Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 16 (1997), No. 2, 435–449

A Uniform Attractor for a Non-Autonomous Generalized Navier-Stokes Equation

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Abstract. A global existence and uniqueness result of a weak solution for a generalized non-autonomous Navier-Stokes equation is given, independently of the dimension $n \geq 2$ of the space. Furthermore, the family of processes associated to the equation is shown to possess a weak uniform attractor with respect to a large class of non-autonomous forcing terms.

Keywords: *Navier-Stokes equations, processes, attractors*

AMS subject classification: 76 D 05, 35 D 05, 35 B 40

1. The generalized Navier-Stokes equation

It is well-known that for the Cauchy-Dirichlet problem for the classical 3-dimensional Navier-Stokes equation no global existence and uniqueness result of a weak solution has yet been proved. To derive the classical n-dimensional equation the following linear relationship between the stress tensor T and the deformation velocity tensor D

$$T = -pI + 2\mu D \qquad \text{where } T = \{\tau_{ij}\} \text{ and } D = \frac{1}{2}\{\partial_i u_j + \partial_j u_i\} \tag{1}$$

is assumed. Here and in the sequel u_i $(i = 1, ..., n)$ represent the components of the velocity vector u, p denotes the pressure of the fluid, $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_t = \frac{\partial}{\partial t}$, I is the identity $(n \times n)$-matrix and $\mu > 0$ is the coefficient of kinematic viscosity. However, there is no theoretical evidence of the general validity of (1), in particular in presence of very high velocities and turbulent flows; it appears therefore natural to modify (1) in these extreme conditions.

A first interesting modification of (1) is due to Ladyzhenskaya [12, 13]. She proved a global existence and uniqueness result for the 3-dimensional Cauchy-Dirichlet problem, assuming that T is a continuous function of the components of D satisfying some further conditions and losing its linear feature for large values of the gradient of the velocity. As the Navier-Stokes equations are not of relativistic nature it is reasonable to suppose that they break down for high velocities. For this motivation, we believe that it is physically more meaningful to modify (1) for large values of the velocity $|u|$ rather than

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ISSN 0232-2064 / $ 2.50

of its gradient. This is precisely the idea of the modification of Prouse [16], who has assumed that (1) only holds when the velocity of the fluid is small. More precisely, the relationship between the stress tensor and the deformation velocity tensor is given by

$$\tau_{ij} = -p\delta_{ij} + \partial_i\varphi_j(u) + \partial_j\varphi_i(u) \tag{2}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a function of u whose properties are quoted in (6) below. Clearly, when $\varphi(u) = \mu u$, (2) reduces to the classical linear law. Introducing (2) into the general equations of conservation of momentum, we obtain the following generalized Navier-Stokes equation for incompressible fluids subject to an external force f:

$$\left.\begin{aligned} \partial_t u - \Delta\varphi(u) + (u\cdot\nabla)u + \nabla p - \nabla(\nabla\cdot\varphi(u)) &= f \\ \nabla\cdot u &= 0. \end{aligned}\right\} \tag{3}$$

Problems relative to the above equation have been studied in [8, 9, 14]. To (3) we associate the following Cauchy-Dirichlet problem in $\Omega\times[\tau,T]$:

$$\left.\begin{aligned} &u(x,t) = 0 && \text{if } (x,t)\in\partial\Omega\times[\tau,T] \\ &u(x,\tau) = u_0(x) && \text{if } x\in\Omega \end{aligned}\right\} \tag{4}$$

where $\Omega\subset\mathbb{R}^n$, $\tau\in\mathbb{R}$ and $T>\tau$.

2. Preliminaries

2.1 Functional setting and notations. We assume that $\Omega\subset\mathbb{R}^n$ is an open bounded set with boundary $\partial\Omega$ of class $C^{1,1}$. We denote by L^p the space of p^{th} power absolutely integrable functions, by $W^{m,p}$ the Sobolev spaces of functions in L^p with their first m generalized derivatives in L^p, by $H^m = W^{m,2}$ the Hilbertian Sobolev spaces, by H_0^m the H^m-closure of the space of smooth functions with compact support in Ω, and by γ_n the normal trace operator. To simplify notations we delete the domain of definition Ω. We also need the Hilbert spaces (see [18])

$$H = \Big\{u\in L^2 : \nabla\cdot u = 0 \text{ and } \gamma_n u = 0\Big\} \quad\text{and}\quad V = \Big\{u\in H_0^1 : \nabla\cdot u = 0\Big\}$$

and the dual space V^* of V, endowed with the scalar products

$$\begin{aligned} (u,v)_H &= (u,v)_{L^2} \\ (u,v)_V &= \big((-\Delta)^{\frac12}u,(-\Delta)^{\frac12}v\big)_{L^2} \\ (u,v)_{V^*} &= (G^{\frac12}u,G^{\frac12}v)_{L^2} \end{aligned}$$

where $G : V^*\to V$ is the Green operator relative to $-\Delta$. By the Poincaré inequality we have

$$\lambda_1^2\,\|u\|_{V^*}^2 \le \lambda_1\,\|u\|_H^2 \le \|u\|_V^2 \qquad \text{for all } u\in V \tag{5}$$

where λ_1 is the first eigenvalue of $-\Delta$ in V. We shall use both the weak and the norm topology on H, and we will denote the space H endowed with its weak topology by H_w.

In the n-dimensional case we assume that the function φ introduced in (2) satisfies the following assumptions:

$$\left.\begin{array}{l}\varphi(u)=\sigma(|u|)u \\ \sigma\in C^1(\mathbb{R}^+),\ \sigma(\xi)\geq\mu>0,\ \text{ and } \sigma'(\xi)\geq 0 \text{ for all } \xi\in\mathbb{R}^+ \\ \beta\,\xi^{s-1}\geq\sigma(\xi)\geq\alpha\,\xi^{s-1} \text{ for all } \xi\geq\xi_0\end{array}\right\} \tag{6}$$

where

$$\alpha,\beta,\xi_0>0 \text{ and } s \text{ are constants,}\quad s\begin{cases}\geq 1 & \text{if } n=2\\ \geq n+1 & \text{if } n\geq 3.\end{cases}$$

Some remarks about (6) are in order. First note that σ is required to behave as a pure power at infinity and that the lower bound of such power is precisely the dimension n. Next observe that the case $n = 2$ is an exception since it only requires a positive constant as lower bound for σ. Therefore, in the 2-dimensional case, (3) is indeed a generalization of the classical Navier-Stokes equations (see [14]). The reason of the different behavior of the 2-dimensional case is that to obtain a uniqueness result we need to prove that the solution u satisfies $u\in L^{n+2}(\tau,T;L^{n+2})$ in order to apply Lemma 3.1 below: in the case $n = 2$ this follows directly from $u\in L^\infty(\tau,T;H)\cap L^2(\tau,T;V)$ and no further assumption is needed. We recall that the form

$$b(u,v,w)=(u\cdot\nabla)v\cdot w \tag{7}$$

is trilinear continuous on $(V\cap L^n)^3$ (see [18: Lemma 1.1/p.161]) and that in the classical case one can only prove that the weak solution u belongs to V for a.e. $t\in[\tau,T]$. As $V\not\subset L^n$ for $n\geq 5$, energy estimates for a weak solution in dimension $n\geq 5$ have not been obtained for the classical equation. In our case assumption (6) enables us to conclude that u also belongs to L^{n+2} and thus to obtain energy estimates and uniqueness results in all dimensions.

If $\sigma(\xi)\equiv\mu$, then (2) becomes the classical linear relationship (1) and it is therefore reasonable (although not necessary) to require that

$$\varphi(u)=\sigma(|u|)u=\big(\mu+\bar{\sigma}(|u|)\big)u,$$

with

$$\bar{\sigma}(\xi)\equiv 0 \qquad \text{when } \xi\leq\xi_1 \qquad \text{for some } \xi_1>0.$$

Let us now define what we mean by a weak solution of problem (3)-(4).

Definition 2.1. Assuming that $u_0\in H$ and

$$f\in L^1(\tau,T;H)+L^2(\tau,T;V^*) \tag{8}$$

we say that u solves problem (3)-(4) if u satisfies the following conditions:

$$\left.\begin{array}{l} u \in L^2(\tau,T;V)\cap L^\infty(\tau,T;H)\cap L^{s+1}(\tau,T;L^{s+1}) \\ \langle \partial_t u - \Delta\varphi(u) + (u\cdot\nabla)u - f, h\rangle = 0 \\ \text{for all } h \in L^2(\tau,T;V)\cap L^\infty(\tau,T;H)\cap L^{s+1}(\tau,T;W^{2,s+1}) \\ u(x,\tau) = u_0(x). \end{array}\right\} \quad \textbf{(P)}$$

As usual, we refer to the norm

$$\|f\|^2_{L^1(\tau,T;H)+L^2(\tau,T;V^*)} = \inf_{f^1+f^2}\left\{\|f^1\|^2_{L^1(\tau,T;H)} + \|f^2\|^2_{L^2(\tau,T;V^*)}\right\}$$

where the infimum is taken over all the possible decompositions of f.

To describe the long-time behavior of the solutions of problem (P) we need to introduce some other spaces. Here and in the sequel, for $\tau \in \mathbb{R}$ we use the notation $\mathbb{R}_\tau = [\tau,+\infty)$. For all $p \in [1,+\infty)$ we define the Banach space of L^p_{loc}*-translation bounded functions* on $\mathbb{R}_\tau$ taking values in a real Banach space X, namely (see, e.g., [7])

$$\mathcal{L}^p_{tb}(\mathbb{R}_\tau;X) := \left\{ f \in L^p_{loc}(\mathbb{R}_\tau;X) : \sup_{\xi\in\mathbb{R}_\tau}\int_\xi^{\xi+1}\|f(s)\|^p_X\,ds < \infty\right\}$$

endowed with the norm

$$\|f\|_{tb,p,X} = \sup_{\xi\in\mathbb{R}_\tau}\left(\int_\xi^{\xi+1}\|f(s)\|^p_X\,ds\right)^{\frac{1}{p}}.$$

Finally, we say that a function f is *translation compact* in $L^1_{loc}(\mathbb{R};H) + L^2_{loc}(\mathbb{R};V^*)$ if the *hull* of f defined by

$$\mathcal{H}(f) = \overline{\left\{f(\cdot+s) : s\in\mathbb{R}\right\}}^{L^1_{loc}(\mathbb{R};H)+L^2_{loc}(\mathbb{R};V^*)}$$

is compact in $L^1_{loc}(\mathbb{R};H) + L^2_{loc}(\mathbb{R};V^*)$. Such a function f necessarily belongs to the space $\mathcal{L}^1_{tb}(\mathbb{R}_\tau;H) + \mathcal{L}^2_{tb}(\mathbb{R}_\tau;V^*)$. Necessary and sufficient conditions for translation compactness may be found in [7] and in the references therein. We just recall that, in particular, the class of translation compact functions in $L^1_{loc}(\mathbb{R};H) + L^2_{loc}(\mathbb{R};V^*)$ contains $L^p(\mathbb{R};H) + L^q(\mathbb{R};V^*)$, for any $p\in[1,+\infty)$ and $q\in[2,+\infty)$, the constant V^*-valued functions, and the class of almost periodic functions in $C_b(\mathbb{R};H)$ and $C_b(\mathbb{R};V^*)$ (cf. [1]).

2.2 Main results. In Section 3 we prove the existence and uniqueness of a weak solution of problem (P) for all $n \geq 2$. More precisely, we prove the following

Theorem 2.2. *Let φ be as in* (6), *assume that f satisfies* (8), *and let $u_0 \in H$. Then there exists a unique vector u solving problem* (P). *Moreover, u satisfies*

$$\Delta\varphi(u) \in L^{1+\frac{1}{s}}(\tau,T;W^{-2,1+\frac{1}{s}})$$
$$\partial_t u \in L^{1+\frac{1}{s}}(\tau,T;(W^{2,s+1}_0\cap V)^*) + L^1(\tau,T;H).$$

This result extends those of [14, 16, 17] where the force f merely has the component in $L^2(\tau, T; V^*)$ and where only the 2-dimensional and 3-dimensional cases have been considered. By Definition 2.1, the solution u of problem (P) belongs to $L^2(\tau, T; V) \cap L^{s+1}(\tau, T; L^{s+1}) \cap L^\infty(\tau, T; H)$ and the initial condition $u(x, \tau) = u_0$ in H need not make sense, as $u(t)$ may not be defined pointwise for all $t \in [\tau, T]$. Due to the presence of the nonlinearity in the Laplacian, we cannot prove that $u \in C(\tau, T; H)$ as in the classical case (see, e.g., [18]). To ensure that the initial condition does make sense, in Proposition 3.3 below we will prove the H_w-continuity of $u(t)$.

In Section 4 we first recall the basic tools needed to investigate the long-time behavior of a family of processes associated to an evolution equation; in the appendix we quote two Gronwall-type lemmas which are used for the analysis of problem (P). Next, we consider a family of solutions of problem (P) obtained letting the non-autonomous forcing term f vary in some functional space, and we prove the existence of a uniform absorbing set and of a weak uniform attractor:

Theorem 2.3. *Assume* (6) *and let f in* (8) *be translation compact in $L^1_{loc}(\mathbb{R}; H) + L^2_{loc}(\mathbb{R}; V^*)$. Then the family of processes $\{U_g(t,\tau)\}_{g \in \mathcal{H}(f)}$ associated to problem* (P) *possesses a weak uniform attractor $\mathcal{A}$ given by*

$$\mathcal{A} = \left\{ u(0) \,\middle|\, \begin{array}{l} u(t) \text{ is a bounded complete trajec-} \\ \text{tory of } U_g(t,\tau) \text{ for some } g \in \mathcal{H}(f) \end{array} \right\}.$$

In particular, Theorem 2.3 extends some results given in [9] and [14], where the existence of a weak attractor with stationary forces f has been proved in the 3-dimensional and in the 2-dimensional case, respectively, and where some estimates for non-autonomous forces have also been obtained [1]. The problem of finding a uniform attractor for a family of processes has been investigated by many authors. In [5, 6, 11] the case of almost periodic symbols has been studied. Here, following the ideas of [7] (see also [10, 15]), we focus our attention on the more general situation of translation compact symbols in $L^1_{loc}(\mathbb{R}; H) + L^2_{loc}(\mathbb{R}; V^*)$.

3. Existence and uniqueness

In this section we prove the existence and uniqueness of a solution of problem (P). Let us begin recalling some known results.

Lemma 3.1. *Let $u, v \in L^{n+2}$ and $w \in H$. Then for all $\nu > 0$ there exist constants $\Lambda_1(\nu), \Lambda_2(\nu) > 0$ such that*

$$\left|\langle (w \cdot \nabla)u, Gw \rangle\right| \le \nu \,\|w\|_H^2 + \Lambda_1(\nu)\|u\|_{L^{n+2}}^{n+2}\|w\|_{V^*}^2$$

and

$$\left|\langle (w \cdot \nabla)u, Gw \rangle + \langle (v \cdot \nabla)w, Gw \rangle\right| \le \nu \,\|w\|_H^2 + \Lambda_2(\nu)\Big(\|u\|_{L^{n+2}}^{n+2} + \|v\|_{L^{n+2}}^{n+2}\Big)\|w\|_{V^*}^2.$$

[1] When time-dependent forces are involved, in [9] the process is often improperly called semigroup.

Proof. To obtain the first estimate we generalize arguments used in [16: Lemma 1]: By the Hölder and Young inequalities we have

$$\begin{aligned}\left|\langle (w\cdot\nabla)u, Gw\rangle\right| &\le \|u\|_{L^{n+2}}\|Gw\|_{W^{1,\frac{2(n+2)}{n}}}\|w\|_{L^2}\\ &\le \frac{\nu}{2}\|w\|_{L^2}^2 + \frac{1}{2\nu}\|u\|_{L^{n+2}}^2\|w\|^2_{W^{-1,\frac{2(n+2)}{n}}}.\end{aligned}$$

Next note that by interpolation between $W^{-1,2}$ and $W^{-1,2n/(n-2)}$ (which is $W^{-1,\infty}$ if $n=2$) and again by the Young inequality, being the injection $H \hookrightarrow W^{-1,2n/(n-2)}$ continuous, we get

$$\begin{aligned}\|w\|^2_{W^{-1,\frac{2(n+2)}{n}}} &\le C_1\Big(\|w\|_{V^*}^{\frac{4}{n+2}}\,\|w\|_H^{\frac{2n}{n+2}}\Big)\\ &\le C_2\Big(\varepsilon\|w\|_{V^*}^2 + \varepsilon^{-\frac{2}{n}}\|w\|_H^2\Big)\end{aligned}$$

for some constants $C_1, C_2 > 0$ depending only on n and Ω, and for any $\varepsilon > 0$. Denoting $C_3 = \big(\frac{C_2}{\nu^2}\big)^{\frac{n}{2}}$, and choosing $\varepsilon = C_3\|u\|_{L^{n+2}}^n$, we get the first inequality, having set $\Lambda_1(\nu) = \frac{C_1C_3}{2\nu}$. The proof of the second estimate is analogous ∎

Lemma 3.2. *Assume* (6). *Then*

$$(\varphi(v), v)_V \ge \mu\|v\|_V^2 \qquad \textit{for all } v \in V$$

and

$$\big(\varphi(u)-\varphi(v), u-v\big)_H \ge \mu\,\|u-v\|_H^2 \qquad \textit{for all } u,v \in H.$$

Proof. It follows directly from [16: Lemmas 2 and 3] ∎

We are now ready to give the

Proof of Theorem 2.2. Since the proof is obtained by slight modifications of the device used in [16], we only give a sketch of it and outline the basic differences. We make use of the standard Faedo-Galerkin method: we consider the complete orthonormal system of V of the eigenfunctions e_j ($j \in \mathbb{N}$) of $-\Delta$ and define the subspace $V_m = \operatorname{span}\{e_j\}_{1\le j\le m}$ for all $m \in \mathbb{N}$. Then, we consider the finite-dimensional problem of finding $u_m(t) \in V_m$ satisfying

$$\big(\partial_t u_m(t), e_j\big)_H - \big(\varphi(u_m(t)), \Delta e_j\big)_H + \big((u_m(t)\cdot\nabla)u_m(t), e_j\big)_H - \langle f(t), e_j\rangle = 0 \qquad (9)$$

for all $j = 1,\dots,m$. By standard methods one can prove that, for all m, problem (9) admits a solution $u_m(t)$ (with $t \in [\tau, T]$, for any $T > \tau$) which, by Lemma 3.2, satisfies the estimates

$$\sup_{t\in[\tau,T]}\|u_m(t)\|_H \le C_4, \quad \int_\tau^T \|u_m(t)\|_V^2\,dt \le C_5, \quad \int_\tau^T \|u_m(t)\|_{L^{s+1}}^{s+1}\,dt \le C_6. \qquad (10)$$

Indeed, if f in (8) has also the component in $L^1(\tau, T; H)$, the first two estimates can be obtained as for (4.8) in [16] by modifying the proof following [18: p. 264]. To obtain the third estimate one can reason as for (4.13) in [16] and replace (4.11) in [16] by

$$\int_\tau^T \left|\left\langle f, \sum_{j=1}^m \frac{\alpha_{jm}}{\lambda_j} e_j \right\rangle\right| dt$$
$$\leq \|f^1\|_{L^1(\tau,T;H)} \|u_m\|_{L^\infty(\tau,T;H^{-2})} + \|f^2\|_{L^2(\tau,T;V^*)} \|u_m\|_{L^2(\tau,T;V^*)}$$

which is uniformly bounded (here $f = f^1 + f^2$ with $f^1 \in L^1(\tau, T; H)$ and $f^2 \in L^2(\tau, T; V^*)$, $u_m = \sum_j \alpha_{jm} e_j$ and λ_j is the eigenvalue relative to e_j).

¿From (6) and (10) we obtain

$$\int_\tau^T \|\varphi(u_m(t))\|_{L^{1+\frac{1}{s}}}^{1+\frac{1}{s}} \, dt \leq C_7 \tag{11}$$

and therefore

$$\int_\tau^T \|\Delta(\varphi(u_m(t)))\|_{W^{-2,1+\frac{1}{s}}}^{1+\frac{1}{s}} \, dt \leq C_8, \tag{12}$$

the constants C_i $(i = 4, \dots, 8)$ being independent of m. Then, there exists

$$u \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \cap L^{s+1}(\tau, T; L^{s+1})$$

such that, up to a subsequence, $u_m \to u$ in the weak topologies of $L^2(\tau, T; V)$ and $L^{s+1}(\tau, T; L^{s+1})$, and in the weak* topology of $L^\infty(\tau, T; H)$.

By (11), the sequence $\{\varphi(u_m)\}$ is bounded in $L^{1+\frac{1}{s}}(\tau, T; L^{1+\frac{1}{s}})$ and, up to a further subsequence, it converges in the $L^{1+\frac{1}{s}}(\tau, T; L^{1+\frac{1}{s}})$ weak topology to some limit χ. To prove that $\chi = \varphi(u)$ one exploits the a.e. pointwise convergence of u_m to u (see again [16]). Replacing e_j in (9) by a smooth function h on $[\tau, T]$, letting $m \to \infty$ and reasoning as in [18: pp. 257-259], one gets that u solves problem (P) in the distributional sense. Finally, note that by (12) we have $\Delta(\varphi(u)) \in L^{1+\frac{1}{s}}(\tau, T; W^{-2,1+\frac{1}{s}})$. Therefore, using a density argument one has that u satisfies the equation according to Definition 2.1 and we obtain

$$\partial_t u \in L^{1+\frac{1}{s}}\big(\tau, T; (W_0^{2,s+1} \cap V)^*\big) + L^1(\tau, T; H).$$

To prove uniqueness, we argue by contradiction and assume that both u and v solve problem the (P). Set $w = u - v$. Then $w \in L^2(\tau, T; V) \cap L^{n+2}(\tau, T; L^{n+2})$ and

$$\Big\langle \partial_t w - \Delta\big(\varphi(u) - \varphi(v)\big) + (w \cdot \nabla)u + (v \cdot \nabla)w, h \Big\rangle = 0 \tag{13}$$

for all test functions h. Let $s \in [\tau, T]$ and choose as particular test function

$$h(t) = \begin{cases} Gw(t) & \text{when } t \in [\tau, s] \\ 0 & \text{when } t > s. \end{cases}$$

Lemma 3.2 entails

$$-\langle\Delta(\varphi(u)-\varphi(v)),h\rangle \geq \mu\int_\tau^s \|w(t)\|_H^2\,dt,$$

and from Lemma 3.1 one gets

$$\begin{aligned}&\left|\langle(w\cdot\nabla)u+(v\cdot\nabla)w,h\rangle\right|\\ &\leq \mu\int_\tau^s \|w(t)\|_H^2\,dt+\Lambda_2(\mu)\int_\tau^s\left(\|u(t)\|_{L^{n+2}}^{n+2}+\|v(t)\|_{L^{n+2}}^{n+2}\right)\|w(t)\|_{V^*}^2\,dt.\end{aligned}$$

Then, as $\langle\partial_t w,h\rangle=\frac{1}{2}\|w(s)\|_{V^*}^2$, equality (13) yields

$$\frac{1}{2}\|w(s)\|_{V^*}^2\leq\Lambda_2(\mu)\int_\tau^s\left(\|u(t)\|_{L^{n+2}}^{n+2}+\|v(t)\|_{L^{n+2}}^{n+2}\right)\|w(t)\|_{V^*}^2\,dt$$

for all $s\in[\tau,T]$. Since $\|u(t)\|_{L^{n+2}}^{n+2}+\|v(t)\|_{L^{n+2}}^{n+2}\in L^1(\tau,T)$, by the Gronwall lemma we obtain $\|w(s)\|_{V^*}=0$ and the uniqueness follows by the arbitrariness of s ∎

Next, we prove that the initial condition $u(x,\tau)=u_0\in H$ makes sense; more precisely, $u(t)$ is a.e. equal to a continuous function in the H_w-topology.

Proposition 3.3. *Under the assumptions of Theorem 2.2, let u be the unique solution of problem* (P). *Then $u\in C(\tau,T;H_w)$ and $u\in C^+(\tau,T;H)$, that is, $u(t)$ is continuous in $[\tau,T]$ in the H_w-topology and it is right-continuous in the H-norm topology.*

Proof. Assume that $t_m\to t_0$; we claim that $u(t_m)\to u(t_0)$ in H_w. Since $u\in L^\infty(\tau,T;H)$, it follows that $u(t_m)\to v$ in H_w for some $v\in H$, up to a subsequence. Moreover, by Theorem 2.2 and [18: Lemma 1.1/p. 250] we get $u\in C\big(\tau,T;(W_0^{2,s+1}\cap V)^*\big)$, hence $v=u(t_0)$. The claim then follows by the arbitrariness of the subsequence.

Assume now that $t_m\to t_0^+$. Using the continuity of the trilinear form (7) and of the other operators involved, we proceed formally: by standard energy estimates (see, e.g., (4.49) in [16]) we obtain

$$\|u(t_m)\|_H^2-\|u(t_0)\|_H^2\leq C_9\int_{t_0}^{t_m}|\langle f(s),u(s)\rangle|\,ds$$

for some constant $C_9>0$. By Definition 2.1 it is clear that $|\langle f(t),u(t)\rangle|\in L^1(\tau,T)$, hence we get

$$\lim_{t_m\to t_0^+}\|u(t_m)\|_H\leq\|u(t_0)\|_H.$$

By the H_w-continuity just proved and by the lower semicontinuity of the H-norm in the H_w-topology we obtain the converse inequality, that is the convergence of the H-norms, and the result readily follows ∎

4. Existence of a uniform attractor

4.1 General results. Let E be a Hilbert space. According to [11], we say that a family of operators $U(t,\tau): E \to E$ $((t,\tau) \in \mathbb{R} \times \mathbb{R}_\tau)$ is a *process* if the following two conditions hold:

(I) $U(\tau,\tau) = I$ (identity on E) for every $\tau \in \mathbb{R}$.

(II) $U(t,s)U(s,\tau) = U(t,\tau)$, for every $t \geq s \geq \tau$.

We consider a family of processes $\{U_f(t,\tau)\}_{f\in F}$ depending on a functional parameter $f \in F$, where F is a suitable topological space. The parameter f is sometimes called the *symbol* of the process. We begin with some definitions.

Definition 4.1. A set $\mathcal{B}_0 \subset E$ is said to be *uniformly absorbing* (with respect to $f \in F$) for $\{U_f(t,\tau)\}_{f\in F}$ if for every bounded set $\mathcal{B} \subset E$ and every $\tau \in \mathbb{R}$ there exists $T = T(\mathcal{B},\tau) \in \mathbb{R}_\tau$ such that $\bigcup_{f\in F} U_f(t,\tau)\mathcal{B} \subset \mathcal{B}_0$ for all $t \geq T$.

We now introduce the notions of a weakly uniformly attracting set and a weak uniform attractor. As before we denote the space E endowed with the weak topology by E_w.

Definition 4.2. A set $\mathcal{A} \subset E$ is said to be *weakly uniformly attracting* for the family $\{U_f(t,\tau)\}_{f\in F}$ if for every bounded set $\mathcal{B} \subset E$, every open set $\mathcal{O} \subset E_w$ such that $\mathcal{O} \supset \mathcal{A}$, and every $\tau \in \mathbb{R}$ there exists $T = T(\mathcal{B},\mathcal{O},\tau) \in \mathbb{R}_\tau$ such that

$$\bigcup_{f\in F} U_f(t,\tau)\mathcal{B} \subset \mathcal{O} \tag{14}$$

for all $t \geq T$.

In the case when $\{U_f(t,\tau)\}_{f\in F}$ has a convex, closed and bounded uniform absorbing set $\mathcal{B}_0$ condition (14) can be replaced by

$$\lim_{t\to+\infty} \left[\sup_{f\in F} \mathrm{d}\big(U_f(t,\tau)\mathcal{B},\mathcal{A}\big)\right] = 0$$

where

$$\mathrm{d}\big(U_f(t,\tau)\mathcal{B},\mathcal{A}\big) = \sup_{u\in U_f(t,\tau)\mathcal{B}} \mathrm{dist}(u,\mathcal{A}) = \sup_{u\in U_f(t,\tau)\mathcal{B}} \left[\inf_{v\in\mathcal{A}} \mathrm{dist}(u,v)\right]$$

and "dist" is the the metric induced by the weak topology of E on $\mathcal{B}_0$. Indeed, $\mathcal{A}$ is contained in the weak closure of $\mathcal{B}_0$, which coincides with its norm closure, being $\mathcal{B}_0$ convex.

Definition 4.3. A closed set $\mathcal{A} \subset E_w$ is said to be a *weak uniform attractor* for $\{U_f(t,\tau)\}_{f\in F}$ if the following two conditions occur:

(I) $\mathcal{A}$ is weakly uniformly attracting.

(II) $\mathcal{A} \subset \mathcal{A}'$, for every weakly uniformly attracting closed set $\mathcal{A}' \subset H_w$.

Let now F be a compact metric space and $\{T(t)\}$ a continuous semigroup acting on it. By well-known results (see, e.g., [2, 19]) $\{T(t)\}$ possesses a global attractor $\mathcal{A}(F)$. In order to prove the existence of a weak uniform attractor for the family $\{U_f(t,\tau)\}_{f\in F}$, we need a variation of some results of Chepyzhov and Vishik (cf. [5: Theorem 3.1 and Corollary 3.1] and [6: Theorem 5.2]). It is easy to check that the proofs directly extend to this case.

Theorem 4.4. *Let $\{U_f(t,\tau)\}_{f\in F}$ be an $(E\times\mathcal{A}(F),E_w)$-continuous (for every fixed t and τ) family of processes possessing a bounded absorbing set $\mathcal{B}_0\subset E$, let the semigroup $\{T(t)\}$ acting on F satisfy the translation equality*

$$U_f(t+s,\tau+s)=U_{T(s)f}(t,\tau) \qquad \textit{for all } f\in F,\ \tau\in\mathbb{R},\ t\in\mathbb{R}_\tau,\ s\geq 0. \tag{15}$$

Then $\{U_f(t,\tau)\}_{f\in F}$ has a weak uniform attractor $\mathcal{A}$ which is weakly compact. Moreover, $\mathcal{A}$ is unique, and it is of the form

$$\mathcal{A}=\left\{u(0)\,\middle|\,\begin{array}{l}u(t)\text{ is a bounded complete trajec-}\\ \text{tory of } U_f(t,\tau)\text{ for some } f\in\mathcal{A}(F)\end{array}\right\}.$$

4.2 Application to Problem (P). We associate to problem (P) a family of processes indexed by a symbol f in a standard way, namely, we write $U_f(t,\tau)u_0$ to denote the solution of problem (P) at time t, with forcing term $f\in L^1_{loc}(\mathbb{R}_\tau;H)+L^2_{loc}(\mathbb{R}_\tau;V^*)$ and initial data u_0 given at time $\tau\in\mathbb{R}$.

We first provide a time-uniform estimate.

Proposition 4.5. *In the hypotheses of Theorem 2.2, there exist two positive constants C_{10} and δ_0, only depending on Ω and n, such that the unique solution u of problem (P) fulfils*

$$\|u(t)\|_H^2\leq C_{10}\Bigg\{\|u_0\|_H^2\,e^{-2\delta(t-\tau)} + \left(\int_\tau^t e^{-\delta(t-s)}\|f^1(s)\|_H\,ds\right)^2+\int_\tau^t e^{-2\delta(t-s)}\|f^2(s)\|_{V^*}^2\,ds\Bigg\} \tag{16}$$

for any $\delta\in[0,\delta_0]$, any $t\in\mathbb{R}_\tau$, and any decomposition $f=f^1+f^2$.

Proof. We proceed formally: take $h=u$ in problem (P) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2+\big(\varphi(u(t)),u(t)\big)_V=\langle f(t),u(t)\rangle \tag{17}$$

where we have used well-known properties of the trilinear form (7). Using the decomposition $f=f^1+f^2$, according to (8), the Hölder and Young inequalities imply that

$$\begin{aligned}|\langle f(t),u(t)\rangle| &\leq |\langle f^1(t),u(t)\rangle|+|\langle f^2(t),u(t)\rangle|\\ &\leq \|f^1(t)\|_H\,\|u(t)\|_H+\frac{1}{2\mu}\|f^2(t)\|_{V^*}^2+\frac{\mu}{2}\|u(t)\|_V^2.\end{aligned}$$

Therefore from Lemma 3.2 and (5), setting $\delta_0 = \frac{\mu\lambda_1}{2}$, equation (17) entails

$$\frac{d}{dt}\|u(t)\|_H^2 + 2\delta\|u(t)\|_H^2 \leq 2\|f^1(t)\|_H\|u(t)\|_H + \frac{1}{\mu}\|f^2(t)\|_{V^*}^2. \tag{18}$$

for any $\delta \in [0,\delta_0]$. Hence, applying Lemma A.1 below to (18) we get

$$\|u(t)\|_H^2 \leq 2\|u_0\|_H^2 e^{-2\delta(t-\tau)} + 4\left(\int_\tau^t e^{-\delta(t-s)}\|f^1(s)\|_H\,ds\right)^2 + \frac{2}{\mu^2}\int_\tau^t e^{-2\delta(t-s)}\|f^2(s)\|_{V^*}^2\,ds$$

for any $\delta \in [0,\delta_0]$. Setting $C_{10} = \max\{4, \frac{2}{\mu^2}\}$, we get the result ∎

By means of Proposition 4.5 we prove the existence of a uniform absorbing set as the forcing term runs in a bounded subset of a certain Banach space.

Theorem 4.6. *Assume (6), and fix a bounded set $F \subset \mathcal{L}^1_{tb}(\mathbb{R}_\tau; H) + \mathcal{L}^2_{tb}(\mathbb{R}_\tau; V^*)$ (endowed with the usual norm). Then there exists a closed ball $\mathcal{B}_0 \subset H$ which is uniformly absorbing for the family of processes $\{U_f(t,\tau)\}_{f\in F}$ associated to problem* (P).

Proof. Set

$$M = \sup_{f\in F}\left[\inf_{f=f^1+f^2}\left\{\|f^1\|_{tb,1,H}^2 + \|f^2\|_{tb,2,V^*}^2\right\}\right]^{\frac{1}{2}}$$

and consider a decomposition $f = f^1 + f^2 \in F$. Take $t \in \mathbb{R}_\tau$ and let $m \in \mathbb{N}$ such that $\tau + m - 1 < t \leq \tau + m$. Then

$$\begin{aligned}\int_\tau^t e^{-\delta(t-s)}\|f^1(s)\|_H\,ds &\leq e^{-\delta t}\sum_{j=0}^{m-1}\int_{\tau+j}^{\tau+j+1} e^{2\delta s}\|f^1(s)\|_H\,ds\\ &\leq e^{-\delta t}\sum_{j=0}^{m-1} e^{\delta(\tau+j+1)}\int_{\tau+j}^{\tau+j+1}\|f^1(s)\|_H\,ds\\ &\leq e^{-\delta(t-\tau)}\|f^1\|_{tb,1,H}\sum_{j=1}^{m} e^{\delta j}\\ &\leq \frac{e^\delta}{1-e^{-\delta}}\|f^1\|_{tb,1,H}.\end{aligned}$$

Similarly we obtain

$$\int_\tau^t e^{-2\delta(s-\tau)}\|f^2(s)\|_{V^*}^2\,ds \leq \frac{e^{2\delta}}{1-e^{-2\delta}}\|f^2\|_{tb,2,V^*}^2.$$

Define

$$C_\delta = \frac{M^2 e^{2\delta}}{(1-e^{-\delta})^2}.$$

Taking now the infimum over all the admissible decompositions of f and using the process notation, inequality (16) can be enhanced to

$$\left\|U_f(t,\tau)u_0\right\|_H^2 \le C_{10}\left\{\|u_0\|_H^2 e^{-\delta(t-\tau)} + C_\delta\right\} \tag{19}$$

for any $\delta \in (0, \delta_0]$ and any $u_0 \in H$. Finally, setting

$$\mathcal{B}_0 = \left\{v \in H : \|v\|_H \le (2C_{10}C_{\delta_0})^{\frac{1}{2}}\right\}$$

it is clear that, for every $\tau \in \mathbb{R}$ and every bounded set $\mathcal{B} \subset H$,

$$\limsup_{t\to+\infty}\left[\sup_{f\in F}\sup_{v\in\mathcal{B}}\left\|U_f(t,\tau)v\right\|_H^2\right] \le C_{10}C_{\delta_0},$$

that is, $\mathcal{B}_0$ is a uniform absorbing set for $\{U_f(t,\tau)\}_{f\in F}$ ∎

In the remaining of the paper, we will assume that the forcing term f in problem (P) is *translation compact* in $L^1_{loc}(\mathbb{R};H) + L^2_{loc}(\mathbb{R};V^*)$.

Remark 4.7. It is easy to show that if f is translation compact in $L^1_{loc}(\mathbb{R};H) + L^2_{loc}(\mathbb{R};V^*)$, then

$$\|g\|_{\mathcal{L}^1_{tb}(\mathbb{R}_\tau;H)+\mathcal{L}^2_{tb}(\mathbb{R}_\tau;V^*)} = \|f\|_{\mathcal{L}^1_{tb}(\mathbb{R}_\tau;H)+\mathcal{L}^2_{tb}(\mathbb{R}_\tau;V^*)} < \infty \tag{20}$$

for all $g \in \mathcal{H}(f)$. Indeed, there exists a real sequence $\{h_m\}$ such that $f(\cdot + h_m) \to g$ in $L^1_{loc}(\mathbb{R};H) + L^2_{loc}(\mathbb{R};V^*)$, and

$$\begin{aligned}
&\|g\|^2_{\mathcal{L}^1_{tb}(\mathbb{R}_\tau;H)+\mathcal{L}^2_{tb}(\mathbb{R}_\tau;V^*)}\\
&= \lim_{m\to\infty}\|f(\cdot + h_m)\|^2_{\mathcal{L}^1_{tb}(\mathbb{R}_\tau;H)+\mathcal{L}^2_{tb}(\mathbb{R}_\tau;V^*)}\\
&= \lim_{m\to\infty}\inf_{f=f^1+f^2}\left[\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\|f^1(s+h_m)\|_H\,ds\right)^2 + \sup_{\xi\in\mathbb{R}}\int_\xi^{\xi+1}\|f^2(s+h_m)\|^2_{V^*}\,ds\right]\\
&= \lim_{m\to\infty}\inf_{f=f^1+f^2}\left[\sup_{\xi\in\mathbb{R}}\left(\int_{\xi+h_m}^{\xi+h_m+1}\|f^1(s)\|_H\,ds\right)^2 + \sup_{\xi\in\mathbb{R}}\int_{\xi+h_m}^{\xi+h_m+1}\|f^2(s)\|^2_{V^*}\,ds\right]\\
&= \|f\|^2_{\mathcal{L}^1_{tb}(\mathbb{R}_\tau;H)+\mathcal{L}^2_{tb}(\mathbb{R}_\tau;V^*)}.
\end{aligned}$$

On $\mathcal{H}(f)$ it is defined the semigroup of translations $T(t)$, acting as $T(t)g = g(\cdot + t)$, for $g \in \mathcal{H}(f)$. It is straightforward to see that the global attractor $\mathcal{A}(\mathcal{H}(f))$ of the semigroup $\{T(t)\}$ coincides with the whole space $\mathcal{H}(f)$. Clearly, the family of processes $\{U_g(t,\tau)\}_{g\in\mathcal{H}(f)}$ associated to problem (P), with forcing term $g \in \mathcal{H}(f)$, fulfils condition (15). In order to prove the existence of a weak uniform attractor we have then to show the continuity property.

Proposition 4.8. *Assume* (6), *and let f be translation compact in $L^1_{loc}(\mathbb{R};H) + L^2_{loc}(\mathbb{R};V^*)$. Then, for every fixed t and τ, the family $\{U_g(t,\tau)\}_{g\in\mathcal{H}(f)}$ associated to problem* (P) *is $(H\times\mathcal{H}(f), H_w)$-continuous.*

Proof. Let u, v be the solutions of problem (P) corresponding to initial data u_0, $v_0 \in H$ and forcing terms g, $h \in \mathcal{H}(f)$, respectively. Set $w = u - v$, $w_0 = u_0 - v_0$ and $k = g - h$. Choosing Gw as test function in problem (P) and subtracting the equations relative to u and v, we get

$$\big\langle \partial_t w, Gw\big\rangle + \big(\varphi(u) - \varphi(v), w\big)_H = -\big\langle (w\cdot\nabla)u, Gw\big\rangle - \big\langle (v\cdot\nabla)w, Gw\big\rangle + \langle k, Gw\rangle.$$

Lemma 3.1 (with $\nu = \frac{\mu}{2}$) and Lemma 3.2 imply

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{V^*} + \frac{\mu}{2}\|w(t)\|^2_H \le K(t)\,\|w(t)\|^2_{V^*} + \big|\big\langle k(t), Gw(t)\big\rangle\big| \tag{21}$$

where

$$K(t) = \Lambda_2\left(\frac{\mu}{2}\right)\left(\|u(t)\|^{n+2}_{L^{n+2}} + \|v(t)\|^{n+2}_{L^{n+2}}\right).$$

¿From Theorem 2.2 we have that $K(t) \in L^1_{loc}(\mathbb{R}_\tau)$. Writing now $k = k^1 + k^2$, with $k^1 \in L^1_{loc}(\mathbb{R}_\tau;H)$ and $k^2 \in L^2_{loc}(\mathbb{R}_\tau;V^*)$, inequality (5), the Hölder and Young inequalities give

$$\begin{aligned}\big|\big\langle k(t), Gw(t)\big\rangle\big| &\le \|k(t)\|_{V^*}\,\|w(t)\|_{V^*}\\ &\le \lambda_1\,\|k^1(t)\|_H\,\|w(t)\|_{V^*} + \frac{1}{\mu\lambda_1}\,\|k^2(t)\|^2_{V^*} + \frac{\mu\lambda_1}{4}\,\|w(t)\|^2_{V^*}.\end{aligned}$$

Setting $C_{11} = \frac{2}{\mu\lambda_1}$, (21) becomes

$$\begin{aligned}&\frac{d}{dt}\,\|w(t)\|^2_{V^*} + \frac{1}{C_{11}}\,\|w(t)\|^2_{V^*}\\ &\quad\le 2K(t)\,\|w(t)\|^2_{V^*} + 2\lambda_1\,\|k^1(t)\|_H\,\|w(t)\|_{V^*} + C_{11}\,\|k^2(t)\|^2_{V^*}.\end{aligned} \tag{22}$$

Denoting

$$J(t) = \exp\left[4\int_\tau^t K(s)\,ds\right],$$

from Lemma A.2 below we finally get

$$\|w(t)\|^2_{V^*} \le J(t)\left(2\,\|w_0\|^2_{V^*} + 4\lambda_1^2\left(\int_\tau^t \|k^1(s)\|_H\,ds\right)^2 + 2C_2\int_\tau^t \|k^2(s)\|^2_{V^*}\,ds\right). \tag{23}$$

Choosing $C_{12} = \max\{2, 4\lambda_1^2, 2C_{11}\}$, since (23) holds for all the decompositions of k, we conclude that

$$\|w(t)\|^2_{V^*} \le C_{12}J(t)\Big(\|w_0\|^2_{V^*} + \|k\|^2_{L^1(\tau,t;H)+L^2(\tau,t;V^*)}\Big). \tag{24}$$

Notice that $w_0 \to 0$ in V^* whenever $w_0 \to 0$ in H_w (and thus, in particular, whenever $w_0 \to 0$ in H). Therefore it is clear that (24) entails the continuity of the family of processes from $H \times \mathcal{H}(f)$ to H_{V^*} (the space H endowed with the topology inherited from V^*). To complete the proof take a sequence $(u_0^m, g^m) \to (u_0, g)$ in $H \times \mathcal{H}(f)$ and let u^m and u be the corresponding solutions of problem (P). From (19) and (20) we know that the sequence $\{u^m\}$ is bounded in H and therefore, up to a subsequence, it converges weakly to some $v \in H$. On the other hand, by (24), the sequence $\{u^m\}$ converges in H_{V^*} to u and thus $v = u$. The $(H \times \mathcal{H}(f), H_w)$-continuity is then proved ∎

Remark 4.9. Notice that in Proposition 4.8 we actually show a stronger continuity property, namely, the family $\{U_f(t,\tau)\}_{g \in \mathcal{H}(f)}$ associated to problem (P) is $(H_w \times \mathcal{H}(f), H_w)$-continuous.

We can now easily obtain the

Proof of Theorem 2.3. It follows directly by Theorem 4.6 and Proposition 4.8, in virtue of Theorem 4.4 ∎

Remark 4.10. Using standard techniques (see, e.g., [19]), one can also prove that the classical 2-dimensional case (i.e., $\sigma \equiv \mu$) displays a uniform attractor in the H-norm topology when f satisfies (8), extending some results of [5, 7].

5. Appendix

The following two Gronwall-type lemmas, which are crucial for our calculations, can be easily deduced from [3] and [4: Lemma A.5/p.157].

Lemma A.1. *Let ϕ be a non-negative, absolutely continuous function on $\mathbb{R}_\tau$, for some $\tau \in \mathbb{R}$, which satisfies for a.e. $t \in \mathbb{R}_\tau$ the differential inequality*

$$\frac{d}{dt}\phi(t) + 2\delta\phi(t) \le m_1(t)\,\phi(t)^{\frac{1}{2}} + m_2(t)$$

for some $\delta \ge 0$, where m_1 and m_2 are non-negative locally summable functions on $\mathbb{R}_\tau$. Then

$$\phi(t) \le 2\phi(\tau)\,e^{-2\delta(t-\tau)} + \left(\int_\tau^t m_1(s)\,e^{-\delta(t-s)}ds\right)^2 + 2\int_\tau^t m_2(s)\,e^{-2\delta(t-s)}ds$$

for any $t \in \mathbb{R}_\tau$.

Lemma A.2. *Let ϕ be a non-negative, absolutely continuous function on $\mathbb{R}_\tau$, for some $\tau \in \mathbb{R}$, which satisfies for a.e. $t \in \mathbb{R}_\tau$ the differential inequality*

$$\frac{d}{dt}\phi(t) \le m_0(t)\,\phi(t) + m_1(t)\,\phi(t)^{\frac{1}{2}} + m_2(t)$$

where m_0, m_1 and m_2 are non-negative locally summable functions on $\mathbb{R}_\tau$. Then

$$\phi(t) \le \left(2\phi(\tau) + \left(\int_\tau^t m_1(s)\,ds\right)^2 + 2\int_\tau^t m_2(s)\,ds\right)\exp\left[2\int_\tau^t m_0(s)\,ds\right]$$

for any $t \in \mathbb{R}_\tau$.

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Received 01.10.1996