma \end{aligned} \tag{2.3}$$

holding for sufficiently smooth vector-valued functions u, v and scalar functions p ∎

Furthermore, we will need some H^s-regularity results on our boundary value problem. For fixed $s \geq 2$, we introduce the spaces

$$\widetilde{X} = (H^1(\Omega))^N \times (H^{s-1}(\Omega))^N \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}})$$
$$\widetilde{Y} = (H^{s-2}(\Omega))^N \times H^{s-1}(\Omega) \times (H^{s-\frac{3}{2}}(\Gamma))^N \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}$$

and the operator

$$\widetilde{L} : \widetilde{X} \longrightarrow \widetilde{Y}$$

defined by

$$\widetilde{L}\begin{bmatrix} u \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} -\Delta u + \nabla p + \lambda_1 \\ -\operatorname{div} u \\ \mathcal{T}(u,p)n + \lambda_2 \otimes n \\ \varphi_1(u) \\ \varphi_2(u) \end{bmatrix}.$$

Lemma 2 (Regularity).

(i) *The operator $\widetilde{L}$ is a homeomorphism between the spaces $\widetilde{X}$ and $\widetilde{Y}$.*

(ii) *Suppose $\widetilde{L}[u\,p\,\lambda]^T = [0\,0\,F_B\,0\,0]^T$. Then*

$$\|\lambda\|_{\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}} \leq C_\sigma \|F_B\|_\sigma^\Gamma \tag{2.4}$$

for all $\sigma \in \mathbb{R}$ and

$$\|u\|_0^\Omega \leq C\|F_B\|_{-\frac{3}{2}}^\Gamma. \tag{2.5}$$

Proof. Note that, according to (1.3) and (1.4),

$$\int_\Gamma v \cdot (\lambda_2 \otimes n)\, d\Gamma = \int_\Gamma \lambda_2 \cdot (n \times v)\, d\Gamma = \lambda_2 \cdot \int_\Omega \operatorname{rot} v\, dx = \lambda_2^T \varphi_2(v).$$

Using this and (2.3) we find from $\widetilde{L}[u\,p\,\lambda]^T = [F_I\,g\,F_B\,h_1\,h_2]^T$ the variational formulation

$$\left.\begin{aligned} a(u,v) - \int_\Omega p \operatorname{div} v\, dx + \lambda_1^T \varphi_1(v) + \lambda_2^T \varphi_2(v) & \\ = \int_\Omega (F_I + \nabla g)\cdot v\, dx + \int_\Gamma F_B \cdot v\, d\Gamma & \\ \text{for all } v \in (H^1(\Omega))^N & \\ -\operatorname{div} u = g & \\ \varphi_1(u) = h_1 & \\ \varphi_2(u) = h_2. & \end{aligned}\right\} \tag{2.6}$$

Lemma 1 yields that this problem has a unique solution $[u\,p\,\lambda]^T \in X$, and from the fact that $a(u,\cdot)$ and div vanish on V_0 we find

$$\lambda_{ij} = \int_\Omega (F_I + \nabla g) \cdot v_{ij}\, dx + \int_\Gamma F_B \cdot v_{ij}\, d\Gamma$$

where $i = 1, 2$, λ_{ij} is the j-th component of λ_i, and the v_{ij} form the dual basis of V_0 with respect to the φ_i, i.e. we have $\varphi_{ij}(v_{kl}) = \delta_{ik}\delta_{jl}$. All v_{ij} are smooth, hence

$$\|\lambda\|_{\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}} \leq C\left(\|F_I\|^{\Omega}_{s-2} + \|g\|^{\Omega}_{s-1} + \|F_B\|^{\Gamma}_{s-\frac{3}{2}}\right)$$

and (2.4) follow.

Let s be integer for the moment, the general result will finally follow by interpolation. We will determine u and p by setting

$$\begin{aligned} u &= u_0 + u_1 + u_2 \\ p &= p_1 + p_2 \end{aligned}$$

where

$$\left.\begin{aligned} -\Delta u_1 + \nabla p_1 &= F_I - \lambda_1 \\ -\operatorname{div} u_1 &= g \end{aligned}\right\} \quad \text{in } \Omega$$
$$u_1 = -\frac{1}{|\Gamma|}\int_\Omega g\,dx \cdot n \quad \text{on } \Gamma, \tag{2.7}$$

$u_0 \in V_0$ such that $\varphi_i(u_0) = -\varphi_i(u_1) + h_i \quad (i = 1, 2)$ and

$$\left.\begin{aligned} -\Delta u_2 + \nabla p_2 &= 0 \\ -\operatorname{div} u_2 &= 0 \end{aligned}\right\} \quad \text{in } \Omega$$
$$\mathcal{T}(u_2, p_2)n = -\mathcal{T}(u_1, p_1)n + F_B - \lambda_2 \times n = \Phi \quad \text{on } \Gamma \tag{2.8}$$
$$\varphi_i(u_2) = 0 \quad (i = 1, 2).$$

Note that

$$\int_\Gamma \Phi \cdot v = 0 \qquad \forall v \in V_0. \tag{2.9}$$

The regularity results for the Dirichlet problem of the Stokes equations yield that (2.7) has precisely one solution $(u_1, p_1) \in (H^s(\Omega))^N \times H^{s-1}(\Omega)$ with $\int_\Omega p_1\,dx = 0$ and an estimate

$$\|u_1\|^{\Omega}_s + \|p_1\|^{\Omega}_{s-1} \leq C\left(\|F_I\|^{\Omega}_{s-2} + \|\lambda_1\|_{\mathbb{R}^N} + \|g\|^{\Omega}_{s-1}\right)$$

holds [12: Theorem IV.6.1]. Thus we have $\Phi \in (H^{s-\frac{3}{2}}(\Gamma))^N$ and

$$\begin{aligned} \|\Phi\|^{\Gamma}_{s-\frac{3}{2}} &\leq C\left(\|u_1\|^{\Omega}_s + \|p_1\|^{\Omega}_{s-1} + \|\lambda_2\|_{\mathbb{R}^{\binom{N}{2}}} + \|F_B\|^{\Gamma}_{s-\frac{3}{2}}\right) \\ &\leq C\left(\|F_I\|^{\Omega}_{s-2} + \|g\|^{\Omega}_{s-1} + \|\lambda\|_{\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}}\right). \end{aligned}$$

It remains to show that, for all Φ that satisfy (2.8), (2.9) has a unique solution $(u_2, p_2) \in (H^s(\Omega))^N \times H^{s-1}(\Omega)$ satisfying an estimate

$$\|u_2\|^{\Omega}_s + \|p_2\|^{\Omega}_{s-1} \leq C\|\Phi\|^{\Gamma}_{s-\frac{3}{2}}. \tag{2.10}$$

From the discussion of the weak formulation we recall that (2.9) is a necessary condition for the solvability of (2.8) and that the solution (u_2, p_2) is unique. From a density argument it follows that we can assume $\Phi \in (C(\Gamma))^N$.

We will apply integral representations from the theory of hydrodynamic potentials. For the sake of brevity the description of the details will be restricted to the case $N = 3$. For $x \in \Omega$ we use the ansatz

$$
\begin{aligned}
u_2(x) &= V(x, \psi) \\
V(x, \psi) &= \frac{1}{8\pi} \int_\Gamma \left(\frac{I}{|x-y|} + \frac{(x-y)(x-y)^T}{|x-y|^3} \right) \psi(y)\, d\Gamma_y \\
p_2(x) &= \frac{1}{4\pi} \int_\Gamma \frac{x-y}{|x-y|^3} \psi(y)\, d\Gamma_y
\end{aligned}
$$

where ψ is a $\mathbb{R}^3$-valued (measurable) function on Γ. It is shown in [17: Chapter III] that (u_2, p_2) satisfies the first three equations in (2.8) if ψ is continuous and satisfies

$$
\left(\frac{1}{2} I + K \right) \psi = \Phi \tag{2.11}
$$

with

$$
(K\psi)(x) = -\frac{3}{4\pi} \int_\Gamma \frac{(x-y)(x-y)^T}{|x-y|^5} (x-y) n(x) \psi(y)\, d\Gamma_y \qquad (x \in \Gamma).
$$

The operator K is a weakly singular integral operator, hence it is compact on $(H^0(\Gamma))^N$ and continuity of Φ implies continuity for all $\psi \in (H^0(\Gamma))^N$ that satisfy (2.11) (see, e.g., [20: Theorems 12.1, 12.7 and 12.8]). Moreover, K is a pseudodifferential operator [19], hence it is compact on $(H^{s-\frac{3}{2}}(\Gamma))^N$ and therefore $(\frac{1}{2} I + K)$ is a Fredholm operator of index 0 on this space. Taking into account that $N(\frac{1}{2} I + K)$ consists of continuous functions one can conclude, using the results about the weak formulation, that $V(\cdot, \psi) \in V_0$ for all $\psi \in N(\frac{1}{2} I + K)$. The mapping $\psi \mapsto V(\cdot, \psi)$ is injective [17], hence $\dim N(\frac{1}{2} I + K) \le 6$. The necessary solvability conditions (2.9) imply $\operatorname{codim} R(\frac{1}{2} I + K) \ge 6$, hence

$$
\dim N(\tfrac{1}{2} I + K) = \operatorname{codim} R(\tfrac{1}{2} I + K) = 6,
$$

i.e. the solvability conditions (2.9) are also sufficient and the mapping $\psi \mapsto V(\cdot, \psi)$ maps $N(\frac{1}{2} I + K)$ onto V_0. Thus we can conclude that (2.11) has precisely one solution such that $\varphi_i(V(\cdot, \psi)) = 0$ $(i = 1, 2)$ satisfying an estimate

$$
\|\psi\|^\Gamma_{s-\frac{3}{2}} \le C \, \|\Phi\|^\Gamma_{s-\frac{3}{2}}.
$$

Finally we use the fact that the singular integral operator that maps ψ to $V(\cdot, \psi)|_\Gamma$ is a pseudodifferential operator of order -1 [19], hence we find that the trace of u_2 on Γ is in $(H^{s-\frac{1}{2}}(\Gamma))^N$ and

$$
\|u_2\|^\Gamma_{s-\frac{1}{2}} \le C \, \|\Psi\|^\Gamma_{s-\frac{3}{2}} \le C \, \|\Phi\|^\Gamma_{s-\frac{3}{2}}.
$$

The proof of (2.10) is completed now by another application of the regularity result on the Dirichlet problem.

To show (2.5), consider the "adjoint" problem

$$\left.\begin{aligned} -\Delta v + \nabla q &= u \\ \operatorname{div} v &= 0 \end{aligned}\right\} \quad \text{in } \Omega$$
$$\mathcal{T}(v,q)n = -\mu \otimes n \quad \text{on } \Gamma = \partial\Omega$$

with $\mu \in \mathbb{R}^{\binom{N}{2}}$ given by

$$\mu_j = \int_\Omega u \cdot v_{2,j}\, dx$$

which implies $\|\mu\|_{\mathbb{R}^{\binom{N}{2}}} \le C\,\|u\|_0^\Omega$. By examining the variational formulation of this problem in the same way as in Lemma 1 we find the existence of a weak solution of it that satisfies $\varphi_1(v) = 0$, $\varphi_2(v) = 0$. By the above regularity results we get

$$\|v\|_2^\Omega \le C\left(\|u\|_0^\Omega + \|\mu \otimes n\|_{\frac{1}{2}}^\Gamma\right) \le C\,\|u\|_0^\Omega.$$

With this, we find by the second Green formula for the Stokes equations

$$\begin{aligned} \|u\|_0^{\Omega\,2} &= (u, -\Delta v + \nabla q)_0 + \int_\Gamma \left(\mathcal{T}(u,p)n \cdot v - \mathcal{T}(v,q)n \cdot u\right) d\Gamma \\ &= \int_\Gamma (F_B \cdot v + \mu \otimes n)\, d\Gamma = \int_\Gamma F_B \cdot v\, d\Gamma - \mu \cdot \varphi_2(u) \\ &\le C\,\|F_B\|_{-\frac{3}{2}}^\Gamma \|v\|_{\frac{3}{2}}^\Gamma \le C\,\|F_B\|_{-\frac{3}{2}}^\Gamma \|v\|_2^\Omega \le C\,\|F_B\|_{-\frac{3}{2}}^\Gamma \|u\|_0^\Omega \end{aligned}$$

which proves (2.5) ∎

3. Perturbations of the domain and analytic expansions

In order to describe the evolution of the domain we consider a fixed domain Ω_0 which is supposed to be bounded, smooth, and locally on one side of its boundary Γ_0. Its outer normal vector will be denoted by n, and we choose a fixed vector-valued function $\zeta \in (C^\infty(\Gamma_0))^N$ such that

$$\gamma(\xi) = \zeta(\xi) \cdot n(\xi) > 0 \qquad \forall \xi \in \Gamma_0 \tag{3.1}$$

and a fixed constant $s_0 > 3 + \frac{N-1}{2}$.

Lemma 3.1 (Description of perturbed domains). *There is a $\delta_0 > 0$ such that for all $r \in B_0(\delta_0, H^{s_0}(\Gamma_0))$ the following holds:*

(i) *The set*

$$\Gamma_r = \left\{\xi + \zeta(\xi) r(\xi) \,|\, \xi \in \Gamma_0\right\}$$

is the boundary of a simply connected domain Ω_r.

(ii) *There is a global diffeomorphism $z = z(r)$ mapping Ω_0 onto Ω_r such that $z \in (H^{s_0+\frac{1}{2}}(\Omega_0))^N$ and*

$$\|z - \mathrm{id}\|^{\Omega_0}_{s_0+\frac{1}{2}} \le C \, \|r\|^{\Gamma_0}_{s_0}$$

with C independent of r.

Proof. Statement (i): The collar manifold theorem implies the existence of a diffeomorphism between $\mathcal{I} \times \Gamma_0$ and an open neighbourhood of Γ_0 in $\mathbb{R}^N$ where $\mathcal{I}$ is a certain open neighbourhood of 0 in $\mathbb{R}$. The assertion follows thus from the embedding $H^{s_0}(\Gamma_0) \hookrightarrow C^0(\Gamma_0)$.

Statement (ii): We construct z by setting $z = \mathrm{Tr}^{-1}(r\zeta) + \mathrm{id}$ where Tr^{-1} is a fixed right inverse of the trace operator Tr from $H^{s_0+\frac{1}{2}}(\Omega_0)$ to $H^{s_0}(\Gamma_0)$. The embedding theorems yield then that $\|z - \mathrm{id}\|_{(C^2(\Omega_0))^N}$ is small which implies the global injectivity of z. (For details see [13].) ∎

Consider now, with the notation of the previous section, $s = s_0 - \frac{1}{2}$, and $\Omega = \Omega_r$ the equations

$$\begin{aligned} L[U \, P \, \Lambda]^T &= [f \; 0]^T \\ \widetilde{L}[U \, P \, \Lambda]^T &= [0 \, 0 \, \tilde{\kappa}_r n_r \, 0 \, 0]^T \end{aligned}$$

with $f \in \left((H^1(\Omega_r))^N\right)'$ defined by

$$f(v) = \int_{\Gamma_r} \tilde{\kappa}_r n_r \cdot v \, d\Gamma_r$$

where $\tilde{\kappa}_r$ and n_r are the double mean curvature and the outer normal vector of Γ_r, respectively. Using r and $z(r)$ it is possible by means of Lemma 3 to transform both equations to Ω_0, and in the sequel we will consider the operators L, $\widetilde{L}$ etc. as acting on function spaces defined on Ω_0 and depending on $r \in B_0(\delta_0, H^{s_0}(\Gamma_0))$. Thus we get

$$\begin{aligned} L(r)[u(r) \, p(r) \lambda(r)]^T &= F(r) \\ \widetilde{L}(r)[u(r) \, p(r) \lambda(r)]^T &= \widetilde{F}(r) \end{aligned} \tag{3.2}$$

with

$$[u(r) \, p(r) \lambda(r)]^T = [U \circ z(r) \, P \circ z(r) \, \Lambda]^T$$

$$L(r) \begin{bmatrix} u \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} A(r)u + (B(r))' \begin{bmatrix} p \\ \lambda \end{bmatrix} \\ B(r)u \end{bmatrix}$$

$$F(r) = [f(r) \, 0]^T$$

$$\widetilde{F}(r) = [0 \, 0 \, \kappa_r \nu(r) \, 0 \, 0]^T$$

$$(A(r)u)v = \int_{\Omega_0} \sum_{i,j,m,l=1}^{N} \left(a^{mj} \frac{\partial u_i}{\partial x_m} + a^{mi} \frac{\partial u_j}{\partial x_m} \right) \left(a^{lj} \frac{\partial v_i}{\partial x_l} + a^{li} \frac{\partial v_j}{\partial x_l} \right) \det \mathcal{A} \, dx$$

$$B(r)u = \begin{bmatrix} -\mathrm{div}_r \det\mathcal{A} \\ \int_{\Omega_0} u \det\mathcal{A}\, dx \\ \int_{\Omega_0} \mathrm{rot}_r \det\mathcal{A}\, dx \end{bmatrix}$$

$$f(r)v = \int_\Gamma \chi_r \kappa_r \nu(r) \cdot v\, d\Gamma_0$$

$$\widetilde{L}(r)\,[u\,p\,\lambda] = \begin{bmatrix} -\Delta_r u + \nabla_r p + \lambda_1 \\ -\mathrm{div}_r u \\ \mathcal{T}_r(u,p)\nu(r) + \lambda_2 \otimes \nu(r) \\ \int_{\Omega_0} u \det\mathcal{A}\, dx \\ \int_{\Omega_0} \mathrm{rot}_r u \det\mathcal{A}\, dx \end{bmatrix}$$

$$(\nabla_r p)_i = \sum_j a^{ji} \frac{\partial p}{\partial x_j}$$

$$\mathrm{div}_r u = \sum_{i,j} a^{ji} \frac{\partial u_i}{\partial x_j}$$

$$(\mathrm{rot}_r u)_{K(i,j)} = \sum_l \left(a^{li} \frac{\partial u_j}{\partial x_l} - a^{lj} \frac{\partial u_i}{\partial x_l} \right)$$

$$(\Delta_r u)_i = \sum_{j,k,l} a^{jl} \frac{\partial}{\partial x_j} \left(a^{kl} \frac{\partial u_i}{\partial x_k} \right)$$

$$(\nabla_r u)_{ij} = \sum_k a^{kj} \frac{\partial u_i}{\partial x_k}$$

$$\mathcal{T}_r(u,p) = (\nabla_r u) + (\nabla_r u)^T - pI$$

where $v \in (H^1(\Omega_0))^N$, $\mathcal{A}$ is the Jacobian $\frac{\partial z(r)}{\partial \xi}$, a^{ij} are the elements of $\mathcal{A}^{-1}$, $\kappa_r = \tilde{\kappa}_r \circ z(r)$, $\nu(r) = n_r \circ z(r)$, and χ_r is a scalar function on Γ_0 describing the "change of the surface element" when Γ_0 is mapped to Γ_r by z.

Let E and F be Banach spaces. An operator T that maps a neighbourhood of $x_0 \in E$ to F is called analytic near x_0 if it has a series representation

$$T(x) = T_0 + \sum_{k=1}^{\infty} T_k(x - x_0, \ldots, x - x_0)$$

with symmetric k-linear operators T_k and positive convergence radius. We will use the well-known facts that the sum, the composition and, if F is a Banach algebra, the (pointwise) product of (locally) analytic operators is (locally) analytic.

Let $\mathcal{I}$ denote the embedding operator of $\tilde{X}$ into X.

Lemma 4 (Analyticity of the perturbation).

(i) *The operators L, $\widetilde{L}$, F, and $\widetilde{F}$ are analytic near* 0 *as functions of $r \in H^{s_0}(\Gamma_0)$ into $\mathcal{L}(X,Y)$, $\mathcal{L}(\widetilde{X},\widetilde{Y})$, Y, and $\widetilde{Y}$, respectively.*

(ii) *The estimates*

$$\|F_k(r_1,\dots,r_k)\|_Y \le C_k \|r_1\|_{s_0}^{\Gamma_0}\cdots\|r_{k-1}\|_{s_0}^{\Gamma_0}\|r_k\|_{\frac{3}{2}}^{\Gamma_0}$$
$$\|L_k(r_1,\dots,r_k)\mathcal{I}\|_{\mathcal{L}(\widetilde{X},Y)} \le C_k \|r_1\|_{s_0}^{\Gamma_0}\cdots\|r_{k-1}\|_{s_0}^{\Gamma_0}\|r_k\|_{\frac{1}{2}}^{\Gamma_0}$$

hold.

Proof. We have, writing $\tilde{r} = \mathrm{Tr}^{-1}(r\zeta)$,

$$\|\mathcal{A}-I\|_{(C^0(\Omega_0)^{N\times N})} = \left\|\frac{\partial\tilde{r}}{\partial x}\right\|_{(C^0(\Omega_0)^{N\times N})} \le C\|r\|_{s_0}^{\Gamma_0},$$

hence the inverse of $\mathcal{A}$ exists for all $x\in\Omega_0$ and can be represented by a Neumann series if $\|r\|_{s_0}^{\Gamma_0}$ is sufficiently small. Thus we get a series representation

$$a^{ij} = \delta_{ij} + \sum_{k=1}^{\infty} a_k^{ij}(r,\dots,r) \tag{3.3}$$

where the a_k^{ij} are linear combinations of first partial derivatives of components of $\tilde{r}$. $\det\mathcal{A}$ can also be written as a (finite) series of this type, and thus, for arbitrary $u,v\in H^1(\Omega_0)$, $(A_k(r_1,\dots,r_k)u)v$ is a linear combination of terms

$$\int_\Omega \frac{\partial u_i}{\partial x_m}\frac{\partial v_j}{\partial x_l}\prod_{\sigma=1}^{k}\frac{\partial(\tilde{r}_\sigma)_{t_\sigma}}{\partial x_{\tau_\sigma}}\,dx$$

with $\tilde{r}_\sigma = \mathrm{Tr}^{-1}(r_\sigma\zeta)$.

We will apply to these terms the estimate

$$\left|\int_\Omega \psi_1\psi_2\psi_3\,dx\right| \le \left|\int_\Omega \psi_1\psi_2\,dx\right| \|\psi_3\|_{C^0(\Omega_0)} \le C\,\|\psi_1\|_0^{\Omega_0}\|\psi_2\|_0^{\Omega_0}\|\psi_3\|_{s_0-2}^{\Omega_0}$$

holding for all $\psi_1,\psi_2\in H^0(\Omega_0)$ and $\psi_3\in H^{s_0-2}(\Omega_0)$. If we set

$$\psi_1 = \frac{\partial u_i}{\partial x_m},\qquad \psi_2 = \frac{\partial v_j}{\partial x_l},\qquad \psi_3 = \prod_{\sigma=1}^{k}\frac{\partial(\tilde{r}_\sigma)_{t_\sigma}}{\partial x_{\tau_\sigma}}$$

and take into account that $H^{s_0-2}(\Omega_0)$ is a Banach algebra we obtain after summation that A is analytic near 0 as a function of $r\in H^{s_0}(\Gamma_0)$ into $\mathcal{L}\big((H^1(\Omega_0))^N,((H^1(\Omega_0))^N)'\big)$. If we assume $u\in H^{s_0-\frac{1}{2}}(\Omega_0)$ and set

$$\psi_1 = \frac{\partial v_j}{\partial x_l},\qquad \psi_2 = \frac{\partial(\tilde{r}_k)_{t_k}}{\partial x_{\tau_k}},\qquad \psi_3 = \frac{\partial u_i}{\partial x_m}\prod_{\sigma=1}^{k-1}\frac{\partial(\tilde{r}_\sigma)_{t_\sigma}}{\partial x_{\tau_\sigma}}$$

we get by the same arguments

$$\left\|A_k(r_1,\ldots,r_k)u\right\|_{((H^1(\Omega_0))^N)'} \leq C_k \left\|u\right\|^{\Omega_0}_{s_0-\frac{1}{2}} \left\|r_1\right\|^{\Gamma_0}_{s_0} \cdots \left\|r_{k-1}\right\|^{\Gamma_0}_{s_0} \left\|r_k\right\|^{\Gamma_0}_{\frac{1}{2}}.$$

Together with analogous considerations on B, B', and the integrals describing the auxiliary conditions this yields the assertions on L.

We introduce regular smooth local parametrizations of $\Gamma_0 = \bigcup_{j=1}^J \Gamma_0^{(j)}$ by

$$\Gamma_0^{(j)}: \quad \xi = \xi^{(j)}(w) \qquad (w \in W_j \subset \mathbb{R}^{N-1})$$

which yield local parametrizations of $\Gamma_r = \bigcup_{j=1}^J \Gamma_r^{(j)}$ by

$$\Gamma_r^{(j)}: \quad x = x_r^{(j)}(w) = \xi^{(j)}(w) + (r(\xi^{(j)}(w))\zeta(\xi^{(j)}(w))$$

whose regularity follows for small $\|r\|^{\Gamma_0}_{s_0}$ from the regularity of the $\xi^{(j)}$. On the j-th coordinate patch, $\nu(r)$ can be characterized by the equations

$$\begin{aligned} \nu(r) &= \frac{\tilde{\nu}(r)}{|\tilde{\nu}(r)|} = \frac{\tilde{\nu}(r)}{\sqrt{\tilde{\nu}(r)^T\tilde{\nu}(r)}} \\ \left(\frac{\partial x_r^{(j)}}{\partial w}\right)^T &\tilde{\nu}(r) = 0 \\ n\cdot\tilde{\nu}(r) &= 1. \end{aligned}$$

The latter two of these equations form a system of N scalar linear equations for $\tilde{\nu}(r)$ which at $r = 0$ has the unique solution $\tilde{\nu}(0) = \nu(0) = n$. Inverting this system for small $\|r\|^{\Gamma_0}_{s_0}$ using the same arguments as above and taking into account that $|\tilde{\nu}(r)|$ is near 1 for small r we get a convergent series representation

$$\nu(r)\circ\xi^{(j)} = n\circ\xi^{(j)} + \sum_{k=1}^{\infty} \nu_k^{(j)}(r,\ldots,r)$$

where the $\nu_k^{(j)}(r_1,\ldots,r_k)$ are sums of products of smooth functions with the $(r_\sigma\circ\xi^{(j)})$ or their first partial derivatives. Hence ν is an analytic mapping near 0 from $H^{s_0}(\Gamma_0)$ to $(H^{s_0-2}(\Gamma_0))^N$.

Moreover, we have in local coordinates on $\Gamma_0^{(j)}$

$$\begin{aligned} \chi_r &= \frac{\sqrt{g_r^{(j)}}}{\sqrt{g_0^{(j)}}} \\ \kappa_r\nu(r) = \Delta_{\Gamma_r}x_r &= \frac{1}{\sqrt{g_r^{(j)}}}\sum_{i,k=1}^{N-1}\frac{\partial}{\partial w_i}\left(g^{ik}_{(j)}\frac{\partial x_r^{(j)}}{\partial w_k}\right) \end{aligned}$$

with

$$g_r^{(j)} = \det G_r^{(j)}, \qquad g_{(j)}^{ik} = [G_r^{(j)}]_{ik}, \qquad G_r^{(j)} = \left(\frac{\partial x_r^{(j)}}{\partial w}\right)^T \left(\frac{\partial x_r^{(j)}}{\partial w}\right)$$

and we find by analogous arguments that the mappings $r \mapsto \chi_r$ and $r \mapsto \kappa_r$ are analytic near 0 from $H^{s_0}(\Gamma_0)$ to $H^{s_0-2}(\Gamma_0)$. Thus we get the assertions on F and $\widetilde{F}$.

The analytic dependence of $\widetilde{L}$ on $r \in H^{s_0}(\Gamma_0)$ follows from the above considerations and the Banach algebra properties of the spaces $H^{s_0-\frac{5}{2}}(\Omega_0)$, $H^{s_0-\frac{3}{2}}(\Omega_0)$, and $H^{s_0-2}(\Gamma_0)$ ∎

Lemma 5 (Analytic dependence of the solution). *Let $u(r)$ be defined by* (3.2). *The mapping $r \mapsto u(r)$ is well-defined on $B_0(\varepsilon, H^{s_0}(\Gamma_0))$ for some $\varepsilon > 0$ into $(H^{s_0-\frac{1}{2}}(\Omega_0))^N$. It is analytic near* 0 *and estimates*

$$\left\|u_k(r_1,\ldots,r_k)\right\|_1^{\Omega_0} \le C_k\|r_1\|_{s_0}^{\Gamma_0}\cdots\|r_{k-1}\|_{s_0}^{\Gamma_0}\|r_k\|_{\frac{3}{2}}^{\Gamma_0} \tag{3.4}$$

hold.

Proof. Writing $\underline{v}(r) = [u(r)\,p(r)\,\lambda(r)]^T \in \widetilde{X}$ we have, due to Lemmas 2/(i) and Lemma 4/(i) by the real-analytic version of the Implicit Function Theorem that the mapping $r \mapsto \underline{v}(r)$ exists and is analytic near 0 from $H^{s_0}(\Gamma_0)$ to $\widetilde{X}$, with estimates

$$\left\|\underline{v}_k(r_1,\ldots,r_k)\right\|_{\widetilde{X}} \le C_k\|r_1\|_{s_0}^{\Gamma_0}\cdots\|r_k\|_{s_0}^{\Gamma_0}. \tag{3.5}$$

The assertions on the mapping $r \mapsto u(r)$ follow immediately from this.

To prove (3.4) it is sufficient to establish

$$\left\|\underline{v}_j(r_1,\ldots,r_j)\right\|_X \le C_j\|r_1\|_{s_0}^{\Gamma_0}\cdots\|r_{j-1}\|_{s_0}^{\Gamma_0}\|r_j\|_{\frac{3}{2}}^{\Gamma_0} \tag{3.6}$$

This will be done by induction. For $j = 1$, we have

$$\begin{aligned}\|\underline{v}_1(r)\|_X &= \left\|L(0)^{-1}F_1(r) - L(0)^{-1}L_1(r)\underline{v}_0\right\|_X\\ &\le C\big(\|F_1(r)\|_Y + \|L_1(r)\mathcal{I}\|_{\mathcal{L}(\widetilde{X},Y)}\big)\\ &\le C\|r\|_{\frac{3}{2}}^{\Gamma_0}\end{aligned}$$

where Lemma 4/(ii) has been used. Suppose now (3.6) holds for all $j \le k-1$. Taking the k-th Fréchet derivative at $r = 0$ on both sides of the equation $L(r)\underline{v}(r) = F(r)$ and applying $L(0)^{-1}$ yields

$$\begin{aligned}\underline{v}_k(r_1,\ldots,r_k) = L(0)^{-1}\Bigg(&F_k(r_1,\ldots,r_k)\\ &-\sum_{j=1}^{k}\frac{1}{j!\,(k-j)!}\sum_{\pi} L_j\big(r_{\pi(1)},\ldots,r_{\pi(j)}\big)\,\underline{v}_{k-j}\big(r_{\pi(j+1)},\ldots,r_{\pi(k)}\big)\Bigg)\end{aligned}$$

where π runs over all permutations of $\{1, \dots, k\}$. We will estimate the terms on the right separately, using Lemma 4/(ii). Thus we get

$$\left\|L(0)^{-1} F_k(r_1, \dots, r_k)\right\|_X \leq C \left\|F_k(r_1, \dots, r_k)\right\|_Y \leq C_k \|r_1\|_{s_0}^{\Gamma_0} \cdots \|r_{k-1}\|_{s_0}^{\Gamma_0} \|r_k\|_{\frac{3}{2}}^{\Gamma_0},$$

for the terms in the sum over j with $\pi^{-1}(k) \leq j$ we find, using (3.5),

$$\begin{aligned}
&\left\|L(0)^{-1} L_j\big(r_{\pi(1)}, \dots, r_{\pi(j)}\big)\, \underline{v}_{k-j}\big(r_{\pi(j+1)}, \dots, r_{\pi(k)}\big)\right\|_X \\
&\quad \leq C \left\|L_j\big(r_{\pi(1)}, \dots, r_{\pi(j)}\big)\mathcal{I}\right\|_{\mathcal{L}(\widetilde{X}, Y)} \left\|\underline{v}_{k-j}\big(r_{\pi(j+1)}, \dots, r_{\pi(k)}\big)\right\|_{\widetilde{X}} \\
&\quad \leq C_k \|r_1\|_{s_0}^{\Gamma_0} \cdots \|r_{k-1}\|_{s_0}^{\Gamma_0} \|r_k\|_{\frac{1}{2}}^{\Gamma_0},
\end{aligned}$$

and for the other terms, using the induction assumption,

$$\begin{aligned}
&\left\|L(0)^{-1} L_j\big(r_{\pi(1)}, \dots, r_{\pi(j)}\big)\, \underline{v}_{k-j}\big(r_{\pi(j+1)}, \dots, r_{\pi(k)}\big)\right\|_X \\
&\quad \leq C \left\|L_j\big(r_{\pi(1)}, \dots, r_{\pi(j)}\big)\right\|_{\mathcal{L}(X, Y)} \left\|\underline{v}_{k-j}\big(r_{\pi(j+1)}, \dots, r_{\pi(k)}\big)\right\|_X \\
&\quad \leq C_k \|r_1\|_{s_0}^{\Gamma_0} \cdots \|r_{k-1}\|_{s_0}^{\Gamma_0} \|r_k\|_{\frac{3}{2}}^{\Gamma_0}.
\end{aligned}$$

Hence (3.6) holds also for $j = k$ ∎

We describe now the moving boundary $\Gamma(t)$ near Γ_0 by

$$\Gamma(t) = \Gamma_{r(t)}.$$

The kinematic boundary condition takes then the form

$$\frac{\partial r}{\partial t} = \frac{\mathrm{Tr}_{\Gamma_0}(u(r)) \cdot \nu(r)}{\zeta \cdot \nu(r)} = \rho(r), \tag{3.7}$$

i.e. our moving boundary problem is reformulated as a nonlinear nonlocal evolution equation for r. Using the inequality

$$\|\psi_1 \psi_2\|_{\frac{1}{2}}^{\Gamma_0} \leq C \|\psi_1\|_{C^1(\Gamma_0)} \|\psi_2\|_{\frac{1}{2}}^{\Gamma_0} \leq C \|\psi_1\|_{s_0-1}^{\Gamma_0} \|\psi_2\|_{\frac{1}{2}}^{\Gamma_0}$$

and the Banach algebra property of $H^{s_0-1}(\Gamma_0)$ we find by arguments similar to the ones given above that ρ is analytic near 0 from $H^{s_0}(\Gamma_0)$ to $H^{s_0-1}(\Gamma_0)$ and we have additional estimates

$$\left\|\rho_k(r_1, \dots, r_k)\right\|_{\frac{1}{2}}^{\Gamma_0} \leq C_k \|r_1\|_{s_0}^{\Gamma_0} \cdots \|r_{k-1}\|_{s_0}^{\Gamma_0} \|r_k\|_{\frac{3}{2}}^{\Gamma_0}. \tag{3.8}$$

Note that in all estimates for k-linear forms the constants C_k can be chosen such that

$$C_k \sim O(M^k). \tag{3.9}$$

4. A chain rule

In the following we suppose additionally that Ω_0 is strictly star-shaped, i.e. there is a smooth positive real-valued function R_0 defined on the unit sphere S^{N-1} such that (after a suitable translation)

$$\Gamma_0 = \{\theta R_0(\theta) \,|\, \theta \in S^{N-1}\}.$$

Note that the mapping $\Phi_0 : S^{N-1} \longrightarrow \Gamma_0$ defined by $\Phi_0(\theta) = \zeta R_0(\theta)$ is a C^∞-diffeomorphism between S^{N-1} and Γ_0, hence the direct image map Φ_0^* defined by $(\Phi_0^*\varphi)(\theta) = \varphi(\Phi_0(\theta))$ is an isomorphism from $C^\infty(\Gamma_0)$ to $C^\infty(S^{N-1})$ and from $H^{s_0}(\Gamma_0)$ to $H^{s_0}(S^{N-1})$.

We choose $\zeta(\xi) = \frac{\xi}{|\xi|}$ and consider a fixed system $\{Q_j \,|\, j = 1, \dots, \binom{N}{2}\}$ of linear independent skew-symmetric $(N \times N)$-matrices. We introduce on S^{N-1} and Γ_0, respectively, the first order linear differential operators $\widetilde{D}_j$ and D_j by

$$\widetilde{D}_j\psi(\zeta) = \frac{d}{d\tau}((\psi \circ \exp^{\tau Q_j})(\zeta))\Big|_{\tau=0}$$
$$D_j\varphi(\xi) = \frac{d}{d\tau}((\varphi \circ \Phi_0 \circ \exp^{\tau Q_j} \circ \Phi_0^{-1})(\xi))\Big|_{\tau=0}$$

and for multiindices $\alpha = (\alpha_1 \dots \alpha_{\binom{N}{2}})$ we set $D^\alpha = D_1^{\alpha_1} \cdots D_{\binom{N}{2}}^{\alpha_{\binom{N}{2}}}$, $\widetilde{D}^\alpha$ is defined analogously.

In the following, $T^{(k)}(x)[\cdot, \dots, \cdot]$ will denote the k-th Fréchet derivative of T at x.

Lemma 6 (Chain rule). *Assume r to be smooth and $\|r\|_{s_0}^{\Gamma_0}$ sufficiently small. Then*

$$D^\alpha(\rho(r)) = \sum_{k=1}^{|\alpha|} \sum_{\beta_1+\dots+\beta_k=\alpha} C_{\beta_1,\dots,\beta_k} \rho^{(k)}(r)\left[D^{\beta_1}(r + \mathcal{R}_0), \dots, D^{\beta_k}(r + \mathcal{R}_0)\right] \tag{4.1}$$

where $\mathcal{R}_0 = \Phi_0^{*-1} R_0$, all occurring β_l are non-zero, and for $k = 1$ we have $C_\alpha = 1$.

Proof. Define the operators $\tilde{\rho}$, $\tilde{u}$, and $\tilde{\nu}$ acting on the smooth functions in a small ball around R_0 in $H^{s_0}(S^{N-1})$ by

$$\tilde{\rho}(R) = \rho(\Phi_0^{*-1}(R - R_0)) \tag{4.2}$$

and

$$\tilde{u}(R) = u(\Phi_0^{*-1}(R - R_0))$$
$$\tilde{\nu}(R) = \nu(\Phi_0^{*-1}(R - R_0)).$$

We will show the equality

$$D^\alpha \tilde{\rho}(R) = \sum_{k=1}^{|\alpha|} \sum_{\beta_1+\dots+\beta_k=\alpha} C_{\beta_1,\dots,\beta_k} \tilde{\rho}^{(k)}(R)\left[\widetilde{D}^{\beta_1} R, \dots, \widetilde{D}^{\beta_k} R\right] \tag{4.3}$$

with the same additional assertions as above. This is equivalent to the lemma because of

$$r = \Phi_0^{*-1}(R - R_0)$$
$$\Phi_0^{*-1}(\widetilde{D}^\alpha \psi) = D^\alpha(\Phi_0^{*-1}\psi)$$
$$\tilde{\rho}^{(k)}(R)[h_1, \ldots, h_k] = \rho^{(k)}(\Phi_0^{*-1}(R - R_0))\left[\Phi_0^{*-1}h_1, \ldots, \Phi_0^{*-1}h_k\right]$$

where the last statement holds for all $h_l \in H^{s_0}(\Gamma_0)$ and is obtained by calculating the k-th Fréchet derivative of both sides of (4.2).

The proof of (4.3) will be given by induction over $|\alpha|$ and rests essentially on the invariance of the problem under rigid body motions, in particular, under rotations around the origin.

1. $|\alpha| = 1$: Choose a fixed $j \in \{1, \ldots, \binom{N}{2}\}$ and consider the one-parameter family of rotations around the origin described by $x \mapsto \exp^{\tau Q_j} x$ with τ varying in a small open interval containing 0. Let $\widetilde{\Omega}_R$ be the bounded domain with boundary $\{\theta R(\theta) | \theta \in S^{N-1}\}$. Then clearly $\exp^{\tau Q_j}[\widetilde{\Omega}_R] = \widetilde{\Omega}_{R^\tau}$ with

$$R^\tau(\zeta) = R(e^{-\tau Q_j}\zeta).$$

Taking into account now the fact that the boundary value problem (2.1) as well as the auxiliary conditions $\varphi_1(u) = 0$, $\varphi_2(u) = 0$ are invariant with respect to rotations, i.e. that the coordinate change $x \mapsto \exp^{\tau Q_j} x$ does not alter the form of the equations for fixed Q_j, we find

$$\tilde{u}(R^\tau)\left((\Phi_0 \circ \exp^{\tau Q_j} \circ \Phi_0^{-1})(\xi)\right) = \exp^{\tau Q_j} \tilde{u}(R)(\xi)$$

and further, using (3.7) and the fact that $\exp^{\tau Q_j}$ is an orthogonal matrix,

$$\tilde{\nu}(R^\tau)\left((\Phi_0 \circ \exp^{\tau Q_j} \circ \Phi_0^{-1})(\xi)\right) = \exp^{\tau Q_j} \tilde{\nu}(R)(\xi)$$
$$\zeta\left((\Phi_0 \circ \exp^{\tau Q_j} \circ \Phi_0^{-1})(\xi)\right) = \exp^{\tau Q_j} \zeta(\xi)$$
$$\tilde{\rho}(R^\tau)((\Phi_0 \circ \exp^{\tau Q_j} \circ \Phi_0^{-1})(\xi)) = \tilde{\rho}(R)(\xi)$$

for all $\xi \in \Gamma_0$. Differentiating the last equation with respect to τ at $\tau = 0$ yields

$$D_j\tilde{\rho}(R) = \tilde{\rho}'(R)[\widetilde{D}_j R] \tag{4.4}$$

which is (4.3) for $|\alpha| = 1$.

Moreover, starting from (4.4), by induction one proves

$$\begin{aligned} D_j\tilde{\rho}^{(k)}(R)\left[h_1, \ldots, h_k\right] = &\sum_{l=1}^{k} \tilde{\rho}^{(k)}(R)\left[h_1, \ldots, h_{l-1}, \widetilde{D}_j h_l, h_{l+1}, \ldots, h_k\right] \\ &+ \tilde{\rho}^{(k+1)}(R)\left[\widetilde{D}_j R, h_1, \ldots, h_k\right] \end{aligned} \tag{4.5}$$

for all $k \in \mathbb{N}$ and $h_1, \ldots, h_k \in H^{s_0+1}(\Gamma_0)$ where the induction step only consists in calculating the Fréchet derivative.

2. Suppose now (4.3) holds for $|\alpha'| = m$, consider α with $|\alpha| = m + 1$. Writing $D^\alpha\tilde{\rho}(R) = D_j D^{\alpha'}\tilde{\rho}(R)$, applying the induction assumption and (4.5), and rearranging the terms according to the order of the Fréchet derivative completes the proof ∎

Expansions of $\rho(r)$ and $\rho^{(k)}(r)$ in (4.1) and "comparison of coefficients" yields

$$
\begin{aligned}
D^\alpha \rho_m\big(r^{(1)},\dots,r^{(m)}\big) &= \frac{1}{m!} \\
&\times \sum_\pi \sum_{l=0}^{m} \sum_{k=\max\{1,m-l\}}^{|\alpha|} \sum_{\beta_1+\dots+\beta_k=\alpha} C_{\beta_1,\dots,\beta_k} \frac{(k+l)!}{l!\,(m-l)!\,(k-m+l)!} \\
&\times \sum_\sigma \rho_{k+l}\Big(r^{(\pi(1))},\dots,r^{(\pi(l))}, D^{\beta_{\sigma(1)}} r^{(\pi(l+1))}, \\
&\qquad \dots, D^{\beta_{\sigma(m-l)}} r^{(\pi(m))}, D^{\beta_{\sigma(m-l+1)}} \mathcal{R}_0, \dots, D^{\beta_{\sigma(k)}} \mathcal{R}_0\Big)
\end{aligned}
\tag{4.6}
$$

where π and σ run over all permutations of $\{1,\dots,m\}$ and $\{1,\dots,k\}$, respectively. Considering the special case $m = 1$ and using that $|\beta_j| \le |\alpha| - 1$ for $j = 1,\dots,k$ if $k \ge 2$ we can prove the commutator estimates

$$\big\|(D^\alpha \rho_1 - \rho_1 D^\alpha) r\big\|_{\frac{1}{2}}^{\Gamma_0} \le C\, \|r\|_{|\alpha|+\frac{1}{2}}^{\Gamma_0} \tag{4.7}$$

$$\big\|(D^\alpha \rho_1 - \rho_1 D^\alpha) r\big\|_{s_0-1}^{\Gamma_0} \le C\, \|r\|_{|\alpha|+s_0-1}^{\Gamma_0} \tag{4.8}$$

In the sequel, let s_1 be the smallest integer such that $s_1 > 3 + \frac{N-1}{2}$. Note that the vector fields on Γ_0 that correspond to the differential operators D_j span the tangent space in any $\xi \in \Gamma_0$, therefore the bilinear forms

$$(\varphi,\psi)_s = \sum_{|\alpha|\le s-1} (D^\alpha\varphi, D^\alpha\psi)_1 \qquad (s \le s_1) \tag{4.9}$$

$$(\varphi,\psi)_s = \sum_{|\alpha|\le s-s_1} (D^\alpha\varphi, D^\alpha\psi)_{s_1} \qquad (s > s_1) \tag{4.10}$$

with

$$(\varphi,\psi)_1 = \int_{\Gamma_0} \big(\varphi\psi + \nabla_{\Gamma_0}\varphi \cdot \nabla_{\Gamma_0}\psi\big)\, d\Gamma_0 \tag{4.11}$$

can and will be used as scalar products on $H^s(\Gamma_0)$ with integer $s > 0$. From elliptic regularity theory it follows the inequality

$$\|u\|_{s+t}^{\Gamma_0\,2} \le C_{s,t} \sum_{|\alpha|\le t} \|D^\alpha u\|_s^{\Gamma_0\,2} \tag{4.12}$$

for arbitrary $s,t \in \mathbb{N}$ and $u \in H^{s+t}(\Gamma_0)$.

We will use the notations $x_n \xrightarrow{X} x$ for norm convergence and $x_n \xrightarrow{X} x$ for weak convergence in the (Banach) space X.

Lemma 7 (Continuity of ρ near 0). *There is an $\varepsilon_0 > 0$ such that for all integers $s \ge s_1$ the mapping*

$$\rho: \quad B_0(\varepsilon_0, H^{s_1}(\Gamma_0)) \cap H^s(\Gamma_0) \longrightarrow H^{s-1}(\Gamma_0)$$

is

(i) *continuous and bounded*

(ii) *weakly sequentially continuous.*

Proof. Statement (i): Set $s_0 = s_1$. If $s = s_1$, then the assertion follows directly from Lemma 5. If $s > s_1$, then because of (4.12) it is sufficient to show that the mappings $D^\alpha \rho$ are continuous and bounded from $B_0(\varepsilon_0, H^{s_1}(\Gamma_0)) \cap H^s(\Gamma_0)$ to $H^{s_1-1}(\Gamma_0)$ for all α with $|\alpha| \le s - s_1$. Using (4.6) we find

$$\begin{aligned}
&\left\|D^\alpha\left(\rho(r) - \rho(v)\right)\right\|_{s_1-1}^{\Gamma_0} \\
&\le \sum_{m=1}^{\infty} \left\|D^\alpha \rho_m(r,\ldots,r) - D^\alpha \rho_m(v,\ldots,v)\right\|_{s_1-1}^{\Gamma_0} \\
&\le \sum_{l=0}^{m} \sum_{k=\max\{1,m-l\}}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1,\ldots,\beta_k} \frac{(k+l)!}{l!\,(m-l)!\,(k-m+l)!} \\
&\quad \times \sum_\sigma \Bigg(\sum_{j=1}^{l} \Big\|\rho_{k+l}\Big(r,\ldots,r,r-v,v,\ldots,v,D^{\beta_{\sigma(1)}}v, \\
&\qquad \ldots, D^{\beta_{\sigma(m-l)}}v, D^{\beta_{\sigma(m-l+1)}}\mathcal{R}_0,\ldots,D^{\beta_{\sigma(k)}}\mathcal{R}_0\Big)\Big\|_{s_1-1}^{\Gamma_0} \\
&\quad + \sum_{j=1}^{m-l} \Big\|\rho_{k+l}\Big(r,\ldots,r,D^{\beta_{\sigma(1)}}r, D^{\beta_{\sigma(j-1)}}r, D^{\beta_{\sigma(j)}}(r-v), D^{\beta_{\sigma(j+1)}}v, \\
&\qquad \ldots, D^{\beta_{\sigma(m-l)}}v, D^{\beta_{\sigma(m-l+1)}}\mathcal{R}_0,\ldots,D^{\beta_{\sigma(k)}}\mathcal{R}_0\Big)\Big\|_{s_1-1}^{\Gamma_0}\Bigg)
\end{aligned}$$

with $r, v \in B_0(\varepsilon_0, H^{s_1}(\Gamma_0)) \cap H^s(\Gamma_0)$ and the understanding that in the first sum over j the difference occurs in the j-th argument. We will estimate the summands of the sums in braces separately for $m > 1$, using (3.9), the interpolation inequality

$$\|D^\beta u\|_{s_1}^{\Gamma_0} \le C\,\|u\|_{s_1+|\beta|}^{\Gamma_0} \le C\,{\|u\|_{s_1}^{\Gamma_0}}^{1-\frac{|\beta|}{|\alpha|}}\,{\|u\|_{s}^{\Gamma_0}}^{\frac{|\beta|}{|\alpha|}}$$

holding for all multiindices β with $|\beta| \le |\alpha|$ and the notations

$$\left.\begin{aligned}
\mu_s &= \max\left\{\|u\|_s^{\Gamma_0}, \|v\|_s^{\Gamma_0}\right\} \\
b &= \frac{\sum_{\tau=1}^{m-l} |\beta_{\sigma(\tau)}|}{|\alpha|} \\
\nu &= \frac{|\beta_{\sigma(j)}|}{|\alpha|}.
\end{aligned}\right\} \tag{4.13}$$

We find for the summands in the first sum

$$\Big\|\rho_{k+l}\Big(r,\dots,r,r-v,v,\dots,v,D^{\beta_{\sigma(1)}}v,\dots,$$
$$D^{\beta_{\sigma(m-l)}}v,D^{\beta_{\sigma(m-l+1)}}\mathcal{R}_0,\dots,D^{\beta_{\sigma(k)}}\mathcal{R}_0\Big)\Big\|_{s_1-1}^{\Gamma_0}$$
$$\leq C_s M^{k+l}\mu_{s_1}^{l-1}\|v\|_{s_1}^{\Gamma_0\,m-l-b}\|v\|_{s}^{\Gamma_0\,b}\|r-v\|_{s_1}^{\Gamma_0}$$
$$\leq C_s M^m \mu_{s_1}^{m-1-b}\mu_s^b\|r-v\|_{s_1}^{\Gamma_0}$$

and in the second sum

$$\Big\|\rho_{k+l}\Big(r,\dots,r,D^{\beta_{\sigma(1)}}r,D^{\beta_{\sigma(j-1)}}r,D^{\beta_{\sigma(j)}}(r-v),D^{\beta_{\sigma(j+1)}}v,\dots,$$
$$D^{\beta_{\sigma(m-l)}}v,D^{\beta_{\sigma(m-l+1)}}\mathcal{R}_0,\dots,D^{\beta_{\sigma(k)}}\mathcal{R}_0\Big)\Big\|_{s_1-1}^{\Gamma_0}$$
$$\leq C_s M^{k+l}\|r\|_{s_1}^{\Gamma_0\,l}\mu_{s_1}^{m-l-1-(b-\nu)}\mu_s^{b-\nu}\|r-v\|_{s}^{\Gamma_0}$$
$$\leq C_s M^m \mu_{s_1}^{m-1-(b-\nu)}\mu_s^{b-\nu}\|r-v\|_{s}^{\Gamma_0}.$$

Carrying out the summations over σ, l, k, and the β_j we have to take into account that because of $l \leq m$ and $k \leq |\alpha|$

$$\frac{(k+l)!}{l!} \leq (k+l)^k \leq (|\alpha|+m)^{|\alpha|} \leq 2^{|\alpha|}(|\alpha|^{|\alpha|}+m^{|\alpha|})$$

and this yields for small ε_0

$$\|D^\alpha\rho_m(r,\dots,r)-D^\alpha\rho_m(v,\dots,v)\|_{s_1-1}^{\Gamma_0}$$
$$\leq C_s M^m(1+m^{s-s_1})\mu_{s_1}^{m-2}(1+\mu_s)\|r-v\|_s^{\Gamma_0}.$$

Demanding now $\varepsilon_0 < \frac{1}{M}$, using

$$\|D^\alpha\rho_1(r-v)\|_{s_1-1}^{\Gamma_0} \leq C_s\|r-v\|_s^{\Gamma_0}$$

and carrying out the summation over $m \geq 2$ yields

$$\|D^\alpha(\rho(r)-\rho(v))\|_{s_1-1}^{\Gamma_0} \leq C_s(1+\mu_s)\|r-v\|_s^{\Gamma_0}, \tag{4.14}$$

and this estimate implies the boundedness and continuity of $D^\alpha\rho$.

(ii) From (i) and Lemma 5 with $s_0 < s_1$ it follows that for any integer $s \geq s_1$ there is a $\tilde{s} < s$ such that the mapping

$$\rho:\ B_0(\varepsilon_0,H^{s_1}(\Gamma_0))\cap H^s(\Gamma_0)\longrightarrow H^{s-1}(\Gamma_0)$$

is bounded and the mapping

$$\rho:\ B_0(\varepsilon_0,H^{s_1}(\Gamma_0))\cap H^{\tilde{s}}(\Gamma_0)\longrightarrow H^{\tilde{s}-1}(\Gamma_0)$$

is continuous. For an arbitrary sequence $\{r_n\}$, $r_n \in B_0(\varepsilon_0, H^{s_1}(\Gamma_0)) \cap H^s(\Gamma_0)$ with $r_n \xrightarrow{H^s(\Gamma_0)} r^*$ we have $r_n \xrightarrow{H^{\bar{s}}(\Gamma_0)} r^*$ and hence

$$\rho(r_n) \xrightarrow{H^{\bar{s}-1}(\Gamma_0)} \rho(r^*). \tag{4.15}$$

On the other hand, $\{\rho(r_n)\}$ is a bounded sequence in $H^{s-1}(\Gamma_0)$ and therefore it has a weakly convergent subsequence. Consider now an arbitrary weakly convergent subsequence $\{\rho(r_{n'})\}$ with $\rho(r_{n'}) \xrightarrow{H^{s-1}(\Gamma_0)} \rho^*$. This implies $\rho(r_{n'}) \xrightarrow{H^{\bar{s}-1}(\Gamma_0)} \rho^*$ and thus, because of (4.15), $\rho^* = \rho(r^*)$. Hence we can conclude (see [22: Satz 10.2]) that $\rho(r_n) \xrightarrow{H^{s-1}(\Gamma_0)} \rho(r^*)$ ∎

5. Linearization

For the further investigation of (3.7) one has to identify the operator ρ_1 more precisely. We find, using the notation of Section 3

$$\rho_1(r) = \left(-\frac{u_0 \cdot n}{\gamma^2}\zeta + \frac{1}{\gamma}u_0\right) \cdot \nu_1(r) + \frac{1}{\gamma} n \cdot u_1(r)$$
$$u_1 = \Pi_1 L(0)^{-1}\big(F_1(r) - L_1(r)L(0)^{-1}F_0\big).$$

Calculating F_1 explicitly and recalling from Lemma 4/(ii) that $\|L_1(r)\|_{\mathcal{L}(X,Y)} \le C\,\|r\|^{\Gamma_0}_{\frac{1}{2}}$ we find that

$$\rho_1(r) = \rho_1^*(r) + \Lambda_1(r) + \Lambda_0(r), \tag{5.1}$$

with

$$\rho_1^*(r) = \frac{1}{\gamma}\,(\mathrm{Tr}_{\Gamma_0}\dot{u}) \cdot n$$

and $[\dot{u}\,\dot{p}\,\dot{\lambda}]^T \in X$ the solution of the variational problem

$$\left.\begin{aligned} a(\dot{u}, v) - \int_{\Omega_0} \dot{p}\,\mathrm{div}\, v\, dx + \dot{\lambda}_1^T\varphi_1(v) + \dot{\lambda}_2^T\varphi_2(v) &= \int_{\Gamma_0} \gamma\delta_{\Gamma_0} r\, n \cdot v\, d\Gamma_0 \\ &\quad \text{for all } \ v \in (H^1(\Omega))^N \\ \mathrm{div}\, \dot{u} &= 0 \\ \varphi_1(\dot{u}) &= 0 \\ \varphi_2(\dot{u}) &= 0, \end{aligned}\right\} \tag{5.2}$$

with

$$\Lambda_1(r) = \left(-\frac{u_0 \cdot n}{\gamma^2}\zeta + \frac{1}{\gamma}u_0\right) \cdot \nu_1(r)$$

a first order *differential* operator and

$$\Lambda_0 \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_0), H^{\frac{1}{2}}(\Gamma_0)).$$

By interpolation we have $\rho_1 \in \mathcal{L}\big(H^s(\Gamma_0), H^{s-1}(\Gamma_0)\big)$ for all (real) $s \ge \frac{3}{2}$.

Lemma 8 (Coercivity of $-\rho_1$). *For all positive integer s there are positive constants c_s and C_s such that*

$$-(\rho_1 r, r)_s \geq c_s \|r\|_{s+\frac{1}{2}}^{\Gamma_0\,2} - C_s \|r\|_{s-\frac{1}{2}}^{\Gamma_0\,2} \qquad \forall r \in H^{s+1}(\Gamma_0).$$

Proof. Step 1: $s = 1$. We have

$$\begin{aligned}
&-(\rho_1 r, r)_{H^1(\Gamma_0)} \\
&= -\left(\int_{\Gamma_0} \nabla_{\Gamma_0}\rho_1 r \cdot \nabla_{\Gamma_0} r \, d\Gamma_0 + \int_{\Gamma_0} \rho_1 r \, r \, d\Gamma_0\right) \\
&\geq -\left(\int_{\Gamma_0} \nabla_{\Gamma_0}(\rho_1^\star r + \Lambda_1 r + \Lambda_0 r)\cdot \nabla_{\Gamma_0} r\right) - C\,\|r\|_{\frac{3}{2}}^{\Gamma_0}\|r\|_{-\frac{1}{2}}^{\Gamma_0} \\
&\geq -\left(\int_{\Gamma_0} \nabla_{\Gamma_0}\rho_1^\star r \cdot \nabla_{\Gamma_0} r \, d\Gamma_0 + \int_{\Gamma_0} \nabla_{\Gamma_0}\Lambda_1 r \cdot \nabla_{\Gamma_0} r \, d\Gamma_0\right) - C\,\|r\|_{\frac{3}{2}}^{\Gamma_0}\|r\|_{\frac{1}{2}}^{\Gamma_0}.
\end{aligned}$$

The two remaining integrals will be treated separately.

Substep 1.1. Problem (5.2) is the weak formulation of the problem

$$\tilde{L}(0)[\dot{u}\,\dot{p}\,\dot{\lambda}]^T = [0\,0\,\gamma\Delta_{\Gamma_0} r n\,0\,0]^T$$

and from Lemma 2/(ii) we get

$$\|\dot{\lambda}_1\|_{\mathbb{R}^N} + \|\dot{\lambda}_2\|_{\mathbb{R}^{\binom{N}{2}}} \leq C\,\|\gamma\Delta_{\Gamma_0} r\, n\|_{-\frac{3}{2}}^{\Gamma_0} \leq C\,\|r\|_{\frac{1}{2}}^{\Gamma_0}. \tag{5.3}$$

In order to give an estimate for $\dot{p}$, consider the following Neumann problem for the Laplacian:

$$\left.\begin{aligned}
\Delta\Phi &= \dot{p} && \text{in } \Omega_0 \\
\frac{\partial\Phi}{\partial n} &= \gamma^{-1}\frac{\int_{\Omega_0}\dot{p}\,dx}{\int_{\Gamma_0}\gamma^{-1}\,d\Gamma_0} = g && \text{on } \Gamma_0.
\end{aligned}\right\}$$

It is solvable because

$$\int_{\Gamma_0} g\, d\Gamma_0 = \int_{\Omega_0} \dot{p}\, dx$$

and because of $\dot{p} \in L^2(\Omega_0)$ and $g \in H^{\frac{1}{2}}(\Gamma_0)$ the regularity theory for this problem yields the existence of a solution $\Phi \in H^2(\Omega_0)$ satisfying the estimate

$$\|\Phi\|_2^{\Omega_0} \leq C\big(\|\dot{p}\|_0^{\Omega_0} + \|g\|_{\frac{1}{2}}^{\Gamma_0}\big) \leq C\,\|\dot{p}\|_0^{\Omega_0}.$$

If we set now $v = v_p = \nabla\Phi$ in the first equation of (5.2) and take into account that $\operatorname{div} v_p = \dot{p}$, $\|v_p\|_1^{\Omega_0} \leq \|\Phi\|_2^{\Omega_0} \leq C\,\|\dot{p}\|_0^{\Omega_0}$, $\gamma n \cdot v_p$ is constant on Γ_0 and thus the boundary integral in (5.2) vanishes, we find

$$\begin{aligned}
\|\dot{p}\|_0^{\Omega_0\,2} &\leq |a(\dot{u}, v_p)| + |\dot{\lambda}_1^T\varphi_1(v_p)| + |\dot{\lambda}_2^T\varphi_2(v_p)| \\
&\leq C\,(\|\dot{u}\|_1^{\Omega_0} + \|r\|_0^{\Gamma_0})\|v_p\|_1^{\Omega_0} \\
&\leq C\,(\|\dot{u}\|_1^{\Omega_0} + \|r\|_0^{\Gamma_0})\|\dot{p}\|_0^{\Omega_0}
\end{aligned}$$

and hence

$$\|\dot{p}\|_0^{\Omega_0} \le C\,(\|\dot{u}\|_1^{\Omega_0} + \|r\|_0^{\Gamma_0}). \tag{5.4}$$

The positive smooth function γ has a positive smooth extension to Ω_0 which will be denoted by the same symbol. So we get for the first integral by the Green formula from (5.2) and the ellipticity of a, using the generalized Schwarz inequality and the estimate

$$\|\dot{u}\|_0^{\Omega_0} \le C\,\|\gamma\Delta_{\Gamma_0} r\, n\|_{-\frac{3}{2}}^{\Gamma_0} \le C\,\|r\|_{\frac{1}{2}}^{\Gamma_0}$$

from Lemma 2/(ii)

$$\begin{aligned}
-\int_{\Gamma_0} &\nabla_{\Gamma_0}\rho_1^{*} r \cdot \nabla_{\Gamma_0} r\, d\Gamma_0 \\
&= \int_{\Gamma_0} \frac{\dot{u}}{\gamma^2}\cdot \gamma n \Delta_{\Gamma_0} r\, d\Gamma_0 \\
&= a\left(\dot{u}, \frac{\dot{u}}{\gamma^2}\right) \\
&= \frac{1}{2}\sum_{i,j=1}^{N}\left(\int_{\Omega_0} \gamma^{-2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right)^2 dx\right. \\
&\qquad + \left.\int_{\Omega_0}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right)\left(\dot{u}_i \frac{\partial(\gamma^{-2})}{\partial x_j} + \dot{u}_j\frac{\partial(\gamma^{-2})}{\partial x_i}\right) dx\right) \\
&\ge c\,\|\dot{u}\|_1^{\Omega_0\,2} - C\,\|\dot{u}\|_1^{\Omega_0}\|\dot{u}\|_0^{\Omega_0} \\
&\ge c\,\|\dot{u}\|_1^{\Omega_0\,2} - C\,\|\dot{u}\|_0^{\Omega_0\,2} \\
&\ge c\,\|\dot{u}\|_1^{\Omega_0\,2} - C\,\|r\|_{\frac{1}{2}}^{\Gamma_0\,2}.
\end{aligned} \tag{5.5}$$

On the other hand,

$$\begin{aligned}
\|r\|_{\frac{3}{2}}^{\Gamma_0} &\le C\,\|\Delta_{\Gamma_0} r\|_{-\frac{1}{2}}^{\Gamma_0} + C\,\|r\|_{\frac{1}{2}}^{\Gamma_0} \\
&\le C \sup\left\{\int_{\Gamma_0} \Delta_{\Gamma_0} r\,\varphi\, d\Gamma_0 \,\middle|\, \varphi \in H^{\frac{1}{2}}(\Gamma_0),\ \|\varphi\|_{\frac{1}{2}}^{\Gamma_0} = 1\right\} + C\,\|r\|_{\frac{1}{2}}^{\Gamma_0}.
\end{aligned} \tag{5.6}$$

For any $\varphi \in H^{\frac{1}{2}}(\Gamma_0)$ with $\|\varphi\|_{\frac{1}{2}}^{\Gamma_0} = 1$ define now the constant

$$\overline{\varphi} = \frac{\int_{\Gamma_0} \gamma^{-1}\varphi\, d\Gamma_0}{\int_{\Gamma_0}\gamma^{-1}\, d\Gamma_0}$$

for which $\|\overline{\varphi}\|_{\frac{1}{2}}^{\Gamma_0} \le C\,|\overline{\varphi}| \le C\,\|\varphi\|_0^{\Gamma_0} \le C$ holds. Consider again a Neumann problem

$$\left.\begin{aligned}
\Delta\Phi &= 0 && \text{in } \Omega_0 \\
\frac{\partial \Phi}{\partial n} &= \gamma^{-1}(\varphi - \overline{\varphi}) && \text{at } \Gamma_0.
\end{aligned}\right\}$$

Because of $\int_{\Gamma_0} \gamma^{-1}(\varphi - \overline{\varphi})\, d\Gamma_0 = 0$ it has a solution $\Phi \in H^2(\Omega_0)$ with

$$\|\Phi\|_2^{\Omega_0} \le C\,\|\gamma^{-1}(\varphi - \overline{\varphi})\|_{\frac{1}{2}}^{\Gamma_0} \le C\,(\|\varphi\|_{\frac{1}{2}}^{\Gamma_0} + \|\overline{\varphi}\|_{\frac{1}{2}}^{\Gamma_0}) \le C.$$

If we define again $v = \nabla\Phi$ we find $v \in (H^1(\Omega_0))^N$, $\|v\|_1^{\Omega_0} \le C$, and with (5.2)

$$\begin{aligned}
\int_{\Gamma_0} \Delta_{\Gamma_0} r\varphi\, d\Gamma_0 &= \int_{\Gamma_0} \Delta_{\Gamma_0} r(\varphi - \overline{\varphi})\, d\Gamma_0 \\
&= \int_{\Gamma_0} \gamma\Delta_{\Gamma_0} rn \cdot v\, d\Gamma_0 \\
&= a(\dot{u}, v) - \int_{\Omega_0} \dot{p} \operatorname{div} v\, dx + \dot{\lambda}_1^T \varphi_1(v) + \dot{\lambda}_2^T \varphi_2(v) \\
&\le C\,\|(\dot{u}, \dot{p}, \dot{\lambda})\|_X \\
&\le C\,(\|\dot{u}\|_1^{\Omega_0} + \|r\|_{\frac{1}{2}}^{\Gamma_0})
\end{aligned} \tag{5.7}$$

where (5.3) and (5.4) have been used. Hence, together with (5.6),

$$\|r\|_{\frac{3}{2}}^{\Gamma_0\,2} \le C\,(\|\dot{u}\|_1^{\Omega_0\,2} + \|r\|_{\frac{1}{2}}^{\Gamma_0})$$

and with (5.5)

$$-\int_{\Gamma_0} \nabla_{\Gamma_0}\rho_1^* r \cdot \nabla_{\Gamma_0} r\, d\Gamma_0 \ge c\,\|r\|_{\frac{3}{2}}^{\Gamma_0\,2} - C\,\|r\|_{\frac{1}{2}}^{\Gamma_0\,2}.$$

Substep 1.2: Next, we have to deal with the integral

$$\begin{aligned}
&\int_{\Gamma_0} \nabla_{\Gamma_0}\Lambda_1 r \cdot \nabla_{\Gamma_0} r\, d\Gamma_0 \\
&\quad = -\int_{\Gamma_0} \Lambda_1 r\, \Delta_{\Gamma_0} r\, d\Gamma_0 \\
&\quad = -\int_{\Gamma_0} r\, \Lambda_1^* \Delta_{\Gamma_0} r\, d\Gamma_0 \\
&\quad = \int_{\Gamma_0} r\, \Lambda_1 \Delta_{\Gamma_0} r\, d\Gamma_0 - \int_{\Gamma_0} r(\Lambda_1 + \Lambda_1^*)\Delta_{\Gamma_0} r\, d\Gamma_0 \\
&\quad = \int_{\Gamma_0} r\, \Delta_{\Gamma_0}\Lambda_1 r\, d\Gamma_0 + \int_{\Gamma_0} r\,(\Lambda_1\Delta_{\Gamma_0} - \Delta_{\Gamma_0}\Lambda_1) r\, d\Gamma_0 - \int_{\Gamma_0} r(\Lambda_1 + \Lambda_1^*)\Delta_{\Gamma_0} r\, d\Gamma_0 \\
&\quad = \int_{\Gamma_0} \Delta_{\Gamma_0} r\, \Lambda_1 r\, d\Gamma_0 + \int_{\Gamma_0} r\, \Lambda_2 r\, d\Gamma_0
\end{aligned}$$

where Λ_1^* denotes the adjoint of Λ_1 in $H^0(\Gamma_0)$ and

$$\Lambda_2 = \Lambda_1\Delta_{\Gamma_0} - \Delta_{\Gamma_0}\Lambda_1 - (\Lambda_1 + \Lambda_1^*)\Delta_{\Gamma_0}.$$

Λ_2 is a second order differential operator due to the well-known facts that the commutator $\Lambda_1 \Delta_{\Gamma_0} - \Delta_{\Gamma_0} \Lambda_1$ is a differential operator of second order only and $\Lambda_1 + \Lambda_1^*$ is given purely by multiplication with a smooth function. Hence

$$\left| \int_{\Gamma_0} \nabla_{\Gamma_0} \Lambda_1 r \cdot \nabla_{\Gamma_0} r \, d\Gamma_0 \right| = \frac{1}{2} \left| \int_{\Gamma_0} r \, \Lambda_2 r \, d\Gamma_0 \right| \leq C \, \|r\|^{\Gamma_0}_{\frac{1}{2}} \, \|\Lambda_2 r\|^{\Gamma_0}_{-\frac{1}{2}} \leq C \, \|r\|^{\Gamma_0}_{\frac{1}{2}} \|r\|^{\Gamma_0}_{\frac{3}{2}}.$$

The assertion for $s = 1$ follows now by summing up and applying the generalized Schwarz inequality again.

Step 2: $1 < s \leq s_1$. We have, using Step 1 of the proof, (4.7) and the generalized Schwarz inequality

$$\begin{aligned}
-(\rho_1 r, r)_s &= - \sum_{|\alpha| \leq s-1} (D^\alpha \rho_1 r, D^\alpha r)_1 \\
&= - \sum_{|\alpha| \leq s-1} \Big(((D^\alpha \rho_1 - \rho_1 D^\alpha) r, D^\alpha r)_1 + (\rho_1 D^\alpha r, D^\alpha r)_1 \Big) \\
&\geq c \sum_{|\alpha| \leq s-1} \|D^\alpha r\|^{\Gamma_0 \, 2}_{\frac{3}{2}} \\
&\quad - C \sum_{|\alpha| \leq s-1} \Big(\|(D^\alpha \rho_1 - \rho_1 D^\alpha) r\|^{\Gamma_0}_{\frac{1}{2}} \|D^\alpha r\|^{\Gamma_0}_{\frac{3}{2}} + \|D^\alpha r\|^{\Gamma_0 \, 2}_{\frac{1}{2}} \Big) \\
&\geq c_s \, \|r\|^{\Gamma_0 \; 2}_{s+\frac{1}{2}} - C_s \, \|r\|^{\Gamma_0 \; 2}_{s-\frac{1}{2}}.
\end{aligned}$$

Step 3: $s > s_1$. The proof can be given as in Step 2, using (4.8) instead of (4.7) ∎

6. Existence and uniqueness for the nonlinear problem

Due to its analyticity, the behavior of the operator ρ is locally governed by its linearization ρ_1. This and the chain rule enable us to show the following estimate.

Lemma 9 (Local a priori estimate). *There is an* $\varepsilon_1 > 0$ *such that for all integer* $s > s_1$ *an inequality*

$$(\rho(r), r)_s \leq -c_s \, \|r\|^{\Gamma_0 \; 2}_{s+\frac{1}{2}} + C_s \big(\|r\|^{\Gamma_0 \; 2}_{s-\frac{1}{2}} + 1 \big)$$

holds for all $r \in B_0(\varepsilon_1, H^{s_1+1}(\Gamma_0)) \cap H^{s+1}(\Gamma_0)$.

Proof. We demand $\varepsilon_1 \leq \varepsilon_0$ and conclude from Lemma 7/(i) that $\rho(r) \in H^s(\Gamma_0)$. We decompose

$$\rho(r) = \rho(0) + \rho_1 r + \sum_{m=2}^{\infty} \rho_m(r, \ldots, r)$$

and use (4.10). For the sake of brevity we will restrict our attention to the case $|\alpha| > 0$, the estimates for $\alpha = 0$ are obvious.

1. Because $\rho(0)$ is smooth we have

$$(D^\alpha \rho(0), D^\alpha r)_{s_1} \le C_s \|r\|^{\Gamma_0}_{s-\frac12} \le C_s\big(\|r\|^{\Gamma_0\,2}_{s-\frac12} + 1\big).$$

2. From the proof of Lemma 8 we recall

$$(D^\alpha \rho_1 r, D^\alpha r)_{s_1} \le -c^* \|D^\alpha r\|^{\Gamma_0\,2}_{s_1-\frac12} + C_s \|r\|^{\Gamma_0\,2}_{s-\frac12}.$$

3. We use (4.6) and estimate

$$\begin{aligned}
&\big(D^\alpha \rho_m(r,\dots,r), D^\alpha r\big)_{s_1} \\
&\quad \le C \,\|D^\alpha \rho_m(r,\dots,r)\|^{\Gamma_0}_{s_1-\frac12} \|D^\alpha r\|^{\Gamma_0}_{s_1+\frac12} \\
&\quad \le C \sum_{l=0}^{m} \sum_{k=\max\{1,m-l\}}^{|\alpha|} \sum_{\beta_1+\dots+\beta_k=\alpha} C_{\beta_1,\dots,\beta_k} \frac{(k+l)!}{l!\,(m-l)!\,(k-m+l)!} \\
&\qquad \times \sum_\sigma T_{k+l,m,\underline{\beta},\sigma} \|D^\alpha r\|^{\Gamma_0}_{s_1+\frac12}
\end{aligned}$$

with the shorthand notations

$$\begin{aligned}
\underline{\beta} &= (\beta_1,\dots,\beta_k) \\
T_{k+l,m,\underline{\beta},\sigma} &= \Big\|\rho_{k+l}\Big(r,\dots,r, D^{\beta_{\sigma(1)}} r,\dots, D^{\beta_{\sigma(m-l)}} r, D^{\beta_{\sigma(m-l+1)}} \mathcal{R}_0, D^{\beta_{\sigma(k)}} \mathcal{R}_0\Big)\Big\|^{\Gamma_0}_{s_1-\frac12}
\end{aligned}$$

which will be continuously used in the sequel. We will estimate the terms $T_{k+l,m,\underline{\beta},\sigma}$ separately and then perform the summations.

Note at first that the sum over σ has $k!$ elements and that due to $k \le |\alpha|$ and $l \le m$ we have

$$\frac{(k+l)!\,k!}{l!\,(m-l)!\,(k-m+l)!} = \frac{(k+l)!}{l!} \binom{k}{m-l} \le (m+|\alpha|)^{|\alpha|} 2^{|\alpha|-1} \le C_\alpha m^{|\alpha|}. \tag{6.1}$$

Take now $m \ge 2$, l, k, $\underline{\beta}$, and σ fixed. We will distinguish several cases and continuously use the estimates for the ρ_k together with (3.9).

Case 3.1: $k + l = m$.

Subcase 3.1.1: $k = 1$. To this choice of the indices there corresponds only the term

$$m\, T_{m,m,(\alpha),(1)} \le C m M^m \|r\|^{\Gamma_0\, m-1}_{s_1+\frac12} \|D^\alpha r\|^{\Gamma_0}_{s_1+\frac12}.$$

If we perform the summation over $m \ge 2$ and choose ε_1 small enough we get

$$\sum_{m=2}^{\infty} m\, T_{m,m,(\alpha),(1)} \|D^\alpha r\|^{\Gamma_0}_{s_1+\frac12} \le \frac{c^*}{4} \|D^\alpha r\|^{\Gamma_0\,2}_{s_1+\frac12}.$$

Subcase 3.1.2: $k > 1$. In this case we have $|\beta_j| < |\alpha|$ for all j and using the interpolation inequalities

$$\|D^\beta r\|^{\Gamma_0}_{s_1+\frac{1}{2}} \le C_s \|r\|^{\Gamma_0}_{s_1+|\beta|+\frac{1}{2}} \le C_s \|r\|^{\Gamma_0}_{s_1+1}{}^{1-\frac{|\beta|}{|\alpha|}} \|r\|^{\Gamma_0}_{s+\frac{1}{2}-\zeta}{}^{\frac{|\beta|}{|\alpha|}}$$
$$\|r\|^{\Gamma_0}_{s+\frac{1}{2}-\zeta} \le \delta \|r\|^{\Gamma_0}_{s+\frac{1}{2}} + C_{s,\delta} \|r\|^{\Gamma_0}_{s_1+1}$$

holding for sufficiently small ζ depending only on s, all positive δ and all β with $|\beta| < |\alpha|$ we find

$$T_{m,m,\underline{\beta},\sigma} \le C_s M^m m^{s-s_1} \|r\|^{\Gamma_0}_{s_1+1}{}^{m-1} \left(\delta \|r\|^{\Gamma_0}_{s+\frac{1}{2}} + C_{s,\delta} \|r\|^{\Gamma_0}_{s_1+1}\right).$$

The convergence radius of the series $\sum_{m=2}^{\infty} M^m m^{|\alpha|} \varepsilon^{m-1}$ is $\frac{1}{M}$ and thus independent of s. Hence, performing the summations over k, $\underline{\beta}$, σ, and m, using (6.1), applying the generalized Schwarz inequality and choosing δ sufficiently small we find

$$\sum_{k>1,\underline{\beta},\sigma,m} C_{\underline{\beta}} T_{m,m,\underline{\beta},\sigma} \|D^\alpha r\|^{\Gamma_0}_{s_1+\frac{1}{2}} \le \frac{c_s C^*}{4\mathcal{N}_s} \|r\|^{\Gamma_0}_{s+\frac{1}{2}}{}^2 + C_s$$

where $\mathcal{N}_s$ is the number of elements of the set $\{\alpha : |\alpha| \le s\}$ and c_s is a small positive constant such that

$$\sum_{|\alpha| \le s-s_1} \|D^\alpha r\|^{\Gamma_0}_{s_1+\frac{1}{2}}{}^2 \ge c_s \|r\|^{\Gamma_0}_{s+\frac{1}{2}}{}^2.$$

Case 3.2: $k + l > m$. In this case

$$b = \frac{1}{|\alpha|} \sum_{j=1}^{m-l} |\beta_{\sigma(j)}| < 1$$

and by interpolation and Youngs inequality

$$T_{k+l,m,\underline{\beta},\sigma} \le C_s M^m \|r\|^{\Gamma_0}_{s_1+1}{}^{m-1} \|r\|^{\Gamma_0}_{s_1+1}{}^{1-b} \|r\|^{\Gamma_0}_{s+\frac{1}{2}}{}^{b}$$
$$\le C_s M^m \|r\|^{\Gamma_0}_{s_1+1}{}^{m-1} \left(\delta \|r\|^{\Gamma_0}_{s+\frac{1}{2}} + C_{s,\delta} \|r\|^{\Gamma_0}_{s_1+1}\right)$$

for any $\delta > 0$. In the same way as in the previous case we find from this

$$\sum_{m=2}^{\infty} \sum_{k,l,\underline{\beta},\sigma} T_{k+l,m,\underline{\beta},\sigma} \|D^\alpha r\|^{\Gamma_0}_{s_1+\frac{1}{2}} \le \frac{c_s C^*}{4\mathcal{N}_s} \|r\|^{\Gamma_0}_{s+\frac{1}{2}}{}^2 + C_s$$

and the lemma is proved by carrying out the remaining summations ∎

Now we can prove a short-time existence result for the solution of our evolution problem. As in [15] we will use the notations IT for the closed interval $[0,T]$ $(T > 0)$, $C_w(IT, X)$ and $C^k_w(IT, X)$ for the spaces of weakly continuous and k times weakly differentiable functions valued in some Banach space X, respectively, i.e. the functions $u : IT \longrightarrow X$ such that, for all $\varphi \in X'$, $\langle \varphi, u(t) \rangle \in C(IT)$ and $\langle \varphi, u(t) \rangle \in C^k(IT)$, respectively.

Proposition 1 (Short-time existence). *Let Ω_0 be smooth and strictly star-shaped. There are positive constants ε_2 and T such that for all integer $s \geq s_1$ and all*

$$r_0 \in B_0(\varepsilon_2, H^{s_1+1}(\Gamma_0)) \cap H^{s+1}(\Gamma_0)$$

the initial value problem

$$\left.\begin{aligned} \frac{\partial r}{\partial t} &= \rho(r) \\ r(0) &= r_0 \end{aligned}\right\} \tag{6.2}$$

has a solution r in $C_w(IT, H^{s+1}(\Gamma_0)) \cap C_w^1(IT, H^s(\Gamma_0))$.

Proof. The proof will be given in essentially the same way as the proof of Theorem A in [15] where $H^{s+2}(\Gamma_0)$, $H^{s+1}(\Gamma_0)$, and $H^s(\Gamma_0)$ will play the roles of V, H, and X, respectively. The necessary modifications are due to the fact that both the estimate

$$(\rho(r), r)_{s+1} \leq C_s \left(1 + \|r\|_{s+1}^{\Gamma_0\,2}\right) \tag{6.3}$$

and the weak continuity of ρ are ensured by the Lemmas 9 and 7/(ii) only if $\|r\|_{s_1+1}^{\Gamma_0} \leq 2\varepsilon_2$ with sufficiently small ε_2. Thus we have to use Galerkin approximations which remain small in $H^{s_1+1}(\Gamma_0)$ and uniformly bounded in $H^{s+1}(\Gamma_0)$.

If $s > s_1$, then there is a self-adjoint operator S on $H^{s_1+1}(\Gamma_0)$ such that

$$(u, v)_{s+1} = (Su, v)_{s_1+1} \qquad \forall u \in D(S),\ v \in H^{s+1}(\Gamma_0).$$

By Rellich's theorem, S has a purely discrete spectrum, i.e. S has a complete orthonormal system of eigenfunctions $\{e_j\}$ in $H^{s_1+1}(\Gamma_0)$. Elliptic regularity theory yields that all e_j are smooth. If $s = s_1$, then we choose an arbitrary orthonormal basis $\{e_j\}$ in $H^{s_1+1}(\Gamma_0) = H^{s+1}(\Gamma_0)$ consisting of functions in $H^{s+2}(\Gamma_0)$. We define now

$$M_k = \operatorname{span}\{e_1, \ldots, e_k\}$$

$$P_k u = \sum_{j=1}^{k} (u, e_j)_{s_1+1}\, e_j$$

and it is easily seen that P_k is the orthogonal projection on M_k both in $H^{s_1+1}(\Gamma_0)$ and $H^{s+1}(\Gamma_0)$.

Consider the unique solution m of the initial value problem

$$\left.\begin{aligned} \dot{m} &= 2C_{s_1}(1 + m) \\ m(0) &= \varepsilon_2^2 \end{aligned}\right\}$$

where C_{s_1} is the constant C_s from (6.3) with $s = s_1$ and choose T to be the (uniquely defined) positive number satisfying $m(T) = 4\varepsilon_2^2$. Note that m is strictly increasing on IT. We will show now that the Galerkin approximations r_j defined by

$$\left.\begin{aligned} \frac{\partial r_j}{\partial t} &= P_j \rho(r_j) \\ r_j(0) &= P_j r_0 \end{aligned}\right\} \tag{6.4}$$

exist at least at IT and satisfy

$$\|r_j(t)\|_{s_1+1}^{\Gamma_0} < 2\varepsilon_2 \qquad \forall j \in \mathbb{N},\ t \in IT. \tag{6.5}$$

Suppose the opposite: this implies by the theory of ordinary differential equations that for a certain j there is a $T^* < T$ such that $\|r_j(T^*)\|_{s_1+1}^{\Gamma_0} = 2\varepsilon_2$ and $\|r_j(t)\|_{s_1+1}^{\Gamma_0} < 2\varepsilon_2$ for all $t \in [0, T^*)$. Note that $T^* > 0$ because of

$$\|r_j(0)\|_{s_1+1}^{\Gamma_0} = \|P_j r_0\|_{s_1+1}^{\Gamma_0} \leq \varepsilon_2. \tag{6.6}$$

For all $t \in IT^*$ we can estimate, by (6.3) and the same arguments as in [15],

$$\frac{d}{dt}\left(\|r_j(t)\|_{s_1+1}^{\Gamma_0\,2}\right) \leq 2C_{s_1}\left(1 + \|r_j(t)\|_{s_1+1}^{\Gamma_0\,2}\right)$$

and from this and (6.6) an elementary comparison result for the solutions of initial value problems of ordinary differential equations in $\mathbb{R}$ yields

$$m(T) = 4\varepsilon_2^2 = \|r_j(T^*)\|_{s_1+1}^{\Gamma_0\,2} \leq m(T^*)$$

in contradiction to the strict increasing of m. Hence (6.5) holds, and therefore, by repeating the above arguments for the $H^{s+1}(\Gamma_0)$-norm,

$$\frac{d}{dt}\left(\|r_j(t)\|_{s+1}^{\Gamma_0\,2}\right) \leq 2C_s\left(1 + \|r_j(t)\|_{s+1}^{\Gamma_0\,2}\right)$$
$$\|r_j(0)\|_{s+1}^{\Gamma_0\,2} \leq \|r_0\|_{s+1}^{\Gamma_0\,2}$$

which implies that $\|r_j(t)\|_{s+1}^{\Gamma_0}$ exists and is bounded independently of j on IT. The existence proof can be given now in strict analogy to the proof in [15] mentioned above ∎

Taking into account that $C_w^1(IT, H^s(\Gamma_0)) \subset C^1(IT, H^{s-1}(\Gamma_0))$ and the embedding theorems we immediately find:

Corollary 1. *Under the assumptions of Proposition* 1, *suppose additionally* $r_0 \in C^\infty(\Gamma_0)$. *Then* (6.2) *has a solution in* $C^1(IT, C^\infty(\Gamma_0))$.

Lemma 10 (Weakened local monotonicity). *For all* $s \geq s_1$ *there are positive constants* c_s, C_s, *and* ε_s *such that*

$$\begin{aligned}&\big(\rho(r) - \rho(v), r - v\big)_{s+1}\\ &\quad \leq -c_s\|r - v\|_{s+\frac{3}{2}}^{\Gamma_0\,2} + C_s\|r - v\|_{s+\frac{1}{2}}^{\Gamma_0\,2}\\ &\qquad + C_s \max\left\{\|r\|_{s+\frac{3}{2}}^{\Gamma_0}, \|v\|_{s+\frac{3}{2}}^{\Gamma_0}\right\}\|r - v\|_{s+\frac{3}{2}}^{\Gamma_0}\|r - v\|_{s+1}^{\Gamma_0}\end{aligned} \tag{6.7}$$

for all $r, v \in B_0(\varepsilon_s, H^{s+1}(\Gamma_0)) \cap H^{s+2}(\Gamma_0)$.

Proof. We proceed similar to the proof of Lemma 9 and use the notation (4.13) again. We find

$$
\begin{aligned}
&(\rho(r)-\rho(v), r-v)_{s+1} \\
&\quad \le -c^* \|r-v\|^{\Gamma_0\,2}_{s+\frac32} + C\|r-v\|^{\Gamma_0\,2}_{s+\frac12} + C\sum(1+m^s) \\
&\quad\quad \times \Bigg(\sum_{j=1}^{l} \Big\| \rho_{k+l}\Big(r,\dots,r,r-v,v,\overset{\bullet}{\dots},v,D^{\beta_{\sigma(1)}}v,\dots,D^{\beta_{\sigma(m-l)}}v, \\
&\quad\quad\quad D^{\beta_{\sigma(m-l+1)}}\mathcal{R}_0,\dots,D^{\beta_{\sigma(k)}}\mathcal{R}_0\Big)\Big\|^{\Gamma_0}_{s_1-\frac12} \\
&\quad\quad + \sum_{j=1}^{m} \Big\| \rho_{k+l}\Big(r,\dots,r,D^{\beta_{\sigma(1)}}r,\dots,D^{\beta_{\sigma(j-1)}}r,D^{\beta_{\sigma(j)}}(r-v),D^{\beta_{\sigma(j+1)}}v,\dots, \\
&\quad\quad\quad D^{\beta_{\sigma(m-l)}}v,D^{\beta_{\sigma(m-l+1)}}\mathcal{R}_0,\dots,D^{\beta_{\sigma(k)}}\mathcal{R}_0\Big)\Big\|^{\Gamma_0}_{s_1-\frac12}\Bigg)\|r-v\|^{\Gamma_0}_{s+\frac32}
\end{aligned}
$$

where the sum has to be taken over α, k, $\beta_1,\dots,\beta_k$, l, m, and σ as in Lemma 7. The summands in brackets can be estimated by

$$CM^m(1+m^s)\mu_{s+1}^{m-2}\mu_{s+\frac32}\|r-v\|^{\Gamma_0}_{s+1}\|r-v\|^{\Gamma_0}_{s+\frac32}$$

or

$$CM^m(1+m^s)\mu_{s+1}^{m-1}\|r-v\|^{\Gamma_0\,2}_{s+\frac32}$$

depending on whether the derivatives of highest order occur in the argument containing $r-v$. Choosing ε_s small enough and carrying out the summations completes the proof ∎

Proposition 2 (Uniqueness). *Let Ω_0 be as in Proposition 1. There are positive constants ε_3 and T such that for all $r_0 \in B_0(\varepsilon_3, H^{s_1+1}(\Gamma_0))$ the problem (6.2) has at most one solution in*

$$C^1\big(IT, H^{s_1+1}(\Gamma_0)\big) \cap L^\infty\big(IT, H^{s_1+\frac32}(\Gamma_0)\big).$$

Proof. Let ε_3 be small enough that, due to Lemmas 9 and 10, (6.3) and (6.7) hold for $s=s_1$ if $\|r\|^{\Omega_0}_{s_1+1} \le 2\varepsilon_3$. Suppose $r,v \in C^1\big(IT,H^{s_1+1}(\Gamma_0)\big)\cap L^\infty\big(IT,H^{s_1+\frac32}(\Gamma_0)\big)$ are solutions of (6.2). From (6.3) one concludes $\|r(t)\|^{\Omega_0}_{s_1+1}$, $\|v(t)\|^{\Omega_0}_{s_1+1} \le 2\varepsilon_3$ for all $t \in IT$ for a certain $T>0$ in the same manner as the corresponding estimates on the r_j in the proof of Proposition 1. Moreover, using the boundedness of $\|r(t)\|^{\Gamma_0}_{s+\frac32}$, $\|v(t)\|^{\Gamma_0}_{s+\frac32}$ and the generalized Schwarz inequality we find from (6.7)

$$
\begin{aligned}
\frac{d}{dt}\Big(\|r(t)-v(t)\|^{\Gamma_0\,2}_{s_1+1}\Big) &= 2\big(\rho(r(t))-\rho(v(t)),r(t)-v(t)\big)_{s_1+1} \\
&\le -c\,\|r(t)-v(t)\|^{\Gamma_0\,2}_{s_1+\frac32} + C\,\|r(t)-v(t)\|^{\Gamma_0\,2}_{s_1+\frac12} \\
&\quad + C_{r,v}\,\|r(t)-v(t)\|^{\Gamma_0}_{s_1+1}\|r(t)-v(t)\|^{\Gamma_0}_{s_1+\frac32} \\
&\le C_{r,v}\,\|r(t)-v(t)\|^{\Gamma_0\,2}_{s_1+1}
\end{aligned}
$$

for almost all $t \in IT$ and from the Gronwall inequality it follows $r(t)=v(t)$ for all $t \in IT$ ∎

In a similar way, under slightly stronger smoothness assumptions on the initial condition, one can prove continuous dependence of $r(t)$ for fixed t on $r(0)$.

7. Global existence and stability of solutions near the ball

From physical reasons and corresponding results in the two-dimensional case (partly in the case of the corresponding problem for an outer domain [3, 4, 18]) one expects that the only stationary solutions of our free boundary problem are given by the balls. This will be proved in the following. We remind that u_0 is the first component of the solution of (2.6) with $\Omega = \Omega_0$.

Lemma 11 (Stationary solutions). *If $u_0 \cdot n \equiv 0$ on Γ_0, then Γ_0 has constant mean curvature, i.e. Ω_0 is a circle if $N = 2$ and a ball if $N = 3$.*

Proof. From

$$\int_{\Gamma_0} \kappa_0 n \cdot u_0 \, d\Gamma_0 = \int_{\Gamma_0} \mathcal{T}(u_0, p_0) n \cdot u_0 \, d\Gamma_0 = 0$$

it follows by setting $u = v = u_0$, $p = p_0$ in (2.3) that, using the notation of Section 2, $a(u_0, u_0) = 0$ and thus $u_0 = 0$. The Stokes equations and the boundary condition on the stress tensor reduce to

$$\left.\begin{aligned} \nabla p &= 0 \quad \text{in} \quad \Omega_0 \\ -pn &= \kappa_0 n \quad \text{on} \quad \Gamma_0 \end{aligned}\right\},$$

hence both p and κ_0 are constant. This completes the proof because the only (bounded) simply connected domains in $\mathbb{R}^3$ whose boundaries have constant mean curvature are the balls (see, e.g., [5]) ∎

In order to investigate the moving boundary problem near the ball we set

$$\Omega_0 = B_0(1, \mathbb{R}^N) \qquad \text{and} \qquad \zeta(\xi) = n(\xi). \tag{7.1}$$

This clearly satisfies all assumptions made above and leads to $\gamma \equiv 1$, $\kappa_0 = -(N-1)$ and $\Gamma_0 = S^{N-1}$. The diffeomorphism Φ_0 can and will be chosen to be the identity and thus we have $R_0 = \mathcal{R}_0 \equiv 1$ on Γ_0. In this case we get from (4.6)

$$D^\alpha \rho_1(r) = \rho_1(D^\alpha r). \tag{7.2}$$

For the sake of technical simplicity, the following considerations are restricted to the case $N = 3$. They can be generalized, however, to the general case without essential changes.

Using spherical coordinates it is not difficult to obtain the expressions

$$V(r) = \frac{1}{3} \int_{\Gamma_0} (1+r)^3 \, d\Gamma_0$$

$$M(r) = \frac{1}{4} \int_{\Gamma_0} (1+r)^4 n \, d\Gamma_0$$

for the volume and the centre of gravity of the domain Ω_r, respectively. We define the function $F: H^{s_0}(\Gamma_0) \longrightarrow \mathbb{R} \times \mathbb{R}^3$ by

$$F(r) = \begin{bmatrix} V(r) - \frac{4}{3}\pi \\ M(r) \end{bmatrix}.$$

Note that F is an analytic function on $H^{s_0}(\Gamma_0)$, $F(0) = 0$, and

$$F'(0)[h] = \begin{bmatrix} \int_{\Gamma_0} h \, d\Gamma_0 \\ \int_{\Gamma_0} hn \, d\Gamma_0 \end{bmatrix}.$$

For all $s \geq s_0$ we define

$$\mathcal{M}_s = \{ r \in H^s(\Gamma_0) | \, F(r) = 0 \}$$

and demand $r_0 \in \mathcal{M}_{s+1}$ in (6.2). (It is obvious that this, as well as the choice of radius 1, is no essential restriction of generality but just a matter of appropriate shifting and scaling.) Because of the incompressibility condition and the demand $\int_\Omega u \, dx = 0$ in the fixed time problem we find simply by integration

$$r(t) \in \mathcal{M}_{s+1} \qquad \forall t \in IT \tag{7.3}$$

for any solution of (6.2).

For the linearization we find after a certain amount of calculation $\rho_1(r) = (\mathrm{Tr}_{\Gamma_0} \dot{u}^\star) \cdot n$ where $(\dot{u}^\star, \dot{p}^\star, \dot{\lambda}^\star)$ is the solution of the variational problem

$$\left.\begin{aligned} a(\dot{u}^\star, v) + b(v, \dot{p}^\star) + \dot{\lambda}_1^{\star\,T} \varphi_1(v) + \dot{\lambda}_2^{\star\,T} \varphi_2(v) &= \int_{\Gamma_0} (\Delta_{\Gamma_0} r + 2r) \, n \cdot v \, d\Gamma_0 \\ &\quad \text{for all } \; v \in (H^1(\Omega))^N \\ b(\dot{u}^\star, q) &= 0 \quad \forall q \in L^2(\Omega) \\ \varphi_1(\dot{u}^\star) &= 0 \\ \varphi_2(\dot{u}^\star) &= 0. \end{aligned}\right\}$$

Note that

$$\dot{\lambda}_1^\star = \dot{\lambda}_2^\star = 0 \tag{7.4}$$

and thus we get, instead of (5.4), the sharper estimate

$$\|\dot{p}^\star\|_0^{\Omega_0} \leq C \, \|\dot{u}^\star\|_1^{\Omega_0}. \tag{7.5}$$

In the following we use series expansions in eigenfunctions of the Laplace-Beltrami operator on S^2 to define Hilbert norms that are adjusted to our needs. Let $\{Y_{kl} | \, l = 0, 1, 2, \ldots; \, k = -l, \ldots, l\}$ be an orthonormal basis of $L^2(S^2)$ satisfying $\Delta_{\Gamma_0} Y_{kl} = -l(l+1) Y_{kl}$. Such a basis is given by choosing an arbitrary L^2-orthonormal basis of the l-th order spherical harmonics for all l. We will write $r_{kl} = (r, Y_{kl})_0$ and introduce (on all $H^s(\Gamma_0)$) the projection $\mathcal{P}$ by

$$\mathcal{P}r = \sum_{l=2}^{\infty} \sum_{k=-l}^{l} r_{kl} Y_{kl}$$

and on the spaces $H^s(\Gamma_0)$ for positive integer s, $s = -\frac{1}{2}$, and $s = \frac{3}{2}$ the scalar products

$$(r, v)_s = r_{00}\, v_{00} + \sum_{k=-1}^{1} r_{k1}\, v_{k1} + \sum_{l=2}^{\infty} \sum_{k=-l}^{l} \left(l(l+1) - 2\right)^s r_{kl}\, v_{kl} \quad (s < 2)$$

$$(r, v)_s = r_{00}\, v_{00} + \sum_{k=-1}^{1} r_{k1}\, v_{k1} + \sum_{|\alpha| \le s-1} (D^\alpha \mathcal{P}r, D^\alpha \mathcal{P}v)_1 \quad (s \ge 2).$$

It is easily seen that $\mathcal{P}$ commutes with all D^α and that $\mathcal{P}$ is orthogonal with respect to all these scalar products. Furthermore, we introduce a semi-scalar product and a seminorm on $H^s(\Gamma_0)$ by

$$[r, v]_s = (\mathcal{P}r, \mathcal{P}v)_s$$
$$|r|_s = [r, r]_s^{\frac{1}{2}}.$$

Lemma 12. *Assume $s \ge s_1$. There are constants $\varepsilon > 0$ and $C > 0$ (depending on s) such that*

$$\|r\|_s^{\Gamma_0} \le C\left(|r|_s + \|F(r)\|_{\mathbb{R}\times\mathbb{R}^3}\right) \tag{7.6}$$

for all $r \in B_0(\varepsilon, H^s(\Gamma_0))$ and, moreover,

$$\|r\|_s^{\Gamma_0} \le (1 + C|r|_s)|r|_s \tag{7.7}$$

for all $r \in \mathcal{M}_s \cap B_0(\varepsilon, H^s(\Gamma_0))$.

Proof. The first inequality is a consequence of the local diffeomorphism theorem applied to the mapping $\Phi : H^s(\Gamma_0) \longrightarrow \mathcal{P}[H^s(\Gamma_0)] \times (\mathbb{R} \times \mathbb{R}^3)$ defined by $\Phi(r) = \left[\begin{smallmatrix} \mathcal{P}r \\ F(r) \end{smallmatrix}\right]$ in the neighbourhood of 0.

Due to the orthogonality of $\mathcal{P}$ we have $\|r\|_s^{\Gamma_0\,2} = |r|_s^2 + \|\bar{r}\|_s^{\Gamma_0\,2}$ with $\bar{r} = (I - \mathcal{P})r$. We consider now $\bar{r} \in \operatorname{span}\{1, x_1, x_2, x_3\}$ as solution of the equation

$$\widetilde{F}(\mathcal{P}r, \bar{r}) = F(\mathcal{P}(r) + \bar{r}) = 0$$

which is satisfied for all $r \in \mathcal{M}_s$. Applying the implicit function theorem to it and using that the Fréchet derivative of $\widetilde{F}$ with respect to the first argument at $(0, 0)$ is the zero operator we find $\|\bar{r}\|_s^{\Gamma_0} \le C|r|_s^2$ if $|r|_s$ is sufficiently small. The estimate (7.7) follows easily from this ∎

The following estimates are parallel to those given in the Lemmas 8 and 9. The key idea here is that, due to the new context and the use of the seminorms $|\cdot|_s$ instead of complete norms, one is able to avoid the occurrence of "lower order terms".

Lemma 13. *Under the additional assumptions* (7.1), *there is a constant* $c > 0$ *such that*

$$-[\rho_1 r, r]_1 \geq c\,|r|^2_{\frac{3}{2}}$$

holds for all $r \in H^2(\Gamma_0)$.

Proof. Taking into account that

$$(\rho_1 r)_{00} = (4\pi)^{-\frac{1}{2}} \int_{\Gamma_0} \dot{u}^\star \cdot n \, d\Gamma_0 = (4\pi)^{-\frac{1}{2}} \int_{\Omega_0} \operatorname{div} \dot{u}^\star \, dx = 0$$

we find

$$\begin{aligned}
-[\rho_1 r, r]_1 &= -\sum_{l=2}^{\infty} \sum_{k=-l}^{l} \big(l(l+1) - 2\big)(\rho_1 r)_{kl}\, r_{kl} \\
&= -\sum_{l=0}^{\infty} \sum_{k=-l}^{l} \big(l(l+1) - 2\big)(\rho_1 r)_{kl}\, r_{kl} \\
&= -\sum_{l=0}^{\infty} \sum_{k=-l}^{l} (\rho_1 r)_{kl}(-\Delta_{\Gamma_0} r - 2r)_{kl} \\
&= \int_{\Gamma_0} \rho_1 r (\Delta_{\Gamma_0} r + 2r)\, d\Gamma_0 \\
&= \int_{\Gamma_0} \dot{u}^\star (\Delta_{\Gamma_0} r + 2r)\, d\Gamma_0 \\
&= a(\dot{u}^\star, \dot{u}^\star) \\
&\geq c\, \|\dot{u}^\star\|_1^{\Omega_0\,2}.
\end{aligned}$$

On the other hand,

$$|r|^2_{\frac{3}{2}} = |\Delta_{\Gamma_0} r + 2r|^2_{-\frac{1}{2}} \leq \|\Delta_{\Gamma_0} r + 2r\|^{\Gamma_0\,2}_{-\frac{1}{2}} \leq C\, \|(\dot{u}^\star, \dot{p}^\star, \dot{\lambda}^\star)\|^2_X$$

where the last inequality can be shown analogously to the general case (cf. (5.7), $\Delta_{\Gamma_0} r$ has to be replaced by $\Delta_{\Gamma_0} r + 2r$). Taking into account now (7.4) and (7.5) completes the proof ∎

Lemma 14. *Under the additional assumptions* (7.1), *for all integer* $s \geq s_1$ *the inequality*

$$[\rho(r), r]_{s+1} \leq -c\,|r|^2_{s+1} + C\,\|F(r)\|^2_{\mathbb{R}\times\mathbb{R}^3} \tag{7.8}$$

holds for all $r \in B_0(\varepsilon, H^{s+1}(\Gamma_0)) \cap H^{s+2}(\Gamma_0)$ *where the positive constants* ε, c, *and* C *depend only on* s.

Proof. In analogy to the proof of Lemma 9, we have

$$[\rho(r), r]_{s+1} = \sum_{|\alpha| \leq s} [D^\alpha \rho(r), D^\alpha r]_1$$

and estimate the summands on the right as in Lemma 10 where we take additionally into account (7.2) and $\rho_0 = \rho(0) = 0$. Thus we get

$$[D^\alpha \rho(r), D^\alpha r]_1 = [D^\alpha \rho_1 r, D^\alpha r]_1 + \sum_{k=2}^{\infty}[D^\alpha \rho_k(r,\dots,r), D^\alpha r]_1$$
$$\leq -c\,|D^\alpha r|^2_{\frac{3}{2}} + \sum_{k=2}^{\infty} C_k \,{\|r\|^{\Gamma_0}_{s+1}}^{k-1} {\|r\|^{\Gamma_0}_{s+\frac{3}{2}}}^2.$$

Summing up and using Lemma 12 we find

$$[\rho(r), r]_{s+1} \leq -c \sum_{|\alpha|\leq s} |D^\alpha r|^2_{\frac{3}{2}} + C\sum_{k=2}^{\infty} C_k \,{\|r\|^{\Gamma_0}_{s+1}}^{k-1} {\|r\|^{\Gamma_0}_{s+\frac{3}{2}}}^2$$
$$\leq -c \sum_{|\alpha|\leq s} |D^\alpha r|^2_{\frac{3}{2}} + C\sum_{k=2}^{\infty} C_k \,{\|r\|^{\Gamma_0}_{s+1}}^{k-1} \left(|r|_{s+\frac{3}{2}} + \|F(r)\|_{\mathbb{R}\times\mathbb{R}^3}\right)^2$$
$$\leq -c \sum_{|\alpha|\leq s} {\|D^\alpha \mathcal{P} r\|^{\Gamma_0}_{\frac{3}{2}}}^2 + C\sum_{k=2}^{\infty} C_k \,{\|r\|^{\Gamma_0}_{s+1}}^{k-1} \left(|r|^2_{s+\frac{3}{2}} + \|F(r)\|^2_{\mathbb{R}\times\mathbb{R}^3}\right)$$

and the proof is completed by using $\sum_{|\alpha|\leq s} {\|D^\alpha \mathcal{P} r\|^{\Gamma_0}_{\frac{3}{2}}}^2 \geq c\,{\|\mathcal{P} r\|^{\Gamma_0}_{s+\frac{3}{2}}}^2$ and making a suitable choice for ε ∎

Proposition 3 (Global existence and exponential stability near equilibrium). *Under the assumptions of Proposition* 1 *and* (7.1), $s \geq s_1$, $r_0 \in \mathcal{M}_{s+1} \cap B_0(\varepsilon, H^{s+1}(\Gamma_0))$ *with* ε *sufficiently small (depending on* s*), the initial value problem* (6.2) *has a solution*

$$r \in C_w(\mathbb{R}_+, H^{s+1}(\Gamma_0)) \cap C^1_w(\mathbb{R}_+, H^s(\Gamma_0))$$

that satisfies the estimate

$$\|r(t)\|^{\Gamma_0}_{s+1} \leq Ce^{-ct}\|r_0\|^{\Gamma_0}_{s+1} \qquad \forall\, t \geq 0 \tag{7.9}$$

with a positive constant c *depending only on* s.

Proof. Note that (7.8) implies (6.3) for all $r \in H^{s+2}(\Gamma_0)$ with $\|r\|^{\Gamma_0}_{s+1}$ sufficiently small. We choose $\tilde{\varepsilon}$ small enough that both (6.3) and (7.7) with s replaced by $s+1$ holds if $\|r\|^{\Gamma_0}_{s+1} \leq 2\tilde{\varepsilon}$. We assume $\|r_0\|^{\Gamma_0}_{s+1} \leq \tilde{\varepsilon}$ and proceed as in the proof of Proposition 1, working only with estimates in $H^{s+1}(\Gamma_0)$. We choose the finite-dimensional subspaces $M_j \subset H^{s+2}(\Gamma_0)$ such that $M_1 = \text{span}\{1, x_1, x_2, x_3\}$ and $\overline{\text{span}\bigcup_{j>1} M_j} = \mathcal{P}[H^{s+2}(\Gamma_0)]$.

Thus, there is a $T > 0$ such that (6.2) has a solution r in $C_w(IT, H^{s+1}(\Gamma_0)) \cap C^1_w(IT, H^s(\Gamma_0))$. We set $\varepsilon = \min(\tilde{\varepsilon}, \frac{e^{\frac{cT}{2}}-1}{C^*})$ where c and C^* are the constants from (7.8) and (7.7), respectively.

The solution r is given by

$$r(t) = w - \lim_{j\to\infty} r_j(t) \qquad \forall t \in IT$$

where w-lim denotes the the weak limit in $H^{s+1}(\Gamma_0)$, the $r_j \in C^1(IT, H^{s+2}(\Gamma_0))$ are the solutions of the Galerkin equations (6.4), and the convergence is uniform in t. Hence $r_j(t) \xrightarrow{H^{s_0}(\Gamma_0)} r(t)$ uniformly in t and thus

$$\|F(r_j(t))\|_{\mathbb{R}\times\mathbb{R}^3} \to 0 \qquad \text{uniformly in } t \in IT \tag{7.10}$$

because, as remarked above, $r(t) \in \mathcal{M}_{s+1}$.

Our choice of the M_j yields that $\mathcal{P}$ and P_j commute for all j, and thus we have for all $t \in IT$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|r_j(t)|_{s+1}^2) &= [P_j\rho(r_j(t)), r_j(t)]_{s+1} \\ &= \big(\mathcal{P}P_j\rho(r_j(t)), \mathcal{P}r_j(t)\big)_{s+1} \\ &= \big(\mathcal{P}\rho(r_j(t)), \mathcal{P}P_jr_j(t)\big)_{s+1} \\ &= \big[\rho(r_j(t)), r_j(t)\big]_{s+1} \\ &\leq -c\,|r_j(t)|_{s+1}^2 + C\,\|F(r_j(t))\|_{\mathbb{R}\times\mathbb{R}^3}^2\end{aligned}$$

because of $P_jr_j = r_j$, $\|r_j(t)\|_{s+1}^{\Gamma_0} \leq 2\tilde{\varepsilon}$, and (7.8), hence

$$|r_j(t)|_{s+1}^2 \leq e^{-ct}|r_0|_{s+1}^2 + C\int_0^t e^{c(\tau-t)}\|F(r_j(\tau))\|_{\mathbb{R}\times\mathbb{R}^3}^2\,d\tau$$

and thus, using (7.10),

$$\begin{aligned}|r(t)|_{s+1} = \|\mathcal{P}r(t)\|_{s+1}^{\Gamma_0} &= \Big\|\mathcal{P}w-\lim_{j\to\infty} r_j(t)\Big\|_{s+1}^{\Gamma_0} = \Big\|w-\lim_{j\to\infty}\mathcal{P}r_j(t)\Big\|_{s+1}^{\Gamma_0} \\ &\leq \liminf_{j\to\infty}\|\mathcal{P}r_j(t)\|_{s+1}^{\Gamma_0} = \liminf_{j\to\infty}|r_j(t)|_{s+1} \leq e^{-ct}|r_0|_{s+1}.\end{aligned}$$

Finally, $r(t) \in \mathcal{M}_{s+1} \cap B_0(2\varepsilon, H^{s+1}(\Gamma_0))$ implies

$$\begin{aligned}\|r(T)\|_{s+1}^{\Gamma_0} &\leq \big(1 + C^*|r(T)|_{s+1}\big)|r(T)|_{s+1} \\ &\leq \big(1 + C^*\,\|r_0\|_{s+1}^{\Gamma_0}\big)e^{-cT}\|r_0\|_{s+1}^{\Gamma_0} \\ &\leq e^{-\frac{c}{2}T}\|r_0\|_{s+1}^{\Gamma_0} \\ &\leq \varepsilon\end{aligned}$$

because of $\varepsilon \leq \frac{e^{\frac{cT}{2}}-1}{C^*}$. Therefore we can continue the solution to $[T, 2T]$ and by induction to $[nT, (n+1)T]$ for all $n \in \mathbb{N}$ with the estimate

$$|r(t)|_{s+1} \leq e^{-ct}|r_0|_{s+1} \qquad \forall t \geq 0$$

from which (7.9) follows by (7.6) ∎

8. Conclusions

The most remarkable feature of the analysis given above is that it does not depend too strongly on special properties of the Stokes operator: The only facts we have used about it are its rotational invariance and ellipticity in the sense of Agmon-Douglis-Nirenberg, together with the regularity and self-adjointness of the corresponding Neumann problem. Therefore it seems to be possible to apply the same methods without essential changes to similar non-local evolution problems, in particular, to the problem of Hele-Shaw flow driven by surface tension.

Based on discussion of perturbations of the liquid domain and linearization of the resulting operator with respect to these perturbations, one can also obtain existence, uniqueness and smoothness results for stationary free boundary problems for the full Navier-Stokes equations [1, 2].

It has to be pointed out that the assumptions of the general existence theorem from [15] that has been applied here does not resemble the parabolic character of the evolution equation, actually, it is more suited to nonlinear hyperbolic equations (and has originally been used for a problem of that kind). This is the reason that our approach provides no proof of the smoothing effect we expect to find in a parabolic problem.

Finally, we remark that due to the local character the analysis given here obviously cannot provide answers to the questions on the occurrence of irregular behaviour like cusp formation or change of connectivity.

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