Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 16 (1997), No. 2, 363–375

Remarks on Analytic Solutions of Leray-Volevich-Coupled Systems

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Abstract. A Banach-scales approach is used for studying analytic solutions of systems of linear partial differential equations satisfying the Leray-Volevich conditions. Results on the Cauchy problem lead to a continuation result which is of the same type as the well known result of Widder [10] on analytic solutions of the heat equation.

Keywords: *Leray-Volevich systems, analytic solutions, continuation, Banach scales*

AMS subject classification: Primary 35 B 60, secondary 35 C 10, 35 G 10

1. Introduction

Recently M. Reissig [4] discussed the Cauchy problem for systems of abstract evolution equations satisfying Leray-Volevich conditions within the framework of Banach scales (in this way generalizing abstract forms of the Cauchy-Kowalewski theorem, see [2, 3, 6]). A simple example of such a system is given by

$$D_t u_1 = D_x^2 u_1 + \alpha D_x^3 u_2 \tag{1.1}$$

$$D_t u_2 = \beta D_x u_1 + D_x^2 u_2 \tag{1.2}$$

if an appropriate scale of Banach spaces of analytic functions is used. (We use the notations $D_t = \frac{\partial}{\partial t}$ and $D_x = \frac{\partial}{\partial x}$.) With $m_{11} = 2$, $m_{12} = 3$, $m_{21} = 1$, $m_{22} = 2$, $q_1 = \frac{3}{2}$, $q_2 = 1$ we have $\frac{1}{2} m_{ij} \leq q_i - q_j + 1$ for $i, j = 1, 2$. These inequalities are Leray-Volevich conditions since the numbers $\frac{m_{ij}}{2}$ have to be considered as orders of the differential operators $D_x^{m_{ij}}$ in the appropriate Banach scale. These conditions allow to get results on the existence of a unique solution for systems which are not of the Cauchy-Kowalewski type.

Of course, the terms $\alpha D_x^3 u_2$ and $\beta D_x u_1$ can be considered as perturbation terms which are added to the (non-coupled) system

$$\left.\begin{aligned} D_t u_1 &= D_x^2 u_1 \\ D_t u_2 &= D_x^2 u_2. \end{aligned}\right\} \tag{1.3}$$

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ISSN 0232-2064 / \$ 2.50

If analytic solutions of the system (1.3) are studied, a result of D. V. Widder [10] on the continuation of analytic solutions of the (scalar) heat equation $D_t u = D_x^2 u$ is of interest. He proved that a solution

$$u(t,x) = \sum_{m,n} a_{m,n} \frac{t^m}{m!} \frac{x^n}{n!} \qquad \text{for } |t| < \sigma \text{ and } |x| < \rho \tag{1.4}$$

of the heat equation can be continued into the strip $|t| < \sigma$, $x \in \mathbb{R}$. This follows from a representation of u in the form

$$u(t,x) = \sum_{n=0}^{\infty} a_{0,n} \frac{1}{n!} h_n(t,x) \tag{1.5}$$

(h_n are heat polynomials, see also Rosenbloom and Widder [5]) and the fact that the series (1.5) is convergent in the strip $|t| < \sigma$, $x \in \mathbb{R}$. (There is no simple generalization of this continuation result to analytic solutions of the higher-dimensional heat equation, see, e.g., [8].)

It is easy to see that such a continuation result cannot hold for arbitrary analytic solutions of the system (1.1), (1.2) without any restrictions on the perturbation terms. The functions

$$\left.\begin{aligned} u_1(t,x) &= -D_x u_2(t,x) \\ u_2(t,x) &= (1-x^2)^{-1} \end{aligned}\right\} \qquad (t \in \mathbb{R},\ x \in (-1,1))$$

give a solution of the system (1.1), (1.2) if $\alpha = \beta = 1$. Of course, a very similar example can be given for the simpler system $D_t u = A D_x^2 u$, where A is a real (2×2)-matrix with $\operatorname{rank} A = 1$.

Results on the continuation of analytic solutions of an equation $P(D_t, D_x)u = 0$ which are based on expansions in terms of polynomial solutions were proved in [9] for two classes of linear partial differential equations. In [9] the structure of the argumentation was the same as it was used by Widder [10], but the details were quite different. The results of Rosenbloom and Widder on the convergence of the series (1.5) were based on sharp estimates for the heat polynomials h_n, in [9] a Banach-scales approach was used which is appropriate for the discussion of the Cauchy problem in analytic classes too. In the following we will use this approach for the discussion of systems which are analogous to the system (1.1), (1.2). For the Cauchy problem we shall get a representation of solutions which immediately leads to an expansion in terms of polynomials. This result is more specific than the existence result which follows from the discussions of Reissig [4]. In addition, the approach is suitable for a discussion of the continuation problem.

The investigations are based on a reduction to a system $D_t v = ALv$, where A is a real matrix and the scalar equation $D_t w = Lw$ has the property that analytic solutions can be continued into a strip. In the Sections 2 and 3 we shall not study the most general situation since the discussion of a system which is similar to (1.1), (1.2) already shows the main features of the approach. In the last section, generalizations are sketched.

2. On the Cauchy problem in analytic classes

Let $\ell_{r,s}$ $(r \in \{2,3,\ldots\},\ s>0)$ be the linear space of (complex) sequences $(a_k)_{k\in\mathbb{N}_0}$ ($\mathbb{N}_0 = \mathbb{N}\cup\{0\}$) with

$$\|(a_k)\|_{r,s} = \sum_{k=0}^{\infty} |a_k|(k!)^{-\frac{1}{r}} s^{\frac{k+1}{r}} < \infty.$$

An injective map $j : \ell_{r,s} \to C^\infty(R)$ can be defined by

$$j((a_k))(x) = \sum_{k=0}^{\infty} a_k \frac{1}{k!} x^k \qquad \text{for } x \in \mathbb{R}$$

and the space $X_{r,s} = j(\ell_{r,s})$ becomes a Banach space if the norm

$$\|f\|_{r,s} = \|j^{-1}(f)\|_{r,s}$$

is used. If $f \in X_{r,s}$ and $m \in \mathbb{N}$, then $f^{(m)} \in X_{r,\sigma}$ for $\sigma \in (0,s)$ and

$$\|f^{(m)}\|_{r,\sigma} \le \left(\frac{m}{e}\right)^{\frac{m}{r}} (s-\sigma)^{-\frac{m}{r}} \|f\|_{r,s} \tag{2.1}$$

(see [7]). This fact implies that the system

$$D_t u_1 = D_x^r u_1 + \alpha D_x^{r+1} u_2 \tag{2.2}$$

$$D_t u_2 = \beta D_x^{r-1} u_1 + D_x^r u_2 \tag{2.3}$$

where $r \in \{2,3,\ldots\}$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \neq 0$, considered in the scale $(X_{r,s} \times X_{r,s})_{s\in(0,s_0]}$ is a special case of the systems which were studied by Reissig [4]. If

$$\left.\begin{aligned} F_1(u_1,u_2) &:= D_x^r u_1 + \alpha D_x^{r+1} u_2 \\ F_2(u_1,u_2) &:= \beta D_x^{r-1} u_1 + D_x^r u_2 \end{aligned}\right\}$$

then according to (2.1) we have for $u_1,\ u_2,\ v_1,\ v_2 \in X_{r,s}$ $(0<\sigma<s<s_0)$

$$\left\|F_j(u_1,u_2) - F_j(v_1,v_2)\right\|_{r,\sigma} \le C_{j,1}\frac{\|u_1-v_1\|_{r,s}}{(s-\sigma)^{p_{j,1}}} + C_{j,2}\frac{\|u_2-v_2\|_{r,s}}{(s-\sigma)^{p_{j,2}}}$$

with $p_{1,1}=1,\ p_{1,2}=1+\frac{1}{r},\ p_{2,1}=1-\frac{1}{r},\ p_{2,2}=1$ and $C_{j,1}>0,\ C_{j,2}>0$ independent of s and σ. Therefore, with $q_1 = 1+\frac{1}{r}$ and $q_2 = 1$ the Leray-Volevich conditions

$$p_{i,j} \le q_i - q_j + 1 \qquad (j=1,2)$$

are satisfied. On the basis of the results of Reissig [4] it can be expected that the system (2.2), (2.3) combined with the initial conditions

$$\left.\begin{aligned} u_1(0) &= \varphi \\ u_2(0) &= \psi \end{aligned}\right\} \tag{2.4}$$

has a unique solution in $X_{r,\sigma} \times X_{r,\sigma}$ for $t \in [0, \text{const}\cdot(s-\sigma))$ if $\varphi,\ \psi \in X_{r,s}$ $(s>\sigma)$. But for the system (2.2), (2.3) it is possible to replace the assumption $\varphi,\ \psi \in X_{r,s}$ by another one which is largely adapted to the perturbation terms $\alpha D_x^{r+1} u_2$ and $\beta D_x^{r-1} u_1$. This will be a consequence of results of [9]. These results show that the sequence $(a_{0,n})_n$ is an element of a space $\ell_{r,s}$ if a certain recursion formula holds for the coefficients $a_{m,n}$ of the function (1.4).

Lemma 2.1. *If $\alpha > 0$, $\beta > 0$ and*

$$u_j(t,x) = \sum_{k,m} a_{j,k,m} \frac{t^k}{k!} \frac{x^m}{m!} \qquad (j = 1,2)$$

is a solution of the system (2.2), (2.3) *for $|x| < x_0$, $|t| < t_0$, then*

$$\Big(a_{1,0,m} + \sqrt{\tfrac{\alpha}{\beta}}\, a_{2,0,m+1}\Big)_m \in \ell_{r,r(1+\sqrt{\alpha\beta})(t_0-\varepsilon)} \tag{2.5}$$

for $\varepsilon \in (0,t_0)$. If $\alpha > 0$, $\beta > 0$ and $\alpha\beta \neq 1$, then in addition

$$\Big(a_{1,0,m} - \sqrt{\tfrac{\alpha}{\beta}}\, a_{2,0,m+1}\Big)_m \in \ell_{r,r|1-\sqrt{\alpha\beta}|(t_0-\varepsilon)} \tag{2.6}$$

for $\varepsilon \in (0,t_0)$.

Proof. A short computation shows that the function

$$v(t,x) = u_1(t,x) + \sqrt{\tfrac{\alpha}{\beta}}\, D_x u_2(t,x) \qquad (|x| < x_0,\ |t| < t_0) \tag{2.7}$$

is a solution of

$$D_t v(t,x) = (1 + \sqrt{\alpha\beta})\, D_x^r v(t,x). \tag{2.8}$$

Since

$$v(t,x) = \sum_{k,m} \Big(a_{1,k,m} + \sqrt{\tfrac{\alpha}{\beta}}\, a_{2,k,m+1}\Big) \frac{t^k}{k!} \frac{x^m}{m!} \qquad (|x| < x_0,\ |t| < t_0),$$

results of Lemmas 1 and 2 of [9] show that (2.5) holds. The function

$$w(t,x) = u_1(t,x) - \sqrt{\tfrac{\alpha}{\beta}}\, D_x u_2(t,x) \qquad (|x| < x_0,\ |t| < t_0) \tag{2.9}$$

is a solution of

$$D_t w(t,x) = (1 - \sqrt{\alpha\beta})\, D_x^r w(t,x). \tag{2.10}$$

If $1 - \sqrt{\alpha\beta} > 0$, the results of [9] again show that (2.6) holds. If $1 - \sqrt{\alpha\beta} < 0$, we consider the function

$$\tilde{w}(t,x) = w(-t,x) \qquad (|x| < x_0,\ |t| < t_0).$$

Since $D_t\tilde{w}(t,x) = |1 - \sqrt{\alpha\beta}\,|\, D_x^r \tilde{w}(t,x)$, (2.6) follows again ∎

Since $\ell_{r,r(1+\sqrt{\alpha\beta})s} \subset \ell_{r,r|1-\sqrt{\alpha\beta}|s}$, Lemma 2.1 has the consequence that an assumption of the type $\varphi, \psi \in X_{r,r|1-\sqrt{\alpha\beta}|s}$ is adequate if the Cauchy problem (2.2) - (2.4) is studied in analytic classes and $\alpha > 0$, $\beta > 0$, $\alpha\beta \neq 1$.

The estimate (2.1) shows that a bounded linear operator $T(t) : X_{r,s} \to X_{r,\sigma}$ (where $0 < \sigma < s$) can be defined by

$$T(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} D_x^{rm}$$

for $|t| < \frac{1}{r}(s-\sigma)$. (Of course, $D_x^0 f = f \in X_{r,\sigma}$ for $f \in X_{r,s}$.) It follows that the function $u(t) = T(t)u_0$ ($|t| < s-\sigma$) is a solution of $D_t u = D_x^r u$ if $u_0 \in X_{r,rs}$.

A bounded linear operator $J : X_{r,s} \to X_{r,s}$ can be defined by

$$(Jf)(x) = \int_0^x f(\xi)\,d\xi \qquad \text{for } x \in \mathbb{R} \text{ and } f \in X_{r,s}. \tag{2.11}$$

This follows from the estimate

$$\|Jf\|_{r,s} = \sum_{k=0}^{\infty} |a_k|((k+1)!)^{-\frac{1}{r}} s^{\frac{k+2}{r}} \le s^{\frac{1}{r}} \|f\|_{r,s}.$$

(We have used that $f = j((a_k))$ for some $(a_k) \in \ell_{r,s}$.)

Theorem 2.1. *Assume* $\alpha > 0$, $\beta > 0$, $\alpha\beta \neq 1$, $\tilde{s} > s > \sigma > 0$, $\varphi \in X_{r,r|1-\sqrt{\alpha\beta}|s}$, $\psi \in X_{r,r|1-\sqrt{\alpha\beta}|\tilde{s}}$ *and*

$$\varphi + \sqrt{\tfrac{\alpha}{\beta}}\, D_x\psi \in X_{r,r(1+\sqrt{\alpha\beta})s}. \tag{2.12}$$

Then a solution $u(t) \in X_{r,r|1-\sqrt{\alpha\beta}|\sigma} \times X_{r,r|1-\sqrt{\alpha\beta}|\sigma}$ *of the Cauchy problem* (2.2)–(2.4) *is given by*

$$u_1(t) = \frac{1}{2}\Big(T\big((1+\sqrt{\alpha\beta})t\big)(\varphi + \sqrt{\tfrac{\alpha}{\beta}}\, D_x\psi) + T\big((1-\sqrt{\alpha\beta})t\big)(\varphi - \sqrt{\tfrac{\alpha}{\beta}}\, D_x\psi)\Big)$$
$$u_2(t) = \frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\,\Big(T\big((1+\sqrt{\alpha\beta})t\big)(J\varphi + \sqrt{\tfrac{\alpha}{\beta}}\,\psi) - T\big((1-\sqrt{\alpha\beta})t\big)(J\varphi - \sqrt{\tfrac{\alpha}{\beta}}\,\psi)\Big)$$

for $|t| < s-\sigma$.

Proof. Combining the assumption (2.12) with

$$J(\varphi + \sqrt{\tfrac{\alpha}{\beta}}\, D_x\psi) = J\varphi + \sqrt{\tfrac{\alpha}{\beta}}\,\psi - \sqrt{\tfrac{\alpha}{\beta}}\,\psi(0)$$

we get $J\varphi + \sqrt{\frac{\alpha}{\beta}}\psi \in X_{r,r(1+\sqrt{\alpha\beta})s}$. Since $\varphi \in X_{r,r|1-\sqrt{\alpha\beta}|s}$ and $D_x\psi \in X_{r,r|1-\sqrt{\alpha\beta}|s}$, we have

$$\varphi - \sqrt{\tfrac{\alpha}{\beta}}\, D_x\psi \in X_{r,r|1-\sqrt{\alpha\beta}|s} \qquad \text{and} \qquad J\varphi - \sqrt{\tfrac{\alpha}{\beta}}\,\psi \in X_{r,r|1-\sqrt{\alpha\beta}|s}.$$

Therefore $u_j(t) \in X_{r,r|1-\sqrt{\alpha\beta}|\sigma}$ ($j = 1,2$) are well defined for $|t| < s-\sigma$. It is easy to see that $u_1(0) = \varphi$ and $u_2(0) = \psi$. The functions

$$\left.\begin{aligned} v(t) &= T\big((1+\sqrt{\alpha\beta})t\big)(J\varphi + \sqrt{\tfrac{\alpha}{\beta}}\,\psi) \\ w(t) &= T\big((1-\sqrt{\alpha\beta})t\big)(J\varphi - \sqrt{\tfrac{\alpha}{\beta}}\,\psi) \end{aligned}\right\}$$

are solutions of the system

$$\left.\begin{aligned} D_t v &= (1+\sqrt{\alpha\beta})\, D_x^r v \\ D_t w &= (1-\sqrt{\alpha\beta})\, D_x^r w \end{aligned}\right\} \tag{2.13}$$

for $|t| < s - \sigma$. (The equations (2.13) have to be interpreted in $X_{r,r|1-\sqrt{\alpha\beta}|\tilde\sigma} \supset X_{r,r|1-\sqrt{\alpha\beta}|\sigma}$ with $\tilde\sigma \in (0,\sigma)$ arbitrary.) It is also easy to see that we may use

$$\left.\begin{aligned} T((1+\sqrt{\alpha\beta})t)(\varphi + \sqrt{\tfrac{\alpha}{\beta}}\, D_x\psi) &= D_x v(t) \\ T((1-\sqrt{\alpha\beta})t)(\varphi - \sqrt{\tfrac{\alpha}{\beta}}\, D_x\psi) &= D_x w(t) \end{aligned}\right\}$$

for $|t| < s - \sigma$. It follows

$$u_1 = \frac{1}{2}(D_x v + D_x w) \qquad \text{and} \qquad u_2 = \frac{1}{2}\sqrt{\tfrac{\beta}{\alpha}}\,(v - w) \tag{2.14}$$

and

$$D_t u_1 = \frac{1}{2}\Big((1+\sqrt{\alpha\beta})D_x^{r+1} v + (1-\sqrt{\alpha\beta})\, D_x^{r+1} w\Big) \tag{2.15}$$

$$D_t u_2 = \frac{1}{2}\sqrt{\tfrac{\beta}{\alpha}}\,\Big((1+\sqrt{\alpha\beta})D_x^r v (1-\sqrt{\alpha\beta})D_x^r w\Big). \tag{2.16}$$

Since (2.14) implies

$$D_x^{k+1} v = D_x^k u_1 + \sqrt{\tfrac{\alpha}{\beta}}\, D_x^{k+1} u_2 \qquad \text{and} \qquad D_x^{k+1} w = D_x^k u_1 - \sqrt{\tfrac{\alpha}{\beta}}\, D_x^{k+1} u_2$$

for $k \in \mathbb{N}_0$, a short computation shows that (2.2) and (2.3) follow from (2.15) and (2.16) ∎

Remark. The case $\alpha < 0$, $\beta < 0$ with $\alpha\beta \neq 1$ can be reduced to the situation which we have discussed in Lemma 2.1 and Theorem 2.1. If $\alpha < 0$, $\beta < 0$, with $\omega(t) := -u_2(t)$ we get the system

$$\left.\begin{aligned} D_t u_1 &= D_x^r u_1 + |\alpha| D_x^{r+1}\omega \\ D_t \omega &= |\beta| D_x^{r-1} u_1 + D_x^r \omega. \end{aligned}\right\}$$

It is easy to adapt Lemma 2.1 and Theorem 2.1.

The representation of u_1, u_2 which we have given in Theorem 2.1 can be interpreted as a representation in the form of an expansion in terms of polynomials. If $e_k(x) = \frac{x^k}{k!}$ $(k \in \mathbb{N}_0)$ and $f \in X_{r,s}$ with $f = j((a_k))$, then it is easy to see that $\|f - \sum_{k=0}^m a_k e_k\|_{r,s} \to 0$ for $m \to \infty$. Since $T(t) : X_{r,s} \to X_{r,\sigma}$ $(\sigma < s)$ is continuous, it follows

$$T(t)f = \sum_{k=0}^{\infty} a_k T(t) e_k.$$

It is easy to see that the functions $T(t)e_n$ $(n \in \mathbb{N}_0)$ are polynomials:

$$(T(t)e_n)(x) = \sum_{rk+m=n} \frac{t^k}{k!}\frac{x^m}{m!}.$$

Since $T(t)$ is depending on r, we will use the notation

$$v_{r,n}(t,x) := (T(t)e_n)(x)$$

for $(t,x) \in \mathbb{R}^2$ $(n \in \mathbb{N}_0)$.

3. On the continuation of analytic solutions

In some sense, the representation which was given in Theorem 2.1 is typical for analytic solutions of the system (2.2), (2.3). We shall prove a corresponding result for locally given analytic solutions. This result will be an appropriate base for a discussion of the continuation problem. If the solution is defined for $|x| < x_0$, $|t| < t_0$, it will be continued to the strip $x \in \mathbb{R}$, $|t| < t_0$. This shows that an analytic solution which is given for $|x| < x_0$, $t \in \mathbb{R}$ can be continued into $\mathbb{R}^2$.

Theorem 3.1. *Let*

$$u_j(t,x) := \sum_{k,m} a_{j,k,m} \frac{t^k}{k!} \frac{x^m}{m!} \qquad (j = 1, 2)$$

be a solution of the problem (2.2), (2.3) *for* $|x| < x_0$, $|t| < t_0$. *Define*

$$b_n = a_{1,0,n} + \sqrt{\tfrac{\alpha}{\beta}}\, a_{2,0,n+1} \qquad \text{and} \qquad c_n = a_{1,0,n} - \sqrt{\tfrac{\alpha}{\beta}}\, a_{2,0,n+1}$$

for $n \in \mathbb{N}_0$.

If $\alpha > 0$, $\beta > 0$, $\alpha\beta \neq 1$, *then*

$$u_1(t,x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} b_n\, v_{r,n}\big((1+\sqrt{\alpha\beta}\,)t, x\big) + \sum_{n=0}^{\infty} c_n\, v_{r,n}\big((1-\sqrt{\alpha\beta}\,)t, x\big)\right) \tag{3.1}$$

$$u_2(t,x) = a_{2,0,0} + \frac{1}{2}\sqrt{\tfrac{\beta}{\alpha}}\left(\sum_{n=1}^{\infty} b_{n-1}\, v_{r,n}\big((1+\sqrt{\alpha\beta}\,)t, x\big) - \sum_{n=1}^{\infty} c_{n-1}\, v_{r,n}\big((1-\sqrt{\alpha\beta}\,)t, x\big)\right). \tag{3.2}$$

If $\alpha > 0$, $\beta > 0$, $\alpha\beta = 1$, *then*

$$u_1(t,x) = \frac{1}{2}\left(u_1(0,x) - \alpha D_x u_2(0,x) + \sum_{n=0}^{\infty} b_n\, v_{r,n}(2t, x)\right) \tag{3.3}$$

$$u_2(t,x) = \frac{1}{2\alpha}\left(-\int_0^x u_1(0,\xi)\,d\xi + \alpha u_2(0,x) + \sum_{n=0}^{\infty} b_{n-1}\, v_{r,n}(2t, x)\right) \tag{3.4}$$

with $b_{-1} = \alpha a_{2,0,0}$.

In both cases the series are absolutely and uniformly convergent on compact subsets of $(-t_0, t_0) \times \mathbb{R}$.

Proof. As in the proof of Lemma 2.1, we consider the functions v and w which are defined by (2.7) and (2.9), respectively. Since v is a solution of (2.8) with $v(0,x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$, results of the first part of [9] imply that

$$v(t,x) = \sum_{n=0}^{\infty} b_n\, v_{r,n}\big((1+\sqrt{\alpha\beta}\,)t, x\big) \tag{3.5}$$

for $|t| < t_0$, $x \in \mathbb{R}$. In addition, the series is absolutely and uniformly convergent on compact subsets of $(-t_0, t_0) \times \mathbb{R}$.

If $\alpha\beta \neq 1$, we use (2.10) and $w(0,x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$. This implies

$$w(t,x) = \sum_{n=0}^{\infty} c_n\, v_{r,n}\big((1-\sqrt{\alpha\beta}\,)t, x\big). \tag{3.6}$$

Obviously, (3.5) and (3.6) lead to the representation (3.1) for u_1 and to

$$D_x u_2(t,x) = \\ \frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\left(\sum_{n=0}^{\infty} b_n\, v_{r,n}\big((1+\sqrt{\alpha\beta}\,)t, x\big) - \sum_{n=0}^{\infty} c_n\, v_{r,n}\big((1-\sqrt{\alpha\beta}\,)t, x\big)\right). \tag{3.7}$$

Lemma 2.1 shows that

$$(b_n)_{n\in\mathbb{N}_0} \in \ell_{r,r(1+\sqrt{\alpha\beta}\,)(t_0-\varepsilon)} \qquad \text{for } \varepsilon \in (0,t_0).$$

If $\alpha\beta \neq 1$, then also

$$(c_n)_{n\in\mathbb{N}_0} \in \ell_{r,r|1-\sqrt{\alpha\beta}\,|(t_0-\varepsilon)} \qquad \text{for } \varepsilon \in (0,t_0).$$

Since the integral operator (2.11) is a bounded linear operator, we also have

$$(b_{n-1})_{n\in\mathbb{N}} \in \ell_{r,r(1+\sqrt{\alpha\beta}\,)(t_0-\varepsilon)} \qquad \text{and} \qquad (c_{n-1})_{n\in\mathbb{N}} \in \ell_{r,r|1-\sqrt{\alpha\beta}\,|(t_0-\varepsilon)}.$$

Hence, results of [9] show that a function $U : (-t_0+\varepsilon, t_0-\varepsilon) \times \mathbb{R} \to \mathbb{C}$ can be defined by

$$U(t,x) = \frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\left(\sum_{n=1}^{\infty} b_{n-1}\, v_{r,n}\big((1+\sqrt{\alpha\beta}\,)t, x\big) - \sum_{n=1}^{\infty} c_{n-1}\, v_{r,n}\big((1-\sqrt{\alpha\beta}\,)t, x\big)\right)$$

where the series are absolutely and uniformly convergent on compact subsets of the strip $(-t_0+\varepsilon, t_0-\varepsilon) \times \mathbb{R}$. Since

$$D_x v_{r,n}(t,x) = v_{r,n-1}(t,x) \qquad \text{for } n \in \mathbb{N},\ (t,x) \in \mathbb{R}^2$$

it follows from (3.7) that

$$D_x U(t,x) = D_x u_2(t,x) \qquad \text{for } (t,x) \in (-t_0+\varepsilon, t_0-\varepsilon) \times \mathbb{R}.$$

Therefore we have $u_2(t,x) = U(t,x) + u_2(t,0) - U(t,0)$. Hence, (3.2) is proved if we can show that $u_2(t,0) - U(t,0) = a_{2,0,0}$ for $|t| < t_0 - \varepsilon$. We define

$$\omega(t) = u_2(t,0) - U(t,0).$$

Since the definition of the polynomials $v_{r,n}$ shows that

$$U(t,0) = \frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\sum_{k=1}^{\infty}\Big(b_{rk-1}(1+\sqrt{\alpha\beta}\,)^k - c_{rk-1}(1-\sqrt{\alpha\beta}\,)^k\Big)\frac{t^k}{k!},$$

we have $\omega(0) = u_2(0,0)$. It is easy to see that with

$$V_n^+(t,x) := v_{r,n}\big((1+\sqrt{\alpha\beta}\,)t,x\big) \quad \text{and} \quad V_n^-(t,x) := v_{r,n}\big((1-\sqrt{\alpha\beta}\,)t,x\big)$$

the pairs $\big(V_n^+,\sqrt{\frac{\beta}{\alpha}}\,V_{n+1}^+\big)$ and $\big(V_n^-,-\sqrt{\frac{\beta}{\alpha}}\,V_{n+1}^-\big)$ $(n \in \mathbb{N}_0)$ are solutions of the problem (2.2), (2.3). It follows that the functions u_1 and $u_2 - \omega$ give a solution of the problem (2.2), (2.3) too. This implies $\omega'(t) = 0$ and consequently $\omega(t) = \omega(0) = a_{2,0,0}$ for $|t| < t_0 - \varepsilon$.

Now assume $\alpha\beta = 1$. If we define

$$U_1(t,x) = u_1(t,x) - \frac{1}{2}\big(u_1(0,x) - \alpha D_x u_2(0,x)\big) \tag{3.8}$$

$$U_2(t,x) = u_2(t,x) - \frac{1}{2\alpha}\Big(-\int_0^x u_1(0,\xi)\,d\xi + \alpha u_2(0,x)\Big) \tag{3.9}$$

a simple calculation shows that the pair (U_1,U_2) is a solution of the problem (2.2), (2.3). Since $\alpha\beta = 1$, we also have

$$\alpha D_x D_t U_2 = D_t U_1. \tag{3.10}$$

The definitions (3.8) and (3.9) show that

$$U_1(0,x) - \alpha D_x U_2(0,x) = 0. \tag{3.11}$$

From (3.10) and (3.11) it follows $U_1(t,x) = \alpha D_x U_2(t,x)$. Since (2.3) holds for (U_1,U_2), we get $D_t U_2 = 2D_x^r U_2$. The definition of the coefficients b_n shows that

$$U_2(0,x) = \frac{1}{2\alpha}\sum_{n=0}^{\infty} b_{n-1}\frac{x^n}{n!}.$$

It follows

$$U_2(t,x) = \frac{1}{2\alpha}\sum_{n=0}^{\infty} b_{n-1}\,v_{r,n}(2t,x) \qquad \text{for } x \in \mathbb{R} \text{ and } |t| < t_0 - \varepsilon$$

($\varepsilon \in (0,t_0)$ arbitrary). Now the representations (3.3) and (3.4) are evident ∎

Remark. The results of Theorem 3.1 lead to the following observations on analytic solutions of the problem (2.2), (2.3) (with $\alpha\beta > 0$):

(I) If $\alpha\beta \neq 1$, "continuation via polynomials" is possible.

(II) If $\alpha\beta = 1$, the question of a possible continuation of a given analytic solution into the strip $(-t_0,t_0) \times \mathbb{R}$ only depends on the properties of $u_1(0,\cdot) - \alpha D_x u_2(0,\cdot)$.

4. Generalizations

The method of reducing a Leray-Volevich-coupled system to non-coupled equations is applicable to more general systems of the form

$$D_t u_j = \sum_{k=1}^{n} a_{jk} D_x^{r_{jk}} u_k \qquad (j = 1, \dots, n) \tag{4.1}$$

if the matrix (r_{jk}) has the properties

$$r_{jk} \in \mathbb{N} \quad \text{for } j,k = 1, \dots, n \tag{4.2}$$

$$r_{jj} = r \geq 2 \quad \text{for } j = 1, \dots, n \tag{4.3}$$

$$r_{1k} - r_{jk} = \rho_j \geq 0 \quad \text{for } j = 2, \dots, n;\ k = 1, \dots, n \tag{4.4}$$

and the real matrix (a_{jk}) is diagonalizable.

The Leray-Volevich conditions are satisfied if this system is studied within the framework of the spaces $X_{r,s}$: The considerations of Section 2 show that we have to use $p_{j,k} = \frac{1}{r} r_{jk}$ for $j,k = 1, \dots, n$, and if we choose

$$q_1 = 1 + \max\left\{\frac{1}{r}\rho_j : j = 2, \dots, n\right\}$$

$$q_k = q_1 - \frac{1}{r}\rho_k \quad \text{for } k = 2, \dots, n,$$

then it is easy to see that $q_k > 0$ for $k = 1, \dots, n$ and $p_{j,k} = q_j - q_k + 1$ for $j,k = 1, \dots, n$.

In the following lemma we use the notations

$$\mu = \max\{r_{1k} : k = 1, \dots, n\}, \qquad A = (a_{jk})_{j,k=1}^{n}, \qquad \rho_1 = 0.$$

Lemma 4.1. *Let $(u_1, \dots, u_n) \in \big(C^\mu((-t_0, t_0) \times (-x_0, x_0))\big)^n$ be a solution of the system (4.1) (with (4.2) – (4.4)). If $T = (t_{jk})$ is an invertible real $(n \times n)$-matrix with*

$$TAT^{-1} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

and

$$w_j := \sum_{k=1}^{n} t_{jk} D_x^{\rho_k} u_k \qquad \text{for } j = 1, \dots, n, \tag{4.5}$$

then

$$D_t w_j = \lambda_j D_x^r w_j \qquad \text{for } j = 1, \dots, n. \tag{4.6}$$

Proof. Because of (4.3) and (4.4) we may notice

$$\left.\begin{aligned} r_{1k} &= r + \rho_k \\ r_{jk} &= r_{1k} - \rho_j \quad (j = 2, \dots, n) \end{aligned}\right\} \qquad \text{for } k = 1, \dots, n.$$

If $(u_1,\dots,u_n) \in \left(C^\mu((-t_0,t_0)\times(-x_0,x_0))\right)^n$ is a solution of system (4.1), it follows

$$D_t(D_x^{\rho_j}u_j) = \sum_{k=1}^n a_{jk}D_x^{r_{1k}}u_k = \sum_{k=1}^n a_{jk}D_x^r(D_x^{\rho_k}u_k) \tag{4.7}$$

for $j=1,\dots,n$. Hence we can write $D_t v = AD_x^r v$ if v is the column vector with the components $v_j = D_x^{\rho_j}u_j$. If the column vector w is defined by (4.5), we can notice $w = Tv$. This leads to

$$D_t w = TD_t v = D_x^r TAv = D_x^r TAT^{-1}w$$

which is identical with (4.6) ∎

It is easy to see how results which are analogous to those of Sections 2 and 3 can be obtained by using Lemma 4.1. We mention the following ones:

(i) If $u_j(t,x) = \sum_{k,m} a_{j,k,m}(\frac{t^k}{k!})\frac{x^m}{m!}$ $(j=1,\dots,n)$ is a solution of the system (4.1) for $|x|<x_0$, $|t|<t_0$ and $\lambda_j \in \mathbb{R}\setminus\{0\}$ is an eigenvalue of the matrix A, then

$$\left(\sum_{k=1}^n t_{jk}\,a_{k,0,m+\rho_k}\right)_m \in \ell_{r,r|\lambda_j|(t_0-\varepsilon)}$$

for $\varepsilon \in (0,t_0)$.

(ii) If $\lambda_j \neq 0$ for all eigenvalues of the matrix A, then corresponding to Theorem 3.1 we get a representation of $D_x^{\rho_k}u_k$ $(k=1,\dots,n)$ in the form of a sum of expansions in terms of the polynomials $v_{r,m}(\lambda_j t,x)$ $(m\in\mathbb{N}_0,\ j=1,\dots,n)$.

(iii) If there is at least one $j_0 \in \{1,\dots,n\}$ with $\lambda_{j_0}=0$, then the question of a possible continuation of $u_1,\dots,u_n$ into the strip $(-t_0,t_0)\times\mathbb{R}$ depends on the properties of the initial values $u_1(0,x),\dots,u_n(0,x)$.

The proof of Lemma 4.1 shows that even more general systems can be treated by the same methods. We consider a system

$$D_t u_j = \sum_{k=1}^n a_{jk}L_{jk}u_k \qquad (j=1,\dots,n) \tag{4.8}$$

where again the real matrix (a_{jk}) is diagonalizable and L_{jk} are linear differential operators (in general not with constant coefficients), which are coupled by the following conditions:

(a) $L_{jj} = L_{11}$ for $j=2,\dots,n$.

(b) There are $\rho_2,\dots,\rho_n \in \mathbb{N}_0$ such that with $\rho_1 = 0$

$$D_x^{\rho_j}L_{jk} = L_{11}D_x^{\rho_k} \qquad \text{for } j,k=1,\dots,n,\ j\neq k.$$

It is evident that these assumptions allow to get an analogue to (4.7). This leads to a reduction to the non-coupled system

$$D_t w_j = \lambda_j L_{11} w_j \qquad (j=1,\dots,n).$$

Therefore it is important to know that a result on "continuation via polynomials" holds for the equation $D_t w = \lambda L_{11} w$ $(\lambda \neq 0)$. It was shown in [9] that this is possible also for operators L_{11} with variable coefficients. For these operators the approach to the continuation result was the same as in the case of constant coefficients. In both cases, an appropriate recursion formula for the coefficients $b_{m,n}$ of a solution $\sum b_{m,n} t^m x^n$ was the essential fact. In the following we mention two examples.

Of special interest is the example of the operator

$$L_\gamma = (1+\gamma)D_x + xD_x^2 \qquad (\gamma > -1)$$

since it is tightly connected with the "Laguerre heat equation"

$$D_t u = xD_x^2 u + (\gamma + 1 - x)D_x u$$

which was discussed by Cholewinski and Haimo [1]. It is easy to see that the transformation $\xi = xe^{-t}$, $\tau = 1 - e^{-t}$ leads to the equation

$$D_\tau u = \xi D_\xi^2 u + (\gamma + 1)D_\xi u$$

for which continuation via polynomials was shown in [9]. Now, constructing a system of the form (4.8) with $n = 2$, $L_{11} = L_\gamma$ and $\rho_2 = 1$ we get the system

$$\left.\begin{aligned} D_t u_1 &= L_\gamma u_1 + \alpha L_\gamma D_x u_2 \\ D_t u_2 &= \beta(xD_x u_1 + \gamma u_1) + L_\gamma u_2 \end{aligned}\right\}$$

which has the same properties with respect to continuation via polynomials as the system (2.2), (2.3).

Results of [9] show that analytic solutions of the equation $D_t u = L_{a,b} u$ with

$$L_{a,b} = ax^2 D_x^4 + 2axD_x^3 + bD_x^2 \qquad (a > 0,\ b > 0)$$

have the appropriate continuation property. Therefore we get a second example for a system of the form (4.8) with $n = 2$ if we choose

$$L_{11} = L_{a,b}, \quad L_{22} = L_{a,b}, \quad L_{12} = L_{a,b} D_x, \quad L_{21} = ax^2 D_x^3 + bD_x.$$

Finally, we mention that the methods of the present paper are applicable to systems with derivatives of higher order with respect to t. Results of [9] show that continuation via polynomials is possible for analytic solutions of certain equations of higher order with respect to t.

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Received 19.08.1996; in revised form 10.03.1997