

# On the $\mathcal{L}$ -Characteristic of the Superposition Operator in Lebesgue Spaces with Mixed Norm

Chen Chur-jen and M. Väth

**Abstract.** We consider the superposition operator  $Fx(t, s) = f(t, s, x(t, s))$  of functions of two variables in spaces with mixed norm  $[L_p \rightarrow L_q]$ . After establishing a necessary and sufficient acting condition, we get some conclusions on the  $\mathcal{L}$ -characteristic of  $F$ . We also prove some theorems, which imply that  $F$  is uniformly continuous on balls in the interior of its  $\mathcal{L}$ -characteristic.

**Keywords:** *Superposition operators, spaces with mixed norm,  $\mathcal{L}$ -characteristics*

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## 0. Introduction

Let  $T$  and  $S$  each be measurable subsets of some Euclidean space with positive and finite measure. A real function  $f(t, s, u)$  of three variables  $-\infty < u < \infty$ ,  $t \in T$ ,  $s \in S$  is said to satisfy a Carathéodory condition, if it is continuous with respect to  $u$  for almost all  $(t, s) \in T \times S$  and measurable with respect to  $(t, s)$  for all  $u$ . We denote by  $Fx(t, s) = f(t, s, x(t, s))$  the superposition operator generated by the function  $f$  which satisfies a Carathéodory condition. In this article we shall be interested in the properties of the operator  $F$  for the case when it acts from  $[L_p \rightarrow L_q]$  into  $[L_r \rightarrow L_\sigma]$ , where  $1 \leq p, q, r, \sigma < \infty$ . Here,  $[L_p \rightarrow L_q]$  denotes the Lebesgue space with mixed norm

$$\|x\|_{[L_p \rightarrow L_q]} = \left( \int_T \left( \int_S |x(t, s)|^p ds \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}.$$

Such spaces occur in a natural way in norm estimates for linear integral operators ('Hille-Tamarkin kernels', see, e.g., [1]) and in the study of partial integral operators [11]. For properties of these spaces see, e.g., [5].

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Chen Chur-jen: Universität Würzburg, Math. Inst., Am Hubland, D - 97074 Würzburg; permanently: Tunhai University, Dept. Math., Taichung, Taiwan. The paper was written while the author was visiting the University of Würzburg as a DAAD fellow. Financial support by the DAAD Bonn (Kz. A/95/08858) and the DFG Bonn (Az. Ap 40/6-2) is gratefully acknowledged.

M. Väth: Universität Würzburg, Math. Inst., Am Hubland, D - 97074 Würzburg

In the case  $p = q$  we have  $[L_p \rightarrow L_q] = L_p(T \times S)$ . In this case many properties of the superposition operator are of course well known (see, e.g., [7, 8]). In general,  $[L_p \rightarrow L_q]$  is a regular ideal space [10]. Thus also many properties of  $F$  are known, e.g. that  $F$  is always continuous, if it maps  $[L_p \rightarrow L_q]$  into  $[L_r \rightarrow L_\sigma]$  (see [2]; if  $T$  and  $S$  are just assumed to be  $\sigma$ -finite measure spaces, this is also shown in [9]). Thus we shall mainly be interested in typical properties of  $F$ , which depend on the ‘additional structure’ of  $[L_p \rightarrow L_q]$ .

### 1. The $\mathcal{L}$ -characteristic

Recall that for a ‘one-dimensional’ superposition operator  $Gx(s) = g(s, x(s))$  in Lebesgue spaces the study of acting conditions is reduced to a study of growth conditions for  $g$  [7]. This leads to characterizations of the  $\mathcal{L}$ -characteristic of such operators [8]. However, for the superposition operator  $Fx(t, s) = f(t, s, x(t, s))$  in Lebesgue spaces with mixed norm already the simple growth condition  $|f(t, s, u)| \leq |u|^{\frac{1}{\lambda}}$  implies that  $F$  acts from  $[L_p \rightarrow L_q]$  into  $[L_{\lambda p} \rightarrow L_{\lambda q}]$ . Thus, if one wants to consider growth conditions of the form

$$|f(t, s, u)| \leq a(t, s) + b|u|^{\frac{1}{\lambda}},$$

one can not expect to find acting conditions for  $F : [L_p \rightarrow L_q] \rightarrow [L_r \rightarrow L_\sigma]$  without the connection  $r = \lambda p$  and  $\sigma = \lambda q$ , i.e.  $\sigma = \frac{qr}{p}$ . In this section we thus restrict to this case. However, the first lemma will be formulated more general, since it will be needed later in this form.

The proofs of the following lemmas follow [7].

**Lemma 1.** *Suppose that the operator  $F$  acts from  $[L_p \rightarrow L_q]$  into  $[L_r \rightarrow L_\sigma]$ . Then it is bounded, i.e. it maps bounded sets into bounded sets.*

**Proof.** Without loss of generality, we can assume that  $F0 = 0$ . By [3: Theorem 3] (see also [2: Theorem 2.6]), the operator  $F$  is continuous at the point 0. This means that there can be found a number  $b > 0$  such that

$$\int_T \left[ \int_S |f(t, s, x(t, s))|^r ds \right]^{\frac{q}{r}} dt \leq 1 \quad \text{if} \quad \int_T \left[ \int_S |x(t, s)|^p ds \right]^{\frac{q}{p}} dt \leq b^q.$$

Suppose now that  $u \in [L_p \rightarrow L_q]$  and

$$nb^q \leq \int_T \left[ \int_S |u(t, s)|^p ds \right]^{\frac{q}{p}} dt \leq (n + 1)b^q,$$

where  $n$  is an integer. Then  $T$  can be divided into parts  $T_1, \dots, T_{n+1}$  such that

$$\int_{T_i} \left[ \int_S |u(t, s)|^p ds \right]^{\frac{q}{p}} dt \leq b^q \quad (i = 1, \dots, n + 1).$$

Hence,

$$\int_T \left[ \int_S |f(t, s, u(t, s))|^r ds \right]^{\frac{q}{r}} dt \leq \sum_{i=1}^{n+1} \int_{T_i} \left[ \int_S |f(t, s, u(t, s))|^r ds \right]^{\frac{q}{r}} dt \leq n + 1.$$

Thus

$$\|Fu\|_{[L_r \rightarrow L_\sigma]} \leq \left[ \left( \frac{\|u\|_{[L_p \rightarrow L_q]}}{b} \right)^q + 1 \right]^{\frac{1}{\sigma}}.$$

This proves the lemma ■

We remark that using a similar method as in the previous proof, it can be shown that  $[L_p \rightarrow L_q]$  is a  $\Delta_2$  space (and then a split space) in the sense of [4]. Thus Lemma 1 is a special case of the results in [4].

Assume  $f$  satisfies the growth condition

$$|f(t, s, u)| \leq a(t, s) + b(s)|u|^{\frac{p}{r}},$$

for some  $a \in [L_r \rightarrow L_\sigma]$  and  $b \in L_r$ , where

$$\tau = \frac{pr\sigma}{p\sigma - qr} \geq 0$$

(which implies  $\tau > r \geq 1$ ). Then  $F$  maps  $[L_p \rightarrow L_q]$  into  $[L_r \rightarrow L_\sigma]$ : Indeed,

$$\begin{aligned} \|Fu\|_{[L_r \rightarrow L_\sigma]} &\leq \|a\|_{[L_r \rightarrow L_\sigma]} + \left( \int_T \left( \int_S |b(s)|^r |u(t, s)|^{\frac{qr}{\sigma}} ds \right)^{\frac{\sigma}{r}} dt \right)^{\frac{1}{\sigma}} \\ &\leq \|a\|_{[L_r \rightarrow L_\sigma]} + \|b\|_{L_r} \|u\|_{[L_p \rightarrow L_q]} \end{aligned}$$

by Minkowski's and Hölder's inequalities.

We notice in the following example that the condition  $\tau \geq 0$  may not be dropped.

**Example 1.** For  $\sigma = q$ ,  $p < r$  and  $f(t, s, u) = u$  the superposition operator  $F$  does not map  $[L_p \rightarrow L_q]$  into  $[L_r \rightarrow L_\sigma]$ .

It is remarkable that in case  $\tau = \infty$  (i.e.  $p\sigma = qr$ ) the condition is even necessary:

**Lemma 2.** *The operator  $F$  acts from  $X = [L_p \rightarrow L_q]$  into  $Y = [L_r \rightarrow L_{\frac{p}{r}r}]$  for  $q \geq p$ , if and only if*

$$|f(t, s, u)| \leq a(t, s) + b|u|^{\frac{p}{r}},$$

where  $a \in Y$  and  $b \geq 0$ . In case  $q < p$  this condition is at least sufficient.

**Proof.** We just prove necessity. Without loss of generality, assume  $F0 = 0$ . By Lemma 1, there can be found a number  $b > 0$  such that

$$\int_T \left[ \int_S |f(t, s, x(t, s))|^r ds \right]^{\frac{\sigma}{p}} dt \leq b^{\frac{qr}{p}} \quad \text{if} \quad \int_T \left[ \int_S |x(t, s)|^p ds \right]^{\frac{\sigma}{p}} dt \leq 1.$$

We define the function

$$\varphi(t, s, u) = \begin{cases} |f(t, s, u)| - b|u|^{\frac{p}{r}} & \text{if } |f(t, s, u)| > b|u|^{\frac{p}{r}}, \\ 0 & \text{if } |f(t, s, u)| \leq b|u|^{\frac{p}{r}}. \end{cases}$$

Obviously, if  $\varphi(t, s, u) \neq 0$ , then

$$|\varphi(t, s, u)|^r \leq |f(t, s, u)|^r - b^r |u|^p.$$

Now, let  $u \in X$  and  $G = \{(t, s) \in T \times S : \varphi(t, s, u(t, s)) > 0\}$ . Suppose that

$$\int_T \left[ \int_S |u(t, s) \chi_G(t, s)|^p ds \right]^{\frac{q}{p}} dt = n + \varepsilon,$$

where  $n$  is an integer and  $0 \leq \varepsilon < 1$ . Then  $T$  can be divided into parts  $T_1, \dots, T_{n+1}$  such that

$$\int_{T_i} \left[ \int_S |u(t, s) \chi_G(t, s)|^p ds \right]^{\frac{q}{p}} dt \leq 1 \quad (i = 1, \dots, n + 1).$$

Then

$$\begin{aligned} & \int_T \left[ \int_S |f(t, s, u(t, s) \chi_G(t, s))|^r ds \right]^{\frac{q}{p}} dt \\ &= \sum_{i=1}^{n+1} \int_{T_i} \left[ \int_S |f(t, s, u(t, s) \chi_G(t, s))|^r ds \right]^{\frac{q}{p}} dt \leq (n + 1) b^{\frac{qr}{p}} \end{aligned}$$

and

$$\begin{aligned} & \int_T \left[ \int_S |\varphi(t, s, u(t, s))|^r ds \right]^{\frac{q}{p}} dt \\ & \leq \int_T \left[ \int_S |\varphi(t, s, u(t, s) \chi_G(t, s))|^r ds \right]^{\frac{q}{p}} dt \\ & \leq \int_T \left[ \int_S (|f(t, s, u(t, s) \chi_G(t, s))|^r - b^r |u(t, s) \chi_G(t, s)|^p) ds \right]^{\frac{q}{p}} dt \\ & \leq \int_T \left[ \int_S |f(t, s, u(t, s) \chi_G(t, s))|^r ds \right]^{\frac{q}{p}} dt - \int_T \left[ \int_S b^r |u(t, s) \chi_G(t, s)|^p ds \right]^{\frac{q}{p}} dt \\ & \leq (n + 1) b^{\frac{qr}{p}} - (n + \varepsilon) b^{\frac{qr}{p}} \\ & \leq b^{\frac{qr}{p}}. \end{aligned}$$

By the Krasnoselskii-Ladyzhenskii lemma [6] (see also [2: Lemma 6.2]), there exists a sequence of measurable functions  $u_k(t, s)$  such that  $|u_k(t, s)| \leq k$  and  $\varphi(t, s, u_k(t, s)) = \max_{|u| \leq k} \varphi(t, s, u)$  ( $k = 1, 2, \dots$ ). Obviously,  $u_k \in X$ . We set

$$a(t, s) = \sup_{-\infty < u < \infty} \varphi(t, s, u) = \lim_{k \rightarrow \infty} \varphi(t, s, u_k(t, s)).$$

By the previous inequality and Fatou's lemma,

$$\int_T \left[ \int_S |a(t, s)|^r ds \right]^{\frac{q}{p}} dt \leq \liminf_{k \rightarrow \infty} \int_T \left[ \int_S |\varphi(t, s, u_k(t, s))|^r ds \right]^{\frac{q}{p}} dt \leq b^{\frac{qr}{p}}.$$

This means that  $a \in Y$ . Since

$$a(t, s) = \sup_{-\infty < u < \infty} \varphi(t, s, u) \geq \sup_{-\infty < u < \infty} \left\{ |f(t, s, u)| - b|u|^{\frac{p}{r}} \right\},$$

then  $|f(t, s, u)| \leq a(t, s) + b|u|^{\frac{p}{r}}$  ■

For a superposition operator  $F$ , let

$$L_F = \left\{ (\alpha, \beta, \gamma) \in I^3 \mid F : [L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\beta}}] \rightarrow [L_{\frac{1}{\beta}} \rightarrow L_{\frac{\alpha}{\beta\gamma}}] \right\}$$

denote the  $\mathcal{L}$ -characteristic of  $F$ , where  $I = (0, 1]$ . We will mainly consider  $L_F$  on the "prism"

$$\Delta = \{ (\alpha, \beta, \gamma) \in I^3 : \alpha \geq \gamma \},$$

since  $q \geq p$  is the condition of Lemma 2.

**Theorem 1.** *If  $(\alpha_0, \beta_0, \gamma_0) \in L_F \cap \Delta$ , then  $L_F$  contains all points  $(\alpha, \beta, \gamma) \in I^3$  with*

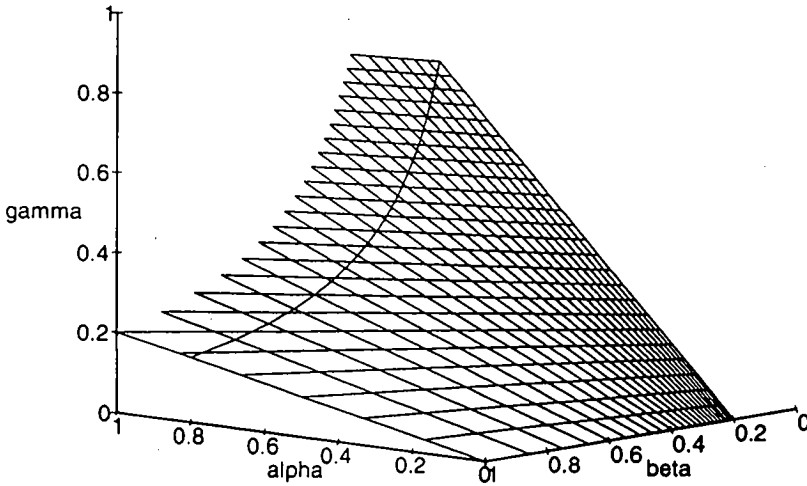
$$\beta \geq \beta_0, \quad \frac{\beta}{\alpha} \geq \frac{\beta_0}{\alpha_0}, \quad \frac{\beta\gamma}{\alpha} \geq \frac{\beta_0\gamma_0}{\alpha_0} \tag{1}$$

or, equivalently,

$$\alpha \leq \lambda\mu^{-1}\alpha_0, \quad \beta \geq \lambda\beta_0, \quad \gamma \geq \mu^{-1}\gamma_0 \quad \text{for some } \lambda, \mu \geq 1. \tag{2}$$

**Proof.** Lemma 2 implies  $|f(t, s, u)| \leq a(t, s) + |u|^{\frac{\beta_0}{\alpha_0}}$  with  $a \in Y_0 = [L_{\frac{1}{\beta_0}} \rightarrow L_{\frac{\alpha_0}{\beta_0\gamma_0}}]$ . Since  $a \in Y_0 \subseteq Y = [L_{\frac{1}{\beta}} \rightarrow L_{\frac{\alpha}{\beta\gamma}}]$ , again by Lemma 2,  $F : [L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\beta}}] \rightarrow Y$  if  $\frac{\beta}{\alpha} \geq \frac{\beta_0}{\alpha_0}$ . To see that (1) implies (2), put  $\lambda = \frac{\beta}{\beta_0}$  and  $\mu = \frac{\lambda\alpha_0}{\alpha}$ . The converse is straightforward ■

As a representative example the following figure shows, how the surface of points of equality in (2) looks like in case  $(\alpha_0, \beta_0, \gamma_0) = (0.8, 0.2, 0.8)$ :



The theorem is sharp in the following sense:

**Example 2.** Given some  $(\alpha_0, \beta_0, \gamma_0) \in I^3$ , there exists some  $f$  such that  $L_F$  consists precisely of all points  $(\alpha, \beta, \gamma) \in I^3$  satisfying (1).

Put  $f(t, s, u) = |a(t)b(s)| + |u|^{\lambda_0}$ , where  $a \in L_{\mu_0} \setminus \bigcup_{\mu > \mu_0} L_{\mu}$  and  $b \in L_{\frac{1}{\beta_0}} \setminus \bigcup_{\beta < \beta_0} L_{\frac{1}{\beta}}$  with

$$\lambda_0 = \frac{\beta_0}{\alpha_0} \quad \text{and} \quad \mu_0 = \frac{\alpha_0}{\beta_0\gamma_0}.$$

On the one hand, Lemma 2 implies that  $L_F$  contains at least all points satisfying (1). On the other hand, since  $\|F0\|_{[L_{\frac{1}{\beta}} - L_{\frac{1}{\beta\gamma}}]} = \|b\|_{L_{\frac{1}{\beta}}} \|a\|_{L_{\frac{1}{\beta\gamma}}}$ ,  $L_F$  may contain no points  $(\alpha, \beta, \gamma)$  with  $\frac{\alpha}{\beta\gamma} > \mu_0$  or  $\beta < \beta_0$ . Furthermore, if  $x \in L_1 \setminus \bigcup_{m>1} L_m$ , we have that  $y(t, s) = |x(s)|^\alpha$  belongs to  $[L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\alpha}}]$ , but  $|Fy(t, s)| \geq |x(s)|^{\lambda_0\alpha}$  implies  $Fy \notin [L_{\frac{1}{\beta}} \rightarrow L_{\frac{1}{\beta\gamma}}]$  for  $\frac{\lambda_0\alpha}{\beta} > 1$ .

For some subset  $L$  of  $I^3$  we define the projections

$$\begin{aligned} L^1(\alpha) &= \{(\beta, \gamma) : (\alpha, \beta, \gamma) \in L\} \\ L^2(\beta) &= \{(\alpha, \gamma) : (\alpha, \beta, \gamma) \in L\} \\ L^3(\gamma) &= \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in L\} \end{aligned}$$

on each component. Now we may reformulate Theorem 1 as

**Theorem 2.**  $L = L_F \cap \Delta$  has the following properties:

1. Each  $L^3(\gamma)$  is zero-concave, i.e.

$$(\alpha, \beta) \in L^3(\gamma) \implies (\lambda\alpha, \lambda\beta) \in L^3(\gamma)$$

for  $\lambda \geq 1$  (and  $\lambda\alpha, \lambda\beta \leq 1$ ). Additionally,

$$(\alpha_0, \beta_0) \in L^3(\gamma) \implies (\alpha, \beta) \in L^3(\gamma)$$

for  $\alpha \leq \alpha_0$  and  $\beta \geq \beta_0$  (and  $\alpha \geq \gamma$ ).

2. Each  $L^2(\beta)$  is zero-convex, i.e.

$$(\alpha, \gamma) \in L^2(\beta) \implies (\lambda^{-1}\alpha, \lambda^{-1}\gamma) \in L^2(\beta)$$

for  $\lambda \geq 1$ . Additionally,

$$(\alpha_0, \gamma_0) \in L^2(\beta) \implies (\alpha, \gamma) \in L^2(\beta)$$

for  $\alpha \leq \alpha_0$  and  $\gamma \geq \gamma_0$  (and  $\alpha \geq \gamma$ ).

3. Each  $L^1(\alpha)$  is zero-hyperbolic, i.e.

$$(\beta, \gamma) \in L^1(\alpha) \implies (\lambda\beta, \lambda^{-1}\gamma) \in L^1(\alpha)$$

for  $\lambda \geq 1$  (and  $\lambda\beta \leq 1$ ). Additionally,

$$(\beta_0, \gamma_0) \in L^1(\alpha) \implies (\beta, \gamma) \in L^1(\alpha)$$

for  $\beta \geq \beta_0$  and  $\gamma \geq \gamma_0$  (and  $\alpha \geq \gamma$ ).

4. We also have the inclusions

$$L^1(\alpha) \cap \Delta^1(\alpha_0) \subseteq L^1(\alpha_0) \quad \text{for } \alpha \geq \alpha_0$$

$$L^2(\beta) \subseteq L^2(\beta_0) \quad \text{for } \beta \leq \beta_0$$

$$L^3(\gamma) \cap \Delta^3(\gamma_0) \subseteq L^3(\gamma_0) \quad \text{for } \gamma \leq \gamma_0.$$

The first three properties are just reformulations of (2) (restricted to  $\Delta$ ). The remaining properties are consequences (or may of course also be verified directly by (1)).

It is yet unknown, whether the zero-concave sets (with the additional property of 1.) coincide with the sets of  $\mathcal{L}$ -characteristics of "one-dimensional" superposition operators  $Gx(s) = g(s, x(s))$  in Lebesgue spaces. However, this is at least 'almost' true (up to some boundary points), see [8]. The next example shows that in this sense the first property in the previous theorem is 'sharp'.

**Example 3.** Given some set  $M$ , which is the  $\mathcal{L}$ -characteristic of some superposition operator of one variable  $Gx(s) = g(s, x(s))$ , there exists a superposition operator  $F$  in spaces with mixed norm with

$$L_F^3(\gamma) = M \cap \left\{ (\alpha, \beta) : \frac{\alpha}{\beta\gamma} \geq 1 \right\} \quad (0 < \gamma \leq 1),$$

i.e.  $L_F^3(\gamma)$  coincides with the given zero-concave set  $M$  wherever possible.

Put  $f(t, s, u) = g(s, u)$ . If  $g : L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\beta}}$ , we have by the well-known acting condition of the 'one-dimensional' superposition operator (see, e.g., [7]) that

$$|f(t, s, u)| = |g(s, u)| \leq a(s) + b|u|^{\frac{\alpha}{\beta}}$$

for some  $a \in L_{\frac{1}{\beta}}$ . Thus Lemma 2 implies  $F : [L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\gamma}}] \rightarrow [L_{\frac{1}{\beta}} \rightarrow L_{\frac{\alpha}{\beta\gamma}}]$ .

Conversely, if  $F : [L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\gamma}}] \rightarrow [L_{\frac{1}{\beta}} \rightarrow L_{\frac{\alpha}{\beta\gamma}}]$ , we have  $G : L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\beta}}$ , since for any  $x \in L_{\frac{1}{\alpha}}$  the function  $y(t, s) = x(s)$  belongs to  $[L_{\frac{1}{\alpha}} \rightarrow L_{\frac{1}{\gamma}}]$ , whence  $Fy \in [L_{\frac{1}{\beta}} \rightarrow L_{\frac{\alpha}{\beta\gamma}}]$  implies  $Gx \in L_{\frac{1}{\beta}}$ .

## 2. Uniform continuity

In general, the superposition operator is not uniformly continuous on balls in spaces with mixed norm (an example will be given later). This is already well-known for the 'one-dimensional' superposition operator  $Gx(s) = g(s, x(s))$ . However, for that operator a useful sufficient condition is given in [8: Theorem 17.4]. We will extend this result for spaces with mixed norm.

The following lemma is implicitly proved in [8: Theorem 17.4], but we give a proof without referring to the Scorza-Dragoni lemma (thus our lemma holds on more general measure spaces  $\Omega$ , see the remarks at the end of the paper).

**Lemma 3.** *Let  $B$  be a set of measurable functions on  $\Omega = T \times S$ , which is bounded in measure, i.e.*

$$\lim_{n \rightarrow \infty} \sup_{x \in B} \text{mes} \{ \omega \in \Omega : |x(\omega)| \geq n \} = 0.$$

*Then the superposition operator  $F$  is uniformly continuous on  $B$  in measure, i.e. for any  $x_n, y_n \in B$  with  $x_n - y_n \rightarrow 0$  in measure, we have  $Fx_n - Fy_n \rightarrow 0$  in measure.*

**Proof.** It suffices to prove that any subsequence of  $z_n = Fx_n - Fy_n$  contains a subsequence, which converges to 0 in measure. Thus we even may assume  $x_n - y_n \rightarrow 0$  a.e. Now, fix  $\varepsilon > 0$  and  $\delta > 0$ . There exists some  $M$  such that the measure of

$$\Omega_n = \left\{ \omega \in \Omega : |x_n(\omega)| \geq M \text{ or } |y_n(\omega)| \geq M \right\}$$

is less than  $\frac{\varepsilon}{2}$  for any  $n$ . Since  $f(\omega, \cdot)$  is for almost all  $\omega$  uniformly continuous in  $[-M, M]$ , the sequence

$$w_n(\omega) = \chi_{\Omega_n^c}(\omega)z_n(\omega) = \chi_{\Omega_n^c}(\omega)(Fx_n(\omega) - Fy_n(\omega))$$

converges to 0 a.e., whence in measure, i.e., for  $n$  big enough, the measure of

$$\Omega_n^* = \left\{ \omega \in \Omega \setminus \Omega_n : |z_n(\omega)| \geq \delta \right\}$$

is less than  $\frac{\varepsilon}{2}$ . Thus

$$\text{mes}\{\omega \in \Omega : |z_n(\omega)| \geq \delta\} \leq \text{mes}(\Omega_n \cup \Omega_n^*) < \varepsilon,$$

which means  $z_n \rightarrow 0$  in measure ■

Recall that a set  $M$  of functions in  $X = [L_p \rightarrow L_q]$  is said to be of *equicontinuous norm*, if

$$\lim_{n \rightarrow \infty} \sup_{x \in M} \|P_{D_n} x\| = 0$$

for any decreasing sequence of measurable sets  $D_n$  with  $\bigcap D_n = \emptyset$ . Here  $P_{D_n} x(t, s) = \chi_{D_n}(t, s)x(t, s)$  denotes the ‘projection’ of  $x$  on  $D_n$ . Since  $X$  is a regular ideal space, Vitali’s convergence theorem holds true [10]:

*A sequence  $x_n \in X$  converges in norm to some  $x \in X$ , if and only if  $x_n \rightarrow x$  in measure, and the set of all  $x_n$  is of equicontinuous norm.*

**Lemma 4.** *Let  $X = [L_p \rightarrow L_q]$  and  $Y = [L_r \rightarrow L_\sigma]$  ( $1 \leq p, q, r, \sigma < \infty$ ). If the superposition operator  $F$  acts from  $X$  into  $Y$ , it maps sets of equicontinuous norm into sets of equicontinuous norm.*

**Proof.** We just apply the fact that  $F$  is continuous at 0: Assume, there is some set  $M \subseteq X$  of equicontinuous norm such that  $FM$  is not of equicontinuous norm. Then there exists a decreasing sequence of sets  $D_n$  with  $\bigcap D_n = \emptyset$  and  $x_n \in M$  with  $\|P_{D_n} Fx_n\|_Y \not\rightarrow 0$ . Since  $M$  is of equicontinuous norm, Vitali’s convergence theorem implies  $\|P_{D_n} x_n\|_X \rightarrow 0$ . Hence the continuity of  $F$  at 0 yields the contradiction  $\|P_{D_n} Fx_n\|_Y = \|F(P_{D_n} x_n) - F0 + P_{D_n} F0\|_Y \leq \|F(P_{D_n} x_n) - F0\|_Y + \|P_{D_n} F0\|_Y \rightarrow 0$  ■

**Lemma 5.** *Each ball of  $[L_p \rightarrow L_q]$  is of equicontinuous norm in  $[L_{p_0} \rightarrow L_{q_0}]$ , if  $p > p_0 \geq 1$  and  $q > q_0 \geq 1$ .*

**Proof.** For the proof just apply Hölder’s inequality on the product  $\chi_{D_n} \cdot x$  ■



We now consider the following situation: Let

$$X_0 = [L_{p_0} \rightarrow L_{q_0}], \quad Y_0 = [L_{r_0} \rightarrow L_{\sigma_0}], \quad X = [L_p \rightarrow L_q], \quad Y = [L_r \rightarrow L_\sigma],$$

where

$$1 \leq p_0 \leq p, \quad 1 \leq q_0 \leq q, \quad 1 \leq r \leq r_0, \quad 1 \leq \sigma \leq \sigma_0.$$

Then it is clear that any operator  $F : X_0 \rightarrow Y_0$  maps  $X$  into  $Y$ . However, the mapping  $F : X \rightarrow Y$  may have 'better' properties than  $F$ .

**Theorem 3.** *Assume the superposition operator  $F$  acts from  $X_0$  into  $Y_0$ . Then the mapping  $F : X \rightarrow Y$  is uniformly continuous on each ball of  $X$ , if at least one of the conditions*

$$p_0 < p \quad \text{and} \quad q_0 < q \tag{3}$$

or

$$r_0 > r \quad \text{and} \quad \sigma_0 > \sigma \tag{4}$$

is true.

**Proof.** Let  $x_n, y_n \in X$  be bounded in norm with  $\|x_n - y_n\|_X \rightarrow 0$ . We have to prove that  $z_n = Fx_n - Fy_n$  satisfies  $\|z_n\|_Y \rightarrow 0$ . By Lemma 3 we have  $z_n \rightarrow 0$  in measure. By Vitali's convergence theorem we thus have to prove that the set  $M$  of all  $z_n$  is of equicontinuous norm. If (4) is satisfied, this is true by Lemma 5, since  $M$  is contained in some ball of  $Y_0$  by Lemma 1. If (3) is true, the set of all  $x_n$  and  $y_n$  is of equicontinuous norm in  $X_0$  by Lemma 5. Thus the set of all  $Fx_n$  and  $Fy_n$  is of equicontinuous norm in  $Y$  by Lemma 4, whence also the set of all  $z_n$  by triangle's inequality ■

We emphasize that in (3) and (4) both inequalities must be strict. In fact, the theorem is sharp in the following sense (we modify an example from [8]):

**Example 4.** Given numbers  $p_0, q_0, r_0$ , and putting  $X_0 = [L_{p_0} \rightarrow L_{q_0}], Y_0 = [L_{r_0} \rightarrow L_{\sigma_0}]$ ,  $\sigma_0 = \frac{q_0 r_0}{p_0}$ , and  $X = [L_p \rightarrow L_q], Y = [L_r \rightarrow L_\sigma]$  for  $T = S = [0, 1]$  there exists an (even autonomous) superposition operator  $F$  acting from  $X_0$  into  $Y_0$  such that  $F : X \rightarrow Y$  is not uniformly continuous on any ball of  $X$ , if either

$$p = p_0, \quad q \geq q_0, \quad r = r_0, \quad \sigma \leq \sigma_0 \tag{5}$$

or

$$p \geq p_0, \quad q = q_0, \quad r \leq r_0, \quad \sigma = \sigma_0. \tag{6}$$

Indeed, put  $f(t, s, u) = |u|^{\frac{p_0}{r_0}} \sin u$ , and in case (5) consider  $Q_n = T \times [0, (\frac{B}{4n\pi})^{p_0}]$  ( $B > 0$  fixed),  $x_n = \frac{4n+1}{2} \pi \chi_{Q_n}$  and  $y_n = \frac{4n-1}{2} \pi \chi_{Q_n}$ . Then we have  $\|x_n\|_X, \|y_n\|_X \leq B$ , and  $|x_n - y_n| = \pi \chi_{Q_n}$  implies  $\|x_n - y_n\|_X \rightarrow 0$ . On the other hand  $|Fx_n - Fy_n| \geq \chi_{Q_n} (2n\pi)^{\frac{p_0}{r_0}}$  implies  $\|Fx_n - Fy_n\|_Y \geq (\frac{B}{2})^{\frac{p_0}{r_0}} \not\rightarrow 0$ . In case (6) consider  $Q_n = [0, (\frac{B}{4n\pi})^{q_0}] \times S$  instead.

### 3. Remarks

Most results hold for more general measure spaces  $T$  and  $S$  (with similar proofs):

If one just assumes that  $T$  and  $S$  are  $\sigma$ -finite measure spaces,  $T$  being atomic free, Lemmas 1 and 2 still hold true. If additionally  $T$  and  $S$  have finite measure, Theorems 1 and 2 remain true.

Lemma 3 holds for arbitrary measure spaces  $\Omega$ , if convergence in measure is replaced by convergence in measure on each set of finite measure (just apply the lemma for  $\Omega$  replaced by such a set).

Since Vitali's convergence theorem may be generalized in the same sense for  $\sigma$ -finite measure spaces [9], Lemma 4 is still true for arbitrary  $\sigma$ -finite measure spaces  $T$  and  $S$ .

Lemma 5 and Theorem 3 just make use of the fact that  $T$  and  $S$  are finite measure spaces (in case (4), additionally  $T$  should be atomic free, since Lemma 1 is needed).

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