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Asymptotic Behaviour of Relaxed Dirichlet Problems Involving a Dirichlet-Poincaré Form

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Abstract. We study the convergence of the solutions of a sequence of relaxed Dirichlet problems relative to Dirichlet forms to the solution of the Γ-limit problem. In particular we prove the strong convergence in $D_0^p[a,\Omega]$ $(1 \le p \le 2)$ and the existence of "correctors" for the strong convergence in $D_0[a,\Omega]$. The above two results are generalizations to our framework of previous results proved in [10] in the usual uniformly elliptic setting.

Keywords: Γ-*convergence, Dirichlet forms, subelliptic equations*

AMS subject classification: 35 B 40, 31 C 25, 35 J 70

1. Introduction

In this paper we are interested in the convergence of the solutions of relaxed Dirichlet problems involving Dirichlet forms. The relaxed Dirichlet problem relative to symmetric uniformly elliptic operators was studied by G. Dal Maso and U. Mosco in [11] and [12]; in particular in [12] the convergence of the solutions is studied in connection with the Γ-convergence of the measures involved in the problems. We recall also that in the two previous papers the connections between relaxed Dirichlet problems and problems of homogenization with holes are emphasized (for the notions concerning homogenization with holes we refer to [1] and [7]).

The aim of the paper [10] is to study the convergence of solutions of relaxed Dirichlet problems in the non-symmetric uniformly elliptic case (in connection with the Γ-convergence of the measures involved). There some results are also given that are new also in the symmetric case; in particular the strong convergence of the solutions in $H^{1,p}$ $(1 \le p < 2)$ is proved using a compact embedding result of F. Murat [25], and the existence of correctors is studied (for previous results on correctors in the symmetric case see also [16]).

Our aim in this paper is to generalize those results to the case of relaxed Dirichlet problems involving Dirichlet forms, with some assumptions on the form which hold in the most of the applications.

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We make precise now the framework and the results. Let X be a locally compact, connected Hausdorff space and m be a positive Radon measure on X with $\text{supp}[m] = X$. We will assume that we are given a *strongly local, regular, symmetric Dirichlet form* $a(\cdot,\cdot)$ in the Hilbert space $L^2(X,m)$, in the sense of M. Fukushima [15], whose domain will be denoted by $D[a]$. Such a form a admits the integral representation $a(u,v) = \int_X d\alpha(u,v)$ for every $u, v \in D[a]$ where $\alpha(u,v)$ is a signed Radon measure on X, uniquely associated with the functions u and v (the energy density of the form). Moreover, for any open subset Ω of X the restriction of $\alpha(u,v)$ to Ω depends only on the restrictions of u and v to Ω. The strong locality allows us to define the domain of the form restricted to Ω denoted by $D_0[a,\Omega]$ as the closure in $D[a]$ (endowed with the norm $\|u\| = (a(u,u) + \|u\|^2_{L^2(X,m)})^{\frac{1}{2}}$) of $D[a] \cap C_0(\Omega)$ and to extend unambigously the definition of the measure $\alpha(u,v)$ in X to all m-measurable functions u and v in X, that coincide m-a.e. on every compact subset of Ω with some function of $D[a]$. The space of these functions will be denoted by $D_{loc}[a,\Omega]$. We refer to [4, 15] for the properties of $\alpha(u,v)$ with respect to Leibnitz, chaine and troncature rules.

Given a as above we assume that the form has a *separating core* [4]. We define a distance d associated with the form by

$$d(x,y) = sup\Big\{\phi(x) - \phi(y) : \ \phi \in D[a] \cap C_0(X) \text{ with } \alpha(\phi,\phi) \le m\Big\}$$

and we denote $B(x,r) = \{y : d(x,y) < r\}$, $B(r)$ will be balls $B(x,r)$ with fixed center x.

We assume the following:

(D) The distance d define a topology on X equivalent to the initial one; moreover, for every $R_0 > 0$ a *duplication property* holds for the balls $B(x,r)$ $(r \le R_0)$, that is

$$m(B(x,2r) \le c_0\, m(B(x,r))$$

where c_0 is a constant independent of x and r, but depending on R_0, i.e. $c_0 = c_0(R_0)$.

(P) For every ball $B(x,r)$ $(r \le \overline{R})$ and every $f \in D_{loc}[a]$ the *Poincaré inequality*

$$\int_{B(x,r)} |f - f_{x,r}|^2 dm \le c_1 r^2 \int_{B(x,kr)} d\alpha(f,f)$$

holds where c_1 and $k \ge 1$ are constants independent of $x, r \le 2R$ and $f_{x,r}$ is the average of f on $B(x,r)$.

From property (P) assuming that $B(x,r) \subseteq B(x,2r) \ne X$ $(r \le \frac{\overline{R}}{2})$ we obtain by standard methods the inequality

$$(\mathbf{P}_0) \qquad \int_{B(x,r)} |f|^2 dm \le c_2 r^2 \int_{B(x,r)} d\alpha(f,f)$$

for every $f \in D_0[a, B(x,r)]$; by a covering argument it is easy to prove that the inequality (P_0) holds also if $r \ge \frac{\overline{R}}{2}$, with a constant c_2, that depends on $\overline{R}$.

We observe that from *duplication property* in (D) the space X acquires the structure of an *homogeneous* space [8] and that, using (D), we can prove by iteration the inequality

$$\textbf{(D')} \qquad m(B(x,r)) \geq \frac{1}{c_0}\left(\frac{r}{R}\right)^{\nu} m(B(x,R))$$

for all $x \in X$ and $R \leq \frac{1}{2}R_0$, where $\nu = \log_2 c_0$, so ν is an estimate of the *homogeneous dimension* of X. Moreover, for any ball $B_R \subseteq B_{2R}$ with $B_{2R} \neq X$ and $R \leq \overline{R}$, we have Sobolev inequalities relative to ν (see [3, 5]); a simple covering argument allow to generalize the Sobolev inequality for functions in $D_0[a, B_R]$ to every $R > 0$ with constants depending on $\overline{R}$.

We recall that under two assumptions (D) and (P) a theory of local regularity of harmonics in $B_R \subseteq B_{2R}$ with $B_{2R} \neq X$ and estimates on the Green function have been given in [4] (see also [24]).

In this paper we have one more assumption:

(A) We assume the existence of the Radon-Nikodym derivative

$$\alpha(u,u)(\cdot) = \frac{d\alpha(u,u)}{dm} \in L^1_{loc}(\Omega, m)$$

and the existence of n linear operators L_i $(i = 1, \dots, n)$ from $D_0[a]$ into $L^2(X,m)$ and two positive constants λ and Λ such that

$$\lambda \sum_{i=1}^{n} |L_i u(x)|^2 \leq \alpha(u,u)(x) \leq \Lambda \sum_{i=1}^{n} |L_i u(x)|^2\, m \qquad \text{a.e. in } X.$$

Moreover, we also assume that the adjoint operators L_i^* restricted to $D[a]$ are bounded from $D[a]$ into $L^2(X,m)$. The operators L_i are closed from $D_0[a,\Omega]$ into $L^2(X,m)$.

We observe that the above assumptions on the Dirichlet form we are considering holds for the following forms:

(a) for forms connected with degenerate elliptic operators with a weight in the A_2 Muckenhoupt's class (here the distance is the usual Euclidean distance and we refer for properties (D) and (P) to [13]);

(b) for forms connected with subelliptic operators both in the case of smooth or non-smooth coefficients (here the distance is defined in relation with the operator and we refer to [21] for the properties (D) and (P));

(c) for forms connected with vector fields satisfying a Hörmander condition both in the case of smooth or non-smooth coefficients, given by a matrix, that is uniformly elliptic with respect to a weight in the A_2 intrinsic Muckenhoupt's class (here the distance is the same as in non-weighted case, the property (D) derives from the definition of the A_2 intrinsic Muckenhoupt's class and we refer to [22] for property (P));

(d) for forms connected with elliptic operators on C^∞ Riemannian manifolds with Ricci curvature bounded from belowe (here the properties (D) and (P) are consequences of analogous properties for elliptic operators on $\mathbb{R}^N$).

For every Borel subset E of an open set Ω in X, let

$$\mathrm{cap}^a(E,\Omega) = \inf\left\{ a(v,v) \;\middle|\; \begin{array}{l} v \in D_0[a,\Omega],\ v \geq 1\ m\text{-a.e.} \\ \text{on a neighbourhood of } E \end{array} \right\}.$$

We refer for all the properties holding for the capacity related to a Dirichlet form defined on X to the book of Fukushima [15], only observing that they hold again due to the validity of property (P). We recall that, if $E \subseteq \overline{E} \subseteq \Omega$, we have $\mathrm{cap}^a(E,\Omega) = 0$ if and only if $\mathrm{cap}^a(E,X) = 0$ and, from the Poincaré inequality (P_0), $\mathrm{cap}^a(E,X) = 0$ implies $m(E) = 0$. Moreover, every function $u \in D_0[a,\Omega]$ has a quasi-continuous representative (for the above introduced capacity). Whenever we have the necessity to take into consideration a quasi-everywhere representative of $u \in D_0[a,\Omega]$, we identify u with its quasi-continuous representative.

Definition 1.1. For a relatively compact open set $\Omega \subset X$, we introduce $\mathcal{M}_0^a(\Omega) \equiv \mathcal{M}_0$ as the space of all non-negative Borel measures on Ω which are absolutely continuous with respect to the capacity related to the form $a(\cdot,\cdot)$, i.e. we say that $\mu \in \mathcal{M}_0$ if $\mathrm{cap}^a(E,\Omega) = 0$ implies $\mu(E) = 0$, where $E \subseteq \overline{E} \subseteq \Omega$.

Definition 1.2. The function u is a solution of a homogeneous relaxed Dirichlet problem in Ω with respect to the form a, the function $f \in D^{-1}[a,\Omega]$ and the measure $\mu \in \mathcal{M}_0$ if

$$\begin{cases} u \in V^a_{\mu,0}(\Omega) \\ a(u,v) + \displaystyle\int_\Omega uv\, d\mu = \langle f, v\rangle \quad \text{for all } v \in V^a_{\mu,0}(\Omega) \end{cases} \tag{1.1}$$

where $V^a_{\mu,0}(\Omega) = L^2(\Omega,\mu) \cap D_0[a,\Omega]$ is a Hilbert space endowed with the scalar product $(u,v)_{V^a_{\mu,0}(\Omega)} = a(u,v) + \int_\Omega uv\, dm + \int_\Omega uv\, d\mu$.

Remark 1.1. By using the Poincaré inequality and (1.1) with $v = u$ it is easy to see that

$$\left(\int_\Omega d\alpha(u,u)\right)^{\frac{1}{2}} \leq C \tag{1.2}$$

where the constant C depends only on Ω and not on μ.

Now we want to recall the definition of Γ-convergence of a sequence of measures in the space $\mathcal{M}_0$. For any measure $\mu \in \mathcal{M}_0$, let us consider the following functional F^μ defined on $L^2(\Omega,m)$:

$$F^\mu(v) = \begin{cases} \int_\Omega d\alpha(v,v) + \int_\Omega v^2 d\mu & \text{if } v \in D_0[a,\Omega] \\ +\infty & \text{elsewhere.} \end{cases} \tag{1.3}$$

Definition 1.3. Let ε be a sequence of positive numbers converging to zero, μ^ε a sequence of measures in the space $\mathcal{M}_0$, and $\mu \in \mathcal{M}_0$. Let F^{μ^ε} and F^μ the functionals associated with μ^ε and μ, as in (1.3). Then

$$\mu^\varepsilon \xrightarrow{\Gamma} \mu$$

if the sequence of functionals F^{μ^ϵ} Γ-converges, in the sense of E. De Giorgi and T. Franzoni [9], to the functional F^μ.

As in the classical case (i.e. $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx$), it is possible to prove that the Γ-convergence of F^{μ^ϵ} to F^μ is equivalent to the $L^2(\Omega, m)$-convergence of the solutions u^ϵ of the homogeneous relaxed Dirichlet problems with respect to the form a, the function f and the sequence of measures μ^ϵ

$$\begin{cases} u^\epsilon \in V^a_{\mu^\epsilon,0}(\Omega) \\ a(u^\epsilon, v) + \int_\Omega u^\epsilon v\, d\mu^\epsilon = \int_\Omega f v\, dm \quad \text{for all } v \in V^a_{\mu^\epsilon,0}(\Omega) \end{cases} \tag{1.4}$$

to the solution u of the homogeneous relaxed Dirichlet problem with respect to the form a, the function f and the measure μ (see (1.1)).

It is easy to prove that

$$u^\epsilon \rightharpoonup u \qquad \text{weakly in } D_0[a,\Omega].$$

Our aim is to give more precise results on the convergence of the sequence u^ϵ. In general, as in the classical case, the convergence will not be strong in $D_0[a,\Omega]$.

Let us recall the definitions of the Dirichlet-Sobolev spaces, introduced by M. Biroli and U. Mosco in [5]. For $p \in [2,\infty)$, set

$$D^p[a,\Omega] = \left\{ u \in D_{loc}[a,\Omega] : \int_\Omega \alpha(u,u)(x)^{\frac{p}{2}} dm + \int_\Omega u^p dm =: \|u\|^p_{D^p[a,\Omega]} < \infty \right\}.$$

In the case $p = 2$ we denote $D^2[a,\Omega] = D[a,\Omega]$. In the case $p \in [1,2)$ we denote by $D^p[a,\Omega]$ the completion of $D[a,\Omega]$ in the norm

$$\|u\|_{D^p[a,\Omega]} := \left(\int_\Omega \alpha(u,u)(x)^{\frac{p}{2}} dm + \int_\Omega u^p dm \right)^{\frac{1}{p}}.$$

In Section 2 we will prove that the operators L_i are closed from $D^p[a,\Omega]$ into $L^p(X,m)$. We observe that the above definition allow us to define $L_i u$ and $\alpha(u,u)$ for $u \in D^p[a,\Omega]$ (see Lemmas 2.3 and 2.4).

Let us define the spaces $D^p_0[a,\Omega]$ as the closure of the space $C_0(\Omega) \cap D^p[a,\Omega]$ with respect to the norm $\|v\|_{D^p[a,\Omega]}$.

Remark 1.2. Let us remark that the space $D^p_0[a,\Omega] \cap L^\infty(\Omega)$ is an algebra and also an ideal in $D^p[a,\Omega] \cap L^\infty(\Omega)$, $p \in [1,+\infty)$. Moreover, we recall that if $g : \mathbb{R} \to \mathbb{R}$ is such that $g \in C^1(\mathbb{R})$ and $v \in D^p[a,\Omega] \cap L^\infty(\Omega)$, then $g(v) \in D^p[a,\Omega] \cap L^\infty(\Omega)$ and, for $p = 2$,

$$\alpha(g(v), w) = g'(v)\alpha(v,w)$$

for every $w \in D[a,\Omega]$. Moreover, if $g(0) = 0$ and $v \in D^p_0[a,\Omega] \cap L^\infty(\Omega)$, then $g(v) \in D^p_0[a,\Omega] \cap L^\infty(\Omega)$. A consequence of the above inequality is that for every $v \in D_{loc}[a,\Omega] \cap L^\infty(\Omega)$ we have, for every constant k, $\inf(v,k) \in D_{loc}[a,\Omega] \cap L^\infty(\Omega)$ and

$$\alpha(inf(v,k), w) = 1_{w<k}\alpha(v,w)$$

for every $w \in D[a,\Omega]$; then

$$\int_{v=k} \alpha(v,v)dm = 0.$$

In the following Ω will be a relatively compact open set in X such that $\Omega \subseteq B_R \subseteq B_{2R}$, with $B_{2R} \neq X$ and $R \leq \frac{\overline{R}}{2}$.

In Section 2 we prove the following strong convergence in D_0^p for $p \in (1,2)$.

Theorem 1.4 (Strong D_0^p-convergence for $p \in (1,2)$). *Let μ^ε be a sequence of measures in the space $\mathcal{M}_0$ and $\mu \in \mathcal{M}_0$ such that*

$$\mu^\varepsilon \xrightarrow{\Gamma} \mu.$$

Let u^ε be the solutions of the homogeneous relaxed Dirichlet problems with respect to the form a, the function $f \in D^{-1}[a,\Omega]$ and the sequence of measures μ^ε (see (1.4)), i.e.

$$\begin{cases} u^\varepsilon \in V^a_{\mu^\varepsilon,0}(\Omega) \\ a(u^\varepsilon, v) + \int_\Omega u^\varepsilon v\, d\mu^\varepsilon = \langle f, v\rangle \quad \textit{for all } v \in V^a_{\mu^\varepsilon,0}(\Omega), \end{cases}$$

and let u be the solution of the homogeneous relaxed Dirichlet problem with respect to the form a, the function f and the measure μ (see (1.1)), i.e.

$$\begin{cases} u \in V^a_{\mu,0}(\Omega) \\ a(u, v) + \int_\Omega uv\, d\mu = \langle f, v\rangle \quad \textit{for all } v \in V^a_{\mu,0}(\Omega). \end{cases}$$

Then, for $p \in (1,2)$,

$$u^\varepsilon \to u \qquad \textit{strongly in } D_0^p[a,\Omega]. \tag{1.5}$$

We also introduce, in Section 4, a sequence of functions independent from f: "*correctors*", which describes more precisely the behaviour of the sequence u^ε (see [10] and [16]) in $D_0[a,\Omega]$. To this aim let us introduce the sequence of solutions w^ε of the homogeneous relaxed Dirichlet problems with respect to the form a, the function $f \equiv 1$ and the sequence of measures μ^ε, i.e.

$$\begin{cases} w^\varepsilon \in V^a_{\mu^\varepsilon,0}(\Omega) \\ a(w^\varepsilon, v) + \int_\Omega w^\varepsilon v\, d\mu^\varepsilon = \int_\Omega v dm \quad \text{for all } v \in V^a_{\mu^\varepsilon,0}(\Omega), \end{cases} \tag{1.6}$$

and their $L^2(\Omega,m)$-limit function w as solution of the homogeneous relaxed Dirichlet problem with respect to the form a, the function $f \equiv 1$ and the measure μ. i.e.

$$\begin{cases} w \in V^a_{\mu,0}(\Omega) \\ a(w, v) + \int_\Omega wv\, d\mu = \int_\Omega v\, dm \quad \text{for all } v \in V^a_{\mu,0}(\Omega). \end{cases} \tag{1.7}$$

We prove, in Section 3, the following result.

Theorem 1.5 (Correctors result). *Let μ^ε be a sequence of measures in the space $\mathcal{M}_0$ and $\mu \in \mathcal{M}_0$ such that*

$$\mu^\varepsilon \xrightarrow{\Gamma} \mu.$$

Let u^ε be the solutions of the homogeneous relaxed Dirichlet problems with respect to the form a, the function $f \in L^\infty(\Omega, m)$ and the sequence of measures μ^ε (see (1.4)), i.e.

$$\begin{cases} u^\varepsilon \in V^a_{\mu^\varepsilon,0}(\Omega) \\ a(u^\varepsilon, v) + \int_\Omega u^\varepsilon v d\mu^\varepsilon = \int_\Omega f v dm \quad \text{for all } v \in V^a_{\mu^\varepsilon,0}(\Omega), \end{cases}$$

and let u be the solution of the homogeneous relaxed Dirichlet problem with respect to the form a, the function f and the measure μ, (see (1.1)), i.e.

$$\begin{cases} u \in V^a_{\mu,0}(\Omega) \\ a(u, v) + \int_\Omega uv\, d\mu = \int_\Omega fv\, dm \quad \text{for all } v \in V^a_{\mu,0}(\Omega). \end{cases} \tag{1.8}$$

Let, be for any $\delta > 0$,

$$r^\varepsilon_\delta = u^\varepsilon - \frac{uw^\varepsilon}{\sup\{w, \delta\}}. \tag{1.9}$$

Then

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \|r^\varepsilon_\delta\|_{D_0[a,\Omega]} = 0 \tag{1.9}$$

where w^ε and w are the solutions of the problems (1.6) and (1.7), respectively.

We end this section by observing that in Section 2, in view of the proof of Theorem 1.4, we prove some preliminary results interesting in itself, in particular we study the Sobolev spaces associated to the form and their dual spaces proving also a generalization of the compact embedding lemma in [25].

2. Proof of Theorem 1.4

At first we give the following result on the existence of a cut-off function of a compact set $K \subseteq \Omega$ with respect to Ω.

Lemma 2.1 (Cut-off function). *Let K be a compact set in Ω and $d_K := d(K, \partial\Omega)$. Then there exists a function $\Phi_K \in D_0[a, \Omega] \cap C^0(\Omega)$ such that*

$$\Phi_K = 1 \quad \text{on } K$$

$$\text{supp}\,[\Phi_K] \subset \left\{x \in \Omega : d(x, K) \leq \frac{d_K}{2}\right\}$$

$$\alpha(\Phi_K, \Phi_K)(x) \leq \frac{C}{d_K^2} \quad \text{a.e. in } \Omega$$

where C is an absolute constant.

Proof. We can cover K by a finite number of balls with center $x_i \in K$ $(i = 1, ..., q)$ and radius $\frac{d_K}{4}$. Let now ϕ_i the cut-off functions of $B(x_i, \frac{d_K}{4})$ and $B(x_i, \frac{d_K}{2})$; we have $\alpha(\phi_i, \phi_i)(x) \leq \frac{C}{d_K^2}$ a.e. in Ω (for the existence of cut-off functions between balls and the estimate on their energy densities see [4]). Choose now $\Phi_K = \sup_i \phi_i$. It is easy to see that Φ_K satisfies the conditions of the lemma ∎

The following result state the Hölder inequality in the spaces $D^p[a,\Omega]$ and prove that the function $\|v\|_{D^p[a,\Omega]}$ defined in Section 1 is a norm on $D^p[a,\Omega]$ if $p \in [2,+\infty)$ and on $D[a,\Omega]$ if $p \in [1,2)$.

Lemma 2.2 (Hölder inequality). *Let $p \in [1,\infty]$ and $p' \in [1,\infty]$ be conjugate exponents (i.e. $\frac{1}{p}+\frac{1}{p'}=1$), let $u,v \in D^p[a,\Omega]$ and let finally $\alpha(u,u)(\cdot)=\frac{d\alpha(u,u)}{dm} \in L^1_{loc}(\Omega,m)$. Assume that*

$$\alpha(v,v)(\cdot)^{\frac{1}{2}} \in L^p(\Omega,m) \qquad \text{and} \qquad \alpha(u,u)(\cdot)^{\frac{1}{2}} \in L^{p'}(\Omega,m).$$

Then $\alpha(u,v)(\cdot) \in L^1(\Omega,m)$ and

$$\int_\Omega |\alpha(u,v)(\cdot)|\,dm \le \left(\int_\Omega \alpha(v,v)(\cdot)^{\frac{p}{2}}dm\right)^{\frac{1}{p}} \left(\int_\Omega \alpha(u,u)(\cdot)^{\frac{p'}{2}}dm\right)^{\frac{1}{p'}}. \tag{2.1}$$

Moreover, if $p \in [2,+\infty)$, then the function $\|v\|_{D^p[a,\Omega]}$ defined in Section 1 is a norm on $D^p[a,\Omega]$.

Proof. The proof is analogous to the classical Hölder inequality for the L^p spaces. For the sake of completeness we sketch it here: The density $\alpha(u,v)(\cdot)$ is a bilinear form, such that $\alpha(u,u)(\cdot) \ge 0$. Then

$$|\alpha(u,v)(\cdot)| \le \frac{1}{2}\alpha(u,u)(\cdot) + \frac{1}{2}\alpha(v,v)(\cdot). \tag{2.2}$$

The function $\alpha(u,v)$ is continuous in $\Omega\setminus E$ where $m(E)=0$. Let us consider (2.2) where we replace v by λv, with $\lambda = (\alpha(v,v)(x))^{\frac{1}{2}}/(\alpha(u,u)(x))^{\frac{1}{2}}$. Let x be fixed in $\Omega\setminus E$. Then

$$\big|\alpha(u,v)(x)\big| \le \big(\alpha(v,v)(x)\big)^{\frac{1}{2}}\big(\alpha(u,u)(x)\big)^{\frac{1}{2}}. \tag{2.3}$$

From the Young inequality we have

$$\big|\alpha(u,v)(x)\big| \le \frac{1}{p}\big(\alpha(v,v)(x)\big)^{\frac{p}{2}} + \frac{1}{p'}\big(\alpha(u,u)(x)\big)^{\frac{p'}{2}} \tag{2.4}$$

for a.e. $x \in \Omega$. By integrating (2.4) in Ω with respect to the measure m we obtain

$$\int_\Omega \big|\alpha(u,v)\big|\,dm \le \frac{1}{p}\int_\Omega \big(\alpha(v,v)\big)^{\frac{p}{2}}dm + \frac{1}{p'}\int_\Omega \big(\alpha(u,u)\big)^{\frac{p'}{2}}dm. \tag{2.5}$$

By replacing v by λv in (2.5) where

$$\lambda = \frac{\left(\int_\Omega (\alpha(u,u))^{\frac{p'}{2}}dm\right)^{\frac{1}{p}}}{\left(\int_\Omega (\alpha(v,v))^{\frac{p}{2}}dm\right)^{\frac{1}{p}}} \tag{2.6}$$

we prove (2.1). The last part of the result is an easy consequence of (2.3) ∎

We remark that from Lemmas 2.1 and 2.2 it follows that $D^p[a,\Omega]$, $p \in [2,+\infty)$, is a Banach space; then the same property for $p \in (1,2)$ follows from the definition. Morover, we observe that if $w\text{-}\lim_{\varepsilon\to 0} f^\varepsilon = f^0$ in $D_0[a,\Omega]$, then $w\text{-}\lim_{\varepsilon\to 0} f^\varepsilon = f^0$ in $D[a,\Omega]$.

Lemma 2.3 ($\alpha(u,u)$ and L_i in $D^p[a,\Omega]$, $p \in [1,2]$). *Let $p \in [1,2]$. Then $D^p[a,\Omega]$ is continuosly embedded into $L^p(\Omega,m)$ and the operator $\alpha(u,u)^{\frac{1}{2}}$ has a unique extension (wich does not depend on Ω) to a continuous operator, denoted again by $\alpha(u,u)^{\frac{1}{2}}$, from $D^p[a,\Omega]$ into $L^p(\Omega,m)$. Moreover, the operators L_i $(i = 1,2,..,n)$ have unique extensions to linear closed operators, denoted again by L_i, from $L^p(\Omega,m)$ into $L^p(\Omega,m)$, with domain $D^p[a,\Omega]$.*

Proof. The condition (A) allows us to extend the operators L_i to closed linear operators from $L^2(\Omega,m)$ into $L^2(\Omega,m)$ with domain $D[a,\Omega]$, denoted again by L_i, such that for every $u \in D[a,\Omega]$ we have

$$\lambda \sum_{i=1}^{n} |L_i u(x)|^2 \le \alpha(u,u)(x) \le \Lambda \sum_{i=1}^{n} |L_i u(x)|^2\, m \qquad \text{a.e. in } \Omega.$$

Let be $u_j \to 0$ in $L^p(\Omega,m)$, where $u_j \in D[a,\Omega]$ and $\operatorname{supp}(u_j) \subseteq K$, with K a compact set in Ω. Assume $L_i u_j \to \chi$ in $L^p(\Omega,m)$. Let $u_j^k = \sup(-k, \inf(u_j,k))$ $(k > 0)$. We have

$$(L_i u_j^k, v)_{L^2(\Omega,m)} = (u_j^k, L_i^* v)_{L^2(\Omega,m)}$$

for every v in $D_0[a,\Omega] \cap L^{p'}(\Omega,m)$, with $\frac{1}{p} + \frac{1}{p'} = 1$. Then $L_i u_j^k$ weakly converges to 0 in $L^p(\Omega,m)$, for any fixed $k > 0$. The functions $|L_i u_j^k|^p$ are equiintegrable; moreover,

$$|L_i(u_j^k - u_j)|^2 \le \alpha(u_j^k - u_j, u_j^k - u_j).$$

Using the truncation rule, we obtain that $L_i u_j^k$ converges to χ a.e. in Ω; then $L_i u_j^k$ converges to χ in $L^p(\Omega,m)$, so we have $\chi = 0$.

Assume now that u_j is a Cauchy sequence in $D^P[a,\Omega]$ and $u_j \to 0$ in $L^p(\Omega,m)$. We assume that $\alpha(u_j,u_j)^{\frac{1}{2}} \to \chi'$ in $L^p(\Omega,m)$. Let K be a compact set in Ω. From Lemma 2.1 there exists a function ϕ with $\alpha(\phi,\phi) \in L^\infty(\Omega,m)$, $\phi = 1$ on K and $\operatorname{supp}(\phi) \subseteq \Omega$. The sequence ϕu_j is again a Cauchy sequence in $D^P[a,\Omega]$ and $\phi u_j \to 0$ in $L^p(\Omega,m)$.

Using the condition (A) and the Leibnitz rule we obtain that $L_i(\phi u_j)$ is a Cauchy sequence in $L^p(\Omega,m)$. From the first part of the proof we have $L_i(\phi u_j) \to 0$ in $L^p(\Omega,m)$; then $\alpha(\phi u_j, \phi u_j)^{\frac{1}{2}} \to 0$ in $L^p(\Omega,m)$. Using the properties of ϕ and the Leibnitz rule we obtain $\chi = 0$ a.e. in K, then $\chi = 0$ a.e. on Ω. The first part of the lemma is so proved. The second part easily follows using assumption (A) ∎

We have also easily the following

Lemma 2.4 ($\alpha(u,u)$ and L_i in $D^p[a,\Omega]$, $p > 2$). *Let $p \in (2,+\infty)$. Then $D^p[a,\Omega]$ is continuosly embedded into $L^p(\Omega,m)$ and $\alpha(u,u)^{\frac{1}{2}}$ is a continuous operator from $D^p[a,\Omega]$ into $L^p(\Omega,m)$. Moreover, the L_i $(i = 1,2,..,n)$ are linear closed operators from $L^p(\Omega,m)$ into $L^p(\Omega,m)$ with domain $D^p[a,\Omega]$.*

Using Lemmas 2.3 and 2.4 we have to define $\alpha(u,u)^{\frac{1}{2}}$ and $L_i u$ from $L^p(\Omega,m)$ to $D^p[a,\Omega]$ (also $\alpha(u,u)$ then can be defined a.e. in Ω) and we have again

$$\lambda^* \sum_{i=1}^n |L_i u(x)|^p \le \alpha(u,u)(x)^{\frac{p}{2}} \le \Lambda^* \sum_{i=1}^n |L_i u(x)|^p \quad m\text{−a.e. in } \Omega \tag{2.7}$$

where the two positive constants λ^* and Λ^* depend only on λ, Λ and n in (A) and from p. Moreover,

$$\|u\|_{D^p[a,\Omega]} = \left(\int_\Omega \alpha(u,u)(x)^{\frac{p}{2}} dm + \int_\Omega u^p dm\right)^{\frac{1}{p}}$$

for every $u \in D^p[a,\Omega]$, $p \in [1,+\infty)$.

Now we prove the following embedding result, wich has an interest in itself.

Lemma 2.5 (Compact embedding property). *Let B_R be a ball in X. Then the property*

(C) *$D_0[a,B_R]$ is compactly embedded into $L^2(B_R,m)$*

is fulfilled.

Proof. We can suppose, without loss of generality, that $4R \le \inf(R_0,\overline{R})$ (in the general case the result follows by a covering argument). Let f_n be a sequence weakly convergent in $D_0[a,B_R]$. Then the sequence f_n is also weakly convergent in $L^2(B_R,m)$, since the embedding of $D_0[a,B_R]$ into $L^2(B_R,m)$ is continuous. We have $\int_{B_R} d\alpha(f_n,f_n) \le C$. We denote again by f_n the prolongation of f_n to X by 0, which belongs to $D[a]$. From [8: p. 69] there exists a covering $B(x_i,r) = B_i$ $(i = 1,\ldots,q)$ of B_R such that $d(x_i,x_j) \ge r$. We have that the number M of the balls $B(x_i,r)$, that cover a point x in B_R, is equal to the number of points x_i in $B(x,r)$. For such a point we have

$$B\left(x_i,\frac{r}{2}\right) \subseteq B(x,2r) \subseteq B(x_i,4r),$$

moreover, the balls $B(x_i,\frac{r}{2})$ are disjoint. Using property (D') we obtain

$$m\left(B\left(x_i,\frac{r}{2}\right)\right) \ge 2^{-(3\nu+1)} m(B(x,2r)).$$

Then

$$\begin{aligned} M2^{-(3\nu+1)} m(B(x,2r)) &\le M \min_{x_i \in B(x,r)} m\left(B\left(x_i,\frac{r}{2}\right)\right) \\ &\le m\left(\cup_{i=1}^M B\left(x_i,\frac{r}{2}\right)\right) \le m(B(x,2r)) \end{aligned}$$

so the point x belongs at most to $M = 2^{4\nu}$ balls $B(x_i,r)$. Again by property (D') we can estimate q from above by $M(\frac{R}{r})^\nu$. Moreover, we can prove, by the same techniques used above, that every point of B_R belongs to at most $k^\nu M$ balls $B(x_i,kr)$.

Let $\varepsilon > 0$ be arbitrary and denote $w_{n,m} = f_n - f_m$. By the same methods as in [5: Proposition 1/p. 315] we have

$$\int_{B_R} w_{n,m}^2 dm \le 2\sum_{i=1}^{q} \int_{B(x_i,r)} |w_{n,m} - (w_{n,m})_i|^2 dm$$
$$+ 2 \sup_{1,\dots,q} \frac{1}{m(B(x_i,r))} \left(\int_{B(x_i,r)} w_{n,m}\, dm\right)^2$$
$$\le 2k^{\nu} M c_1 r^2 \int_{B_R} d\alpha(u,u) + 2 \sup_{1,\dots,q} \frac{1}{m(B(x_i,r))} \left(\int_{B(x_i,r)} w_{n,m} dm\right)^2$$
$$\le 2k^{\nu} M c_1 r^2 C + 2^{7\nu+1} \left(\frac{R}{r}\right)^{2\nu} \frac{1}{m(B_R)} \sup_i \left(\int_{B_i} w_{n,m}\, dm\right)^2$$

where $(w_{n,m})_i$ is the average on $B(x_i,r)$ of $w_{n,m}$ and we take into account that from property (D') we have $m(B_i) \ge 2^{2\nu+1} m(B_R)$. We choose $r = (\frac{\varepsilon}{4k^{\nu} M c_1 C})^{\frac{1}{2}}$. Taking into account that f_n is weakly convergent in $L^2(B_R, m)$ we can choose n_ε such that for $n, m > n_\varepsilon$

$$\sup_i \left(\int_{B_i} w_{n,m}\, dm\right)^2 \le \frac{\varepsilon r^{2\nu} m(B_R)}{2^{7\nu+1} R^{2\nu}}.$$

Then for $n, m > n_\varepsilon$ we have

$$\int_{B_R} w_{n,m}^2 dm \le \varepsilon,$$

i.e. f_n is a Cauchy sequence in $L^2(B_R, m)$; then f_n converges strongly in $L^2(B_R, m)$ ∎

The method used in the proof of Lemma 2.5 is a refinement of the one used in [17] to prove the same result in the usual elliptic case (see also [14] and [5], where similar techniques are used).

Lemma 2.6 (Reflexivity). *Let $p \in (1, \infty)$. Then under the assumption* (A), $D^p[a,\Omega]$ *and* $(D_0^p[a,\Omega])$ *are reflexive Banach spaces.*

Proof. The proof is analogous to the classical case. To prove the reflexivity it is enough to prove that $D^p[a,\Omega]$ endowed with the norm (equivalent to $\|v\|_{D^p[a,\Omega]}$)

$$\left(\int_\Omega u^p dm + \sum_{i=1}^{n} \int_\Omega |L_i u(x)|^p dm\right)^{\frac{1}{p}} \tag{2.8}$$

is reflexive. We observe that the linear operator

$$Tu = [u,\, L_1 u,\, \dots,\, L_n u] \tag{2.9}$$

is an isometry from $D^p[a,\Omega]$, endowed with the norm (2.8), into $(L^p(\Omega,m))^{n+1}$; then $T(D^p[a,\Omega]) = Y$ is a closed subspace of $(L^p(\Omega,m))^{n+1}$. We recall that $(L^p(\Omega,m))^{n+1}$ is reflexive and then Y is reflexive, so $D^p[a,\Omega]$ is reflexive. The space $D_0^p[a,\Omega]$ is a closed subspace of $D^p[a,\Omega]$; then it is reflexive ∎

We denote by $D^{-1}[a,\Omega]$ the dual space of $D_0[a,\Omega]$, by $D_{p'}^{-1}[a,\Omega]$ the dual space of $D_0^p[a,\Omega]$ (where $\frac{1}{p}+\frac{1}{p'}=1$), and by $\langle\cdot,\cdot\rangle$ the duality between $D_{p'}^{-1}[a,\Omega]$ and $D_0^p[a,\Omega]$. Let us remark that we have $D_{p'}^{-1}[a,\Omega]\subset D^{-1}[a,\Omega]$ for $1<p<2$.

Lemma 2.7 (Dual Spaces). *Let Ω be an open subset of X, $p\in(1,\infty]$, and let us assume property* (A). *Then $F\in D_{p'}^{-1}[a,\Omega]$ if and only if there exist $(f_0,\, f_1,\dots,\, f_n)\in(L^{p'}(\Omega,m))^{n+1}$ such that*

$$\langle F,u\rangle=\int_\Omega f_0u\,dm+\sum_{i=1}^n\int_\Omega f_iL_iu(x)\,dm\qquad \textit{for any } u\in D_0^p[a,\Omega].$$

Proof. We fix $(f_0,\, f_1,\dots,\, f_n)\in(L^{p'}(\Omega))^{n+1}$ and define

$$\langle F,u\rangle=\int_\Omega f_0u\,dm+\sum_{i=1}^n\int_\Omega f_iL_iu(x)\,dm\qquad \text{for any } u\in D_0^p[a,\Omega].$$

By the Hölder inequality we deduce that F is a bounded linear functional on $D^p[a,\Omega]$.

Let now $F\in D_{p'}^{-1}[a,\Omega]$. Consider the embedding T of $D_0^p[a,\Omega]$ into $L^p(\Omega,m)$ defined in (2.9); T is an isometry and $T(D_0^p[a,\Omega])=Y_0$ is a closed subspace of $(L^p(\Omega,m))^{n+1}$. Let us consider the inverse operator $T^{-1}:Y_0\to D_0^p[a,\Omega]$ and $F\cdot T^{-1}:Y_0\to\mathbb{R}$ which is a bounded linear functional on Y_0. Then from the Hahn Banach theorem, there exists a unique extension G of $F\cdot T^{-1}$ to $(L^p(\Omega,m))^{n+1}$ as a bounded linear functional, with the same norm as $F\cdot T^{-1}$. Let now $v=(v_0,\, v_1,\dots,v_n)\in(L^p(\Omega,m))^{n+1}$. From the Riesz representation theorem there exists $(f_0,f_1,\dots,f_n)\in(L^{p'}(\Omega,m))^{n+1}$ such that

$$\langle G,v\rangle_{p',p}=\int_\Omega f_0v_0\,dm+\sum_{i=1}^n\int_\Omega f_iv_i\,dm \tag{2.10}$$

where $\langle\cdot,\cdot\rangle_{p',p}$ is the duality between $(L^{p'}(\Omega,m))^{n+1}$ and $(L^p(\Omega,m))^{n+1}$. Then, for any $u\in D_0^p[a,\Omega]$,

$$\langle F,u\rangle=\langle G,Tu\rangle_{p',p}=\int_\Omega f_0u\,dm+\sum_{i=1}^n\int_\Omega f_iL_iu(x)\,dm$$

and the assertion is proved ∎

Lemma 2.8 (Convergence of integral terms). *Let us assume that the function ϕ and the sequence $\psi^\varepsilon\in D_0[a,\Omega]$ verifies*

$$\psi^\varepsilon\xrightarrow{D[a,\Omega]}\psi$$

$$\phi\in D[a,\Omega]\qquad \textit{and}\qquad \alpha(\phi,\phi)(x)\le C\ \ \textit{a.e. with respect to } m \textit{ in } \Omega.$$

Then $\alpha(\psi^\varepsilon,\phi)\in L^2(\Omega,m)$ and

$$\alpha(\psi^\varepsilon,\phi)\xrightarrow{L^2(\Omega,m)}\alpha(\psi,\phi).$$

Proof. Using (2.4) we have

$$|\alpha(\psi^\varepsilon,\phi)(\cdot)| \le \left(\alpha(\psi^\varepsilon,\psi^\varepsilon)(\cdot)\right)^{\frac{1}{2}}\left(\alpha(\phi,\phi)(\cdot)\right)^{\frac{1}{2}} \le C\left(\alpha(\psi^\varepsilon,\psi^\varepsilon)(\cdot)\right)^{\frac{1}{2}}.$$

Then

$$\int_\Omega |\alpha(\psi^\varepsilon,\phi)(\cdot)|^2 dm \le C\int_\Omega \left(\alpha(\psi^\varepsilon,\psi^\varepsilon)(\cdot)\right) dm \le C.$$

Thus $\alpha(\psi^\varepsilon,\phi) \in L^2(\Omega,m)$ is uniformly bounded and, at least after extraction of subsequences, there exists a function $\chi \in L^2(\Omega,m)$ such that

$$\alpha(\psi^\varepsilon,\phi) \xrightarrow{L^2(\Omega,m)} \chi.$$

On the other hand, applying the Mazur lemma, it is easy to prove that there exists a sequence of non-negative coefficients γ_i^ε such that $\sum_\varepsilon \gamma_i^\varepsilon = 1$ and such that the sequence

$$g_i = \sum_\varepsilon \gamma_i^\varepsilon \psi^{\varepsilon+i}$$

strongly converges to $\psi \in D[a,\Omega]$. This implies that

$$\begin{aligned}\int_\Omega \left(\alpha(g_i,\phi) - \alpha(\psi,\phi)\right)^2 dm &= \int_\Omega \alpha(g_i-\psi,\phi)^2 dm \\ &\le C\int_\Omega \alpha(g_i-\psi,g_i-\psi)\,dm \\ &\to 0\end{aligned}$$

and then $\alpha(g_i,\phi) \xrightarrow{L^2(\Omega,m)} \alpha(\psi,\phi)$. Then for any $v \in L^2(\Omega,m)$ we have

$$\begin{aligned}\int_\Omega \alpha(\psi,\phi)v\,dm &= \int_\Omega \lim_{i\to\infty} \alpha(g_i,\phi)v\,dm \\ &= \lim_{i\to\infty}\int_\Omega \alpha(g_i,\phi)v\,dm \\ &= \lim_{i\to\infty}\int_\Omega \alpha\left(\sum_\varepsilon \gamma_i^\varepsilon\psi^{\varepsilon+i},\phi\right)v\,dm \\ &= \lim_{i\to\infty}\sum_\varepsilon \gamma_i^\varepsilon\int_\Omega \alpha(\psi^{\varepsilon+i},\phi)v\,dm \\ &= \lim_{i\to\infty}\sum_\varepsilon \gamma_i^\varepsilon\int_\Omega \left(\alpha(\psi^{\varepsilon+i},\phi)-\chi\right)v\,dm + \int_\Omega \chi v\,dm.\end{aligned}$$

We have to prove that

$$\lim_{i\to\infty}\sum_\varepsilon \gamma_i^\varepsilon\int_\Omega \left(\alpha(\psi^{\varepsilon+i},\phi)-\chi\right)v\,dm = 0.$$

From the definition of $L^2(\Omega,m)$-weakly convergence we have that for any $\eta > 0$ there exists an ε^* such that for any $\varepsilon > \varepsilon^*$ we have

$$\left|\sum_\varepsilon \gamma_i^\varepsilon\int_\Omega \left(\alpha(\psi^{\varepsilon+i},\phi)-\chi\right)v\,dm\right| \le \sum_\varepsilon \gamma_i^\varepsilon\eta \le \eta,$$

so the assertion is proved ∎

Let us denote by $\mathcal{R}(\Omega)$ the set of Radon measures on Ω. We say that a sequence $\mu^\varepsilon \in \mathcal{R}(\Omega)$ is *w^*-bounded* if for every compact set K in Ω there exists a constant C_K such that

$$\left|\int_\Omega \phi\, d\mu^\varepsilon\right| \leq C_K \|\phi\|_{L^\infty(\Omega;m)}$$

for every $\phi \in C^0(\Omega)$ with $\operatorname{supp}[\phi] \subset \Omega$ where $C^0(\Omega)$ denotes the space of functions that are continuous in Ω. We say that a sequence $\mu^\varepsilon \in \mathcal{R}(\Omega)$ w^*-converges to μ if

$$\lim_{\varepsilon\to 0} \int_\Omega \phi\, d\mu^\varepsilon = \int_\Omega \phi\, d\mu$$

for every $\phi \in C^0(\Omega)$ with $\operatorname{supp}[\phi] \subset \Omega$.

Lemma 2.9 (Convergence of Radon measures). *Let K be a compact set in Ω, and $\mu^\varepsilon \in \mathcal{R}(\Omega)$ with*

$$w^* - \lim_{\varepsilon\to 0} \mu^\varepsilon = \mu.$$

Let us assume that V is a compact set in $C^0(\Omega)$. Then

$$\lim_{\varepsilon\to 0} \int_\Omega \Phi_K v\, d\mu^\varepsilon = \int_\Omega \Phi_K v\, d\mu,$$

uniformly for $v \in V$, where Φ_K is the cut-off function defined in Lemma 2.1.

Proof. It is enough to observe that $\Phi_K v$ is a compact set in $C^0(\Omega)$ and

$$\operatorname{supp}[\Phi_K v] \subset \cup_{x\in K} B\left(x, \frac{d_K}{2}\right) = \left\{x \in \Omega : d(x,K) \leq \frac{d_K}{2}\right\}$$

where $\{x \in \Omega : d(x,K) \leq \frac{d_K}{2}\}$ is a closed set contained in Ω and then a compact set in Ω ∎

Definition 2.10. We say that $f \geq 0$, with $f \in D^{-1}[a,\Omega]$, if

$$\langle f, v\rangle \geq 0 \qquad \text{for any } v \in D_0[a,\Omega] \text{ with } v \geq 0 \text{ a.e. in } \Omega.$$

Lemma 2.11 (Convergence in $D^{-1}[a,\Omega]$). *Let f^ε be a sequence in $D^{-1}[a,\Omega]$, such that*

$$w - \lim_{\varepsilon\to 0} f^\varepsilon = f \quad \text{in } D^{-1}[a,\Omega] \qquad \text{and} \qquad f^\varepsilon \geq 0.$$

Then f^ε and f belong to $\mathcal{R}(\Omega)$ and

$$w^* - \lim_{\varepsilon\to 0} f^\varepsilon = f \qquad \text{in } \mathcal{R}(\Omega).$$

Proof. Let K be a compact set in Ω and $\phi \in D_0[a,\Omega] \cap C^0(\Omega)$ with $\operatorname{supp}[\phi] \subset K$. Then

$$-\|\phi\|_{L^\infty(\Omega,m)}\Phi_K \leq \phi \leq \|\phi\|_{L^\infty(\Omega,m)}\Phi_K$$

where Φ_K is as in Lemma 2.1. Then, using the assumption $f^\varepsilon \geq 0$,

$$|\langle f^\varepsilon, \phi\rangle|, |\langle f, \phi\rangle| \leq \sup_\varepsilon |\langle f^\varepsilon, \Phi_K\rangle| \, \|\phi\|_{L^\infty(\Omega,m)} \leq C_K \|\phi\|_{L^\infty(\Omega,m)}.$$

Taking into account the density of $D_0[a,\Omega] \cap C^0(\Omega)$ in $C^0(\Omega)$ we have that f^ε and f define some Radon measures. From the relation

$$\lim_{\varepsilon \to 0} |\langle f^\varepsilon - f, \phi \rangle| = 0$$

for any $\phi \in D_0[a,\Omega] \cap C^0(\Omega)$ and taking into account that, using Lemma 2.1, every $\phi \in C^0(\Omega)$ with support in a compact set K in Ω can be approximated by a sequence $\{\phi^\varepsilon\} \in D_0[a,\Omega] \cap C^0(\Omega)$ with $\operatorname{supp}[\phi^\varepsilon] \subset \cup_{x\in K} B(x, \frac{d_K}{2})$, we have $w^*\text{-}\lim_{\varepsilon\to 0} f^\varepsilon = f$ in $\mathcal{R}(\Omega)$ ∎

Let us consider now the problem

$$\begin{cases} u \in D_0[a,\Omega] \\ a(u,v) = \langle F, v\rangle \quad \text{for all } v \in D_0[a,\Omega] \end{cases} \tag{2.11}$$

where $F \in D^{-1}[a,\Omega]$. We want to prove some properties of the solution of problem (2.11).

Theorem 2.12 (L^∞-estimates). *Let $F \in D_q^{-1}[a,\Omega]$, and $q > \nu \vee 2$ where ν is as in Section 1. Then the solution of problem (2.11) verifies*

$$\sup_\Omega |u| \le C m(B_R)^{\frac{1}{\nu'} - \frac{1}{q}} \|F\|$$

where $\Omega \subset B_R$ with $R \le \overline{R}$ and $\nu \vee 2 < \nu' < q$.

Proof. From Lemma 2.7 there exists $(f_0, f_1, \dots, f_n) \in (L^q(\Omega, m))^{n+1}$ such that

$$\langle F, v\rangle = \int_\Omega f_0 v \, dm + \sum_{i=1}^n \int_\Omega f_i L_i v(x) \, dm \qquad \text{for any } v \in D^{q'}[a,\Omega]$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\|F\| = \left(\int_\Omega f_0^q \, dm + \sum_{i=1}^n \int_\Omega f_i^q \, dm \right)^{\frac{1}{q}}.$$

We use Stampacchia's method [26]. Let

$$\beta(\tau) = (\operatorname{sign} \tau)\big(\max(|\tau| - k, 0)\big) = \begin{cases} \tau - k & \text{if } \tau > k \\ 0 & \text{if } \tau \le k \\ \tau + k & \text{if } \tau < -k. \end{cases}$$

As in [3] we use $\beta(u) \in D_0[a,\Omega]$ as test function. By the Sobolev-Poincaré inequality [3, 5] we obtain

$$\left(\int_\Omega (|u| - k)^{\frac{2\nu'}{\nu'-2}} dm \right)^{\frac{\nu'-2}{\nu'}} \le C \|F\|^2 m(A(k))^{1-\frac{2}{q}}$$

where C denotes here and in the following possibly different structural constants and $A(k) = \{x \in \Omega : |u(x)| > k\}$. Then if $h > k > 0$, we obtain

$$m(A(h)) \le (h-k)^{\frac{2\nu'}{\nu'-2}} m(A(k))^l \|F\|^{\frac{2\nu'}{\nu'-2}}$$

where $l = (1 - \frac{2}{q})/(1 - \frac{2}{\nu'}) > 1$. Then the result follows from Lemma 4.1 in [26] ∎

Theorem 2.13 (Hölder-continuity). *Let $F \in D_q^{-1}[a,\Omega]$, $q > \nu \vee 2$, and u be the solution of problem* (2.11). *Then u is locally Hölder continuous in Ω. Moreover, let $B(x,k^*R) \subset\subset \Omega$, where $k^* \geq 2$ is a suitable structural constant depending on k only. Then for $k^*r \leq R \leq \frac{\overline{R}}{k^*}$ we have*

$$\operatorname{osc}_{B(x,r)} u \leq C \left[\left(\frac{r}{R}\right)^{\gamma} + m(B(x;R))^{\frac{1}{\nu'}-\frac{1}{q}}\right] \|F\|$$

where $\gamma \in (0,1)$ is a structural constant and $\nu \vee 2 < \nu' < q$.

Proof. We represent u in $B(x,k^*R)$ as $v + w$ where v and w are solutions of the problems

$$\begin{cases} v - u \in D_0[a, B(x,k^*R)] \\ \displaystyle\int_\Omega \alpha(v,\zeta)\,dm = 0 \quad \text{for all } \zeta \in D_0[a, B(x,k^*R)] \end{cases} \tag{2.12}$$

and

$$\begin{cases} w \in D_0[a, B(x,k^*R)] \\ \displaystyle\int_\Omega \alpha(w,\zeta)\,dm = \langle F, v\rangle \quad \text{for all } \zeta \in D_0[a, B(x,k^*R)], \end{cases} \tag{2.13}$$

respectively. From [5: Proposition 7.1 and Theorem 7.3] we obtain for $r < R$

$$\operatorname{osc}_{B(x,r)} v \leq C \left(\frac{r}{R}\right)^{\gamma} \int_{B(x,k^*R)} |v|^2\,dm \leq C \left(\frac{r}{R}\right)^{\gamma} \|F\|$$

where $\gamma \in (0,1)$ is a structural constant. From Theorem 2.12 we have

$$\sup_{B(x,k^*R)} |w| \leq C\, m(B(x,k^*R))^{\frac{1}{\nu'}-\frac{1}{q}} \|F\|.$$

Then, also using the duplication property,

$$\operatorname{osc}_{B(x,r)} u \leq C \left[\left(\frac{r}{R}\right)^{\gamma} + m(B(x,R))^{\frac{1}{\nu'}-\frac{1}{q}}\right] \|F\|$$

and the assertion is proved ∎

Corollary 2.14 (Locally uniform convergence). *Let u^ε be the sequence of solutions of problem* (2.11) *associated with $F = G^\varepsilon \in D_q^{-1}[a,\Omega]$, $q > \nu \vee 2$, and assume that the sequence G^ε is bounded in $D_q^{-1}[a,\Omega]$. Then there exists a subsequence $u^{\varepsilon'}$ of u^ε which converges uniformly locally in Ω.*

Now we can prove the result that will be the fundamental tool for the proof of convergence in $D_0^p[a,\Omega]$ where $p \in (1,2)$.

Theorem 2.15 (Strong convergence). *Let u^ε be the sequence of solutions of problem* (2.11) *associated with $F^\varepsilon = f^\varepsilon + f$ where*

$$f^\varepsilon, f \in D^{-1}[a,\Omega], \qquad f^\varepsilon \geq 0, \qquad w-\lim_{\varepsilon\to 0} f^\varepsilon = f^0 \ \text{in } D^{-1}[a,\Omega].$$

Then

$$w-\lim_{\varepsilon\to 0} u^\varepsilon = u^0 \quad in\ D_0[a,\Omega] \qquad and \qquad \lim_{\varepsilon\to 0} u^\varepsilon = u^0 \quad in\ D_0^q[a,\Omega]$$

for every $q \in (1,2)$*, where* u^0 *is the solution of problem* (2.11) *associated to* $F = f^0 + f$*.*

Proof. It is easy to see that

$$w-\lim_{\varepsilon\to 0} u^\varepsilon = u^0 \qquad \text{in } D_0[a,\Omega].$$

By the compact embedding property of $D_0[a,\Omega]$ into $L^2(\Omega,m)$ (Lemma 2.5) we have

$$\lim_{\varepsilon\to 0} u^\varepsilon = u^0 \qquad \text{in } L^2(\Omega,m).$$

Take now $\psi \in D_{q'}^{-1}[a,\Omega]$ with $\|\psi\|_{-1,q'} \le 1$ where $q' > \nu \vee 2$, $q \in (1,2)$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\|\cdot\|_{-1,q'}$ denotes the norm in $D_{q'}^{-1}[a,\Omega]$. Denote $w^\varepsilon = u^\varepsilon - u^0$ and by u_ψ the solution of problem (2.11) associated with $F = \psi$. Let $K \subset \Omega$ be a compact set, and let Φ_K be as in Lemma 2.1. We want to prove that

$$\lim_{\varepsilon\to 0} \Phi_K w^\varepsilon = 0 \qquad \text{in } D_0^q[a,\Omega].$$

We have that

$$\begin{aligned} \|\Phi_K w^\varepsilon\|_{D_0^q[a,\Omega]} &= \sup_{\psi\in D_{q'}^{-1}[a,\Omega],\ \|\psi\|_{-1,q'}\le 1} \langle \Phi_K w^\varepsilon, \psi\rangle \\ &= \sup_{\psi\in D_{q'}^{-1}[a,\Omega],\ \|\psi\|_{-1,q'}\le 1} a(\Phi_K w^\varepsilon, u_\psi). \end{aligned}$$

So we have to prove that $a(\Phi_K w^\varepsilon, u_\psi)$ converges to 0 uniformly with respect to $\psi \in D_{q'}^{-1}[a,\Omega]$, with $\|\psi\|_{-1,q'} \le 1$. We have

$$\begin{aligned} &a(\Phi_K w^\varepsilon, u_\psi) \\ &= \int_\Omega \alpha(\Phi_K w^\varepsilon, u_\psi)\, dm \\ &= \int_\Omega \alpha(\Phi_K, u_\psi) w^\varepsilon dm + \int_\Omega \alpha(w^\varepsilon, u_\psi)\Phi_K\, dm \\ &= \int_\Omega \alpha(\Phi_K, u_\psi) w^\varepsilon dm + \int_\Omega \alpha(w^\varepsilon, u_\psi\Phi_K)\, dm - \int_\Omega \alpha(w^\varepsilon, \Phi_K) u_\psi\, dm \\ &= \int_\Omega \alpha(\Phi_K, u_\psi) w^\varepsilon dm + \int_\Omega u_\psi \Phi_K (f^\varepsilon - f^0)\, dm - \int_\Omega \alpha(w^\varepsilon, \Phi_K) u_\psi\, dm. \end{aligned}$$

The first term in the right-hand side converges to 0 uniformly with respect to $\psi \in D_{q'}^{-1}[a,\Omega]$, with $\|\psi\|_{-1,q'} \le 1$. Thanks to Corollary 2.14 and to the bound $\|\psi\|_{-1,q'} \le 1$, we have that $u_\psi \Phi_K$ belongs to a compact set of $C^0(\Omega)$. Then we use Lemma 2.11 and we have that the second term also converges to 0, uniformly with respect to $\psi \in D_{q'}^{-1}[a,\Omega]$, with $\|\psi\|_{-1,q'} \le 1$. Finally, from Lemma 2.8 we have that $\alpha(w^\varepsilon, \Phi_K)$ converges weakly to 0 in $L^2(\Omega)$; moreover, u_ψ is in a compact set in $L^2(\Omega, m)$. Then also the third term converges to 0 uniformly with respect to $\psi \in D_{q'}^{-1}[a,\Omega]$, with $\|\psi\|_{-1,q'} \leq 1$, and we have proved that

$$\lim_{\varepsilon\to 0} \Phi_K w^\varepsilon = 0 \text{ in } D^q[a,\Omega] \qquad \text{and then} \qquad \lim_{\varepsilon\to 0} \Phi_K w^\varepsilon = 0 \text{ a.e. in } \Omega.$$

Taking into account that $\alpha(w^\varepsilon, w^\varepsilon)^{\frac{1}{2}}$ is bounded in $L^2(\Omega, m)$, then $\alpha(w^\varepsilon, w^\varepsilon)^{\frac{q}{2}}$ is equi-integrable in Ω where $q \in (1,2)$. So we have, for every $q \in (1,2)$,

$$\lim_{\varepsilon\to 0} \alpha(w^\varepsilon, w^\varepsilon)^{\frac{1}{2}} = 0 \qquad \text{in } L^q(\Omega, m).$$

Taking into account that $\lim_{\varepsilon\to 0} w^\varepsilon = 0$ in $L^2(\Omega, m)$, we obtain $\lim_{\varepsilon\to 0} w^\varepsilon = 0$ in $D_0^q[a,\Omega]$ ∎

The following corollary of Theorem 2.15 gives a generalization to our framework of a previous result of F. Murat [25] relative to the usual Sobolev spaces.

Corollary 2.16. *Let*

$$f^\varepsilon \in D^{-1}[a,\Omega], \qquad f^\varepsilon \geq 0, \qquad w\text{-}\lim_{\varepsilon\to 0} f^\varepsilon = f^0 \quad \textit{in } D^{-1}[a,\Omega].$$

Then

$$\lim_{\varepsilon\to 0} f^\varepsilon = f^0 \qquad \textit{in } D_q^{-1}[a,\Omega]$$

for every $q \in (1,2)$.

Proposition 2.17 (Inequality for positive data). *Let u be the solution of the homogeneous relaxed Dirichlet problem* (1.1) *in Ω with respect to the form a, the function $0 \leq f \in D^{-1}[a,\Omega]$ and the measure $\mu \in \mathcal{M}_0$. Then*

$$\int_\Omega \alpha(u,v)\, dm \leq \int_\Omega v\, df \tag{2.14}$$

for all $v \in D_0[a,\Omega] \cap C^0(\Omega)$ with compact support in Ω and $v \geq 0$.

Let us remark that the hypothesis $0 \leq f \in D^{-1}[a,\Omega]$ thanks to Lemma 2.12 implies that $f \in \mathcal{R}(\Omega)$.

Proof. The proof is the same as given in [11] for the usual elliptic setting taking into account the "chaine" rule for the density of our form (see [4] and [15]) ∎

We are now in position to prove Theorem 1.4.

Proof of Theorem 1.4. It is enough to prove the result for $f \in L^2(\Omega)$, and by the linearity of the problem for $f \geq 0$. First of all we have the weak convergence of the sequence u^ε in $D_0[a,\Omega]$ to u from the definition of Γ-convergence. Then we can define $f^\varepsilon \in D^{-1}[a,\Omega]$ and $f^0 \in D^{-1}[a,\Omega]$ by the relations

$$\int_\Omega \alpha(u^\varepsilon, v)\, dm = \langle f - f^\varepsilon, v\rangle \qquad \text{for all } v \in D_0[a,\Omega]$$

and

$$\int_\Omega \alpha(u,v)\, dm = \langle f - f^0, v\rangle \qquad \text{for all } v \in D_0[a,\Omega].$$

By Proposition 2.17 $f^\varepsilon > 0$ and $f^0 \geq 0$; moreover, we have also that $w\text{-}\lim_{\varepsilon\to 0} f^\varepsilon = f^0$ in $D^{-1}[a;\Omega]$. Then (1.5) follows from Theorem 2.15 ∎

3. Proof of Theorem 1.5

First we give some preliminary results.

Lemma 3.1 (Convergence of integral terms). *Let us assume that $\phi \in D[a,\Omega]$ and that there exists a constant C such that the sequences ψ^ε and v^ε verify*

$$\int_\Omega (\alpha(\psi^\varepsilon,\psi^\varepsilon)(x))\,dm \le C < \infty. \tag{3.1}$$

$$\|v^\varepsilon\|_{L^\infty(\Omega,m)} \le C < \infty \qquad \textit{and} \qquad v^\varepsilon \to 0 \;\; \textit{a.e. with respect to } m. \tag{3.2}$$

Then

$$\int_\Omega \alpha(\psi^\varepsilon,\phi)v^\varepsilon\,dm \to 0. \tag{3.3}$$

Proof. As in the proof of Lemma 2.2, it is easy to see that

$$|\alpha(\psi^\varepsilon,\phi)|\,|v^\varepsilon| \le \alpha(\psi^\varepsilon,\psi^\varepsilon)^{\frac12}\alpha(\phi,\phi)^{\frac12}|v^\varepsilon|.$$

By integrating in Ω with respect to m and by using the standard Hölder inequality, we obtain

$$\begin{aligned}\left|\int_\Omega \alpha(\psi^\varepsilon,\phi)v^\varepsilon dm\right| &\le \int_\Omega |\alpha(\psi^\varepsilon,\phi)v^\varepsilon|\,dm \\ &\le \int_\Omega \alpha(\psi^\varepsilon,\psi^\varepsilon)^{\frac12}\alpha(\phi,\phi)^{\frac12}|v^\varepsilon|\,dm \\ &\le \left(\int_\Omega \alpha(\psi^\varepsilon,\psi^\varepsilon)\,dm\right)^{\frac12}\left(\int_\Omega \alpha(\phi,\phi)(v^\varepsilon)^2 dm\right)^{\frac12} \\ &\le C\left(\int_\Omega \alpha(\phi,\phi)(v^\varepsilon)^2 dm\right)^{\frac12}.\end{aligned}$$

Now, $\alpha(\phi,\phi)(v^\varepsilon)^2 \le C\alpha(\phi,\phi) \in L^1(\Omega,m)$ and $\alpha(\phi,\phi)(v^\varepsilon)^2 \to 0$ a.e. with respect to m, and we conclude by using the Lebesgue convergence theorem ∎

Lemma 3.2 (Convergence of integral terms). *Let us assume that*

$$\phi \in D[a,\Omega]\cap L^\infty(\Omega,m) \qquad \textit{and} \qquad v \in D[a,\Omega]\cap L^\infty(\Omega,m)$$

and that there exists a constant C such that the sequence $\psi^\varepsilon \in D_0[a,\Omega]$ verifies

$$\|\psi^\varepsilon\|_{L^\infty(\Omega,m)} \le C \qquad \textit{and} \qquad \psi^\varepsilon \xrightarrow{D_0[a,\Omega]} \psi.$$

Then

$$\int_\Omega \alpha(\psi^\varepsilon,\phi)v\,dm \to \int_\Omega \alpha(\psi,\phi)v\,dm. \tag{3.4}$$

Proof. From the compact embedding property (C) we have that $\psi^\varepsilon \to \psi$ into $L^2(\Omega,m)$. Let us recall that the space $D_0[a,\Omega]\cap L^\infty(\Omega,m)$ is an algebra and also an ideal in $D[a,\Omega] \cap L^\infty(\Omega,m)$. Then the sequence $\psi^\varepsilon v \in D_0[a,\Omega] \cap L^\infty(\Omega,m)$, it is uniformly bounded in $D_0[a,\Omega]$, and we obtain that $\psi^\varepsilon v \xrightarrow{D[a,\Omega]} \chi$. Again by the compact embedding property (C) we have that

$$\psi^\varepsilon v \to \chi \qquad \text{strongly in } L^2(\Omega,m). \tag{3.5}$$

Then $\chi = \psi v$ and

$$\psi^\varepsilon v \xrightarrow{D[a,\Omega]} \psi v. \tag{3.6}$$

We have

$$\int_\Omega \alpha(\psi^\varepsilon, \phi) v \, dm = \int_\Omega \alpha(\psi^\varepsilon v, \phi)\, dm - \int_\Omega \alpha(v,\phi)\psi^\varepsilon dm. \tag{3.7}$$

Because of (3.6) the first term in (3.7) is such that

$$\int_\Omega \alpha(\psi^\varepsilon v, \phi)\, dm \to \int_\Omega \alpha(\psi v, \phi)\, dm.$$

By applying Lemma 3.1 to the second term in (3.7) we obtain

$$\int_\Omega \alpha(\psi^\varepsilon, \phi) v \, dm \to \int_\Omega \alpha(\psi v, \phi)\, dm - \int_\Omega \alpha(v,\phi)\psi \, dm = \int_\Omega \alpha(\psi,\phi) v \, dm$$

and the assertion is proved ∎

Using Lemmas 3.1 and 3.2 it is easy to prove the following one.

Lemma 3.3 (Convergence of integral terms). *Let us assume that $\phi \in D[a,\Omega] \cap L^\infty(\Omega,m)$ and that*

(i) $\|v^\varepsilon\|_{L^\infty(\Omega,m)} \le C$ *for some constant* C

(ii) $v^\varepsilon \to v$ *a.e. with respect to* m

(iii) $v \in D[a,\Omega] \cap L^\infty(\Omega,m)$

and that the sequence $\psi^\varepsilon \in D_0[a,\Omega]$ *verifies*

$$\|\psi^\varepsilon\|_{L^\infty(\Omega,m)} \le C \qquad \text{and} \qquad \psi^\varepsilon \xrightarrow{D_0[a,\Omega]} \psi.$$

Then

$$\int_\Omega \alpha(\psi^\varepsilon, \phi) v^\varepsilon dm \to \int_\Omega \alpha(\psi,\phi) v \, dm. \tag{3.8}$$

Remark 3.1 ("Comparison principle"). Let u^ε, w^ε, u and w as in (1.4), (1.6), (1.7) and (1.8), and $f \in L^\infty(\Omega)$. Then using the properties of the strongly local regular Dirichlet form a we obtain the existence of a constant C independent on ε (but dependent on $\|f\|_{L^\infty(\Omega)}$) such that

$$|u^\varepsilon| \le C w^\varepsilon \qquad \text{and} \qquad |u| \le C w. \tag{3.9}$$

Lemma 3.4 (Convergence in the set $w \neq 0$). *Let u^ε, w^ε, u, w and f be as in Remark* 3.1 *and let the assumptions in Theorem* 1.5 *hold. Denote for any $\delta > 0$*

$$r_\delta^\varepsilon = u^\varepsilon - \frac{uw^\varepsilon}{\sup\{w,\delta\}} \tag{3.10}$$

and

$$\Omega_\delta = \{x \in \Omega : w(x) > \delta\}. \tag{3.11}$$

Then

$$\lim_{\varepsilon \to 0} \int_{\Omega_{2\delta}} \alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\, dm = 0.$$

Proof. The weak maximum principle for the strongly local regular Dirichlet form a, the positivity of the measures μ^ε and μ, and Proposition 2.16 imply the $L^\infty(\Omega)$ and $D_0[a,\Omega]$ uniform bounds

$$\begin{array}{ccc} \|u^\varepsilon\|_{L^\infty(\Omega,m)} \leq C & & \|w^\varepsilon\|_{L^\infty(\Omega,m)} \leq C \\ & \text{and} & \\ \|u^\varepsilon\|_{D_0[a,\Omega]} \leq C & & \|w^\varepsilon\|_{D_0[a,\Omega]} \leq C. \end{array} \tag{3.12}$$

Moreover, it is easy to remark that

$$u^\varepsilon \rightharpoonup u,\ w^\varepsilon \rightharpoonup w \quad \text{weakly in } D_0[a,\Omega] \tag{3.13}$$

$$u^\varepsilon \to u,\ w^\varepsilon \to w \quad \text{strongly in } L^2(\Omega,m) \tag{3.14}$$

$$u^\varepsilon \to u,\ w^\varepsilon \to w \quad \text{a.e. with respect to } m \text{ in } \Omega. \tag{3.15}$$

By using (3.12) and (3.15) we obtain the following properties on the convergence of the sequence r_δ^ε:

$$\left.\begin{array}{l} r_\delta^\varepsilon \in D_0[a,\Omega] \\ \lim\limits_{\varepsilon\to 0} r_\delta^\varepsilon = r_\delta \quad \text{a.e. with respect to } m \text{ in } \Omega \\ \|r_\delta^\varepsilon\|_{L^\infty(\Omega,m)} \leq C \ \text{ and } \ \|r_\delta^\varepsilon\|_{D_0[a,\Omega]} \leq C \\ r_\delta^\varepsilon \rightharpoonup r_\delta \quad \text{weakly in } D_0[a,\Omega] \end{array}\right\} \tag{3.16}$$

where C is a constant independent of ε and

$$r_\delta = u\left(1 - \frac{w}{\sup\{w,\delta\}}\right). \tag{3.17}$$

If $\Phi^{1,\delta}(t) = \frac{1}{\delta}\inf\{(t-\delta)^+, \delta\}$, let us define the function ϕ by $\phi(x) = \Phi^{1,\delta}(w(x))$. It has the following properties:

$$\phi \in D_0[a,\Omega] \cap L^\infty(\Omega), \quad \phi(x) \in [0,1], \quad \phi = \begin{cases} 1 & \text{in } \Omega_{2\delta} \\ \phi = 0 & \text{in } \Omega \setminus \Omega_\delta \end{cases}. \tag{3.18}$$

We have to prove that $\int_{\Omega_{2\delta}} \alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\, dm$ tends to zero. Using (3.18) and the positivity of $\alpha(\cdot,\cdot)$ and μ^ε we have

$$\int_{\Omega_{2\delta}} \alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\, dm \leq \int_\Omega \alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\phi\, dm + \int_\Omega (r_\delta^\varepsilon)^2 \phi\, d\mu^\varepsilon. \tag{3.19}$$

We shall prove that the right-hand side in (3.19), denoted by I in the following, tends to zero. Thanks to the definition of r_δ^ε, the bilinearity of the form and the Leibnitz rule

$$\begin{aligned}
I &= \int_\Omega \alpha(u^\varepsilon, r_\delta^\varepsilon)\phi\, dm - \int_\Omega \alpha\left(\frac{uw^\varepsilon}{\sup\{w,\delta\}}, r_\delta^\varepsilon\right)\phi\, dm \\
&\quad + \int_\Omega u^\varepsilon r_\delta^\varepsilon \phi\, d\mu^\varepsilon - \int_\Omega \left(\frac{uw^\varepsilon}{\sup\{w,\delta\}}\right) r_\delta^\varepsilon \phi\, d\mu^\varepsilon \\
&= \int_\Omega \alpha(u^\varepsilon, r_\delta^\varepsilon)\phi\, dm - \int_\Omega \alpha(w^\varepsilon, r_\delta^\varepsilon)\frac{u}{\sup\{w,\delta\}}\phi\, dm \\
&\quad - \int_\Omega \alpha\left(\frac{u}{\sup\{w,\delta\}}, r_\delta^\varepsilon\right) w^\varepsilon \phi\, dm + \int_\Omega u^\varepsilon r_\delta^\varepsilon \phi\, d\mu^\varepsilon - \int_\Omega \frac{uw^\varepsilon}{\sup\{w,\delta\}} r_\delta^\varepsilon \phi\, d\mu^\varepsilon \\
&= \left[\int_\Omega \alpha(u^\varepsilon, r_\delta^\varepsilon\phi)\, dm + \int_\Omega u^\varepsilon r_\delta^\varepsilon\phi\, d\mu^\varepsilon\right] \\
&\quad - \left[\int_\Omega \alpha\left(w^\varepsilon, r_\delta^\varepsilon \frac{u}{\sup\{w,\delta\}}\phi\right) dm + \int_\Omega w^\varepsilon \frac{u}{\sup\{w,\delta\}} r_\delta^\varepsilon \phi\, d\mu^\varepsilon\right] \\
&\quad - \int_\Omega \alpha(u^\varepsilon, \phi) r_\delta^\varepsilon\, dm + \int_\Omega \alpha\left(w^\varepsilon, \frac{u\phi}{\sup\{w,\delta\}}\right) r_\delta^\varepsilon\, dm \\
&\quad - \int_\Omega \alpha\left(\frac{u}{\sup\{w,\delta\}}, r_\delta^\varepsilon\right) w^\varepsilon \phi\, dm.
\end{aligned} \tag{3.20}$$

We have $r_\delta^\varepsilon\phi$, $r_\delta^\varepsilon\phi\frac{u}{\sup\{w,\delta\}} \in D_0[a,\Omega] \cap L^\infty(\Omega, m)$ and we can use the two functions as test functions in (1.4) and (1.6), respectively, to get

$$\begin{aligned}
I &= \int_\Omega f r_\delta^\varepsilon \phi\, dm + \int_\Omega r_\delta^\varepsilon \phi \frac{u}{\sup\{w,\delta\}}\, dm - \int_\Omega \alpha(u^\varepsilon, \phi) r_\delta^\varepsilon\, dm \\
&\quad + \int_\Omega \alpha\left(w^\varepsilon, \frac{u\phi}{\sup\{w,\delta\}}\right) r_\delta^\varepsilon\, dm - \int_\Omega \alpha\left(\frac{u}{\sup\{w,\delta\}}, r_\delta^\varepsilon\right) w^\varepsilon \phi\, dm.
\end{aligned} \tag{3.21}$$

Let us consider the first two terms. From (3.16) we obtain that $r_\delta^\varepsilon \to r_\delta$ strongly in $L^p(\Omega, m)$ for any $p \in [1,\infty)$ and then

$$\begin{aligned}
\lim_{\varepsilon\to 0} &\left[\int_\Omega f r_\delta^\varepsilon \phi\, dm + \int_\Omega r_\delta^\varepsilon \phi \frac{u}{\sup\{w,\delta\}}\, dm\right] \\
&= \int_\Omega f r_\delta \phi\, dm + \int_\Omega r_\delta \phi \frac{u}{\sup\{w,\delta\}}\, dm \\
&= \int_{\Omega_\delta} f r_\delta \phi\, dm + \int_{\Omega\setminus\Omega_\delta} f r_\delta \phi\, dm \\
&\quad + \int_{\Omega_\delta} r_\delta \phi \frac{u}{\sup\{w,\delta\}}\, dm + \int_{\Omega\setminus\Omega_\delta} r_\delta \phi \frac{u}{\sup\{w,\delta\}}\, dm.
\end{aligned} \tag{3.22}$$

From (3.18) $\phi = 0$ in $\Omega \setminus \Omega_\delta$. Otherwise, in Ω_δ, $w(x) > \delta$ and $r_\delta = u(1 - \frac{w}{\sup\{w,\delta\}}) = 0$, which implies that the right-hand side in (3.22) is zero.

Let us consider the third term in (3.21). Applying Lemma 3.3, with $\psi^\varepsilon = u^\varepsilon$, $\phi = \phi$ and $v^\varepsilon = r_\delta^\varepsilon$, thanks to (3.12) - (3.15) u^ε verifies the assumptions of the lemma.

Analogously, $\phi \in D^p[a,\Omega] \cap L^\infty(\Omega,m)$ and r_δ^ε verify the assumptions of the lemma thanks to (3.16). Then we obtain

$$\int_\Omega \alpha(u^\varepsilon,\phi) r_\delta^\varepsilon \, dm \to \int_\Omega \alpha(u,\phi) r_\delta \, dm = 0.$$

The last equality has been obtained using the Leibnitz rule. In fact, we have

$$\alpha(u,\phi)r_\delta = \alpha(u,\phi r_\delta) - \alpha(u,r_\delta)\phi = \alpha(u,r_\delta)\phi$$

a.e. in Ω where we take into account that $\phi r_\delta = 0$; the result follows from the observation that the term in the left-hand side is zero where ϕ or r_δ are zero.

Analogously, for the fourth and fifth terms in (3.21) we have

$$\int_\Omega \alpha\left(w^\varepsilon, \frac{u\phi}{\sup\{w,\delta\}}\right) r_\delta^\varepsilon \, dm - \int_\Omega \alpha\left(\frac{u}{\sup\{w,\delta\}}, r_\delta^\varepsilon\right) w^\varepsilon \phi \, dm \to$$
$$\int_\Omega \alpha\left(w, \frac{u\phi}{\sup\{w,\delta\}}\right) r_\delta \, dm - \int_\Omega \alpha\left(\frac{u}{\sup\{w,\delta\}}, r_\delta\right) w\phi \, dm = 0$$

and the assertion is proved ∎

Lemma 3.5 (Convergence in the set $w = 0$). *Let u^ε and w be as Section 1. Then*

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \int_{w\le\delta} \alpha(u^\varepsilon,u^\varepsilon)\, dm = 0. \tag{3.23}$$

Proof. Let us consider the function

$$\Phi^{2,\delta}(t) = 1 - \Phi^{1,\delta}(t) = 1 - \frac{1}{\delta}\inf\{(t-\delta)^+,\delta\}$$

and denote

$$\psi^\delta(x) = \Phi^{2,\delta}(w(x)). \tag{3.24}$$

The function ψ^δ has the following properties (thanks to Remark 1.2):

$$\psi^\delta \in D[a,\Omega] \cap L^\infty(\Omega,m), \quad \psi^\delta(x) \in [0,1], \quad \psi^\delta = \begin{cases} 0 & \text{in } \Omega_{2\delta} \\ 1 & \text{in } \Omega \setminus \Omega_\delta. \end{cases} \tag{3.25}$$

Moreover, we observe that $1 - \psi^\delta \in D_0[a,\Omega]$. Then, as in (3.19), from (3.25) we obtain

$$\begin{aligned}
&\int_{w\le\delta} \alpha(u^\varepsilon,u^\varepsilon)\, dm \\
&\le \int_\Omega \alpha(u^\varepsilon,u^\varepsilon)\psi^\delta dm \\
&\le \int_\Omega \alpha(u^\varepsilon,u^\varepsilon)\psi^\delta dm + \int_\Omega (u^\varepsilon)^2\psi^\delta d\mu^\varepsilon \\
&= \int_\Omega \alpha(u^\varepsilon,u^\varepsilon\psi^\delta)\, dm + \int_\Omega u^\varepsilon(u^\varepsilon\psi^\delta)\, d\mu^\varepsilon - \int_\Omega \alpha(u^\varepsilon,\psi^\delta)u^\varepsilon dm \\
&= \int_\Omega f(u^\varepsilon\psi^\delta)\, dm - \int_\Omega \alpha(u^\varepsilon,\psi^\delta)u^\varepsilon dm.
\end{aligned} \tag{3.26}$$

Easily, because (3.12) and (3.15),

$$\int_\Omega f(u^\varepsilon \psi^\delta)\,dm \to \int_\Omega (u\psi^\delta)\,dm. \tag{3.27}$$

Let us consider the second term in (3.26) and apply Lemma 3.3 where $\psi^\varepsilon = u^\varepsilon$, $\phi = \psi^\delta$ and $v^\varepsilon = u^\varepsilon$. Thanks to (3.12) - (3.15) u^ε verifies the assumptions of the lemma and $\psi^\delta \in D^p[a,\Omega] \cap L^\infty(\Omega, m)$. Then

$$\int_\Omega \alpha(u^\varepsilon, \psi^\delta) u^\varepsilon dm \to \int_\Omega \alpha(u, \psi^\delta) u\,dm. \tag{3.28}$$

By using (3.27) and (3.28) in (3.26) we obtain that

$$\limsup_{\varepsilon \to 0} \int_{w<\delta} \alpha(u^\varepsilon, u^\varepsilon)\,dm \le \int_\Omega f(u\psi^\delta)\,dm - \int_\Omega \alpha(u, \psi^\delta) u\,dm. \tag{3.29}$$

Now we have to let δ converge to zero in (3.29). First let us consider the second term. Thanks to the Hölder inequality,

$$\begin{aligned}\int_\Omega \alpha(u,\psi^\delta)u\,dm &\le \left(\int_\Omega \alpha(\psi^\delta,\psi^\delta)u^2 dm\right)^{\frac12}\left(\int_\Omega \alpha(u,u)\,dm\right)^{\frac12}\\ &\le C\left(\int_\Omega \alpha(\psi^\delta,\psi^\delta)u^2 dm\right)^{\frac12}.\end{aligned}$$

Using the strong locality of the form a, the truncature rule and Remark 3.1 on the comparison principle, we obtain

$$\begin{aligned}&\int_\Omega \alpha(\psi^\delta,\psi^\delta)u^2 dm\\ &= \int_\Omega \chi_{\psi^\delta \in (0,1)} \alpha(\psi^\delta,\psi^\delta)u^2 dm \le \frac{1}{\delta^2}\int_\Omega \chi_{w\in(\delta,2\delta)}\alpha(w,w)u^2 dm\\ &\le C\frac{1}{\delta^2}\int_\Omega \chi_{w\in(\delta,2\delta)}\alpha(w,w)w^2 dm \le C\frac{1}{\delta^2}\int_\Omega \chi_{w\in(\delta,2\delta)}\alpha(w,w)w^2 dm\\ &\le C\frac{1}{\delta^2}\delta^2\int_\Omega \chi_{w\in(\delta,2\delta)}\alpha(w,w)\,dm \le C\int_\Omega \chi_{w\in(\delta,2\delta)}\alpha(w,w)\,dm\end{aligned} \tag{3.30}$$

where χ_A is the characteristic function of the set A. But $\chi_{w\in(\delta,2\delta)} \to 0$ a.e. in Ω, as $\delta \to 0$. Moreover, $\chi_{w\in(\delta,2\delta)}\alpha(w,w) \le \alpha(w,w)$ and by applying the Lebesgue convergence theorem

$$\int_\Omega \chi_{w\in(\delta,2\delta)}\alpha(w,w)\,dm \to 0. \tag{3.31}$$

Let us consider the first term in the right-hand side of (3.29). Using (3.25) and Remark 3.1 on the comparison principle we get

$$\begin{aligned}\left|\int_\Omega f(u\psi^\delta)\,dm\right| &\le \int_\Omega |fu\psi^\delta|\,dm \le C\int_{w\le 2\delta}|u|\,dm\\ &\to C\int_{w=0}|u|\,dm \le \int_{w=0}|w|\,dm = 0.\end{aligned} \tag{3.32}$$

Then the convergence of (3.29) to zero as $\delta \to 0$ follows from (3.31) and (3.32) ∎

We are now in position to prove Theorem 1.5.

Proof of Theorem 1.5. Thanks to (3.16) and the Poincaré inequality we have only to prove that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_\Omega \alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\,dm = 0.$$

From Lemma 3.4 we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_{2\delta}} \alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\,dm = 0$$

where we recall that $\Omega_\delta = \{x\in\Omega : w(x) > \delta\}$. Then we have to prove that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_{w\leq 2\delta} \alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\,dm = 0. \tag{3.35}$$

From the definition of r_δ^ε and the bilinearity of the form,

$$\int_{w\leq 2\delta}\alpha(r_\delta^\varepsilon, r_\delta^\varepsilon)\,dm \leq 2\left(\int_{w\leq 2\delta}\alpha(u^\varepsilon,u^\varepsilon)\,dm + \int_{w\leq 2\delta}\alpha\left(\frac{uw^\varepsilon}{\sup\{w,\delta\}},\frac{uw^\varepsilon}{\sup\{w,\delta\}}\right)dm\right).$$

From Lemma 3.5 we have

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_{w\leq 2\delta}\alpha(u^\varepsilon,u^\varepsilon)\,dm = 0.$$

Then, to prove (3.35), we have to prove that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_{w\leq 2\delta}\alpha\left(\frac{uw^\varepsilon}{\sup\{w,\delta\}},\frac{uw^\varepsilon}{\sup\{w,\delta\}}\right)dm = 0. \tag{3.36}$$

But

$$\begin{aligned}\int_{w\leq 2\delta}&\alpha\left(\frac{uw^\varepsilon}{\sup\{w,\delta\}},\frac{uw^\varepsilon}{\sup\{w,\delta\}}\right)dm\\ &= \int_{w\leq 2\delta}\left(\frac{u}{\sup\{w,\delta\}}\right)^2\alpha(w^\varepsilon,w^\varepsilon)\,dm\\ &\quad + 2\int_{w\leq 2\delta}\frac{uw^\varepsilon}{\sup\{w,\delta\}}\alpha\left(\frac{u}{\sup\{w,\delta\}},w^\varepsilon\right)dm\\ &\quad + \int_{w\leq 2\delta}(w^\varepsilon)^2\alpha\left(\frac{u}{\sup\{w,\delta\}},\frac{u}{\sup\{w,\delta\}}\right)dm.\end{aligned} \tag{3.37}$$

Let us recall that, thanks to the bilinearity and the Leibnitz rule, we have

$$\alpha(u,u) = \alpha\left(\frac{u}{z}z,\frac{u}{z}z\right) = \alpha\left(\frac{u}{z},\frac{u}{z}\right)z^2 + \alpha(z,z)\left(\frac{u}{z}\right)^2 + 2\alpha\left(\frac{u}{z},z\right).$$

Denoting by R the right-hand side in (3.37),

$$\begin{aligned} R = &\int_{w\leq 2\delta} \left(\frac{u}{\sup\{w,\delta\}}\right)^2 \alpha(w^\varepsilon, w^\varepsilon)\, dm \\ &+ \int_{w\leq 2\delta} 2\frac{uw^\varepsilon}{\sup\{w,\delta\}}\alpha\left(\frac{u}{\sup\{w,\delta\}}, w^\varepsilon\right) dm \\ &+ \int_{w\leq 2\delta} \frac{(w^\varepsilon)^2}{(\sup\{w,\delta\})^2}\alpha(u,u)\, dm \\ &- \int_{w\leq 2\delta} \frac{(w^\varepsilon)^2}{(\sup\{w,\delta\})^2}\left(\frac{u}{\sup\{w,\delta\}}\right)^2 \alpha(\sup\{w,\delta\},\sup\{w,\delta\})\, dm \\ &- \int_{w\leq 2\delta} 2\frac{(w^\varepsilon)^2}{(\sup\{w,\delta\})^2}\, u\, \alpha\left(\frac{u}{\sup\{w,\delta\}}, \sup\{w,\delta\}\right) dm \\ = &\, I + II + III + IV + V. \end{aligned} \tag{3.38}$$

Let us consider separately the five integrals I, II, III, IV and V.

The integral I: To prove the convergence of the first term we use Remark 3.1 on the comparison principle:

$$I = \int_{w\leq 2\delta} \left(\frac{u}{\sup\{w,\delta\}}\right)^2 \alpha(w^\varepsilon, w^\varepsilon)\, dm \leq C\int_{w\leq 2\delta} \alpha(w^\varepsilon, w^\varepsilon)\, dm.$$

By Lemma 3.5 (applied to w^ε where $f \equiv 1$; see (3.5)),

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \int_{w\leq 2\delta} \alpha(w^\varepsilon, w^\varepsilon)\, dm = 0.$$

Then

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} I = 0. \tag{3.39}$$

The integral $II + V$: We have

$$\begin{aligned} II + V = \int_{w\leq 2\delta} &\frac{uw^\varepsilon}{\sup\{w,\delta\}}\alpha\left(\frac{u}{\sup\{w,\delta\}}, w^\varepsilon\right) \\ &- \frac{(w^\varepsilon)^2}{(\sup\{w,\delta\})^2}\, u\, \alpha\left(\frac{u}{\sup\{w,\delta\}}, \sup\{w,\delta\}\right) dm. \end{aligned}$$

First we let $\varepsilon \to 0$. Then the lim sup is obtained by using arguments analogous to those in the proof of Lemma 3.4. More precisely, by Lemma 3.3, then

$$\begin{aligned} \limsup_{\varepsilon\to 0} &\int_{w\leq 2\delta} \frac{uw^\varepsilon}{\sup\{w,\delta\}}\, \alpha\left(\frac{u}{\sup\{w,\delta\}}, w^\varepsilon\right) dm \\ &- \int_{w\leq 2\delta} \frac{(w^\varepsilon)^2}{(\sup\{w,\delta\})^2}\, u\, \alpha\left(\frac{u}{\sup\{w,\delta\}}, \sup\{w,\delta\}\right) dm \\ = &\int_{w\leq 2\delta} \frac{uw}{\sup\{w,\delta\}}\, \alpha\left(\frac{u}{\sup\{w,\delta\}}, w\right) \\ &- \frac{(w)^2}{(\sup\{w,\delta\})^2}\, u\, \alpha\left(\frac{u}{\sup\{w,\delta\}}, \sup\{w,\delta\}\right) dm. \end{aligned} \tag{3.40}$$

Now letting $\delta \to 0$ in (3.40) we get

$$
\begin{aligned}
\lim_{\delta\to 0} \int_{w\le 2\delta} & \frac{uw}{\sup\{w,\delta\}}\,\alpha\left(\frac{u}{\sup\{w,\delta\}}, w\right) \\
& - \frac{(w)^2}{(\sup\{w,\delta\})^2}\,u\,\alpha\left(\frac{u}{\sup\{w,\delta\}}, \sup\{w,\delta\}\right) dm \\
&\le \lim_{\delta\to 0}\left(\int_{\delta\le w\le 2\delta}\left(u\,\alpha\left(\frac{u}{\sup\{w,\delta\}}, w\right)\right.\right. \\
&\quad \left.\left. -u\,\alpha\left(\frac{u}{w}, w\right)\right) dm + \int_{w<\delta} \frac{uw}{\delta}\,\alpha\left(\frac{u}{\delta}, w\right) dm\right) \\
&= \lim_{\delta\to 0}\int_{w<\delta} \frac{uw}{\delta}\,\alpha\left(\frac{u}{\delta}, w\right) dm \\
&= \lim_{\delta\to 0}\int_{w<\delta} \frac{u}{\delta}\frac{w}{\delta}\,\alpha(u,w)\,dm
\end{aligned}
$$

where we have used the strong locality of the form. Moreover, from Remark 3.1, there exists a constant C such that $|u| \le Cw \le C\delta$ in the set $w < \delta$. Then

$$
\begin{aligned}
\lim_{\delta\to 0}\left|\int_{w<\delta} \frac{u}{\delta}\frac{w}{\delta}\alpha(u,w)\,dm\right| &\le C \lim_{\delta\to 0}\int_{w<\delta} |\alpha(u,w)|\,dm \\
&= C\int_{w=0} |\alpha(u,w)|\,dm = 0.
\end{aligned}
$$

For the last relation we can use the Hölder inequality and the equality $\int_{w=0} |\alpha(w,w)|\,dm = 0$ (see Remark 1.2).

The integral III: As in the preceding term let first $\varepsilon \to 0$ and apply the Lebesgue convergence theorem to get

$$
\limsup_{\varepsilon\to 0}\int_{w\le 2\delta} \frac{(w^\varepsilon)^2}{(\sup\{w,\delta\})^2}\,\alpha(u,u)\,dm = \int_{w\le 2\delta} \frac{(w)^2}{(\sup\{w,\delta\})^2}\,\alpha(u,u)\,dm. \tag{3.41}
$$

Now we let $\delta \to 0$ in (3.41) to get

$$
\begin{aligned}
0 &\le \lim_{\delta\to 0}\int_{w\le 2\delta} \frac{(w)^2}{(\sup\{w,\delta\})^2}\,\alpha(u,u)\,dm \\
&\le \lim_{\delta\to 0}\int_{w\le 2\delta} \alpha(u,u)\,dm = \int_{w=0} \alpha(u,u)\,dm \le \int_{u=0} \alpha(u,u)\,dm \\
&= 0
\end{aligned}
$$

(see Remark 1.2 for the last equality). Indeed, thanks to Remark 3.1, on the comparison principle $u = 0$ on the set $w = 0$.

The integral IV: As in the preceding term let first $\varepsilon \to 0$ and apply the remark to Lemma 3.3 to get

$$
\begin{aligned}
\limsup_{\varepsilon\to 0}\int_{w\le 2\delta} & \left(\frac{w^\varepsilon}{\sup\{w,\delta\}}\right)^2 \left(\frac{u}{\sup\{w,\delta\}}\right)^2 \alpha\big(\sup\{w,\delta\}, \sup\{w,\delta\}\big)\,dm \\
&= \int_{w\le 2\delta} \left(\frac{w}{\sup\{w,\delta\}}\,\frac{u}{\sup\{w,\delta\}}\right)^2 \alpha\big(\sup\{w,\delta\}, \sup\{w,\delta\}\big)\,dm.
\end{aligned} \tag{3.42}
$$

Now we let $\delta \to 0$ in (3.42). Using as in the preceding terms Remark 3.1 on the comparison principle and the troncature rule, we get

$$0 \le \lim_{\delta\to 0} \int_{w\le 2\delta} \left(\frac{w}{\sup\{w,\delta\}} \frac{u}{\sup\{w,\delta\}} \right)^2 \alpha(\sup\{w,\delta\}, \sup\{w,\delta\})\, dm$$
$$\le C \lim_{\delta\to 0} \int_{w\le 2\delta} \alpha(w,w)\, dm = C \int_{w=0} \alpha(w,w)\, dm = 0$$

where the last equality follows from Remark 1.2.

Thus we have proved that the $\lim_{\delta\to 0} \limsup_{\epsilon\to 0}(I + II + III + IV + V) = 0$ ∎

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