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A General Approach to the Min-Max Principle

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Abstract. Following the abstract approach of [11], we give a general min-max principle in critical point theory which covers the classical results and applies to a variety of settings. Especially, thanks to the notion of weak slope introduced in [16] and results of [12], this principle applies to continuous functionals and some classes of lower semicontinuous functionals considered in the literature.

Keywords: *Deformation property, min-max principle, weak d-slope.*

AMS subject classification: 58 E 05

1. Introduction

In this work, we propose an abstract and systematic approach to the min-max method in critical point theory. Our interest in this matter was first motivated by the *exposé* of Eilenberg [17] which introduced us to critical point theory. Further motivated by a result of Pucci and Serrin [26] extending to the "limit-case" the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1], we were led to the deformation property and min-max principle of [11]. Allowing to "locate" min-max critical points on an appropriately given set, this principle is in the same spirit as the corresponding results of Ghoussoub and Preiss [22] and Ghoussoub [21]. Our approach, however, is different from the one followed in [21, 22], based on Ekeland's Variational Principle and ideas from [29]. Our point of view is to obtain an abstract min-max principle by assuming the verification of a simple deformation property, in analogy with the classical approach using the Deformation Theorem [25]. This deformation property should then be verified, under a local Palais-Smale type condition, in a variety of settings considered in the literature.

Here, we improve the deformation property of [11] essentially by the use of the so-called *graph metric*. This allows to perform a more systematic treatment of the abstract theory and to enlarge the range of application of the principle, including situations where the min-max values are defined via families of non-compact sets, or where the functionals are not smooth. In fact, the introduction of the graph metric in the present work was motivated by its use in critical point theory for some classes of lower semicontinuous functions (see [14, 15]).

Degiovanni and Marzocchi introduced in [16] the notion of *weak slope* which permits to do critical point theory for continuous functions defined on metric spaces, as well as for

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some classes of lower semicontinuous functions, via their *epigraph function*. This theory has been further developed in [12], including the classical theory for C^1 functionals while extending it to functionals defined on C^1 Finsler manifolds. Using a slightly modified version of the weak slope and a deformation result of [12], it turns out that also our deformation property is verified, in a natural way, under the "local Palais-Smale condition", for a continuous function defined on a complete metric space.

This paper is organized as follows.

Section 2 contains the deformation property and the min-max principle. Like the whole paper, they are formulated in an equivariant setting, and the principle has the form of a multiplicity result. Indeed, it is intended to yield, as direct consequences, the known results on existence and multiplicity of min-max critical points. After recalling the notion of *index*, some corollaries are given as an intermediate step in direction of more specialized results. For the sake of brevity, we did not consider other possible refinements, such as the use of *relative indices*. For the same reason, the bibliography is limited up to the examples given in Section 4. More references can be found in the ones we give.

Section 3 deals with critical point theory for non-smooth functions. We introduce the *weak d-slope* and give the deformation results for continuous functions mentioned above. We also show how one can adapt standard constructions to define suitable (sequences of) min-max values of the epigraph function, in order that the theory be applicable to some classes of lower semicontinuous functions.

In Section 4 , we give several examples of settings in which the deformation property is verified under the local Palais-Smale condition. This is achieved either via the classical method of *steepest descent* (and requires only slight modifications of the usual proofs) or by estimating the weak d-slope and using the results of Section 3 . We also give, as special cases of those of Section 2 , a few examples of results (or improvements of results) which can be found in the literature.

This paper was written in the first half of 1992. Since then, however, further developments and applications of the theory based on the notion of weak slope have appeared. We refer to A. Canino and M. Degiovanni: *Nonsmooth critical point theory and quasilinear elliptic equations* (Top. Meth. in Diff. Equ. and Incl. (Montréal 1994); NATO ASI Series C. Dortrecht: Kluwer, 1995, pp. 1–50) and the references therein, for a more recent account.

2. The min-max principle

In this section, (X,d) will denote a metric space and G a group (for the composition of applications) of isometries from X to X, i.e. $d(g(x),g(y)) = d(x,y)$ for all $x,y \in X$ and $g \in G$. In this situation we say that X is a *G-space*. As usual, we say that

$A \subset X$ is *G-invariant* if $g(A) = A$ for all $g \in G$

$h : X \to \mathbb{R}$ is *G-invariant* if $h \circ g = h$ for all $g \in G$

$h : X \to X$ is *G-equivariant* if $h \circ g = g \circ h$ for all $g \in G$.

We shall suppose given a G-invariant function $f : X \to \mathbb{R}$ and a G-invariant set $K = K(f) \subset X$, called the *set of critical points* of f. As usual, we let

$$Gx = \{g(x) : \ g \in G\}$$

denote the G-orbit of $x \in X$. If $x \in K$, Gx is called a *critical G-orbit*.

We shall mainly consider, in place of the metric d, the so-called *graph metric* d_f defined by

$$d_f(x,y) = d(x,y) + |f(x) - f(y)|$$

and denote by X_f the metric space (X, d_f). For $a \in \mathbb{R}$ we shall use the notations $f{\leq}a = \{x \in X : \ f(x) \leq a\}$ ($f{<}a, f{\geq}a$ and $f{>}a$ being defined accordingly) and $K_a = K \cap f^{-1}(a)$. If $\delta \geq 0$ and $A, B \subset X$, then $B(A;\delta)$ and $B_f(A;\delta)$ are the closed δ-neighbourhood of A in (X,d) and in X_f, respectively, with the convention that $B(\emptyset;\delta) = B_f(\emptyset;\delta) = \emptyset$, and $d(A,B) = \inf\{d(a,b) : \ a \in A \ \text{ and } \ b \in B\}$, with the convention that $d(A,\emptyset) = +\infty$ ($d_f(A,B)$ being defined accordingly).

Since f is G-invariant, G is a group of isometries of X_f. Hence, setting

$$E_{f,G} = \Big\{ A \subset X : \ A \text{ is closed in } X_f \text{ and } G\text{–invariant} \Big\}$$

we have that $A \cap B, A \cup B, \overline{A \setminus B}$ (closure in X_f), $f{\leq}a, f{\geq}a, B_f(A;\delta), B_f(K_a;\delta)$ belong to $E_{f,G}$ whenever $A, B \in E_{f,G}$. Also, the application ρ_A defined by $\rho_A(x) = d_f(x,A)$ $(x \in X)$ is G-invariant if $A \subset X$ is G-invariant.

We shall denote by $\mathcal{D}_G$ and $\mathcal{D}_{f,G}$ the sets of G-invariant deformations of $X = (X,d)$ and X_f, respectively, i.e. $\eta \in \mathcal{D}_G$ means that $\eta : \ X \times [0,1] \to X$ is continuous, $\eta(x,0) = 0$ for all x and $\eta(\cdot,t)$ is G-invariant for each fixed t (the same for $\mathcal{D}_{f,G}$, replacing X by X_f). Of course, $\mathcal{D}_G = \mathcal{D}_{f,G}$ if f is continuous (still, the metrices d and d_f are not equivalent – though *topologically* equivalent – unless f is Lipschitz continuous).

Finally, we shall indicate by id_X both the identity of X and the trivial deformation $\eta(x,t) = x$ for all $(x,t) \in X \times [0,1]$, which belongs to both $\mathcal{D}_G$ and $\mathcal{D}_{f,G}$.

Definition 2.1 (*Property* (P)). For $\eta_1, \eta_2 \in \mathcal{D}_{f,G}$ and $g : \ X_f \times [0,1] \to [0,1]$ continuous with $g(\cdot,0) \equiv 0$ set $\eta(x,t) = \eta_1\big(\eta_2(x,t), g(x,t)\big)$. We say that $\mathcal{D}^* \subset \mathcal{D}_{f,G}$ verifies *property* (P) if $id_X \in \mathcal{D}^*$ and the implication

$$\left.\begin{array}{l} \eta_1, \eta_2 \in \mathcal{D}^* \\ g(\cdot,t) \ G\text{-invariant for each } \ t \in [0,1] \end{array}\right\} \implies \eta = \eta_1(\eta_2, g) \in \mathcal{D}^*$$

is true.

Of course, $\mathcal{D}_{f,G}$ verifies property (P).

Definition 2.2 (*Deformation property* $(\mathcal{D}^*)_{K,a}$). Given $a \in \mathbb{R}$ and $\mathcal{D}^* \subset \mathcal{D}_{f,G}$ verifying property (P), we say that f possesses *property* $(\mathcal{D}^*)_{K,a}$ if given $\delta > 0$, there exists $\varepsilon > 0$ and $\eta \in \mathcal{D}^*$ such that

(i) $d_f(\eta(x,t), x) \leq \delta$ for all $(x,t) \in X \times [0,1]$

(ii) $\eta(f{\leq}a{+}\varepsilon \setminus B_f(K_a;\delta), 1) \subset f{\leq}a{-}\varepsilon$

is fulfilled.

Whenever f is continuous and $\mathcal{D}^* = \mathcal{D}_{f,G} \equiv \mathcal{D}_G$, then the deformation property will still be referred to as $(\mathcal{D}_{f,G})_{K,a}$ since it involves the metric d_f (see, e.g., Theorem 3.6) — unless f is Lipschitz continuous.

For the remainder of this section, we fix a subclass $\mathcal{D}^*$ of $\mathcal{D}_{f,G}$ verifying property (P). All subsets of X are endowed with the topology of X_f.

Definition 2.3 ($\mathcal{D}^*$*-admissibility*). We say that (Γ, S) *is a* $\mathcal{D}^*$*-admissible pair* in X_f if

(i) $\Gamma \subset E_{f,G}$ and $S \in E_{f,G}$

(ii) $\eta \in \mathcal{D}^*, \eta(x,t) = x$ for all $(x,t) \in S \times [0,1] \implies \overline{\eta(U,1)} \in \Gamma$ for all $U \in \Gamma$

is fulfilled.

Definition 2.4 (*min-max value*). If (Γ, S) is a $\mathcal{D}^*$-admissible pair in X_f, define by

$$c = \inf_{U \in \Gamma} \sup_{x \in U} f(x)$$

the min-max value $c = c(f,\Gamma) \in [-\infty, +\infty]$ of f over Γ.

Definition 2.5 (*Property* (E)). Given a function $\mathcal{J} : E_{f,G} \to \mathbb{Z}_+ \cup \{+\infty\}$, we say that a sequence $\{\Gamma_i\}_{1 \leq i \leq M}$ $(M \in \mathbb{N} \cup \{+\infty\})$ possesses *property* (E) with respect to $\mathcal{J}$ if for all i

(i) $\emptyset \neq \Gamma_{i+1} \subset \Gamma_i \subset E_{f,G}$

(ii) $U \in \Gamma_{i+p}$ $(p \geq 0)$ and $Y \subset E_{f,G}$ with $\mathcal{J}(Y) \leq p \implies \overline{U \setminus Y} \in \Gamma_i$

is fulfilled.

The following lemma will be useful when deriving corollaries from our min-max principle.

Lemma 2.6. *Let* $a, \alpha, \lambda \in \mathbb{R}$ *with* $a > \alpha$ *and* $\lambda > 0$. *Assume that* f *possesses property* $(\mathcal{D}^*)_{K,a}$. *Then there exists* $\varepsilon > 0$ *such that, if* (Γ, S) *is a* $\mathcal{D}^*$*-admissible pair verifying*

$$\alpha \geq \sup\{f(S)\} \qquad \text{and} \qquad d_f(U, f{\geq}a) > \lambda \ \text{ for some } \ U \in \Gamma,$$

then $c(f,\Gamma) \leq a - \varepsilon$ *holds.*

Proof. Let $0 < \delta = \min\{\lambda, \frac{a-\alpha}{3}\}$. Let $0 < \varepsilon = \varepsilon(\delta) \le \delta$ and $\eta \in \mathcal{D}^*$ be as in $(\mathcal{D}^*)_{K,a}$, such that

$$d_f(\eta(x,t), x) \le \delta \tag{2.1}$$

$$\eta(U,1) \subset f^{\le a-\varepsilon} \tag{2.2}$$

where $U \in \Gamma$ is such that $d_f(U, f^{\ge a}) > \lambda$, so that $U \subset f^{<a} \setminus B_f(K_a;\delta)$. Define

$$\rho(x) = \min\{d_f(x,S), \delta\} \qquad (x \in X)$$

$$h(x,t) = \eta\Big(x, \frac{\rho(x)t}{\delta}\Big) \qquad ((x,t) \in X \times [0,1])$$

Then $h \in \mathcal{D}^*$ by property (P) (ρ is G-invariant) and $\overline{h(U,1)} \in \Gamma$ by $\mathcal{D}^*$-admissibility. If $x \in U$ is such that $f(h(x,1)) \ge a - \delta$, it follows from (2.1) that $f(x) \ge a - 2\delta > \alpha + \delta$. Hence $\rho(x) = \delta$ and $f(h(x,1)) = f(\eta(x,1)) < a - \varepsilon$, using (2.2). In conclusion, we have $c(f,\Gamma) \le \sup\{f(\overline{h(U,1)})\} \le a - \varepsilon$ ∎

Suppose now given $\mathcal{J} : E_{f,G} \to \mathbb{Z}_+ \cup \{+\infty\}$ and let $\{(\Gamma_i, S_i)\}_{1\le i\le M}$ $(M \in \mathbb{N} \cup \{+\infty\})$ be such that (Γ_i, S_i) is $\mathcal{D}^*$-admissible for each i and $\{\Gamma_i\}$ possesses property (E) with respect to $\mathcal{J}$. Let $\{c_i\} = \{c_i(f,\Gamma_i)\}$ be the corresponding sequence of min-max values of f over Γ_i. For $A \subset X$, $N \le M$ and $\varepsilon > 0$ set

$$k_{N,\varepsilon}(A, \{\Gamma_i\}) = \#\{i \le N : \ d_f(A, \Gamma_i) \le \varepsilon\}$$

where $\#$ denotes cardinality and $d_f(A, \Gamma_i) = \sup\{d_f(A,U) : \ U \in \Gamma_i\}$. Observe that since $\Gamma_{i+1} \subset \Gamma_i$, $\{c_i\}$ is a non-decreasing sequence and $\{d_f(A,\Gamma_i)\}$ is a non-increasing sequence. Thus, we can define

$$k_N(A, \{\Gamma_i\}) = \lim_{\varepsilon\to 0} k_{N,\varepsilon}(A, \{\Gamma_i\}) \qquad (k \in \mathbb{N} \cup \{+\infty\}).$$

In fact, it holds

$$k_N(A, \{\Gamma_i\}) = N - \max\{j \le N : \ d_f(A, \Gamma_i) > 0\} \tag{2.3}$$

if N is finite and

$$k_{+\infty}(A, \{\Gamma_i\}) = \begin{cases} +\infty & \text{if } d_f(A,\Gamma_i) \to 0 \text{ as } i \to +\infty \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The following is the main result of this section.

Theorem 2.7 (Min-Max Principle). *Let $\mathcal{J}$, $\{(\Gamma_i, S_i)\}_{1\le i\le M}$ and $\{c_i\}_{1\le i\le M}$ be as above. Assume that, for some $A \in E_{f,G}$ and some $N \le M$,*

$$\alpha := \inf\{d_f(A, S_i) : \ i \le N\} > 0 \quad \textit{and} \quad c := \inf\{f(A)\} = \sup\{c_i : \ i \le N\}.$$

Assume also that f possesses property $(\mathcal{D}^*)_{K,c}$. Then

$$\mathcal{J}\big(B_f(K_c;\lambda)\cap B_f(A;\lambda)\big) \geq k_N(A,\{\Gamma_i\}) \qquad \text{for all } \lambda>0.$$

Proof. We may suppose that $k_N(A,\{\Gamma_i\})>0$, otherwise there is nothing to prove. Suppose further, by contradiction, that

$$p := \mathcal{J}\big(B_f(K_c;\lambda)\cap B_f(A;\lambda)\big) < k_N(A,\{\Gamma_i\})$$

for some $\lambda>0$. Let $0<\delta<\min\{\lambda,\frac{\alpha}{2}\}$, $\varepsilon>0$ and $\eta\in\mathcal{D}^*$, as in $(\mathcal{D}^*)_{K,c}$, be such that

$$d_f\big(\eta(x,t),x\big) \leq \delta \tag{2.5}$$

$$\eta\big(f{\leq}c{+}\varepsilon \setminus B_f(K_c;\lambda),1\big) \subset f{\leq}c{-}\varepsilon. \tag{2.6}$$

According to (2.3) and (2.4) and since $k_N(A,\{\Gamma_i\})>p$, given $\gamma>0$ there exists $i\leq N$ such that

$$c_{i+p}\leq c \qquad \text{and} \qquad d_f(A,\Gamma_i)<\gamma. \tag{2.7}$$

Choose $\gamma<\min\{\lambda-\delta,\frac{\alpha}{2}-\delta,\varepsilon\}$ and let $i\leq N$ such that (2.7) holds. Let $U\in\Gamma_{i+p}\cap f{\leq}c{+}\varepsilon$. Then $V=\overline{U\setminus\big(B_f(K_c;\lambda)\cap B_f(A;\lambda)\big)}\in\Gamma_i$ by property (E). Define, for $(x,t)\in X\times[0,1]$,

$$\rho(x)=\min\{d_f(x,S_i),\delta\} \qquad \text{and} \qquad h(x,t)=\eta\Big(x,\frac{\rho(x)t}{\delta}\Big).$$

Then, similarly as in the proof of Lemma 2.6, $h\in\mathcal{D}^*$ and $\overline{\{\eta(x,\frac{\rho(x)}{\delta}) : x\in V\}}\in\Gamma_i$.

Let $x\in V$ be such that $\eta(x,\frac{\rho(x)}{\delta})\in B_f(A;\gamma)$. In particular,

$$f\Big(\eta\Big(x,\frac{\rho(x)}{\delta}\Big)\Big)\geq c-\gamma>c-\varepsilon.$$

On the other hand, (2.5) implies that $d_f(x,A)<\lambda$ so that $d_f(x,K_c)>\delta$, and that $d_f(x,S_i)>\delta$ so that $\rho(x)=\delta$. Hence, it follows from (2.6) that

$$f\Big(\eta\Big(x,\frac{\rho(x)}{\delta}\Big)\Big)=f\big(\eta(x,1)\big)\leq c-\varepsilon,$$

which is a contradiction ∎

We now wish to give two corollaries of Theorem 2.7 which correspond to situations encountered in applications. In the first one, we consider the simple case of a single $\mathcal{D}^*$-admissible pair (Γ,S), with $G=\{id_X\}$ and $\mathcal{J}:2^X\to\{0,1\}$ defined by $\mathcal{J}(A)=0$ if and only if $A=\emptyset$.

Corollary 2.8. *Let (Γ, S) be a $\mathcal{D}^*$-admissible pair in X_f. Assume the following:*

(i) $c = c(f, \Gamma) \in \mathbb{R}$

(ii) *There exists $F \subset X$ such that $d(F, S) > 0$, $U \cap F \neq \emptyset$ for all $U \in \Gamma$ and* $\beta := \inf\{f(F)\} \geq \sup\{f(S)\}$

(iii) *f possesses property $(\mathcal{D}^*)_{K,c}$ and K_c is compact.*

Then the following assertions are true:

(a) $K_c \neq \emptyset$

(b) $c = \beta$ *implies* $K_\beta \cap \overline{F} \neq \emptyset$.

Proof. First, condition (ii) implies that $c \geq \beta$. If $c = \beta$, assertion (b) is the conclusion of Theorem 2.7 with $A = \overline{F}$ and $N = 1$ (i.e. $\Gamma_1 = \Gamma$), taking into account that K_c is compact. If $c > \beta$, then $c > \sup\{f(S)\}$ and we can apply Theorem 2.7 with $A = f{\geq}c$ and $N = 1$, since it follows from Lemma 2.6 that $k_1(f{\geq}c, \Gamma) = 1$ ∎

Remark 2.9. Observe that we do not need to assume in the above result that S and the sets in Γ are *closed* in X_f (in fact, this is irrelevant since G and $\mathcal{J}$ are chosen to be the trivial group and *"index"*). However, the assumption of closedness in X_f can be made without loss of generality. (Of course, K_c is compact in X if and only if it is compact in X_f, and if F is closed in X, then it is closed in X_f.)

For the second corollary of we want to give, we shall restrict our attention to a particularly relevant choice of the group G and of the function $\mathcal{J}$. First, we recall the notion of *index.*

Definition 2.10 (*Index*). Let X be a metric G-space, E_G the set of its closed G-invariant subsets and $\mathcal{D}_G$ the set of its G-invariant deformations. A function $\mathcal{J} : E_G \to \mathbb{Z}_+ \cup \{+\infty\}$ is said to be an *index associated to* $(E_G, \mathcal{D}_G)$ if it possesses the following properties:

(i_1) $\mathcal{J}(A) = 0$ if and only if $A = \emptyset$

(i_2) $\mathcal{J}(A_1) \leq \mathcal{J}(A_2)$ if $A_1 \subset A_2$

(i_3) $\mathcal{J}(A_1 \cup A_2) \leq \mathcal{J}(A_1) + \mathcal{J}(A_2)$

(i_4) $\mathcal{J}(A) \leq \mathcal{J}(\overline{\eta(A,1)})$ if $\eta \in \mathcal{D}_G$

(i_5) $\mathcal{J}(B(A;\delta)) = \mathcal{J}(A)$ for some $\delta > 0$ if $A \in E_G$.

Of course, property (i_2) is a special case of property (i_4).

Remark 2.11. Whenever the function $\mathcal{J}$ in Theorem 2.7 is an index associated to $(E_{f,G}, \mathcal{D}_{f,G})$ and K_c is compact, the conclusion of the theorem reads

$$\mathcal{J}(K_c \cap A) \geq k_N(A, \{\Gamma_i\}).$$

Indeed, we may suppose that $k_N(A, \{\Gamma_i\}) > 0$ and the result obtains from conditions (i_1), (i_2) and (i_5): we have $K_c \cap A \neq \emptyset$ and

$$\mathcal{J}(K_c \cap A) = \mathcal{J}\big(B_f(K_c \cap A; \lambda_1)\big) \geq \mathcal{J}\big(B_f(K_c; \lambda_2) \cap B_f(A; \lambda_2)\big)$$

for $\lambda_1 \geq \lambda_2 > 0$ sufficiently small.

Assume now that G is a *representation of a compact Lie group acting on* X (see [5]). We may assume that G is a group of isometries (averaging d over G by means of the Haar integral yields an equivalent G-invariant metric). Any G-orbit Gx is a compact subset in X and X_f. If $G \neq \{id_X\}$, let

$$\mathrm{Fix}_X G = \left\{ x \in X : \ g(x) = x \text{ for all } g \in G \right\}$$

be the *fixed-point set* of G in X.

Let $\mathcal{J}$ be an index associated to $(E_{f,G}, \mathcal{D}_{f,G})$. In order to estimate the number of critical points of f via the min-max principle, the index $\mathcal{J}$ may be required to verify also the (standard) normalization condition

(i_6) If $x \notin \mathrm{Fix}_X G$, then $\mathcal{J}(Gx) = 1$.

We thus have, as a consequence of conditions (i_2), (i_3), (i_5) and (i_6):

(i_7) If $A \in E_{f,G}$ is compact and $A \cap \mathrm{Fix}_X G = \emptyset$, then $\mathcal{J}(A) < +\infty$ and A contains at least $\mathcal{J}(A)$ G-orbits.

Here is now the announced corollary of Theorem 2.7.

Corollary 2.12. *Let* $\{(\Gamma_i, S_i)\}_{1 \leq i \leq M}$ *and* $\{c_i\}_{1 \leq i \leq M}$ *be as in* (2.7), G *a representation of a compact Lie group, and* $\mathcal{J}$ *an index associated to* $(E_{f,G}, \mathcal{D}_{f,G})$ *and verifying condition* (i_6)*. Assume the following:*

(i) *There exists* $F \in E_{f,G}$ *such that, for each* i, $d(F, S_i) > 0, F \cap U \neq \emptyset$ *for all* $U \in \Gamma_i$ *and* $\beta := \inf\{f(F)\} \geq \sup\{f(S_i)\}$.

(ii) $c_i < b$ *for each* i, *for some* $b \in (\beta, +\infty]$.

(iii) f *possesses property* $(\mathcal{D}^*)_{K,a}$ *and* K_a *is compact for all* $\beta \leq a < b$.

Then the following assertions are true:

(a) *If* $c_j = \beta$ *for some* j, *then* $\mathcal{J}(K_\beta \cap F) \geq j$.

(b) *If* $c_j = c_{j+p} = c$ *for some* $j \leq j + p$, *then* $\mathcal{J}(K_c) \geq p + 1$.

(c) *If* $(F \cup f{>}\beta) \cap K \cap \mathrm{Fix}_X G = \emptyset$, *then* f *possesses at least* $\#\{c_i\} = M$ *critical* G*-orbits. Furthermore, if* $M = +\infty$, *then* $c_i \to b$ *as* $i \to +\infty$.

Proof. Taking Remark 2.11 into account, assertions (a) and (b) are the conclusion of Theorem 2.7 with $A = F$ and $A = f{\geq}c$, respectively, observing that in the latter case we have

$$k_{j+p}(f{\geq}c, \{\Gamma_i\}) \geq p + 1$$

thanks to Lemma 2.6. Assume that $\{c_i\}$ is an infinite sequence converging to $c < b$. Then, either $\mathcal{J}(K_\beta \cap F) = +\infty$ in case $c = \beta$, according to assertion (a), or

$$\mathcal{J}(K_c) = k_{+\infty}(f{\geq}c, \{\Gamma_i\}) = +\infty$$

if $c > \beta$ since $d_f(f{\geq}b, \Gamma_i) \to 0$, according to Lemma 2.6 (see also (2.4)). In any case we have a contradiction with condition (i_7) since $(F \cup f{>}\beta) \cap K \cap \mathrm{Fix}_X G = \emptyset$.

If $M \in \mathbb{N}$, assertions (a) and (b) and condition (i_7) also show that f has at least M critical G-orbits and assertion (c) is proved ∎

Remark 2.13. **(i)** Arguing as in the proof of assertion (c) shows that if $M = +\infty$ and $b \in \mathbb{R}$, then f does not possess property $(\mathcal{D}^*)_{K,b}$.

(ii) Properties (i_3) and (i_4) of $\mathcal{J}$ are not used in proving Corollary 2.12. These properties allow, in practice, to *define* sequences of $\mathcal{D}_G$-admissible families possessing property (E). In Section 3 we shall consider a setting for which Corollary 2.12 holds, involving a function $\widetilde{\mathcal{J}}$ defined via an index but which is not necessarily an index itself.

(iii) Suppose that for each $1 \le j \le M$ $(M \in \mathbb{N} \cup \{+\infty\})$ we are given a sequence $\{(\Gamma_{i,j}, S_{i,j})\}_{1\le i\le M_j}$ $(M_j \in \mathbb{N})$ of $\mathcal{D}^*$-admissible pairs in X_f such that $\{\Gamma_{i,j}\}_{1\le i\le M_j}$ possesses property (E) with respect to a fixed function $\mathcal{J}$. Define

$$\mathbf{\Gamma}_i = \bigcup_{i\le j\le M} \Gamma_{i,j} \qquad \text{and} \qquad \mathbf{S} = \bigcup_{i,j} S_{i,j}.$$

Then $\{(\mathbf{\Gamma}_i, \mathbf{S})\}_{1\le i\le \widetilde{M}}$ with $\widetilde{M} = \sup_j\{M_j\}$ is a sequence of $\mathcal{D}^*$-admissible pairs in X_f and $\{\mathbf{\Gamma}_i\}_{1\le i\le \widetilde{M}}$ possesses property (E) with respect to $\mathcal{J}$ (as is easily verified). Hence, Theorem 2.7 and Corollary 2.12 hold with $\{(\Gamma_i, S_i)\}$ and $\{c_i\}$ replaced by $\{(\mathbf{\Gamma}_i, \mathbf{S})\}$ and $\{\mathbf{c}_i\} = \{\mathbf{c}_i(f, \mathbf{\Gamma}_i)\}$, respectively, if we assume in Theorem 2.7 that

$$\inf\left\{d_f(A, S_{i,j}):\ 1\le i\le N\le \widetilde{M}\ \text{ and }\ 1\le j\le M\right\} > 0$$

and if we replace condition (i) of Corollary 2.12 by the following one:

(i)′ There exists $F \in E_{f,G}$ such that $d(F, \mathbf{S}) > 0$ (resp. [nothing]), $F \cap U \neq \emptyset$ for all $U \in \mathbf{\Gamma}_i$ and $\inf\{f(F)\} \ge \sup\{f(S_{i,j})\}$ for all i, j (resp. $\inf\{f(F)\} > \sup\{f(\mathbf{S})\}$).

In particular, if $M = +\infty$ and $M_j \to +\infty$ as $j \to +\infty$, $\{\mathbf{c}_i\}$ is an infinite sequence and the last conclusion of Corollary 2.12 can be obtained.

Let us now consider a sequence of $\mathcal{D}^*$-admissible pairs of a particular type. Set

$$\widetilde{\Gamma}_i = \left\{U \in E_{f,G}:\ \mathcal{J}(U) \ge i\right\} \qquad (1 \le i \le \mathcal{J}(X))$$

$(i \in \mathbb{N}$ if $\mathcal{J}(X) = +\infty)$ where $\mathcal{J}$ is an index associated to $(E_{f,G}, \mathcal{D}_{f,G})$. That $(\widetilde{\Gamma}_i, \emptyset)$ is $\mathcal{D}_{f,G}$-admissible for each i follows immediately from property (i_4) of $\mathcal{J}$, and property (E) for the sequence $\{\widetilde{\Gamma}_i\}$ follows from properties (i_1) – (i_3) in a standard way.

We have the following proposition.

Lemma 2.14. *Let* $\{\widetilde{\Gamma}_i\}$ *as above and* $\{\tilde{c}_i\}$ *the corresponding sequence of min-max values of* f. *Assume that* f *possesses property* $(\mathcal{D}_{f,G})_{K,a}$ *for all* $a \in f(X)$. *Then*

$$\tilde{c}_i \le \sup\{f(K)\} \qquad (1 \le i \le \mathcal{J}(X))$$

with the convention that $\sup\{f(\emptyset)\} = -\infty$.

Proof. We may suppose that there exists $b \in f(X)$ with $b > \sup\{f(K)\}$ (otherwise there is nothing to prove). Assume also that for any such b we have

$$\eta(X, 1) \subset f^{\le b} \qquad \text{for some } \eta \in \mathcal{D}_{f,G}. \tag{2.8}$$

Then $\mathcal{J}(X) \le \mathcal{J}(f{\le}b)$ according to property (i_4), hence $\tilde{c}_i \le b$ for each i and the conclusion of the lemma follows.

We now show that property (2.8) holds. For this suppose first that $\beta = \sup\{f(X)\}$ is achieved. From deformation property $(\mathcal{D}_{f,G})_{K,a}$ we can cover $[b, \beta]$ by the family of intervals $\{[b_i - \varepsilon_i, b_i + \varepsilon_i]\}_{1 \le i \le k}$, where $b_i \le b_{i+1}$, no interval in this family is included in another one, and there exist $\eta_i : X \times [0,1] \to X$ continuous such that $\eta_i(f{\le}b_i{+}\varepsilon_i, 1) \subset f{\le}b_i{-}\varepsilon_i$. Define inductively

$$\eta_i' : X \times \left[\frac{i-1}{k}, \frac{i}{k}\right] \to X \qquad \text{by } \eta_i'(x,t) = \eta_i(\eta_{i-1}'(x,1), kt - (i-1))$$

$$\eta : X \times [0,1] \to X \qquad \text{by } \eta(x,t) = \eta_i'(x,t) \quad \text{if } t \in \left[\frac{i-1}{k}, \frac{i}{k}\right].$$

Then η satisfies (2.8).

Suppose now that $\beta = \sup\{f(X)\}$ is not achieved. Let $\{b_n\} \subset f(X)$ with $b_n \to \beta$ be a strictly increasing sequence, with $b_1 = b, b_0 = b - \varepsilon$ $(\varepsilon > 0)$ and $\eta_n \in \mathcal{D}_{f,G}$ such that

$$\left.\begin{aligned} &\eta_n(f{\le}b_{n+1}, 1) \subset f{\le}b_n \\ &\eta(x,t) = x \quad \text{for all } (x,t) \in f{\le}b_{n-1} \times [0,1]. \end{aligned}\right\} \tag{2.9}$$

The existence of η_n and ε follows easily from the previous construction and property (i) of the deformation property $(\mathcal{D}_{f,G})_{K,a}$ (see Definition 2.2 and the agreement following it). Indeed, all the deformations η_i above can be chosen so as to keep $f{\le}b_i{-}\varepsilon$ fixed, with $\varepsilon > 0$ arbitrarily but fixed (one then finds $\varepsilon_i < \varepsilon$). Finally, let $\{t_n\} \subset (0,1]$ with $t_1 = 1$ be a strictly decreasing sequence converging to 0 and define $\eta = \eta(x,t)$ if $f(x) \in [b_n, b_{n+1}]$ by

$$\eta(x,t) = \begin{cases} x & \text{if } t \in [0, t_{n+2}] \\ \eta_{n+1}\Big(x, (t - t_{n+2})(t_{n+1} - t_{n+2})^{-1}\Big) & \text{if } t \in [t_{n+2}, t_{n+1}] \\ \cdots & \\ \eta_m\Big(\eta_{m+1}(\ldots\eta_{n+1}(x,1)\ldots,1)(t - t_{m+1})(t_m - t_{m+1})^{-1}\Big) & \text{if } t \in [t_{m+1}, t_m] \end{cases}$$

where $1 \le m \le n$. One easily verifies that $\eta : X \times [0,1] \to X$ is well-defined, that it belongs to $\mathcal{D}_{f,G}$ and satisfies property (2.8) ∎

Remark 2.15. A statement similar to the implication (2.9) ⇒ (2.8) is made in [7: p. 606].

The following result is a corollary of Theorem 2.7 and Lemma 2.14.

Corollary 2.16. *Let $\{\tilde{\Gamma}_i\}_{1 \le i \le \mathcal{J}(X)}$ and $\{\tilde{c}_i\}_{1 \le i \le \mathcal{J}(X)}$ be as in Lemma 2.14, with G being a representation of a compact Lie group and the index $\mathcal{J}$ verifying condition*

(i_6)' *$\mathcal{J}(Gx) = 1$ for all $x \in X$.*

Assume that, for each $b \in f(X)$, f possesses deformation property $(\mathcal{D}_{f,G})_{K,a}$ and K_a is compact for all $a \le b$. Then the following assertions are true:

(a) $\tilde{c}_1 = \inf\{f(X)\}$.

(b) *If* $\tilde{c}_j = \tilde{c}_{j+p} = c \leq b$ *for some* $1 \leq j \leq j+p \leq \mathcal{J}(X)$ *and some* $b \in f(X)$, *then* $\mathcal{J}(K_c) \geq p+1$.

(c) *If* f *is bounded below, if it has at least* $\mathcal{J}(X)$ *critical* G*-orbits and if moreover* $\mathcal{J}(X) = +\infty$, *then* $\sup\{f(K)\} = \sup\{f(X)\}$ *and this sup is not achieved.*

Proof. (a) According to property $(i_6)'$ we have $\tilde{c}_1 \leq f(x) = \sup\{f(Gx)\}$ for all $x \in X$, so that $\tilde{c}_1 \leq \inf\{f(X)\}$. The other inequality is obvious.

(b) This assertion is deduced from (2.7) in the same way as Corollary 2.12/(b) is, letting $A = X$ in Theorem 2.7 (deformation property $(\mathcal{D}_{f,G})_{K,c}$ holds).

(c) Note that $\tilde{c}_1 = \min\{f(X)\}$. If $\tilde{c}_j = \sup\{f(X)\}$ for some j and this supremum is not achieved, Lemma 2.14 implies that K is an infinite set and $\sup\{f(K)\} = \sup\{f(X)\}$. Otherwise $\tilde{c}_i \in f(X)$ for all i. Observe that properties (i_2), (i_3) and $(i_6)'$ imply the following one:

$(i_7)'$ If $A \in E_{f,G}$ is compact, then it contains at least $\mathcal{J}(A)$ G-orbits.

Applying Theorem 2.7 as in Corollary 2.12/(c) then shows that f has at least $\#\{\tilde{c}_i\} = \mathcal{J}(X)$ critical G-orbits, and that if $\mathcal{J}(X) = +\infty$, then $\tilde{c}_i \to \sup\{f(X)\}$ as $i \to +\infty$ and this supremum is not achieved ∎

Remark 2.17. If we assume in Corollary 2.16 that f possesses deformation property $(\mathcal{D}_{f,G})_{K,a}$ and K_a is compact for all $a < b$, for some $b \in \mathbb{R} \cup \{+\infty\}$, then $c_i \to b$ if $\mathcal{J}(X) = +\infty$ like in Corollary 2.12.

3. The weak slope and the epigraph function

In this section, (X, d) will denote a metric space, G a representation of a compact Lie group acting on X by isometries, $\mathcal{D}_G$ the set of G-equivariant deformations of X, E_G the set of closed G-invariant subsets of X and $f : X \to \mathbb{R}$ a G-invariant function.

Definition 3.1 (*Weak d-slope*). Let the function $f : X \to \mathbb{R}$ be continuous and G-invariant. Given $x \in X$ and $\sigma \geq 0$, say that $x \in \mathcal{A}(f, \sigma)$ if there exists $\delta > 0$ and $\eta : B(Gx; \delta) \times [0, \delta] \to X$ continuous such that

(i) $\eta(\cdot, t)$ is G-equivariant for each $t \in [0, \delta]$

(ii) $d(\eta(y, t), y) \leq t$

(iii) $f(\eta(y,t)) - f(\eta(y,s)) \leq -\sigma(t-s)$ if $0 \leq s \leq t \leq \delta$.

We define and denote the *G-weak d-slope* of f at x by

$$|\widetilde{df}|_G(x) = \sup\{\sigma : \; x \in \mathcal{A}(f, \sigma)\} \qquad (\in [0, +\infty]).$$

The definition of the *weak slope of* f *at* x as introduced in [16] (without consideration of symmetry) is the same as (3.1) but with $s = 0$ only in (iii). The terminology "weak slope" was chosen with respect to the "(strong) slope" (see [13, 16]); the weak slope is denoted by $|df|$. Adding the symmetry hypothesis (i), we shall naturally use the notation

$|df|_G(x)$ and obviously, $|df|_G \geq |\widetilde{df}|_G$ (of course, we also write $|\widetilde{df}|$ when $G = \{id_X\}$). We do not know whether equality holds in general. However equality does hold for various important special cases as we shall see in Section 4 (see also Remark 3.8/(i)). Clearly, $|\widetilde{df}|_G$ (as well as $|df|_G$) is G-invariant and lower semicontinuous.

Remark 3.2. It is not known whether or not $|df|_G = |df|$ in general. But if $G_x = \{g \in G : g(x) = x\}$, the *isotropy group* of x, is trivial, then $|d|_G(x) = |df|(x)$ and $|\widetilde{df}|_G(x) = |\widetilde{df}|(x)$. Indeed, let $\delta > 0$ and $\eta : B(x;\delta) \times [0,\delta] \to X$ continuous satisfying properties (ii) and (iii) of Definition 3.1 (for some $\sigma \geq 0$). Let S_x be a *slice* at x (see [5]). Unless reducing δ, if $y \in B(Gx;\delta)$, then there exists a unique $g_y \in G$ such that $g_y^{-1}(y) \in S_x$. Of course, $y \mapsto g_y$ is continuous. Defining

$$\eta'(y,t) = g_y\big(\eta(g_y^{-1}(y),t)\big),$$

we see that $\eta' : B(Gx;\delta) \times [0,\delta] \to X$ verifies all the conditions of Definition 3.1.

Definition 3.3 (*Palais-Smale condition*). Let $f : X \to \mathbb{R}$ and $\hat{f} : X \to \mathbb{R}_+ \cup \{+\infty\}$ be two G-invariant functions and $a \in \mathbb{R}$. We say that f verifies condition $(PS)_{\hat{f},a}$ if the implication

$$\left.\begin{matrix} f(x_n) \to a \\ \hat{f}(x_n) \to 0 \end{matrix} \quad (\{x_n\} \subset X)\right\} \quad \Longrightarrow \quad \{x_n\} \text{ has a convergent subsequence}$$

is true.

Set

$$K = \big\{x \in X : |\widetilde{df}|_G(x) = 0\big\} \in E_G.$$

If f is continuous and verifies the Palais-Smale condition $(PS)_{|\widetilde{df}|_G,a}$, then any accumulation point of a sequence $\{x_n\} \subset X$ such that $|\widetilde{df}|_G(x_n) \to 0$ and $f(x_n) \to a$ belongs to K_a, which is compact (recall that $|\widetilde{df}|_G$ is lower semicontinuous).

Our motivation for introducing the weak d-slope lies in the following result and Theorem 3.6 below.

Lemma 3.4. *Assume that the function $f : X \to \mathbb{R}$ is continuous and denote by f^* the function f considered as a function from X_f to $\mathbb{R}$. Then*

$$|\widetilde{df^*}|_G(x) = \begin{cases} \dfrac{|\widetilde{df}|_G(x)}{1 + |\widetilde{df}|_G(x)} & \text{if } |\widetilde{df}|_G(x) < +\infty \\ 1 & \text{if } |\widetilde{df}|_G(x) = +\infty \end{cases}$$

for each $x \in X$.

Proof. (a) Note that since f^* is Lipschitz continuous of constant 1, $|\widetilde{df^*}|_G \leq 1$. Let $x \in X$, $0 \leq \sigma < 1$ such that $x \in \mathcal{A}(f^*,\sigma)$ with corresponding $\delta > 0$, $\eta : B_f(Gx;\delta) \times [0,\delta] \to X_f$ continuous and such that whenever $y \in B_f(Gx;\delta)$ and $0 \leq s \leq t \leq \delta$, then $\eta(\cdot,t)$ is G-equivariant,

$$d_f\big(\eta(y,t),y\big) \leq t \qquad \text{and} \qquad f(\eta(y,t)) - f(\eta(y,s)) \leq -\sigma(t-s).$$

Then $d(\eta(y,t),y) \leq (1-\sigma)t$. Defining $\tilde{\eta}(y,t) = \eta(y, \frac{t}{1-\sigma})$, it is readily seen that $x \in \mathcal{A}(f, \frac{\sigma}{1-\sigma})$ with corresponding $\tilde{\delta} > 0$ such that $B(Gx;\tilde{\delta}) \subset B_f(Gx;\delta)$, where $\tilde{\delta} \leq (1-\sigma)\delta$ and $\tilde{\eta} : B(Gx;\tilde{\delta}) \times [0,\tilde{\delta}] \to X$ as in Definition 3.1. Hence, $|\widetilde{df}|_G(x) \geq \frac{\sigma}{1-\sigma}$ and we conclude that $|\widetilde{df^*}|_G(x) \leq |\widetilde{df}|_G(x)\big(1+|\widetilde{df}|_G(x)\big)^{-1}$.

(b) Let $x \in X$ and $0 \leq \sigma < +\infty$ such that $x \in \mathcal{A}(f,\sigma)$. We show that $x \in \mathcal{A}(f^*, \frac{\sigma}{1+\sigma})$. We may suppose that $\sigma > 0$. Let $\delta > 0$ and $\eta : B(Gx;\delta) \times [0,\delta] \to X$ continuous be as in Definition 3.1. For $(y,t) \in B(Gx;\delta) \times [0,\delta]$, define $\alpha(y,t)$ by

$$f\big(\eta(y,\alpha(y,t))\big) - f(y) = -\frac{\sigma}{1+\sigma}t.$$

Using the continuity of η and f and condition (iii) of Definition 3.1 – in particular, the fact that $t \mapsto f(\eta(y,t))$ is decreasing on $[0,\delta]$ (which explains the "d" in "weak d-slope") for each fixed y – one sees that $\alpha(y,t)$ is well-defined and α is continuous. Indeed, $\alpha(y,0) = 0$, $t \mapsto \alpha(y,t)$ is increasing for each fixed y and $\alpha(y,t) - \alpha(y,s) \leq (1+\sigma)^{-1}(t-s)$ for $0 \leq s \leq t \leq \delta$. Furthermore, $\alpha(\cdot,t)$ is G-invariant for each t since f is G-invariant and η is G-equivariant. Hence, defining

$$\tilde{\eta}(y,t) = \eta\big(y,\alpha(y,t)\big)$$

shows that $x \in \mathcal{A}(f^*, \frac{\sigma}{1+\sigma})$ with corresponding δ and $\tilde{\eta} : B_f(Gx;\delta) \times [0,\delta] \to X_f$.

We conclude that $|\widetilde{df^*}|_G(x) \geq |\widetilde{df}|_G(x)\big(1+|\widetilde{df}|_G(x)\big)^{-1}$ if $|\widetilde{df}|_G(x) < +\infty$ and $|\widetilde{df^*}|_G(x) = 1$ if $|\widetilde{df}|_G(x) = +\infty$ ∎

The following is a symmetric version of a basic result from [12].

Theorem 3.5. *Assume that (X,d) is a complete metric G-space and $f : X \to \mathbb{R}$ a continuous G-invariant function. Let $A \in E_G$ and $\gamma, \sigma > 0$ be such that $|df|_G(x) > \sigma$ for all $x \in B(A;\gamma)$. Then there exists $\eta : X \times [0,\gamma] \to X$ continuous such that:*

(a) $d(\eta(x,t),x) \leq t$

(b) $f(\eta(x,t)) \leq f(x)$

(c) $f(\eta(x,t)) - f(x) \leq -\sigma t$ *if* $x \in A$

(d) $\eta(\cdot,t)$ *is G-equivariant for each* $t \in [0,\gamma]$.

Proof. That of [12: Theorem 2.11] transposes immediately to the symmetric situation we consider here. That is, starting with invariant sets and equivariant deformations produces, by construction, an equivariant deformation ∎

As a consequence of Theorem 3.5 and Lemma 3.4, we have now

Theorem 3.6. *Let X be a complete metric G-space, $f : X \to \mathbb{R}$ a continuous G-invariant function and $a \in \mathbb{R}$. If f verifies the Palais-Smale condition $(PS)_{|\widetilde{df}|_G,a}$, then function f possesses the deformation property $(\mathcal{D}_{f,G})_{K,a}$.*

Proof. Let $a \in \mathbb{R}$ and $\delta > 0$ and assume the Palais-Smale condition $(PS)_{|\widetilde{df}|_G,a}$. Consider the function f^* as in Lemma 3.4 and let $K^* = \{x \in X : |\widetilde{df^*}|_G(x) =$

0}. Lemma 3.4 tells that $K^* = K$ and that f^* verifies the Palais-Smale condition $(PS)_{\widetilde{|df^*|}_G,a}$. For $\alpha, \beta > 0$ we shall denote

$$A_{\alpha,\beta} = \Big\{x \in X \Big| \ f(x) \in [a-\alpha, a+\alpha] \ \text{ and } \ d_f(x, K_a) \geq \beta\Big\} \in E_{f,G} \quad (= E_G).$$

The Palais-Smale condition $(PS)_{\widetilde{|df^*|}_G,a}$ implies that there exist $\alpha, \sigma > 0$ such that

$$|df^*|_G(x) \geq \widetilde{|df^*|}_G(x) > \sigma \qquad \text{for all } \ x \in A_{\alpha,\frac{\delta}{2}}.$$

Let $\gamma = \min\{\frac{\delta}{2}, \frac{\alpha}{2}\}$. Since f^* is Lipschitz continuous of constant 1, $B_f(A_{\gamma,\delta}; \frac{\delta}{2}) \subset A_{\alpha,\frac{\delta}{2}}$. Since X is complete and f is continuous, X_f is complete and we can apply Theorem 3.5 to X, f and $A_{\alpha,\frac{\delta}{2}}$ to obtain $\eta' : X \times [0,\gamma] \to X$ continuous and such that

$$d_f\big(\eta'(x,t), x\big) \leq t \qquad \text{and} \qquad f(\eta'(x,t)) \leq f(x)$$

and

$$f(\eta'(x,t)) - f(x) \leq -\sigma t \qquad \text{if } x \in A_{\alpha,\frac{\delta}{2}}.$$

Defining $\eta : X \times [0,1] \to X$ by $\eta(x,t) = \eta'(x, \gamma t)$ and letting $\varepsilon = \min\{\gamma, \frac{\sigma\gamma}{2}\}$, we have

$$d_f\big(\eta(x,t), x\big) \leq \gamma t < \delta \qquad \text{and} \qquad f(\eta(x,t)) \leq a - \varepsilon \ \text{ if } f(x) \leq a - \varepsilon$$

and, if $f(x) \in [a-\varepsilon, a+\varepsilon]$ and $d_f(x, K_a) > \delta$,

$$f(\eta(x,1)) \leq f(x) - \sigma\gamma \leq a + \varepsilon - \sigma\gamma \leq a - \varepsilon$$

so that the deformation property $(\mathcal{D}_{f,G})_{K,a}$ is verified ∎

Recall that the *epigraph* of a function $f : X \to \mathbb{R}$ is the set

$$\text{epi} f = \Big\{(x,\xi) : \ x \in X \ \text{ and } \ \xi \geq f(x)\Big\}.$$

Consider $(\text{epi} f, \tilde{d})$ as a metric space with the metric

$$\tilde{d}\big((x,\xi),(y,\mu)\big) = d(x,y) + |\xi - \mu|.$$

It is complete if X is complete and f is lower semicontinuous. For $g \in G$ and $(x,\xi) \in \text{epi} f$ we let $g(x,\xi) := (gx, \xi)$ so that, slightly abusing notation, we shall consider G as a group of isometries of epi$f$ too (epi$f$ as a G-space).

The following definition appeared in [13].

Definition 3.7 (*Epigraph function*). The *epigraph function* of a function $f : X \to \mathbb{R}$ is the function

$$\mathcal{G}_f : \mathrm{epi} f \to \mathbb{R}, \qquad \mathcal{G}_f(x,\xi) = \xi.$$

Obviously, the epigraph function $\mathcal{G}_f$ is Lipschitz continuous of constant 1, and is G-invariant since f is.

Remark 3.8. **(i)** For the epigraph function, weak d-slope and weak slope coincide. That is, for any G-invariant function $f : X \to \mathbb{R}$ and any $(x,\xi) \in \mathrm{epi} f$ it holds

$$|\widetilde{d\mathcal{G}_f}|_G(x,\xi) = |d\mathcal{G}_f|_G(x,\xi).$$

For, if $|d\mathcal{G}_f|_G(x,\xi) > \sigma > 0$, $\delta > 0$ and $\eta : B((x,\xi);\delta) \times [0,\delta] \to \mathrm{epi} f$, $\eta = (\eta_1, \eta_2)$ are such that

$$\tilde{d}(\eta((y,\mu),t),(y,\mu)) \le t \qquad \text{and} \qquad \eta_2((y,\mu),t) - \mu \le -\sigma t,$$

it suffices to use $\tilde{\eta} = (\eta_1, \tilde{\eta}_2)$, where $\tilde{\eta}_2((y,\mu),t) = \mu - \sigma t$, to see that $|\widetilde{d\mathcal{G}_f}|_G(x,\xi) \ge \sigma$, and the conclusion follows.

(ii) If $f : X \to \mathbb{R}$ is continuous, then

$$|d\mathcal{G}_f|_G(x,\xi) = \begin{cases} \dfrac{|df|_G(x)}{1 + |df|_G(x)} & \text{if } \xi = f(x) \text{ and } |df|_G(x) < +\infty \\ 1 & \text{if } \xi > f(x) \text{ or } |df|_G(x) = +\infty \end{cases}$$

(see [16: Proposition 2.3]). Indeed, the different form is only due to a different choice of metric on $\mathrm{epi} f$, and passage to the symmetric case is obvious. Hence, assuming only the Palais-Smale condition $(PS)_{|df|_G,a}$, one may use $\mathcal{G}_f$ in order to obtain a deformation property for f. This is the procedure used in [12: Theorem 2.14].

By applying the results of Section 2 to $\mathcal{G}_f$ via Theorem 3.6, one can obtain results in critical point theory for some classes of lower semicontinuous functions. To this aim, we give in the following simple procedures in order to define appropriate sequences of min-max values of $\mathcal{G}_f$. More concrete examples will be given at the end of Section 4.

Let $\Pi : \mathrm{epi} f \to X$ be the projection on X: $\Pi(x,\xi) = x$. Denote by $\widetilde{E}_G$ the set of *strongly closed* G-invariant subsets of $\mathrm{epi} f$, i.e.

$$A \in \widetilde{E}_G \iff A \subset \mathrm{epi} f \text{ is closed}, G\text{-invariant and } \Pi(A) \text{ is closed}$$

(whence $\Pi(A) \in E_G$). Note that $\widetilde{E}_G$ is not empty, containing the compact G-invariant subsets of $\mathrm{epi} f$, as well as the uniform neighbourhoods of such sets. Let $\widetilde{\mathcal{D}}_G$ be the set of G-equivariant deformations of $\mathrm{epi} f$, that is $\eta \in \widetilde{\mathcal{D}}_G$ if $\eta = (\eta_1, \eta_2)$, where $\eta_1 : \mathrm{epi} f \times [0,1] \to X$ and $\eta_2 : \mathrm{epi} f \times [0,1] \to \mathbb{R}$ are continuous and verify, for $(x,\xi) \in \mathrm{epi} f$, $t \in [0,1]$ and $g \in G$:

$$f(\eta_1((x,\xi),t)) \le \eta_2((x,\xi),t)$$

$$\eta_1((x,\xi),0) = x, \quad \eta_2((x,\xi),0)$$

$$\eta_1((gx,\xi),t) = g\eta_1((x,\xi),t), \quad \eta_2((gx,\xi),t) = \eta_2((x,\xi),t).$$

Suppose now given an index $\mathcal{J}$ associated to $(E_G, \mathcal{D}_G)$ and define

$$\widetilde{\mathcal{J}} : \widetilde{E}_G \to \mathbf{Z}_+ \cup \{+\infty\}, \qquad \widetilde{\mathcal{J}}(A) = \mathcal{J}(\Pi(A))$$

with the convention that $\widetilde{\mathcal{J}}(\emptyset) = 0$. It is easy to show that $\widetilde{\mathcal{J}}$ verifies properties (i_1) - (i_3) and (i_5) of Definition 2.9 (using the fact that $\mathcal{J}$ does), but $\widetilde{\mathcal{J}}$ is not, in general, an index associated to $(\widetilde{E}_G, \widetilde{\mathcal{D}}_G)$ that is, it may not verify property (i_4). $\widetilde{\mathcal{J}}$ also verifies properties (i_6) and (i_7) whenever $\mathcal{J}$ does (clearly, $(x, \xi) \in \operatorname{epi} f$ belongs to $\mathrm{Fix}_{\operatorname{epi} f} G$ if and only if $x \in \mathrm{Fix}_X G$).

We shall assume that $\mathcal{J}$ verifies the following stronger form of property (i_4):

$(\mathbf{i}_4)'$ If $A, B \in E_G$ and $h : A \to B$ is continuous and G-invariant, then $\mathcal{J}(A) \leq \mathcal{J}(B)$.

Let $j \in \mathbb{N}$ and set

$$\Gamma_j = \Big\{ U \in E_G : U \text{ compact and } \mathcal{J}(U) \geq j \Big\}.$$

Assume that

$$\Gamma_j \neq \emptyset \qquad \text{and} \qquad c_j = c_j(f, \Gamma_j) \in \mathbb{R}. \tag{3.1}$$

Define

$$\widetilde{\Gamma}_j = \Big\{ h(U) : U \in \Gamma_j \text{ and } h : U \to \operatorname{epi} f \text{ continuous and } G\text{-equivariant} \Big\} \tag{3.2}$$

and $\tilde{c}_j = \tilde{c}_j(\mathcal{G}_f, \widetilde{\Gamma}_j)$.

Proposition 3.9. *It holds:*

(i) $\widetilde{\Gamma}_j \neq \emptyset$, $\widetilde{\Gamma}_j$ *is* $\widetilde{\mathcal{D}}_G$*-admissible and* $\tilde{c}_j = c_j$.

(ii) *If* (3.1) *holds for* $1 \leq j \leq p$, *then* $\{\widetilde{\Gamma}_j\}_{1 \leq j \leq p}$ *possesses property* (E) *with respect to* $\widetilde{\mathcal{J}}$.

Proof. (i) Let $U \in \Gamma_j$ be be such that $\beta = \sup\{f(U)\} < +\infty$ and define $h : U \to \operatorname{epi} f$ by $h(x) = (x, \beta)$. Clearly, $h(U) \in \widetilde{\Gamma}_j$, and this also shows that $\tilde{c}_j \leq c_j$. Conversely, if $h(U) \in \widetilde{\Gamma}_j$ and $h = (h_1, h_2)$, then $h_1(U) \in \Gamma_j$ by property $(i_4)'$ and $\sup\{f(h_1(U))\} \leq \sup\{\mathcal{G}_f(h(U))\}$, showing that $c_j \leq \tilde{c}_j$. That $\widetilde{\Gamma}_j$ is $\widetilde{\mathcal{D}}_G$-admissible is obvious.

(ii) The inclusion $\widetilde{\Gamma}_{j+1} \subset \widetilde{\Gamma}_j$ is obvious. Let $1 \leq j \leq j + k \leq p$, $h(U) \in \widetilde{\Gamma}_{j+k}$ and $Y \in E_G$ such that $\widetilde{\mathcal{J}}(Y) \leq k$. Then, $\overline{h(U) \setminus Y} = h(\overline{U \setminus Y'})$ where $Y' = \{x \in U : h(x) \in Y\}$ (see also the proof of Proposition 3.10/(ii)). Set $h = (h_1, h_2)$. Then clearly $Y' \subset h_1^{-1}(\Pi(Y))$ so that

$$\mathcal{J}(Y') \leq \mathcal{J}\big(h_1^{-1}(\Pi(Y))\big) \leq \mathcal{J}(\Pi(Y)) = \widetilde{\mathcal{J}}(Y) \leq k$$

using properties (i_2) and $(i_4)'$. It follows in a standard way, using properties (i_2) and (i_3) that $\mathcal{J}(\overline{U \setminus Y'}) \geq j$ whence $\overline{h(U) \setminus Y} \in \widetilde{\Gamma}_j$ ∎

Assume now that $(X, \|\cdot\|)$ is a Banach space and G a group of linear isometries of X. Set $(B,S) = (B_R \cap \widetilde{E}, S_R \cap \widetilde{E})$, where B_R and S_R are the closed ball and sphere of radius $R > 0$, respectively, and $\widetilde{E}$ is a finite-dimensional linear subspace of X. We shall assume that $\beta = \sup\{f(B)\} < +\infty$ and set $\alpha = \sup\{f(S)\}$. Fix $p \in \mathbb{N}$ and set

$$\Gamma = \left\{ h : B \to \operatorname{epi} f \;\middle|\; \begin{array}{l} h \text{ continuous, } G\text{-equivariant} \\ \text{and } h(x) = (x, \alpha) \text{ for all } x \in S \end{array} \right\} \tag{3.3}$$

$$\Gamma_i = \left\{ h(\overline{B \setminus Y}) \;\middle|\; h \in \Gamma,\ Y \in E_G,\ \mathcal{J}(Y) \le p - i \right\} \quad (1 \le i \le p). \tag{3.4}$$

Finally, let $c_i = c_i(\mathcal{G}_f, \Gamma_i)$ for $1 \le i \le p$.

Proposition 3.10. *It holds:*

(i) $(\Gamma_i, S \times \{\alpha\})$ *is* $\widetilde{\mathcal{D}}_G$*-admissible for each* $1 \le i \le p$.

(ii) $\{\Gamma_i\}_{1 \le i \le p}$ *possesses property* (E) *with respect to* $\widetilde{\mathcal{J}}$.

(iii) $c_i \in \mathbb{R}$ *for each* $1 \le i \le p$.

Proof. Assertion (i) is obvious. Assertion (ii): Clearly, $\Gamma_{i+1} \subset \Gamma_i$ for $1 \le i \le p-1$. We show that $\Gamma_p \neq \{\emptyset\}$. Define, for $x \in B$,

$$h_1(x) = \begin{cases} 2x & \text{for } x \in \frac{1}{2}B \\ \dfrac{R}{\|x\|}x & \text{for } x \in \overline{B \setminus \frac{1}{2}B} \end{cases}$$

and

$$h_2(x) = \begin{cases} \beta & \text{for } x \in \frac{1}{2}B \\ \left(2\dfrac{\|x\|}{R} - 1\right)\alpha + 2\left(1 - \dfrac{\|x\|}{R}\right)\beta & \text{for } x \in \overline{B \setminus \frac{1}{2}B}. \end{cases}$$

It is readily verified that h_1 and h_2 are well-defined and continuous, that h_1 is G-equivariant and h_2 is G-invariant, because G is a group of linear isometries. Also, $f(h_1(x)) \le h_2(x)$ for all $x \in B$, $h_1(x) = x$ and $h_2(x) = \alpha$ whenever $x \in S$. Hence, $h = (h_1, h_2) \in \Gamma$ so that $h(B) \in \Gamma_p$. This proves Assertion (iii). Indeed, $\alpha \le c_i \le \beta$ for $i \in \{1, \dots, p\}$.

We now turn to verify that if $U = h(\overline{B \setminus Y}) \in \Gamma_{i+k}$ for $1 \le i \le i + k \le p$ and if $Z \in \widetilde{E}_G$ is such that $\widetilde{\mathcal{J}}(Z) = \mathcal{J}(\Pi(Z)) \le k$, then $\overline{U \setminus Z} \in \Gamma_i$. Indeed, it is easy to see that $\overline{U \setminus Z} = h(\overline{B \setminus (Y \cup Z')})$ where $Z' = \{x \in B : h(x) \in Z\}$ (see also [27: Proposition 9.18]). Now, $h_1(Z') \subset \Pi(Z)$, whence

$$\mathcal{J}(Z') \le \mathcal{J}(h_1(Z')) \le \mathcal{J}(\Pi(Z)) \le k$$

and

$$\mathcal{J}(Y \cup Z') \le \mathcal{J}(Y) + \mathcal{J}(Z') \le p - (i + k) + k = p - i$$

and the conclusion follows ∎

Remark 3.11. Of course, whenever f is continuous one may consider directly the construction:

$$\Gamma = \left\{ h : B \to X \,\middle|\, \begin{array}{l} h \text{ continuous, } G\text{-equivariant} \\ \text{and } h(x) = x \text{ for all } x \in S \end{array} \right\}$$

$$\Gamma_i = \left\{ h(\overline{B \setminus Y}) \,\middle|\, h \in \Gamma,\ Y \in E_G,\ \mathcal{J}(Y) \le p - i \right\} \qquad (1 \le i \le p)$$

and use Theorem 3.6 to obtain $c_i(f, \Gamma_i)$ as critical values for f. It is not clear whether $c_i(f, \Gamma_i) = c_i(\mathcal{G}_f, \Gamma_i)$ in general. However, if we define

$$\widetilde{\mathit{\Gamma}}_i = \left\{ h(\overline{B \setminus Y}) :\ h \in \mathit{\Gamma},\ Y \in E_G,\ \mathcal{J}(Y) \le p - i \text{ and } Y \cap S = \emptyset \right\}$$

$$\widetilde{\Gamma}_i = \left\{ h(\overline{B \setminus Y}) :\ h \in \Gamma,\ Y \in E_G,\ \mathcal{J}(Y) \le p - i \text{ and } Y \cap S = \emptyset \right\}$$

(so that $\widetilde{\mathit{\Gamma}}_i \subset \mathit{\Gamma}_i$ and $\widetilde{\Gamma}_i \subset \Gamma_i$), then we can show that $c_i(f, \widetilde{\mathit{\Gamma}}_i) = c_i(\mathcal{G}_f, \widetilde{\Gamma}_i)$ for each $1 \le i \le p$ (assuming that $\sup\{f(B)\} < +\infty$).

Observe that Theorem 2.7 still holds but with the thesis being valid for λ small enough (and the proof is the same), if item (ii) of property (E) is modified to read:

(ii)′ If $U \in \Gamma_{i+p}$ for $p \ge 0$ and $Z \subset E_{f,G}$ is such that $\mathcal{J}(Z) \le p$ and $d_f(Z, S_i) > 0$, then $\overline{U \setminus Z} \in \Gamma_i$.

We would then say that $\{(\Gamma_i, S_i)\}_{1 \le i \le M}$ possesses property (E) with respect to $\mathcal{J}$. This modified property (ii)′ is similar to property (E) in [21: Theorem 3]. However, for the sequence $\{(\widetilde{\Gamma}_i, S \times \{\alpha\})\}_{1 \le i \le p}$ the following holds:

(ii)″ If $U \in \widetilde{\Gamma}_{i+k}$ for $k \ge 0$ and $Z \subset \widetilde{E}_G$ is such that $\widetilde{\mathcal{J}}(Z) \le k$ and $\Pi(Z) \cap S = \emptyset$, then $\overline{U \setminus Z} \in \widetilde{\Gamma}_i$.

Property (ii)″ is more restrictive than property (ii)′ since $\Pi(Z) \cap S = \emptyset$ implies $Z \cap (S \times \{\alpha\}) = \emptyset$ (S and $S \times \{\alpha\}$ are compact). It follows that Theorem 2.7 holds (for λ small enough) for $\mathcal{G}_f$ and $\{(\widetilde{\Gamma}_i, S \times \{\alpha\})\}_{1 \le i \le p}$ if the set $A \in \widetilde{E}_G$ is such that $\Pi(A) \cap S = \emptyset$.

In order to apply Corollary 2.12 to $\mathcal{G}_f$ and $\{(\widetilde{\Gamma}_i, S \times \{\alpha\})\}_{1 \le i \le p}$, we thus need to know, in particular, that $\Pi(\widetilde{K}_c) \cap S = \emptyset$ for $c > \alpha$, where $\widetilde{K} = \{(x, \xi) :\ |d\mathcal{G}_f|_G(x, \xi) = 0\}$. This condition is verified if $|d\mathcal{G}_f|_G(x, \xi) > 0$ whenever $\xi > f(x)$, and it is clearly necessary in order to obtain "critical points" for f from critical points of $\mathcal{G}_f$. However, elementary examples show that it is not fulfilled in general if f is only lower semicontinuous.

On the other hand, if $|d\mathcal{G}_f|_G$ is *bounded away from* 0 on $\{(x, \xi) :\ \xi > f(x)\}$, then a complete transfer from f to $\mathcal{G}_f$ can be carried out, and it turns out that this property holds for some classes of lower semicontinuous functions f (see Section 4).

To conclude this section let us observe that, if we suppose given for each $j \in \mathbb{N}$ a pair (B_j, S_j) of finite-dimensional ball B_j and sphere S_j of radius $R_j > 0$ in X such that $\sup\{f(B_j)\} < +\infty$, and a corresponding sequence $\{\Gamma_{i,j}\}_{1 \le i \le p_j}$ of the type of the sequence $\{\Gamma_i\}_{1 \le i \le p}$ above with $p_j \to +\infty$ as $j \to +\infty$, then we may define $\mathbf{\Gamma}_i = \bigcup\{\Gamma_{i,j} :\ j \in \mathbb{N}\}$ and $\mathbf{c}_i = \mathbf{c}_i(\mathcal{G}_f, \mathbf{\Gamma}_i)$ $(i \in \mathbb{N})$ as in Remark 2.13/(iii), in order to obtain (eventually) an infinite sequence of critical values of $\mathcal{G}_f$. This is the type of construction to be used in Theorem 4.8.

Remark 3.12. Let us observe that when dealing with the epigraph function $\mathcal{G}_f$, Lemma 2.6 is not needed to derive a result like Corollary 2.12 from Theorem 2.7, since whenever $(x,\xi) \in \operatorname{epi} f$ and $\mu \geq \xi$, then $\tilde{d}((x,\xi), \mathcal{G}_f \geq \mu) = \mu - \xi$.

4. Some particular cases and examples

In this section, we give various examples of triplets $(X; f, K)$ for which the deformation property $(\mathcal{D}^*)_{K,a}$ is implied by the verification of the Palais-Smale condition at level a. To do this, we shall often use the weak d-slope, which reduces the problem to a local one, via the results of Section 3. We also give some examples of results which can then be obtained as special cases of those of Section 2. We shall use the same notations X, X_f, E_G, $E_{f,G}$, $\mathcal{D}_G$ and $\mathcal{D}_{f,G}$ introduced before.

Let $(E, \|\cdot\|)$ be a (real) Banach space and $(E^*, \|\cdot\|_*)$ its dual. In this section, when saying that E is a G-space we mean that G is an isometric representation of a compact Lie group acting on E; each $g \in G$ is a linear isometry of E. Then E^* may (and will) be considered as a G-space, defining $g(\alpha)$ for $g \in G$ and $\alpha \in E^*$ by

$$\langle g(\alpha), x\rangle = \langle \alpha, g^{-1}(x)\rangle \qquad \text{for all } x \in E.$$

When saying that $(H, \langle\cdot,\cdot\rangle)$ is a G-Hilbert space, with associated norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$, we mean that G is an orthogonal representation of a compact Lie group acting on H:

$$\langle g(x), g(y)\rangle = \langle x, y\rangle \qquad \text{for all } g \in G \text{ and } x, y \in H.$$

Now, let X be a Finsler manifold of class C^1 (and without boundary). As usual, let $T(X)$ denote the tangent bundle of X and $T_x(X)$ the tangent space at $x \in X$. Further, $\|\cdot\| : T(X) \to \mathbb{R}$ will denote the Finsler structure and $\|\cdot\|_x$ its restriction to $T_x(X)$ (which is a norm). When saying that X is a G-manifold, we mean that G is a representation of a compact Lie group acting differentiably on X. $T(X)$ is a G-space defining, for $g \in G$ and $x \in X$,

$$g : T_x(X) \to T_x(X) \qquad \text{by} \quad g(y) = dg(x)(y),$$

each g being linear and isometric. The cotangent bundle $T(X)^*$ is also a G-space letting, for $g \in G$, $x \in X$ and $\alpha \in T_x(X)^*$,

$$\langle g(\alpha), y\rangle = \langle \alpha, g(y)\rangle = \langle \alpha, dg^{-1}(x)(y)\rangle \qquad \text{for all } y \in T_x(X).$$

For various notions about G-manifolds to be used in the sequel we refer to [5].

A metric d is well-defined on each connected component of X by

$$d(x,y) = \inf\left\{ L(\sigma) = \int_0^1 \|\sigma'(s)\|_{\sigma(s)}\, ds \,\middle|\, \sigma \in C_{x,y} \right\}$$

where $C_{x,y}$ is the set of C^1 paths $\sigma : [0,1] \to X$ between x and y (see [24]); $d(x,y)$ is the geodesic distance between x and y and d is G-invariant. It follows from the structure of Finsler manifold that given $x \in X$ and a number $k > 1$, there exists a chart $\varphi : U \to T_x(X)$ at x ($x \in U$, φ a diffeomorphism) such that

$$\frac{1}{k}\|\varphi(y) - \varphi(z)\|_x \leq d(y,z) \leq k\|\varphi(y) - \varphi(z)\|_x \qquad \text{for all } y, z \in U. \tag{4.1}$$

Now, let $f : X \to \mathbb{R}$ be a G-invariant function of class C^1, $f'(x)$ the differential of f at $x \in X$ and, of course,

$$K = \Big\{x \in X : \; f'(x) = 0 \in T_x(X)^*\Big\}.$$

Then f' is continuous and G-equivariant, $\|f'(\cdot)\|$ is G-invariant and $K \in E_G$.

Proposition 4.1. *The following assertions are true:*

(a) *For each $x \in X$, $|\widetilde{df}|_G(x) = |df|(x) = \|f'(x)\|_{x^*}$, where $\|\cdot\|_{x^*}$ is the norm in $T_x(X)^*$.*

(b) *If X is complete and f verifies for some $a \in \mathbb{R}$ the Palais-Smale condition $(PS)_{\|f'(\cdot)\|_*,a}$, then f possesses the deformation property $(\mathcal{D}_{f,G})_{K,a}$.*

Proof. Assertion (b) is a consequence of assertion (a) and Theorem 3.6. To prove assertion (a) we first show that $|df| \leq \|f'(\cdot)\|$. Let $x \in X$, $k > 1$ and (U, φ) such that (4.1) holds; we may suppose that $|df|(x) > 0$. Fix $\varepsilon > 0$ and let $\gamma > 0$ be such that

$$\|(f \circ \varphi^{-1})'(z)\|_{x^*} \leq \|f'(x)\|_{x^*} + \varepsilon \qquad \text{for all } z \in B(\varphi(x);\gamma) \cap \varphi(U).$$

Let $\delta, \sigma > 0$ and $\eta : B(x;\delta) \times [0,\delta] \to X$ be such that

$$d(\eta(y,t),y) \leq t \qquad \text{and} \qquad f(\eta(y,t)) - f(y) \leq -\sigma t.$$

We may suppose that δ is so small that $B(x;2\delta) \subset U$ and $\varphi(B(x;2\delta)) \subset B(\varphi(x);\delta)$. Now, for arbitrary $(y,t) \in B(x;\delta) \times [0,\delta]$ fixed we have, using (4.1) and the Mean Value Theorem,

$$\begin{aligned}
\sigma t &\leq f(y) - f(\eta(y,t)) \\
&= \langle (f \circ \varphi^{-1})'(z), \varphi(y) - \varphi(\eta(y,t)) \rangle \\
&\leq \left(\|f'(x)\|_{x^*} + \varepsilon\right) \left\|\varphi(y) - \varphi(\eta(y,t))\right\|_x \\
&\leq \left(\|f'(x)\|_{x^*} + \varepsilon\right) k\, d(\eta(y,t),y) \\
&\leq \left(\|f'(x)\|_{x^*} + \varepsilon\right) kt
\end{aligned}$$

where z belongs to the segment $[\varphi(y), \varphi(\eta(y,t))]$. Since k can be chosen arbitrarily close to 1 and ε arbitrarily close to 0 (taking δ as small as needed), the conclusion follows from the definition of $|df|(x)$.

We now show that $|\widetilde{df}|_G \geq \|f'(\cdot)\|$. Let $x \in X$ and assume that $\|f'(x)\|_{x^*} > 0$. We may take as a chart around Gx a G-equivariant diffeomorphism $\varphi : U \to \mathcal{N}^\varepsilon(Gx)$

(a so-called *tubular neighborhood*), where $\varepsilon > 0$, $\mathcal{N}(Gx)$ is the normal bundle of Gx, $\mathcal{N}_y(Gx)$ the normal space to Gx at $y \in Gx$ and

$$\mathcal{N}_y^{\varepsilon}(Gx) = \{v \in \mathcal{N}_y(Gx) : \ \|v\| < \varepsilon\}$$

$(\varphi(y) = 0 \in \mathcal{N}_y(Gx))$. $\mathcal{N}_y(Gx)$ is a G_y-space, where $G_y = \{g \in G : g(y) = y\}$ is the isotropy group of y; since f is G-invariant, $f'(y) = f'(y)_{|\mathcal{N}_y(Gx)}$. Given $k > 1$, we may choose ε so small that (4.1) holds for all $y, z \in U$. Fix $0 < \alpha < 1$ and let $0 < \gamma < \frac{\varepsilon}{2}$ and $v(y) \in \mathcal{N}_y(Gx)$ (a pseudo-gradient) such that $v = v(y)$ is continuous on Gx, G-equivariant and

$$\left\langle (f \circ \varphi^{-1})'(u), \frac{v(y)}{\|v(y)\|_y} \right\rangle \geq (1 - \alpha)\|f'(x)\|_{x^*} \qquad \text{for all } \ u \in \mathcal{N}_y^{2\gamma}(Gx). \tag{4.2}$$

Such $v = v(y)$ exists. Indeed, there exists a G-equivariant pseudo-gradient vector field $v = v(y)$ defined on $X \setminus K$ such that

$$\left\langle (f \circ \varphi^{-1})'(y), \frac{v(y)}{\|v(y)\|_y} \right\rangle \geq (1 - \alpha)\|f'(y)\|_{y^*} \qquad \text{for all } \ y \in X \setminus K$$

and (4.2) follows using also the continuity of f', $(f \circ \varphi^{-1})'$ and the compactness of Gx. Let $0 < \delta < \gamma$ be such that $\varphi(B(Gx;\delta)) \subset \mathcal{N}_y^{\gamma}(Gx)$ and set, for $z \in B(Gx;\delta)$, $w(z) = v(y)$ if $\varphi(z) \in \mathcal{N}_y(Gx)$ (that is, if z belongs to the *slice* at $y \in Gx$ defined via φ). Define $\eta : B(Gx;\delta) \times [0,\delta] \to X$ by

$$\eta(y,t) = \varphi^{-1}\left(\varphi(y) - \frac{tw(y)}{k\|w(y)\|}\right).$$

Then, being w continuous, η is continuous and $\eta(\cdot, t)$ is G-equivariant for each t. For $y \in B(Gx;\delta)$ and $0 \leq s \leq t \leq \delta$ we have, using (4.1),

$$d\big(\eta(y,t), \eta(y,s)\big) \leq k \left\|\varphi(\eta(y,t)) - \varphi(\eta(y,s))\right\|_x = t - s$$

and, according to (4.2),

$$\begin{aligned} f(\eta(y,t)) - f(\eta(y,s)) &= (f \circ \varphi^{-1})\left(\varphi(y) - \frac{tv}{k\|v\|}\right) - (f \circ \varphi^{-1})\left(\varphi(y) - \frac{sv}{k\|v\|}\right) \\ &= \left\langle (f \circ \varphi^{-1})'\left(\varphi(y) - \frac{rv}{k\|v\|}\right), -(t-s)\frac{v}{k\|v\|} \right\rangle \\ &\leq -\frac{1-\alpha}{k}\|f'(x)\|_{x^*}(t-s) \end{aligned}$$

where $v = v(y)$ and $s \leq r \leq t$. Since k can be chosen arbitrarily close to 1 and α arbitrarily close to 0, the conclusion follows from the definition of $|\widetilde{df}|_G(x)$ ∎

The fact that $|df|(x) = \|f'(x)\|_{x^*}$ can also be found in [16: Corollary 2.13].

If X is a Finsler manifold of class at least C^2, one can use the fact that there exists a G-equivariant locally Lipschitz pseudo-gradient vector field defined on $X \setminus K$ (see [24]) in order to prove Proposition 4.1/(a) in a more classical way. Namely, the desired deformation can be obtained via the (negative) flow associated to this vector field. This can be achieved by means of a slight modification of [27: Theorem A.4] (this result naturally extending to the G-manifold case).

The same method can be applied to treat the following special case being of particular interest in applications (to Hamiltonian systems and wave equations). Assume that $(X, \langle\cdot,\cdot\rangle)$ is a G-Hilbert space and that the function $f : X \to \mathbb{R}$ is of the form

$$f(x) = \frac{1}{2}\langle Lx, x\rangle + b(x) \tag{4.3}$$

where $L : X \to X$ is a G-equivariant linear continuous self-adjoint operator and $b : X \to \mathbb{R}$ is a G-invariant C^1 function such that $b' : X \to X$ is completely continuous. To the class of functions of the form (4.3) there is associated (see Proposition 4.2 below) the following class of G-equivariant deformations of X:

$$\mathcal{D}^* = \Big\{\eta \in \mathcal{D}_G : \eta(x,t) = e^{\theta(x,t)L}x + h(x,t)\Big\}. \tag{4.4}$$

Here $\theta = \theta_\eta : X \times [0,1] \to \mathbb{R}$ is continuous, G-invariant and $\theta(\cdot,0) \equiv 0$, and $h = h_\eta : X \times [0,1] \to X$ is G-equivariant, continuous, completely continuous and $h(\cdot,0) \equiv 0$. The set $\mathcal{D}^*$ verifies property (P) (see Definition 2.1): if $\eta, \xi \in \mathcal{D}^*$ and $g : X \times [0,1] \to [0,1]$ is continuous, G-invariant and such that $g(\cdot,0) \equiv 0$, then $\eta \circ (\xi, g) \in \mathcal{D}^*$ with

$$\theta_{\eta\circ(\xi,g)} = \theta_\eta \circ (\xi,g) + \theta_\xi \qquad \text{and} \qquad h_{\eta\circ(\xi,g)} = e^{\theta_\eta\circ(\xi,g)L}h_\xi + f_\eta \circ (\xi,g).$$

With a slight modification of the proof of (a symmetric version of) [27: Proposition A.18] (see also [3: Theorem 3.4] for the primitive idea, with consideration of symmetry) one shows the following

Proposition 4.2. *Let X be a G-Hilbert space and $f : X \to \mathbb{R}$ a G-invariant C^1 function of the form* (4.3) *verifying for some $a \in \mathbb{R}$ the Palais-Smale condition $(PS)_{\|f'(\cdot)\|,a}$. Then f possesses the deformation property $(\mathcal{D}^*)_{K,a}$, where $\mathcal{D}^*$ is defined in* (4.4).

According to Proposition 4.1, Corollary 2.8 contains as special cases the Mountain Pass Theorem (MPT; see [1]) and its generalization (GMPT), and the Saddle Point Theorem (SPT) (see [27: Theorems 2.2, 5.3 and 4.6], respectively). Indeed, Corollary 2.8/(b) is a generalization of these results to the *limit case* and to continuous functions (via Theorem 3.6).

The limit-case for the Mountain Pass Theorem has been treated in various places, starting with [26]. In [21] the limit-case is treated in general, for invariant C^1 functions defined on C^1 Finsler manifolds and min-max values defined through families of compact sets. There as in [29] (see also [16, 22]) only a weaker deformation property is shown to hold (the construction of that deformation inspired our proof of Proposition 4.1), and the results are obtained by combining this property with the ε-Variational Principle of Ekeland.

Thanks to Proposition 4.2, Corollary 2.8 generalizes [27: Theorem 5.29] where the min-max value of f is defined via a family of possibly *non-compact* subsets of X; [27: Examples 5.22 and 5.26] give examples of such situations (infinite-dimensional linking of the type of the Generalized Mountain Pass Theorem (GMPT) and the Saddle Point Theorem (SPT) [4].

Of course, multiplicity results for C^1 functions in the presence of symmetry are also available as special cases of Corollaries 2.12 and 2.16. We shall give some examples of such results below, for other choices of the function f.

For a second particular case (see [6, 9]) let $(X, \|\cdot\|)$ be a G-Banach space, $(X^*, \|\cdot\|_*)$ its dual and $f : X \to \mathbb{R}$ a G-invariant locally Lipschitz continuous function. Define, for $x, y, z \in X$ and $t > 0$,

$$f^0(x,y) = \limsup_{z\to 0, t\to 0} \frac{f(x+z+ty) - f(x+z)}{t}$$

for each x. The function $f^0(x,\cdot) : X \to \mathbb{R}$ is continuous, subadditive and positively homogeneous, therefore convex. Define

$$\partial f(x) = \Big\{\alpha \in X^* : \ f^0(x,y) \geq \langle \alpha, y\rangle \quad \text{for all} \ \ y \in X\Big\}.$$

Here ∂f is *Clarke's subdifferential* (see [9]). For each $x \in X$, $\partial f(x)$ is a non-empty weak*-compact subset of X^*, so that $\lambda(x) = \min\{\|\alpha\|_* : \ \alpha \in \partial f(x)\}$ is well-defined. Furthermore, $\lambda : X \to \mathbb{R}$ is lower semicontinuous and G-invariant (since f is). Thus, setting $K = \{x \in X : \ \lambda(x) = 0\}$ we see that $K \in E_G$.

Proposition 4.3. *Let X be a G-Banach space and $f : X \to \mathbb{R}$ be G-invariant and locally Lipschitz. Then the following assertions are true:*

(a) *For each $x \in X$, $|\widetilde{df}|_G(x) \geq \lambda(x)$.*

(b) *If f satisfies for some $a \in \mathbb{R}$ the Palais-Smale condition $(PS)_{\lambda,a}$, then f possesses the deformation property $(\mathcal{D}_{f,G})_{K,a}$.*

Proof. Assertion (b) is a consequence of assertion (a) and Theorem 3.6. It is shown in [16: Theorem 2.7] that $|df|(x) \geq \lambda(x)$, using [28: Lemma 1.3] applied to the function $f^0(x,\cdot)$; it is not difficult to see from the proof that in fact $|\widetilde{df}|(x) \geq \lambda(x)$. To treat the symmetric case, we can proceed like in Proposition 4.1. Let $x \in X$ such that $\lambda(x) > 0$ and $0 < \beta < 1$. Using [6: Lemma 3.3] (existence of pseudo-gradients) and after symmetrization we can find $\delta > 0$ and $z : Gx \to X$ continuous and G-equivariant such that $\|z(y)\| \leq 1$ and

$$\langle \alpha, z(y)\rangle \geq (1-\beta)\,\lambda(x) \qquad \text{for all} \ \ \alpha \in \partial f(y'), \ y' \in B(Gx; 2\delta) \cap \mathcal{N}_y(Gx).$$

Defining $z(y') = z(y)$ if $y' \in B(Gx; 2\delta) \cap \mathcal{N}_y(Gx)$ and $\eta : B(Gx;\delta) \times [0,\delta] \to X$ by $\eta(y,t) = y - tz(y)$, we have for $0 \leq s \leq t \leq \delta$, using [6: Proposition (9)],

$$f(\eta(y,t)) - f(\eta(y,s)) = \int_s^t \frac{d}{dt} f(\eta(y,\tau))\, d\tau \leq (1-\beta)\,\lambda(x)\,(t-s)$$

and the conclusion follows ∎

We now consider a class of lower semicontinuous functions (see [14, 15]). Recall first that if $(E, \|\cdot\|)$ is a Banach space, the function $f : E \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $x \in E$ with $f(x) < +\infty$, then the *Fréchet subdifferential* of f at x is defined as the (possibly empty) closed and convex subset

$$\partial^- f(x) = \left\{ \alpha \in E^* \,\middle|\, \liminf_{y \to x} \frac{f(y) - f(x) - \langle \alpha, y - x \rangle}{\|y - x\|} \geq 0 \right\}.$$

Now, let $(H, \langle \cdot, \cdot \rangle)$ be a G-Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ be a G-invariant lower semicontinuous (proper) function. Further, let $X = \{x \in H : f(x) < +\infty\} \in E_G$ denote the effective domain of f. We assume that f has a *φ-monotone subdifferential of order* 2, which means (see [15]) that there exists a continuous function $\chi : X^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ such that

$$\langle \alpha - \beta, x - y \rangle \geq \chi\big(x, y, f(x), f(y)\big)\,\big(1 + \|\alpha\|^2 + \|\beta\|^2\big)\|x - y\|^2$$

whenever $\alpha \in \partial^- f(x)$ and $\beta \in \partial^- f(y)$.

Denote (as before) $\lambda(x) = \min\{\|\alpha\|_* : \alpha \in \partial^- f(x)\}$. Then $\lambda : X_f \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous (as follows from [15: Theorem 1.18]) and G-invariant (since f is). Hence $K := \{x \in X : \lambda(x) = 0\} \in E_{f,G}$. The set K is called the set of *critical points from below* for f. Furthermore, using also [15: Remark 1.14], a convergent sequence $\{x_n\} \subset X$ such that $f(x_n) \to a$ and $\lambda(x_n) \to 0$ converges in X_f, hence to a point in K_a. In particular, the Palais-Smale condition $(PS)_{\lambda,a}$ implies that K_a is compact.

Proposition 4.4. *Let H be a G-Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous G-invariant funcion with a φ-monotone subdifferential of order 2. Assume that f satisfies for some $a \in \mathbb{R}$ the Palais-Smale condition $(PS)_{\lambda,a}$. Then f possesses the deformation property $(\mathcal{D}_{f,G})_{K,a}$.*

The result can be easily deduced from [14: Theorem 3.8]. The basic (non-symmetric) results are in [15: Section 3]. The deformation η in the deformation property $(\mathcal{D}_{f,G})_{K,a}$ is obtained from the flow associated to an evolution problem of the form

$$-\mathcal{U}'(t) \in \partial^- f(\mathcal{U}(t)),$$

generalizing the analogous classical problem.

Let us point out that the introduction of the graph metric in the present work was induced by its use, in the above cited papers, in connection with some classes of lower semicontinuous functions, in particular those having a φ-monotone subdifferential of order 2.

Now, it is shown in [14: Theorem 3.14] that X_f is a G-ANR (while, in general, X is not (see [14: Remark 3.15])). Hence the *Lusternik-Schnirelman G-category* G-cat: $E_{f,G} \to \mathbb{Z} \cup \{+\infty\}$ is an index associated to $(E_{f,G}, \mathcal{D}_{f,G})$ (see [14, 24]). Indeed, by definition, $A \in E_{f,G}$ is *categorical* in X_f if there exists $\eta \in \mathcal{D}_{f,G}$ such that $\eta(A, 1) = Gx$ for some $x \in X$ and G-cat(A) is then defined as the least integer n such that A can be covered by n categorical sets in X_f, with the conventions that $G\text{-cat}_{X_f}(A) = +\infty$ if no such integer exists and that $G\text{-cat}_{X_f}(\emptyset) = 0$.

As an example of a special case of Corollary 2.16 we thus have the following theorem which is essentially [14: Theorem 4.9] and reminds [24: Theorem 7.2]. Observe that the latter extends to the C^1-manifold case, according to Proposition 4.1, which improves the main result of [29].

Theorem 4.5. *Let H be a G-Hilbert space and $f : H \to \mathbf{R} \cup \{+\infty\}$ a lower semicontinuous G-invariant function with a φ-monotone subdifferential of order 2. Denote by X the effective domain of f and assume that f is bounded below and satisfies the Palais-Smale condition $(PS)_{\lambda,a}$ for all $a \leq b$ whenever $b \in f(X)$. Then f has at least G-cat(X_f) critical G-orbits from below. If moreover G-cat$(X_f) = +\infty$, then* $\sup\{f(K)\} = \sup\{f(X)\}$ *and this supremum is not achieved.*

As a final example, we consider another class of lower semicontinuous functions. Let E be a G-Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ a function of the form

$$f = \phi + \psi \tag{4.5}$$

where

$\phi \in C^1(E, \mathbb{R})$ is G-invariant

$\psi : E \to \mathbb{R} \cup \{+\infty\}$ is G-invariant, convex and lower semicontinuous.

This class of functions has been studied in [28]. Let X denote the effective domain of ψ (hence of f). For $x \in X$ consider $\partial^- f(x)$ as defined above. In the special case considered here we have the equivalence

$$\alpha \in \partial^- f(x) \iff \psi(z) - \psi(x) \geq \langle \alpha - \phi'(x), z - x \rangle \text{ for all } z \in X.$$

$\partial^- f(x)$ is a (possibly empty) convex weak*-closed subset of E^*. Hence

$$\lambda(x) = \begin{cases} \min\{\|\alpha\|_* : \alpha \in \partial^- f(x)\} & \text{if } \partial^- f(x) \neq \emptyset \\ +\infty & \text{if } \partial^- f(x) = \emptyset \end{cases}$$

is well-defined and $\lambda : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and G-invariant. Set, as before, $K = \{x \in X : \lambda(x) = 0\} \in E_G$. We shall make the following assumption:

$$0 \in K \quad \text{and the isotropy group } G_x \text{ is trivial for each } x \neq 0. \tag{4.6}$$

In connection with the following proposition, recall Remark 3.11.

Proposition 4.6. *Let E be a G-Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ a G-invariant function of the form* (4.5). *Assume* (4.6). *Then, for each $(x, \xi) \in \operatorname{epi} f$,*

$$|d\mathcal{G}_f|_G(x,\xi) = \begin{cases} \dfrac{\lambda(x)}{1+\lambda(x)} & \text{if } \xi = f(x) \text{ and } \partial^- f(x) \neq \emptyset \\ 1 & \text{if } \xi > f(x) \text{ or } \partial^- f(x) = \emptyset \end{cases}$$

holds.

Proof. The result with $|d\mathcal{G}_f|$ instead of $|d\mathcal{G}_f|_G$ is obtained in [16] (combining various definitions and results, and modulo obvious modifications due to the choice of a different metric on $\operatorname{epi} f$), and this does not require that $0 \in K$. That $|d\mathcal{G}_f|_G = |d\mathcal{G}_f|$ follows from (4.6) and Remark 3.2 ∎

We now assume that ϕ and ψ are *even* functions, i.e. they are G-invariant with $G = \{id_E, -id_E\}$, a representation of the group $\mathbf{Z}_2$. Notice that (4.6) is verified in that case: $\phi'(0) = 0$ and 0 is a minimum of ψ, so that $0 \in K$.

E_G is now the set of closed symmetric subsets of X and $\mathcal{D}_G$ the set of odd deformations of X. An index associated to $(E_G, \mathcal{D}_G)$ verifying property $(i_4)'$ (see Section 3) is the *genus* γ of Krasnoselski (see [10, 23] and [27: Section 7]): for $A \in E_G$, $\gamma(A)$ is the smallest $n \in \mathbb{N}$ such that there exists $\varphi : A \to \mathbb{R}^n \setminus \{0\}$ continuous and odd, with the conventions that $\gamma(A) = +\infty$ if no such n exists and that $\gamma(\emptyset) = 0$. The genus γ also verifies property (i_6) and indeed, if $0 \notin A \in E_G$ and $\gamma(A) > 1$, then A is an infinite set.

In what follows, B_r and S_r denote respectively the closed ball and sphere in E, of radius $r > 0$ and centered at the origin. The following assertion is a version of [8: Theorem 8] and is similar to [28: Theorem 4.3].

Theorem 4.7. *Let E be a Banach space and $f : E \to \mathbf{R} \cup \{+\infty\}$ a function of the form* (4.5) *with ϕ and ψ even. Assume that $f(0) = 0$, f is bounded below, satisfies the Palais-Smale condition $(PS)_{\lambda,a}$ for all $a < 0$, and that there exist $r > 0$ and a finite-dimensional subspace $\widetilde{E}$ of E such that* $\sup\{f(S_r \cap \widetilde{E})\} < 0$. *Then f has at least* $\dim\widetilde{E}$ *pairs of critical points.*

Proof. For $1 \le j \le p := \dim\widetilde{E}$ define

$$\Gamma_j = \left\{U \in E_G : \ U \text{ is compact and } \mathcal{J}(U) \ge j\right\}$$

and $c_j = c_j(f, \Gamma_j)$. Clearly, we have $S_r \cap \widetilde{E} \in \Gamma_p$. Since f is bounded below and $\sup\{f(S_r \cap \widetilde{E})\} < 0$, it follows that $-\infty < c_j < 0$ for $1 \le j \le p$.

According to Proposition 4.6, the epigraph function $\mathcal{G}_f$ verifies the Palais-Smale condition $(PS)_{|d\mathcal{G}_f|_G, a}$ for $a < 0$ and, defining $\widetilde{\Gamma}_j$ $(1 \le j \le p)$ as in (3.2), we can use Theorem 3.6 and Proposition 3.9 to obtain from Corollary 2.12 (letting $F = \operatorname{epi} f$) that $\mathcal{G}_f$ possesses at least p pairs of critical points $(\pm x_j, \tilde{c}_j)$ with $\tilde{c}_j = c_j$ for each j. By Proposition 4.6 again, this yields p pairs of critical points $\pm x_j$ for f, with $f(x_j) = c_j < 0$ ∎

The following theorem is a version of the Symmetric Mountain Pass Theorem [27: Theorem 9.12], which improves [28: Theorem 4.4 and Corollary 4.8].

Theorem 4.8. *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ a function of the form* (4.5) *with ϕ and ψ even. Assume that $f(0) = 0$, f is bounded below, satisfies the Palais-Smale condition $(PS)_{\lambda,a}$ for all $a \ge 0$, and that*

(i) *there is a subspace E_1 of $\bar{E}$ of finite codimension and $r, \alpha > 0$ such that* $\inf\{f(S_r \cap E_1)\} \ge \alpha$

(ii) *for each finite-dimensional subspace $\widetilde{E}$ of E there exists $R = R(\widetilde{E}) > 0$ such that* $\sup\{f(\widetilde{E} \setminus B_r)\} \le 0$.

Then f possesses $\dim E_1$ *pairs of critical points. Moreover, if* $\dim E = +\infty$, *f possesses an unbounded sequence of critical values.*

Proof. Set $k = \operatorname{codim} E_1$, let $\widetilde{E} \subset E$ be a finite-dimensional subspace with $\dim \widetilde{E} = p > k$ and $R > r$ such that $\sup\{f(S_R \cap \widetilde{E})\} \le 0$. It is easy to see that $\sup\{f(B_R \cap \widetilde{E})\} < +\infty$. Thus, we may define consistently Γ and Γ_i $(1 \le i \le p-k)$ as in (3.3) and (3.4) in Section 3 (the notations correspond). If $h = (h_1, h_2) \in \Gamma$ and $U := h(B_R \cap \widetilde{E} \setminus Y) \in \Gamma_i$, then $h_1(B_R \cap \widetilde{E} \setminus Y) \cap S_r \cap E_1 \neq \emptyset$ (see [27: Proposition 9.23], observing that $0 \notin Y$ since $\gamma(Y) < +\infty$). If we set

$$F = [(S_r \cap E_1) \times \mathbb{R}] \cap \operatorname{epi} f,$$

then $U \cap F \neq \emptyset$. Using Proposition 4.6, Theorem 3.6 and Proposition 3.10, Corollary 2.12 yields $p-k$ pairs of critical points of $\mathcal{G}_f$ and hence $p-k$ pairs of critical points of f, by Proposition 4.6 again. If E is finite-dimensional, the result is thus proved letting $\widetilde{E} = E$.

If E is infinite-dimensional, for each j-dimensional subspace E_j with $j > k$ one can define (as already mentioned at the end of Section 3) $\Gamma_{i,j}$ like Γ_i above for $1 \le i \le j$, choosing $R_j = R_j(E_j)$ such that $\inf\{R_j - r : j > k\} > 0$ (this restriction is not necessary if $\alpha > 0$). Defining $\mathbf{\Gamma}_i$ $(i \in \mathbb{N})$ as in Remark 2.13/(iii) (similarly as in the proof of [27: Theorem 9.12]) we obtain the result from Corollary 2.12 applied to $\mathcal{G}_f$ again ∎

Remark 4.9. **(i)** This result improves [28, Theorem 4.4, Corollary 4.8] since we have defined actual critical values of the function f, and we can conclude on the behaviour of the sequence of critical values whenever $\dim E = +\infty$.

(ii) From previous results and remarks, one can see that Theorem 4.8 holds with E_1 of possibly infinite codimension, for E a Hilbert space and f of the form (4.3). Also, analogous results hold using other groups G and related indices $\mathcal{J}$ possessing the *dimension property*, for example the S^1-index of Benci [2] (see [3: Theorem 4.2 and Corollary 4.5]). The cohomological index theories of [19, 20] can be used as well.

(iii) The analogue of Proposition 4.6 holds if E is a Hilbert space and f is of the form $f = \phi + \psi$ with $\phi \in C^1(E, \mathbb{R})$ and $\psi : E \to \mathbb{R} \cup \{+\infty\}$ lower semicontinuous with a φ-monotone subdifferential of order 2 (see [12, 16]); indeed, this result holds without the restriction (4.6).

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