Hardy Inequalities for Overdetermined Classes of Functions

A. **Kufner and** C. G. **Simader**

Abstract. Conditions on weights w_0 and w_k are given for the k-th order Hardy inequality $(f_0^1 |u(t)|^q w_0(t) dt)^{1/q} \leq c(f_0^1 |u^{(k)}(t)|^p w_k(t) dt)^{1/p}$ to hold for two special classes of functions *u* satisfying 2k and *k* + 1 boundary conditions, respectively. The conditions are sufficient and partially also necessary. For one class, a hypothesis is formulated describing necessary and sufficient conditions on w_0 and w_k .

Keywords: *Hardy's inequality, weighted norm inequalities* **AMS subject classification:** 26 D 10

0. Introduction

The k-th order Hardy inequality

k + 1 boundary conditions, respectively. The conditions are sufficient and
sary. For one class, a hypothesis is formulated describing necessary and
on
$$
w_0
$$
 and w_k .
is inequality, weighted norm inequalities
sification: 26 D 10
On

$$
\left(\int_0^1 |u(t)|^q w_0(t) dt\right)^{\frac{1}{q}} \leq c \left(\int_0^1 |u^{(k)}(t)|^p w_k(t) dt\right)^{\frac{1}{p}}
$$
 (1)
so, q (0 < q \leq \infty, 1 \leq p \leq \infty) and weight functions w_0, w_k (i.e.,
the and positive a.e. in (0,1)) is meaningful if we consider classes of
ing certain boundary conditions
 $u^{(i)}(0) = 0$ for $i \in M_0$
 $u^{(j)}(1) = 0$ for $j \in M_1$ (2)

with parameters p, q ($0 < q \le \infty$, $1 \le p \le \infty$) and weight functions w_0, w_k (i.e., functions measurable and positive a.e. in $(0, 1)$ is meaningful if we consider classes of functions *u* satisfying certain boundary conditions $\begin{aligned} dt & \Bigg\} & \leq c \left(\int_0^a |u^{(k)}(t)|^p w_k(t) \right) \[1mm] & \Bigg\{ \begin{aligned} \infty, & 1 \leq p \leq \infty \big) & \text{ and weight} \[1mm] \text{dary conditions} \end{aligned} \[1mm] \begin{aligned} \infty, & 1 \leq p \leq \infty \big) & \text{ and weight} \[1mm] \text{dary conditions} \end{aligned} \] \[1mm] \begin{aligned} \infty, & 0 & \text{ is meaningful} \end{aligned} \] \[1mm] \begin{aligned} \infty, & 0 & \text{ is the probability of the function } \$

$$
u^{(i)}(0) = 0 \text{ for } i \in M_0
$$

\n
$$
u^{(j)}(1) = 0 \text{ for } j \in M_1
$$
\n(2)

with $M_0, M_1 \subset \{0, 1, ..., k-1\}$. It is reasonable to consider sets M_0 and M_1 such that

$$
\#M_0 + \#M_1 = k;\t\t(3)
$$

the *admissible* sets M_0 , M_1 are described in [1, 2, 4].

The paper deals with the problem of finding conditions (sufficient, or necessary and sufficient) if the number of boundary conditions exceeds *k.* Such classes of functions will be called *overdetermined*.

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For the case $k = 1$, the only overdetermined class is that of functions satisfying

$$
u(0) = u(1) = 0 \tag{4}
$$

rdetermined class is that of functions satisfying
 $u(0) = u(1) = 0$ (4)

dition for (1) to hold for such functions has for the and the necessary and sufficient condition for (1) to hold for such functions has for the case rdetermined class is that of functions satisfying
 $u(0) = u(1) = 0$ (4)

dition for (1) to hold for such functions has for the
 $1 < p \le q < \infty$ (5)

$$
1 < p \le q < \infty \tag{5}
$$

the following form:

8 A. Kufner and C. G. Simader
\nFor the case
$$
k = 1
$$
, the only overdetermined class is that of functions satisfying
\n $u(0) = u(1) = 0$ (4)
\nd the necessary and sufficient condition for (1) to hold for such functions has for the
\nsee
\n $1 < p \le q < \infty$ (5)
\nfollowing form:
\n
$$
\sup_{0 < a < b < 1} \left(\int_a^b w_0(t) dt \right)^{\frac{1}{q}} \min \left\{ \left(\int_0^a w_1^{1-p'}(t) dt \right)^{\frac{1}{p'}} , \left(\int_a^1 w_1^{1-p'}(t) dt \right)^{\frac{1}{p'}} \right\} < \infty
$$
 (6)
\n(e [3: Section 8]). For $k \in \mathbb{N}, k > 1$, we have more possibilities concerning the choice
\noverdetermined classes. Here, we will deal with two special cases:
\nCase I. We will consider the maximal class of overdetermined functions, i.e.,
\nactions, satisfying the maximal number of conditions:
\n $u^{(i)}(0) = 0$
\n $u^{(i)}(1) = 0$
\n $u^{(i)}(1) = 0$
\n $u^{(i)}(0) = 0$ ($i = 0, 1, ..., k - 1$).
\n $u^{(k-1)}(1) = 0$ (i = 0, 1, ..., $k - 1$)
\n $u^{(k-1)}(1) = 0$
\n $u^{(k-1$

(see [3: Section 8]). For $k \in \mathbb{N}, k > 1$, we have more possibilities concerning the choice

of overdetermined classes. Here, we will deal with two special cases:
 Case I. We will consider the *maximal* class of overdetermi

functions, satisfying the maximal number of conditions:
 $u^{(i)}(0) = 0$
 $u^{(i)}(1) = 0$
 Case I. We will consider the *maximal* class of overdetermined functions, i.e., functions, satisfying the maximal number of conditions:

$$
k \in \mathbb{N}, k > 1
$$
, we have more possibilities concerning the choice. Here, we will deal with two special cases:
\nonsider the *maximal* class of overdetermined functions, i.e.,
\nmaximal number of conditions:
\n
$$
u^{(i)}(0) = 0
$$
\n
$$
u^{(i)}(1) = 0
$$
\n
$$
u^{(i)}(1) = 0
$$
\n
$$
k - 1
$$
 and
$$
\#M_0 + \#M_1 = 2k
$$
.
\nmsider the class of functions satisfying
\n
$$
u^{(i)}(0) = 0 \quad (i = 0, 1, ..., k - 1)
$$
\n
$$
u^{(k-1)}(1) = 0
$$
\n
$$
k - 1
$$
 and
$$
\#M_0 + \#M_1 = k + 1
$$
.
\n(8)

Here $M_0 = M_1 = \{0, 1, \ldots, k-1\}$ and $\#M_0 + \#M_1 = 2k$.

Case II. We will consider the class of functions satisfying

$$
k - 1
$$
 and $\#M_0 + \#M_1 = 2k$.
\nhsider the class of functions satisfying
\n $u^{(i)}(0) = 0$ $(i = 0, 1, ..., k - 1)$
\n $u^{(k-1)}(1) = 0$ $\left.\begin{matrix} \n\end{matrix}\right\}$, (8)
\n $\left.\begin{matrix} \n\end{matrix}\right\}$, $M_1 = \{k - 1\}$ and $\#M_0 + \#M_1 = k + 1$.
\nFor all functions *u* satisfying the conditions
\n $u(0) = u'(0) = ... = u^{(k-1)}(0) = 0$ (9)
\n $u(1) = u'(1) = ... = u^{(k-1)}(1) = 0$ (10)
\nas

i.e., $M_0 = \{0, 1, \dots, k-1\}, M_1 = \{k-1\}$ and $\#M_0 + \#M_1 = k+1$.

1. The Case I

The inequality (1) holds for all functions u satisfying the conditions

$$
u(0) = u'(0) = \ldots = u^{(k-1)}(0) = 0 \tag{9}
$$

or

$$
u(1) = u'(1) = \ldots = u^{(k-1)}(1) = 0 \tag{10}
$$

if and only if the functions

$$
u(0) = u'(0) = \dots = u^{(k-1)}(0) = 0
$$
(9)

$$
u(1) = u'(1) = \dots = u^{(k-1)}(1) = 0
$$
(10)
the functions

$$
B_1(x) = \left(\int_{x}^{1} (t-x)^{(k-1)q} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{0}^{x} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}
$$

$$
B_2(x) = \left(\int_{x}^{1} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{0}^{x} (x-t)^{(k-1)p'} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}
$$
(11)

are bounded on $(0, 1)$ (in the case (9)) or if and only if the functions

1 1 *(Ox Ba(x)* = *J(x - ^t) ' wo (t) dt) (Z^I'wL"(t) dt)* (12) **1 ^I** *B, (X) = (Iwo(t)dt) (I x) _1Iw_P'(i)dt) u(k_1)(l)* = 0 J

are bounded on $(0, 1)$ (in the case (10)). [Let us recall that throughout the paper, we assume (5) - i.e. $1 < p \le q < \infty$ - and that $p' = \frac{p}{p-1}$ is the number adjoint to p.]

Consequently, we can conclude that for functions satisfying (7) - i.e., (9) and (10) simultaneously - the boundedness of the pair of functions B_1, B_2 or the boundedness of the pair of functions B_3 , B_4 is a sufficient condition for the validity of inequality (1).

But we can proceed in another way: The inequality (1) holds for all functions
 u(0) = *u'*(0) = \cdots = *u*^(k-2)(0) = 0) (13) satisfying the conditions

$$
u(0) = u'(0) = \dots = u^{(k-2)}(0) = 0
$$

$$
u^{(k-1)}(1) = 0
$$
 (13)

or the conditions

$$
q < \infty - \text{ and that } p' = \frac{p}{p-1} \text{ is the number adjoint to p.}
$$
\n: conclude that for functions satisfying (7) - i.e., (9) and (10)
\nndedness of the pair of functions B_1, B_2 or the boundedness of
\n B_4 is a sufficient condition for the validity of inequality (1).
\nin another way: The inequality (1) holds for all functions
\n
$$
u(0) = u'(0) = \dots = u^{(k-2)}(0) = 0
$$
\n
$$
u^{(k-1)}(1) = 0
$$
\n
$$
u^{(k-1)}(0) = 0
$$
\n
$$
u^{(k-1)}(0) = 0
$$
\n
$$
u(1) = u'(1) = \dots = u^{(k-2)}(1) = 0
$$
\n
$$
\text{as}
$$
\n
$$
\int_{0}^{1} t^{(k-2)q} w_0(t) dt
$$
\n
$$
\int_{0}^{\frac{1}{q}} t^{p'} w_1^{1-p'}(t) dt
$$

if and only if the functions

$$
u(1) = u'(1) = \dots = u^{(k-1)}(0) = 0
$$
\n
$$
\tilde{B}_1(x) = \left(\int_{x}^{1} t^{(k-2)q} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{0}^{x} t^{p'} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}
$$
\n
$$
\tilde{B}_2(x) = \left(\int_{0}^{x} t^{(k-1)q} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{x}^{1} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}
$$
\n
$$
(15)
$$
\n
$$
\tilde{B}_2(x) = \left(\int_{0}^{x} t^{(k-1)q} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{x}^{1} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}
$$
\n
$$
(15)
$$
\n
$$
0,1) \text{ (in the case (13)) or if and only if the functions}
$$
\n
$$
0 = \left(\int_{0}^{x} (1-t)^{(k-2)q} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{x}^{1} (1-t)^{p'} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}
$$
\n
$$
0 = \left(\int_{0}^{1} (1-t)^{k-1} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{x}^{x} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}} \tag{16}
$$

are bounded on $(0,1)$ (in the case (13)) or if and only if the functions

$$
\widetilde{B}_3(x) = \left(\int_0^x (1-t)^{(k-2)q} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_x^1 (1-t)^{p'} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}
$$
\n
$$
\widetilde{B}_4(x) = \left(\int_x^1 (1-t)^{k-1} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_0^x w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}} \tag{16}
$$

are bounded on $(0,1)$ (in the case (14)). Since the conditions (13) and (14) together indicate that *u* satisfies conditions (7), the boundedness of the pair of functions $\widetilde{B}_1, \widetilde{B}_2$ or of the pair of functions \widetilde{B}_3 , \widetilde{B}_4 is another sufficient condition for the validity of inequality (1) in the Case I.

Using some other combinations of boundary conditions which together guarantee the validity of (7), we can derive also other types of sufficient conditions for (1) to hold. But first, let us formulate a conjecture motivated by the condition (6), which is necessary and sufficient for (1) to hold in the Case I if $k = 1$.

Hypothesis. Inequality (1) holds for all functions *u* satisfying (7) if and only if the conditions

The first, let us formulate a conjecture motivated by the condition (0), which is
by and sufficient for (1) to hold in the Case I if
$$
k = 1
$$
.
both
is. Inequality (1) holds for all functions u satisfying (7) if and only if
itions

$$
\sup_{0 < a < b < 1} \left(\int_a^b w_0(t) dt \right)^{\frac{1}{q}} \min \left\{ \left(\int_0^a (a - t)^{(k-1)p'} w_k^{1-p'}(t) dt \right)^{\frac{1}{p'}} \right\},
$$
(17)

$$
\left(\int_b^1 (t - b)^{(k-1)p'} w_k^{1-p'}(t) dt \right)^{\frac{1}{p'}}
$$

and

$$
\left(\int_{b} (t-b)^{(k-1)p} w_{k}^{1-p} (t) dt\right) \quad \searrow \infty
$$
\n
$$
\sup_{0 < a < b < 1} \min \left\{ \left(\int_{0}^{a} (a-t)^{(k-1)q} w_{0}(t) dt \right)^{\frac{1}{q}}, \left(\int_{a}^{b} (t-b)^{(k-1)q} w_{0}(t) dt \right)^{\frac{1}{q}} \right\} \left(\int_{a}^{b} w_{k}^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty
$$
\n
$$
(18)
$$

are satisfied.

We will not prove this hypothesis and, instead, some other sufficient conditions will be given. Let us also remark that the conditions connected with the functions B_i and concerning functions u satisfying (9) or (10) are due to V. Stepanov while the conditions expressed by the functions $\tilde{B}_{\bf i}$ and concerning functions u satisfying (13) or (14) are due to H. P. Heinig and A. Kufner. The results are collected, e.g., in [2] or [3). conditions connected

conditions connected

cerning functions u s

cerning functions u s

esults are collected,

Let w_0 and w_k be u

> 0 such that the es is and, instead, some other sufficient conditions will
the conditions connected with the functions B_i and
oncerning functions u satisfying (13) or (14) are due
results are collected, e.g., in [2] or [3].
 ∞ . Let w_0

Theorem 1. Let $1 < p \le q < \infty$. Let w_0 and w_k be weight functions on $(0,1)$ and *assume that there exists a constant C > 0 such that the estimates*

satisfying (9) or (10) are due to V. Stepanov while the conditions
\nons
$$
\tilde{B}_i
$$
 and concerning functions u satisfying (13) or (14) are due
\nKufner. The results are collected, e.g., in [2] or [3].
\n $\langle p \le q \langle \infty. Let w_0 and w_k be weight functions on (0,1) and\na constant $C > 0$ such that the estimates
\n
$$
\left(\int_0^a w_0(t) dt\right)^{\frac{1}{q}} \left(\int_a^1 w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}} \le C \qquad (19)
$$
\n
$$
\left(\int_b^1 w_0(t) dt\right)^{\frac{1}{q}} \left(\int_0^b w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}} \le C \qquad (20)
$$$

$$
\left(\int_{b}^{1} w_0(t) dt\right)^{\frac{1}{q}} \left(\int_{0}^{b} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}} \leq C \tag{20}
$$

Hardy Inequalities 391
\n
$$
\left(\int_a^b w_0(t) dt\right)^{\frac{1}{q}} \min\left\{\left(\int_0^a (a-t)^{(k-1)p'} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}\right\}
$$
\n
$$
\left(\int_b^1 (t-b)^{(k-1)p'} w_k^{1-p'}(t) dt\right)^{\frac{1}{p'}}\right\} \leq C
$$
\n
$$
y a, b \in (0,1) \ (a < b \ in (21)). \ \text{Then the Hardy inequality (1) holds for}
$$
\n
$$
w \in AC^{(k-1)}(0,1) \ \text{satisfying the conditions}
$$
\n
$$
u^{(i)}(0) = u^{(i)}(1) = 0 \qquad (i = 0,1,\ldots,k-1).
$$
\n
$$
(22)
$$
\n
$$
x.
$$
 Notice that condition (21) is in fact condition (17) in the Hypothesis and\n
$$
x^2 + y^2 = 1.
$$
\n
$$
y^2 + z^2 = 1.
$$
\n
$$
y^2 +
$$

hold for every $a, b \in (0, 1)$ $(a < b$ in (21)). Then the Hardy inequality (1) holds for *every function* $u \in AC^{(k-1)}(0, 1)$ *satisfying the conditions*

$$
u^{(i)}(0) = u^{(i)}(1) = 0 \qquad (i = 0, 1, \dots, k - 1). \tag{22}
$$

Remark. Notice that condition *(21) is* in fact condition (17) in the Hypothesis and coincides with (6) for $k = 1$. In fact, we follow in the proof the idea of P. Gurka used for $k = 1$ (see, e.g., [3: Section 8]).

Proof of Theorem 1. There is an integer $m \in \mathbb{Z}$ such that

$$
2^m \le \sup_{0 < x < 1} |u(x)| < 2^{m+1}
$$

For $j \in \mathbb{Z}, j \leq m$, let x_j and y_j be the smallest and greatest numbers from $(0, 1)$ such For $j \in \mathbb{Z}, j \leq m$, let x_j and y_j be the smallest and greatest numbers from $(0, 1)$ such that $|u(x_j)| = 2^j$ and $|u(y_j)| = 2^j$, respectively. Obviously, $x_{j-1} < x_j \leq y_j < y_{j-1}$, $(0,1) = \bigcup_{j \le m} [x_{j-1}, x_j] \cup [x_m, y_m] \bigcup_{j \le m} [y_j, y_{j-1}]$ and

\n- \n 8: Section 8].\n
\n- \n 8: Section 8].\n
\n- \n 9:
$$
2^m \leq \sup_{0 < x < 1} |u(x)| < 2^{m+1}
$$
.\n
\n- \n 1: x_j and y_j be the smallest and greatest number $|u(y_j)| = 2^j$, respectively. Obviously, $x_{j-j} \cup [x_m, y_m] \bigcup_{j < m} [y_j, y_{j-1}]$ and\n
\n- \n
$$
\int_0^1 |u(x)|^q w_0(x) \, dx \leq \sum_{j \leq m} 2^{(j+1)q} \int_{x_j}^{y_j} w_0(x) \, dx.
$$
\n We have\n
\n

Since *u* satisfies (22), we have

$$
(k-1)!\,u(x_j) = \int\limits_{0}^{x_j} (x_j - t)^{k-1} u^{(k)}(t) dt
$$

$$
(k-1)!\,u(x_{j-1}) = \int\limits_{0}^{x_{j-1}} (x_{j-1} - t)^{k-1} u^{(k)}(t) dt
$$

and

$$
(k-1)! u(x_j) = \int_{0}^{x_j} (x_j - t)^{k-1} u^{(k)}(t) dt
$$

\n
$$
(k-1)! u(x_{j-1}) = \int_{0}^{x_{j-1}} (x_{j-1} - t)^{k-1} u^{(k)}(t) dt
$$

\nd
\n
$$
(k-1)! |u(x_j) - u(x_{j-1})|
$$

\n
$$
= \left| \int_{0}^{x_{j-1}} [(x_j - t)^{k-1} - (x_{j-1} - t)^{k-1}] u^{(k)}(t) dt + \int_{x_{j-1}}^{x_j} (x_j - t)^{k-1} u^{(k)}(t) dt \right|
$$

\n
$$
\leq (k-1)(x_j - x_{j-1}) \int_{0}^{x_{j-1}} |u^{(k)}(t)| dt + \int_{x_{j-1}}^{x_j} (x_j - t)^{k-1} |u^{(k)}(t)| dt.
$$
\n(23)

Using the Hölder inequality and the estimate

There and C. G. Simader

\nlder inequality and the estimate

\n
$$
|u(x_j) - u(x_{j-1})| \ge |u(x_j)| - |u(x_{j-1})| = 2^j - 2^{j-1} = 2^{j-1},
$$

we obtain

the Hölder inequality and the estimate
\n
$$
|u(x_j) - u(x_{j-1})| \ge |u(x_j)| - |u(x_{j-1})| = 2^j - 2^{j-1} = 2^{j-1},
$$
\n
$$
2^{j-1} \le c(x_j - x_{j-1}) \left(\int_0^{x_{j-1}} |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{1}{p}} \left(\int_0^{x_{j-1}} w_k^{1-p'}(t) dt \right)^{\frac{1}{p'}}
$$
\n
$$
+ c \left(\int_{x_{j-1}}^{x_j} |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{1}{p}} \left(\int_{x_{j-1}}^{x_j} (x_j - t)^{(k-1)p'} w_k^{1-p'}(t) dt \right)^{\frac{1}{p'}} ,
$$
\n
$$
=
$$
\n
$$
y_j
$$

and consequently,

$$
+ c \left(\int_{z_{j-1}} |u^{(k)}(t)|^p w_k(t) dt \right) \left(\int_{z_{j-1}} (x_j - t)^{(k-1)p} w_k^{k-p} (t) dt \right),
$$

\nconsequently,
\n
$$
2^{(j-1)q} \int_{z_j}^{y_j} w_0(t) dt \leq 2^{q-1} c^q (x_j - x_{j-1})^q \left(\int_0^{z_{j-1}} |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p}}
$$
\n
$$
\times \left(\int_{z_j}^{y_j} w_0(t) dt \right) \left(\int_0^{z_{j-1}} w_k^{1-p'}(t) dt \right)^{\frac{q}{p'}} + 2^{q-1} c^q \left(\int_{z_{j-1}}^{z_j} |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p}}
$$
\n
$$
\times \left(\int_{z_j}^{y_j} w_0(t) dt \right) \left(\int_{z_{j-1}}^{z_j} (x_j - t)^{(k-1)p'} w_k^{1-p'}(t) dt \right)^{\frac{q}{p'}}.
$$
\n
$$
(\text{24})
$$

\nreover, using condition (20) (with $b = x_j$), the first term on the right-hand side can

Moreover, using condition (20) (with $b = x_j$), the first term on the right-hand side can be estimated by

$$
\left(\int_{x_j}^{f} \int_{x_{j-1}}^{f} \int_{x_{j-1}}^{f} \int_{x_{j-1}}^{x_{j-1}} \int_{x_{j-1}}^{x_{j}} \int_{y_{j}}^{y_{j}} \int_{y_{j}}^{y_{j}} \int_{y_{j}}^{x_{j}} dx \right)
$$
\n
$$
(x_j - x_{j-1})^q \left(\int_0^1 |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p}} \left(\int_0^1 w_0(t) dt \right) \left(\int_0^{x_j} w_k^{1-p'}(t) dt \right)^{\frac{q}{p}}
$$
\n
$$
\leq (x_j - x_{j-1})^q \left(\int_0^1 |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p}} C^q,
$$

 \boldsymbol{i}

so that we have finally

2.

Hardy Inequality 393\n
$$
2^{(j-1)q} \int_{x_j}^{y_j} w_0(t) dt \le 2^{q-1} c^q C^q (x_j - x_{j-1})^q \left(\int_0^1 |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p}}
$$
\n
$$
+ 2^{q-1} c^q \left(\int_{x_{j-1}}^{x_j} |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p}}
$$
\n
$$
\times \left(\int_{x_j}^{y_j} w_0(t) dt \right) \left(\int_{x_{j-1}}^{x_j} (x_j - t)^{(k-1)p'} w_k(t) dt \right)^{\frac{q}{p'}}.
$$
\nSince u satisfies (22), we have also $(-1)^k (k-1)! u(y) = \int_0^1 (t-y)^{k-1} u^{(k)}(t) dt$. Putting there $y = y_j$ and $y = y_{j-1}$, we obtain immediately the following analogy of (24):\n
$$
2^{(j-1)q} \int_{x_j}^{y_j} w_0(t) dt \le 2^{q-1} c^q \left[(y_{j-1} - y_j)^q \left(\int_{y_{j-1}}^1 |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p}}
$$
\n
$$
y_j = \left[(y_j - t)^{q/2} \left((y_{j-1} - y_j)^q \left(\int_{y_{j-1}}^1 |u^{(k)}(t)|^p w_k(t) dt \right)^{\frac{q}{p'}} \right)^{\frac{q}{p'}} \right]
$$

here $y = y_j$ and $y = y_{j-1}$, we obtain immediately the following analogy of (24):

$$
\begin{pmatrix}\ny_j \\
x_{j-1}\n\end{pmatrix}\n\times\n\left(\int_{x_j}^{y_j} w_0(t) dt\right)\n\left(\int_{x_{j-1}}^{x_j} (x_j - t)^{(k-1)p'} w_k(t) dt\right)^{\frac{q}{p'}}
$$
\nSince *u* satisfies (22), we have also $(-1)^k (k-1)! u(y) = \int_{y_j}^{1} (t-y)^{k-1} u^{(k)}(t)$
\nhere $y = y_j$ and $y = y_{j-1}$, we obtain immediately the following analogy of\n
$$
2^{(j-1)q} \int_{x_j}^{y_j} w_0(t) dt \leq 2^{q-1} c^q \left[(y_{j-1} - y_j)^q \left(\int_{y_{j-1}}^1 |u^{(k)}(t)|^p w_k(t) dt\right)\right)
$$
\n
$$
\times \left(\int_{x_j}^{y_j} w_0(t) dt\right)\n\left(\int_{y_{j-1}}^1 w_k^{1-p'}(t) dt\right)^{\frac{q}{p'}} +\n\left(\int_{y_j}^{y_{j-1}} |u^{(k)}(t)|^p w_k(t) dt\right)^{\frac{q}{p'}}\n\left(\int_{y_j}^{y_j} w_0(t) dt\right)
$$
\n
$$
\times \left(\int_{y_j}^{y_{j-1}} |t-y_j|^{(k-1)p'} w_k(t) dt\right)^{\frac{q}{p'}}\n\left(\int_{x_j}^{y_j} w_0(t) dt\right)^{\frac{q}{p'}}\right],
$$
\nand condition (19) (with $a = y_{j-1}$) leads finally to the estimate

leads finally to the estimate

l,

$$
\begin{pmatrix}\ny_j & y_j & y_k \\
\sqrt{x_j} & y_k & y_k \\
\sqrt{x_j} & y_k & y_k \\
\frac{y_j}{y_j} & y_k & y_k\n\end{pmatrix}\n\right\}
$$
\n
$$
\text{condition (19) (with } a = y_{j-1}) \text{ leads finally to the estimate}
$$
\n
$$
2^{(j-1)q} \int_{x_j}^{y_j} w_0(t) \, dt \leq 2^{q-1} c^q C^q (y_{j-1} - y_j)^q \left(\int_0^1 |u^{(k)}(t)|^p w_k(t) \, dt \right)^{\frac{q}{p}}
$$
\n
$$
+ 2^{q-1} c^q \left(\int_{y_j}^{y_{j-1}} |u^{(k)}(t)|^p w_k(t) \, dt \right)^{\frac{q}{p}}
$$
\n
$$
\times \left(\int_{x_j}^{y_j} w_0(t) \, dt \right) \left(\int_{y_j}^{y_{j-1}} (t - y_j)^{(k-1)p'} w_k^{1-p'}(t) \, dt \right)^{\frac{q}{p'}}
$$
\n
$$
(26)
$$

If

For
$$
d
$$
 and C . G . G . G . G and G is a function of d and d and d is a function of d and d is a function of d and d is a function of d and d and d is a function of d and d and d is a function of d and d are a function of $$

then according to condition (21) (with $a = x_j$ and $b = y_j$) we have

$$
\begin{aligned}\n\text{so condition (21) (with } a = x_j \text{ and } b = y_j \text{) we have} \\
\left(\int\limits_{0}^{x_j} (x_j - t)^{(k-1)p'} w_k^{1-p'}(t) dt\right)^{\frac{q}{p'}} \left(\int\limits_{x_j}^{y_j} w_0(t) dt\right) &\leq C^q,\n\end{aligned}
$$

and (25) yields

$$
\left(\int_{0}^{t} (x_{j} - t)^{(k-1)p'} w_{k}^{1-p'}(t) dt\right) \left(\int_{x_{j}} w_{0}(t) dt\right) \leq C^{q},
$$
\n1 (25) yields

\n
$$
2^{(j-1)q} \int_{x_{j}}^{y_{j}} w_{0}(t) dt
$$
\n
$$
\leq c_{1} \left[(x_{j} - x_{j-1})^{q} \left(\int_{0}^{1} |u^{(k)}(t)|^{p} w_{k}(t) dt \right)^{\frac{q}{p}} + \left(\int_{x_{j-1}}^{x_{j}} |u^{(k)}(t)|^{p} w_{k}(t) dt \right)^{\frac{q}{p}} \right].
$$
\nIn (27) the reverse inequality holds, we have – again according to (21) – that

\n
$$
\left(\int_{0}^{1} (t - y_{j})^{(k-1)q'} w_{k}^{1-p'}(t) dt \right)^{\frac{q}{p'}} \left(\int_{0}^{y_{j}} w_{0}(t) dt \right) \leq C^{q},
$$

If in (27) the reverse inequality holds, we have $-$ again according to (21) $-$ that

$$
\left(\int_{y_j}^1 (t-y_j)^{(k-1)q'} w_k^{1-p'}(t) dt\right)^{\frac{q}{p'}} \left(\int_{x_j}^{y_j} w_0(t) dt\right) \leq C^q,
$$

and (26) yields

$$
\left(\frac{1}{y_{j}}\right) \left(\frac{1}{z_{j}}\right)
$$
\n
$$
2^{(j-1)q} \int_{x_{j}}^{y_{j}} w_{0}(t) dt
$$
\n
$$
\leq c_{1} \left[(y_{j-1} - y_{j})^{q} \left(\int_{0}^{1} |u^{(k)}(t)|^{p} w_{k}(t) dt \right)^{\frac{q}{p}} + \left(\int_{y_{j}}^{y_{j-1}} |u^{(k)}(t)|^{p} w_{k}(t) dt \right)^{\frac{q}{p}} \right]
$$

But then we have from (23) that

$$
\left[\int_{0}^{1} |u(x)|^{q} w_{0}(x) dx\right]^{1} \left(\int_{0}^{1} |u(x)|^{q} w_{0}(x) dx\right]
$$
\n
$$
\leq 4c_{1} \sum_{j \leq m} \left[(x_{j} - x_{j-1})^{q} + (y_{j-1} - y_{j})^{q} \right] \left(\int_{0}^{1} |u^{(k)}(t)|^{p} w_{k}(t) dt\right)^{\frac{q}{p}}
$$
\n
$$
+ 4c_{1} \sum_{j \leq m} \left[\left(\int_{x_{j-1}}^{x_{j}} |u^{(k)}(t)|^{p} w_{k}(t) dt\right)^{\frac{q}{p}} + \left(\int_{y_{j}}^{y_{j-1}} |u^{(k)}(t)|^{p} w_{k}(t) dt\right)^{\frac{q}{p}} \right]
$$

and the Hardy inequality (1) follows as $\frac{q}{p} \geq 1$

2. The case II

Now, let us consider inequality (1) on the class of functions satisfying the conditions

uality (1) on the class of functions satisfying the conditions
\n
$$
u(0) = u'(0) = \ldots = u^{(k-1)}(0) = 0
$$
\n
$$
u^{(k-1)}(1) = 0.
$$
\n(28)

Similarly as in the Case I, we can derive sufficient conditions combining some welldetermined classes. E.g., the boundedness of the functions B_1, B_2 from (11) or the boundedness of the functions \widetilde{B}_1 , \widetilde{B}_2 from (15) is such a sufficient condition, since the boundedness of B_1, B_2 guarantees the validity of (1) for functions satisfying conditions (9), and the boundedness of \widetilde{B}_1 , \widetilde{B}_2 guarantees the validity of (1) for functions satisfying conditions (13). Now, conditions (9) together with (13) cover the case (28). boundedness of the functions B_1, B_2 from (11) or the \tilde{H}_1, \tilde{B}_2 from (15) is such a sufficient condition, since the velocity of (1) for functions satisfying conditions \tilde{B}_2 guarantees the validity of (1) fo

Here, we will proceed in another way. We will reduce the investigation of inequality (1) for functions satisfying (28) to the investigation of a weighted norm inequality for a special (integral) operator. For this purpose, let us introduce, for a fixed number *z,* $0 < z < 1$, the operator T_z by the formula

$$
(T_x f)(x) = \sum_{i=1}^{4} (T_i f)(x)
$$
 (29)

where

z 1 *(T1 f)(x)* = X(o,) (k-i)! (X) *f(x_s)f(s)ds* **0** *z* ¹*(T2 f)(x) -* (k i)! X(.,,) (X) *f* [(x - *s)' -* (x - *z)'-'] f (s) ds -* **0** *z* **(30)** *(T3 f)(x)* = ______X(Z,l)(x) *J* [(x - *z)k_1 - (- S)k_11f(5) ds (k-i)!* z ¹*(T4 f)(x)* = (k i)! X(z,1)(X) *P X - z) k_h f(s) ds -* I *I Tzf(x)w0(x) dx)* <c (/ *If(^x)I wk(x) dx)* **(31)**

where $\chi_{(a,b)}$ denotes the characteristic function of the interval (a,b) . It is easy to see that the operators T_i are positive, i.e., that $(T_i f) \geq 0$ for $f \geq 0$ $(i = 1, \ldots, 4)$.

Lemma 1. *The problem of investigating inequality* (1) *for functions u satisfying* **(28)** *is equivalent to the problem of investigating the weighted norm inequality*

notes the characteristic function of the interval
$$
(a, b)
$$
. It is easy to see
\nors T_i are positive, i.e., that $(T_i f) \ge 0$ for $f \ge 0$ $(i = 1, ..., 4)$.
\nThe problem of investigating inequality (1) for functions u satisfying
\nat to the problem of investigating the weighted norm inequality
\n
$$
\left(\int_0^1 |T_z f(x)|^q w_0(x) dx\right)^{\frac{1}{q}} \le c \left(\int_0^1 |f(x)|^p w_k(x) dx\right)^{\frac{1}{p}}
$$
\n(31)

for functions f satisfying the additional condition

er
\n
$$
\int_{0}^{z} f(t) dt = \int_{z}^{1} f(t) dt.
$$
\n(32)
\n1. Then we have
\n
$$
\int_{0}^{z} (x-t)^{j-1} u^{(j)}(t) dt \qquad (j = 1, 2, ..., k)
$$
\n(33)
\n2. If $(x-t)^{j-1} u^{(j)}(t) dt$ \n(34)
\n4. If $u^{(k)}(s) ds$ for $t \le z$ \n(35)
\n(36)
\n-1, we obtain after easy calculations that, with (34),

Proof. (i) Let *u* satisfy (28). Then we have

satisfying the additional condition
\n
$$
\int_{0}^{z} f(t) dt = \int_{z}^{1} f(t) dt.
$$
\n(32)
\nLet *u* satisfy (28). Then we have
\n
$$
u(x) = \frac{1}{(j-1)!} \int_{0}^{x} (x-t)^{j-1} u^{(j)}(t) dt \qquad (j = 1, 2, ..., k)
$$
\n(33)
\nend *z*, 0 < *z* < 1, we can write
\n
$$
u^{(k-1)}(t) = \begin{cases} \int_{0}^{t} u^{(k)}(s) ds & \text{for } t \leq z \\ -\int_{t}^{1} u^{(k)}(s) ds & \text{for } t \geq z. \end{cases}
$$
\n(34)
\n*x* > *z* and *j* = *k* - 1, we obtain after easy calculations that, with (34),

and for any fixed $z, 0 < z < 1$, we can write

$$
u^{(k-1)}(t) = \begin{cases} \int_0^t u^{(k)}(s) \, ds & \text{for } t \le z \\ -\int_t^1 u^{(k)}(s) \, ds & \text{for } t \ge z. \end{cases} \tag{34}
$$

Using (33) for $x > z$ and $j = k - 1$, we obtain after easy calculations that, with (34),

ing (33) for
$$
x > z
$$
 and $j = k - 1$, we obtain after easy calculations that, with (34)
\n
$$
u(x) = \frac{1}{(k-2)!} \int_{0}^{x} (x-t)^{k-2} u^{(k-1)}(t) dt
$$
\n
$$
= \frac{1}{(k-2)!} \int_{0}^{z} (x-t)^{k-2} \int_{0}^{t} u^{(k)}(s) ds dt - \frac{1}{(k-2)!} \int_{z}^{x} (x-t)^{k-2} \int_{t}^{1} u^{(k)}(s) ds dt
$$
\n
$$
= \frac{1}{(k-1)!} \left\{ \int_{0}^{z} [(x-s)^{k-1} - (x-z)^{k-1}] u^{(k)}(s) ds - (x-z)^{k-1} \int_{x}^{1} u^{(k)}(s) ds \right\}.
$$
\n
$$
- \int_{z}^{x} [(x-z)^{k-1} - (x-s)^{k-1}] u^{(k)}(s) ds - (x-z)^{k-1} \int_{x}^{1} u^{(k)}(s) ds \right\}.
$$
\nfinding

\n
$$
f(x) = \begin{cases} u^{(k)}(x) & \text{for } x \leq z \\ -u^{(k)}(x) & \text{for } x > z \end{cases}
$$
\n(3

\nimmediately have $u(x) = (T_z f)(x)$. Since $|f(x)| = |u^{(k)}(x)|$ and $\int_{0}^{z} f(x) dx$

Denoting

$$
f(x) = \begin{cases} u^{(k)}(x) & \text{for } x \leq z \\ -u^{(k)}(x) & \text{for } x > z \end{cases}
$$
 (35)

we immediately have $u(x) = (T_x f)(x)$. Since $|f(x)| = |u^{(k)}(x)|$ and $\int_0^x f(x) dx =$
 $\int_x^1 f(x) dx$, we see that (1), (28) imply (31), (32).

(ii) Let f satisfy (31) and (32) and define F as
 $F(x) = \begin{cases} f(x) & \text{for } x \leq z \\ -f(x) & \text{for } x$ $\int_z^1 f(x) dx$, we see that (1), (28) imply (31), (32).

(ii) Let *f* satisfy *(31)* and *(32)* and define *F* as

$$
F(x) = \begin{cases} f(x) & \text{for } x \leq z \\ -f(x) & \text{for } x > z. \end{cases}
$$

Then the function
$$
u = T_z f
$$
 is a solution of the Cauchy problem
\n
$$
u^{(k)} = F \quad \text{in} \quad (0,1)
$$
\n
$$
u^{(i)}(0) = 0 \quad \text{for} \quad i = 1,2,...,k-1,
$$

and due to (32), *u* satisfies also the condition $u^{(k-1)}(1) = 0$. Inequality (1) now follows from (31)

Remark. The idea of reducing inequality (1) with condition (28) to the weighted norm inequality for the operator T_z was submitted to us by Prof. R. Oinarov which is here gratefully acknowledged.

Theorem 2. Let $1 < p \le q < \infty$. Let w_0 and w_k be weight functions on $(0,1)$ and *let* $z \in (0,1)$ *be determined by the condition*

Hardy Inequalities 397
\neducing inequality (1) with condition (28) to the weighted
\nator
$$
T_z
$$
 was submitted to us by Prof. R. Oinarov which is
\n
$$
\leq q < \infty. \text{ Let } w_0 \text{ and } w_k \text{ be weight functions on } (0,1) \text{ and}
$$
\n
$$
\int_0^z w_k^{1-p'}(t) dt = \int_z^1 w_k^{1-p'}(t) dt. \qquad (36)
$$
\n
$$
\text{unders are bounded:}
$$
\n
$$
-x)^{(k-1)q} w_0(t) dt \bigg|_0^{\frac{1}{q}} \left(\int_z^z w_k^{1-p'}(t) dt \right)^{\frac{1}{p'}} \qquad (37)
$$

Assume that the following numbers are bounded:

$$
B_{1} = \sup_{0 < x < z} \left(\int_{x}^{z} (t - x)^{(k-1)q} w_{0}(t) dt \right)^{\frac{1}{q}} \left(\int_{0}^{x} w_{k}^{1-p'}(t) dt \right)^{\frac{1}{p'}} \qquad (37)
$$

\n
$$
B_{2} = \sup_{0 < x < z} \left(\int_{x}^{z} w_{0}(t) dt \right)^{\frac{1}{q}} \left(\int_{0}^{x} (x - t)^{(k-1)p'} w_{k}^{1-p'}(t) dt \right)^{\frac{1}{p'}} \qquad (38)
$$

\n
$$
B_{3} = \sup_{x \to z} \left(\int_{0}^{1} (t - z)^{(k-2)q} w_{0}(t) dt \right)^{\frac{1}{q}} \left(\int_{0}^{x} (t - z)^{p'} w_{k}^{1-p'}(t) dt \right)^{\frac{1}{p'}} \qquad (39)
$$

$$
B_2 = \sup_{0 < x < z} \left(\int_z^{z} w_0(t) \, dt \right)^{\frac{1}{q}} \left(\int_0^{z} (x - t)^{(k-1)p'} w_k^{1-p'}(t) \, dt \right)^{\frac{1}{p'}} \tag{38}
$$

$$
B_2 = \sup_{0 < x < z} \left(\int_z^z w_0(t) \, dt \right)^{\frac{1}{q}} \left(\int_0^x (x - t)^{(k-1)p'} w_k^{1-p'}(t) \, dt \right)^{\frac{1}{p'}} \tag{38}
$$
\n
$$
B_3 = \sup_{z < z < 1} \left(\int_z^1 (t - z)^{(k-2)q} w_0(t) \, dt \right)^{\frac{1}{q}} \left(\int_z^x (t - z)^{p'} w_k^{1-p'}(t) \, dt \right)^{\frac{1}{p'}} \tag{39}
$$
\n
$$
B_4 = \sup_{z < z} \left(\int_z^z (t - z)^{(k-1)q} w_0(t) \, dt \right)^{\frac{1}{q}} \left(\int_z^1 w_k^{1-p'}(t) \, dt \right)^{\frac{1}{p'}} \tag{40}
$$

$$
B_4 = \sup_{z < z < 1} \left(\int_z^z (t-z)^{(k-1)q} w_0(t) \, dt \right)^{\frac{1}{q}} \left(\int_z^1 w_k^{1-p'}(t) \, dt \right)^{\frac{1}{p'}} \tag{40}
$$

$$
B_4 = \sup_{z < z < 1} \left(\int_z (t-z)^{(k-1)q} w_0(t) dt \right) \left(\int_x w_k^{1-p'}(t) dt \right) \tag{40}
$$
\n
$$
B_5 = \left(\int_z^1 t^{(k-2)q} w_0(t) dt \right)^{\frac{1}{q}} \left(\int_0^z (z-t)^{p'} w_k^{1-p'}(t) dt \right)^{\frac{1}{p'}}. \tag{41}
$$
\n
$$
Hardy inequality (1) holds for every function u satisfying conditions (28).
$$
\n
$$
f. Due to Lemma 1, it suffices to show that $B_i < \infty$ ($i = 1, ..., 5$) implies (31) for all f satisfying (32). Inequality (31) can be rewritten by using the $|v||_{r,w} = (\int_0^1 |v(x)|^r w(x) dx)^{1/r}$ in the form
$$
||T_x f||_{q,w_0} \leq c ||f||_{p,w_k}. \tag{42}
$$
\n
$$
u = \text{pair of conditions } B_1 < \infty, B_2 < \infty \text{ is necessary and sufficient for the inequality}
$$
$$

Then the Hardy inequality (1) holds for every function u satisfying conditions (28).

Proof. Due to Lemma 1, it suffices to show that $B_i < \infty$ ($i = 1, ..., 5$) implies inequality (31) for all *f* satisfying (32). Inequality (31) can be rewritten by using the notation $||v||_{r,w} = (\int_0^1 |v(x)|^r w(x) dx)^{1/r}$ in the form

$$
||T_z f||_{q,w_0} \le c ||f||_{p,w_k}.\tag{42}
$$

(i) The pair of conditions $B_1 < \infty, B_2 < \infty$ is necessary and sufficient for the validity of the inequality

$$
|v||_{r,w} = (J_0 | v(x)|^2 w(x) dx)^{1/2} \text{ in the form}
$$
\n
$$
||T_x f||_{q,w_0} \le c ||f||_{p,w_k}.
$$
\ne pair of conditions $B_1 < \infty, B_2 < \infty$ is necessary and suffice the inequality

\n
$$
\left(\int_0^x w_0(x) \left| \int_0^x (x-t)^{k-1} f(t) dt \right|^q dx \right)^{\frac{1}{q}} \le c_1 \left(\int_0^x |f(t)|^p w_k(t) dt \right)^{\frac{1}{p}}
$$
\nIt is due to V. Stepanov; see, e.g., [3: Section 10]). But the last

\n
$$
||T_1 f||_{q,w_0} \le c_1 ||f||_{p,w_k}
$$

(this result is due to V. Stepanov; see, e.g., [3: Section 10]). But the last inequality implies

$$
||T_1 f||_{q,w_0} \leq c_1 ||f||_{p,w_k} \tag{43}
$$

due to the definition of T_1 (see (30)).

(ii) The inequality

and C. G. Simader
ion of
$$
T_1
$$
 (see (30)).
ality

$$
(x-z)^{k-1} - (x-s)^{k-1} \le (s-z)(k-1)(x-z)^{k-2}
$$

for $0 \leq z \leq s \leq x$ implies that

Kufner and C. G. Simader
\ndefinition of
$$
T_1
$$
 (see (30)).
\ne inequality
\n
$$
(x-z)^{k-1} - (x-s)^{k-1} \le (s-z)(k-1)(x-z)^{k-2}
$$
\n
$$
s \le x \text{ implies that}
$$
\n
$$
||T_3 f||_{q,w_0} \le c_2 \left(\int_z^1 w_0(x)(x-z)^{(k-2)q} \left| \int_z^z (s-z)f(s) ds \right|^q dx \right)^{\frac{1}{q}}
$$

inequality for the function $F(s) = (s - z)f(s)$:

The condition
$$
B_3 < \infty
$$
 is necessary and sufficient for the validity of the following Hardy
inequality for the function $F(s) = (s - z)f(s)$:

$$
\left(\int_z^1 W_0(x) \left| \int_z^x F(s) ds \right|^q dx \right)^{\frac{1}{q}} \leq c_3 \left(\int_z^1 |F(s)|^p (s - z)^{-p} w_k(s) ds \right)^{\frac{1}{p}}
$$

$$
= c_3 \left(\int_z^1 |f(s)|^p w_k(s) ds \right)^{\frac{1}{p}}
$$
with $W_0(x) = w_0(x)(x - z)^{(k-2)q}$, and the last two inequalities imply
 $||T_3 f||_{q,w_0} \leq c_4 ||f||_{p,w_k}$. (44)
(iii) Similarly, we have

with $W_0(x) = w_0(x)(x - z)^{(k-2)q}$, and the last two inequalities imply

$$
||T_3 f||_{q,w_0} \le c_4 ||f||_{p,w_k}.
$$
\n(44)

(iii) Similarly, we have

$$
= c_3 \left(\int_{z} |f(s)|^p w_k(s) ds \right)
$$

\n
$$
w_0(x)(x - z)^{(k-2)q}
$$
, and the last two inequalities imply
\n
$$
||T_3 f||_{q,w_0} \le c_4 ||f||_{p,w_k}.
$$

\n(44)
\n
$$
||T_4 f||_{q,w_0} \le \left(\int_{z}^{1} w_0(x)(x - z)^{(k-1)q} \left| \int_{z}^{1} f(s) ds \right|^q dx \right)^{\frac{1}{q}}
$$

\n
$$
\le c_5 \left(\int_{z}^{1} |f(s)|^p w_k(s) ds \right)^{\frac{1}{p}}
$$

\n
$$
\le c_5 ||f||_{p,w_k}
$$

\n(45)

if and only if $B_4 < \infty$, since it is in fact a Hardy inequality on $(z, 1)$ with weight $w_0(x)(x-z)^{(k-1)q}$ at the left-hand side.

(iv) Using the inequality

$$
(x-s)^{k-1}-(x-z)^{k-1}\leq (z-s)(k-1)(x-s)^{k-2}\leq (z-s)(k-1)x^{k-2}
$$

for $0\leq s\leq z\leq x$ and then Hölder's inequality, we obtain

Hardy Inequality, we obtain
\n
$$
||T_2 f||_{q,w_0} \leq \left(\int_z^1 w_0(x)\left|\int_0^z \left[(x-s)^{k-1}-(x-z)^{k-1}\right]f(s) ds\right|^q dx\right)^{\frac{1}{q}}
$$
\n
$$
\leq c_6 \left(\int_z^1 w_0(x)x^{(k-2)q} \left|\int_0^z f(s)(z-s) ds\right|^q\right)^{\frac{1}{q}}
$$
\n
$$
= c_6 \left(\int_z^1 x^{(k-2)q} w_0(x) dx\right)^{\frac{1}{q}} \left|\int_0^z f(s)(z-s) ds\right|^q
$$
\n
$$
\leq c_6 \left(\int_z^1 x^{(k-2)q} w_0(x) dx\right)^{\frac{1}{q}}
$$
\n
$$
\leq c_6 \left(\int_z^1 x^{(k-2)q} w_0(x) dx\right)^{\frac{1}{q}}
$$
\n
$$
\times \left(\int_0^z |f(s)|^p w_k(x) dx\right)^{\frac{1}{p}}
$$
\n
$$
= c_6 B_5 ||f||_{p,w_k}
$$
\nvided $B_5 < \infty$.

\n(v) Since $||T_1 f||_{q,w_0} \leq \sum_{i=1}^4 ||T_i f||_{q,w_0}$, the inequality (42) – and consequently, the
trion of Theorem 2 – follows from (43) · (46) **ii**

\nMoreover, some of the conditions $B_i < \infty$ are also necessary for (1) to hold with

provided $B_5 < \infty$.

assertion of Theorem 2 − follows from (43) \cdot (46) **L**

Moreover, some of the conditions $B_i < \infty$ are also *necessary* for (1) to hold with functions *u* satisfying (28). This follows from the following assertion.

Lemma 2. Let $1 < p \le q < \infty$. Let w_0 and w_k be weight functions and let z be *determined by* (36). If the inequality (42), i.e. $||T_xf||_{q,w_0} \leq C||f||_{p,w_k}$ holds for functions *f satisfying* (32), then $B_i < \infty$ for $i = 1, 2, 3, 4$ with B_i given by formulas (37) – (40).

Proof. (i) For every $t \in [0, z]$, there is a $t_1 \in [z, 1]$ such that

$$
\sum_{i=1}^{4} ||T_i f||_{q,w_0},
$$
 the inequality (42) – and consequently, the
lows from (43) - (46) \blacksquare
conditions $B_i < \infty$ are also necessary for (1) to hold with
This follows from the following assertion.
 $\leq q < \infty$. Let w_0 and w_k be weight functions and let z be
equality (42), i.e. $||T_z f||_{q,w_0} \leq C ||f||_{p,w_k}$ holds for functions
 ∞ for $i = 1, 2, 3, 4$ with B_i given by formulas (37) – (40).
 $\in [0, z]$, there is a $t_1 \in [z, 1]$ such that

$$
\int_0^t w_k^{1-p'}(s) ds = \int_{t_1}^1 w_k^{1-p'}(s) ds.
$$
 (47)

Define

3). This follows from the following assertion.
\n
$$
p \le q < \infty
$$
. Let w_0 and w_k be weight functions and let z be
\ne inequality (42), i.e. $||T_z f||_{q,w_0} \le C ||f||_{p,w_k}$ holds for functions
\n $i < \infty$ for $i = 1, 2, 3, 4$ with B_i given by formulas (37) – (40).
\n $t \in [0, z]$, there is a $t_1 \in [z, 1]$ such that
\n
$$
\int_0^t w_k^{1-p'}(s) ds = \int_{t_1}^1 w_k^{1-p'}(s) ds.
$$
\n(47)
\n
$$
f_1(s) = \begin{cases} w_k^{1-p'}(s) & \text{for } s \in (0, t) \\ 0 & \text{for } s \in [t, t_1] \\ w_k^{1-p'}(s) & \text{for } s \in (t_1, 1). \end{cases}
$$
\n(48)
\n $||T_1 f_1||_{q,w_0} \le ||T_z f_1||_{q,w_0} \le C ||f_1||_{p,w_k}.$ \n(49)

Then f_1 satisfies (32), and since $f_1 \ge 0$ and the operators T_i are positive, we have from *(42)* that

$$
||T_1 f_1||_{q,w_0} \leq ||T_z f_1||_{q,w_0} \leq C||f_1||_{p,w_k}.
$$
\n(49)

400 A. Kufner and C. G. Simader

 $\text{Now, } (w_k^{1-p'})^p w_k = w_k^{1+p(1-p')} = w_k^{1-p'} \text{ and }$

$$
||f_1||_{p,w_k} = \left(\int_0^t w_k^{1-p'}(s) \, ds + \int_{t_1}^1 w_k^{1-p'}(s) \, ds\right)^{\frac{1}{p}} = 2^{\frac{1}{p}} \left(\int_0^t w_k^{1-p'}(s) \, ds\right)^{\frac{1}{p}}
$$

while

$$
(k-1)!\|T_1f\|_{q,w_0} \ge \left(\int_0^z w_0(x) \left| \int_0^x (x-s)^{k-1} f_1(s) ds \right|^q dx \right)^{\frac{1}{q}}
$$

\n
$$
\ge \left(\int_0^z w_0(x) \left(\int_0^z (x-s)^{k-1} f_1(s) ds \right)^q dx \right)^{\frac{1}{q}}
$$

\n
$$
= \left(\int_t^z w_0(x) \left(\int_0^t (x-s)^{k-1} w_k^{1-p'}(s) ds \right)^q dx \right)^{\frac{1}{q}}
$$

\n
$$
\ge \left(\int_t^z w_0(x) \left(\int_0^t (x-t)^{k-1} w_k^{1-p'}(s) ds \right)^q dx \right)^{\frac{1}{q}}
$$

\n
$$
= \left(\int_t^z w_0(x) (x-t)^{(k-1)q} dx \right)^{\frac{1}{q}} \left(\int_0^t w_k^{1-p'}(s) ds \right).
$$

\n(49), we obtain
\n
$$
\left(\int_t^z (x-t)^{(k-1)q} w_0(x) dx \right)^{\frac{1}{q}} \left(\int_0^t w_k^{1-p'}(s) ds \right) \le C2^{\frac{1}{p}} \left(\int_0^t w_k^{1-p'}(s) ds \right)^{\frac{1}{p}},
$$

\n
$$
\text{nce } t \in (0, z) \text{ was arbitrary, we have } B_1 \le C2^{\frac{1}{p}}.
$$

From (49), we obtain

$$
\left(\int\limits_t^z(x-t)^{(k-1)q}w_0(x)\,dx\right)^{\frac{1}{q}}\left(\int\limits_0^t w_k^{1-p'}(s)\,ds\right)\leq C2^{\frac{1}{p}}\left(\int\limits_0^t w_k^{1-p'}(s)\,ds\right)^{\frac{1}{p}},
$$

and since $t \in (0, z)$ was arbitrary, we have $B_1 \leq C2^{\frac{1}{p}}$.

(ii) For the same function f_1 from (48), we have

$$
||f_1||_{p,w_k} = 2^{\frac{1}{p}} \left(\int\limits_{t_1}^1 w_k^{1-p'}(s) \, ds \right)^{\frac{1}{p}}
$$

and

$$
(k-1)!\,||T_4f_1||_{q,w_0} = \left(\int_z^1 w_0(x)(x-z)^{(k-1)q} \left(\int_z^1 f_1(s)\,ds\right)^q dx\right)^{\frac{1}{q}}
$$

$$
\text{Hardy Inequalities}
$$
\n
$$
\geq \left(\int_{t}^{t_{1}} (x - z)^{(k-1)q} w_{0}(x) \left(\int_{t_{1}}^{1} w_{k}^{1-p'}(s) ds \right)^{q} dx \right)^{\frac{1}{q}}
$$
\n
$$
= \left(\int_{t}^{t_{1}} (x - z)^{(k-1)q} w_{0}(x) dx \right)^{\frac{1}{q}} \left(\int_{t_{1}}^{1} w_{k}^{1-p'}(s) ds \right)
$$
\nne inequality

\n
$$
\|T_{4}f_{1}\|_{q,w_{0}} \leq C \|f_{1}\|_{p,w_{k}} \text{ which follows from (42) implies that}
$$
\n
$$
\left(\int_{z}^{t_{1}} (x - z)^{(k-1)q} w_{0}(x) dx \right)^{\frac{1}{q}} \left(\int_{t_{1}}^{1} w_{k}^{1-p'}(s) ds \right) \leq C 2^{\frac{1}{p}} \left(\int_{t_{1}}^{1} w_{k}^{1-p'}(s) ds \right)^{\frac{1}{p}}
$$

and the inequality $||T_4f_1||_{q,w_0} \leq C||f_1||_{p,w_k}$ which follows from (42) implies that

P, *^f² (s)=* 0 for t<s<tj

and since $t_1 \in (z, 1)$ was arbitrary, we have $B_4 \leq C2^{\frac{1}{p}}$.

(iii) Let t and t_1 be given by (47) and define

$$
f_2(s) = \begin{cases} (t-s)^{(k-1)(p'-1)} w_k^{1-p'}(s) & \text{for } 0 < s < t \\ 0 & \text{for } t \le s \le t_1 \\ \tilde{c} w_k^{1-p'}(s) & \text{for } t_1 < s < 1 \end{cases}
$$

where \tilde{c} is choosen so that

$$
h_2(s) = \begin{cases} (t-s)^{1/2} & \text{for } 0 < s < t \\ 0 & \text{for } t \le s \le t_1 \\ \tilde{c}w_k^{1-p'}(s) & \text{for } t_1 < s < t_2 \end{cases}
$$

so that

$$
\int_0^t (t-s)^{(k-1)(p'-1)} w_k^{1-p'}(s) ds = \tilde{c} \int_{t_1}^1 w_k^{1-p'}(s) ds.
$$

32) and the Hölder inequality yields

Then f_2 satisfies (32) and the Hölder inequality yields

$$
\int_{0}^{t} (t-s)^{(k-1)(p'-1)} w_k^{1-p'}(s) ds
$$
\n
$$
= \int_{0}^{t} (t-s)^{(k-1)(p'-1)} w_k^{(1-p')/p}(s) w_k^{(1-p')/p'}(s) ds
$$
\n
$$
\leq \left(\int_{0}^{t} (t-s)^{(k-1)p'} w_k^{1-p'}(s) ds \right)^{\frac{1}{p}} \left(\int_{0}^{t} w_k^{1-p'}(s) ds \right)^{\frac{1}{p'}},
$$

i.e., due to (47),

$$
\left(\int_{0}^{t} (t-s)^{(k-1)p'} w_{k}^{1-p'}(s) ds\right)^{j} \left(\int_{0} w_{k}^{1-p'}(s)\right)^{j}
$$
\n
$$
\int_{0}^{t} (t-s)^{(k-1)p'} w_{k}^{1-p'}(s) ds \geq \tilde{c}^{p} \int_{t_{1}}^{1} \tilde{w}_{k}^{1-p'}(s) ds.
$$

Now,

$$
||f_2||_{p,w_k} = \left(\int_0^t (t-s)^{(k-1)p'} w_k^{1-p'}(s) ds + \tilde{c}^p \int_{t_1}^1 w_k^{1-p'}(s) ds\right)^{\frac{1}{p}}
$$

$$
\leq 2^{\frac{1}{p}} \left(\int_0^t (t-s)^{(k-1)p'} w_k^{1-p'}(s) ds\right)^{\frac{1}{p'}}
$$

and

$$
(k-1)!\|T_1f_2\|_{q,w_0} = \left(\int_0^z \left(\int_0^x (x-s)^{k-1} f_2(s) \, ds\right)^q w_0(x) \, dx\right)^{\frac{1}{q}}
$$

\n
$$
\geq \left(\int_t^z w_0(x) \left(\int_0^t (x-s)^{k-1} (t-s)^{(k-1)(p'-1)} w_k^{1-p'}(s) \, ds\right)^q \, dx\right)^{\frac{1}{q}}
$$

\n
$$
\geq \left(\int_t^z w_0(x) \, dx\right)^{\frac{1}{q}} \left(\int_0^t (t-s)^{(k-1)p'} w_k^{1-p'}(s) \, ds\right).
$$

\nThe inequality $||T_1f_2||_{q,w_0} \leq C ||f_2||_{p,w_k}$ yields finally $B_2 \leq C 2^{\frac{1}{p}}$.
\n(iv) For t and t_1 given by (47), define
\n
$$
f_3(s) = \begin{cases} \tilde{c} w_k^{1-p'}(s) & \text{for } t < s < z \\ (s-z)^{p'-1} w_k^{1-p'}(s) & \text{for } z < s < t_1 \\ 0 & \text{otherwise in } (0,1) \end{cases}
$$

\nwhere \tilde{c} is chosen so that

(iv) For t and t_1 given by (47), define

$$
f_3(s) = \begin{cases} \tilde{c} w_k^{1-p'}(s) & \text{for } t < s < z \\ (s-z)^{p'-1} w_k^{1-p'}(s) & \text{for } z < s < t_1 \\ 0 & \text{otherwise in } (0,1) \end{cases}
$$
\nwhere \tilde{c} is chosen so that

\n
$$
\int_{z}^{t_1} (s-z)^{p'-1} w_k^{1-p'}(s) \, ds = \tilde{c} \int_{t}^{z} w_k^{1-p'}(s) \, ds.
$$
\nThen f_3 satisfies (32), and similarly as in part (iii), the Hölder inequality yields

where \tilde{c} is choosen so that

(0) otherwise in (0)

\nthat

\n
$$
\int_{z}^{t_1} (s-z)^{p'-1} w_k^{1-p'}(s) ds = \tilde{c} \int_{t}^{z} w_k^{1-p'}(s) ds.
$$
\nand similarly as in part (iii), the Hölder in

\n
$$
\int_{0}^{t_1} (s-z)^{p'} w_k^{1-p'}(s) ds \geq \tilde{c}^p \int_{0}^{z} w_k^{1-p'}(s) ds.
$$

$$
\int_{0}^{t_1} (s-z)^{p'-1} w_k^{1-p'}(s) ds = \tilde{c} \int_{t}^{z} w_k^{1-p'}(s) ds
$$

and similarly as in part (iii), the Hölder is

$$
\int_{z}^{t_1} (s-z)^{p'} w_k^{1-p'}(s) ds \geq \tilde{c}^p \int_{t}^{z} w_k^{1-p'}(s) ds.
$$

Now,

$$
||f_3||_{p,w_k} = \left(\int_{z}^{t_1} (s-z)^{p'} w_k^{1-p'}(s) ds + \tilde{c}^p \int_{t}^{z} w_k^{1-p'}(s) ds\right)^{\frac{1}{p}}
$$

$$
\leq 2^{\frac{1}{p}} \left(\int_{z}^{t_1} (s-z)^{p'} w_k^{1-p'}(s) ds\right)^{\frac{1}{p}}
$$

j.

and, due to the inequality $(x - z)^{k-1}$ $-(x-s)^{k-1} \geq (s-z)(x -$

and

\n
$$
\text{Hardy Inequalities}
$$
\nand

\n
$$
\text{Hardy Inequalities}
$$
\n
$$
\text{Cardy in the equation}
$$
\n $$

The inequality $||T_3 f_3||_{q,w_0} \leq C ||f_3||_{p,w_k}$ yields finally $B_3 \leq C2^{\frac{1}{p}}$

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