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Equivalence of Oscillation of a Class of Neutral Differential Equations and Ordinary Differential Equations

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Abstract. In this paper, we establish the equivalence of the oscillation of the two equations

$$(x(t) - x(t-\tau))^{(n)} + p(t)\,x(t-\sigma) = 0 \qquad \text{and} \qquad x^{(n+1)}(t) + \frac{p(t)}{\tau}x(t) = 0$$

where $p(t) \geq 0$ and $n \geq 1$ is odd, from which we obtain some new oscillation conditions and comparison theorems for the first of these equations.

Keywords: *Oscillation, positive solutions, neutral differential equations*

AMS subject classification: 34 K 15

1. Introduction

In the past decade, the oscillation of neutral differential equations has attracted the attention of many mathematicians. A testimony of this is that in the last years three monographs on the oscillation of neutral differential equations has come off the press on end, written by Gyori and Ladas [5], Bainov and Mishev [1] and Erbe, and Kong and Zhang [4], respectively. In most work on the higher order neutral differential equation

$$(x(t) - c\,x(t-\tau))^{(n)} + p(t)\,x(t-\sigma) = 0 \tag{1.1}$$

$c \neq 1$ is assumed. In fact [4], the properties of solutions of equation (1.1) in the case $c \in (0,1)$ are essentially different from those of the case $c > 1$. This means that $c = 1$ is a critical case. In recent years, attention has been payed to this critical case. Chuanxi and Ladas [3] first studied the oscillation of equation (1.1) in the case $n = c = 1$. They proved that if $p \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\int^\infty p(t)\,dt = \infty$, then the equation

$$(x(t) - x(t-\tau))' + p(t)\,x(t-\sigma) = 0 \tag{1.2}$$

is oscillatory. They also put forward the open problem whether $\int^\infty p(t)\,dt = \infty$ is also necessary for the oscillation of equation (1.2). Yu [9] solved this problem by giving a

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counterexample. Zhang and Gopalsamy [12], and Ming-Po Chen and Yu [11] proved independently that the condition $\int_{t_0}^{\infty} sp(s)\left(\int_s^{\infty} p(u)\,du\right)ds = \infty$ is sufficient for equation (1.2) to oscillate. Zhang and Yu [13] gave a necessary and sufficient condition for equation (1.2) to have a bounded positive solution. Recently,Yang and Zhang [2] proved that if

$$\liminf_{t\to\infty} t \int_t^{\infty} p(s)\,ds > \frac{\tau}{4},$$

then equation (1.2) is oscillatory.

In this paper, we consider the neutral differential equation

$$(x(t) - x(t-r))^{(n)} + p(t)\,x(t-\sigma) = 0 \tag{1.3}$$

where $r > 0$ and $\sigma \in R$ are constants, $n \geq 1$ is an odd integer and $p : [0,\infty) \to [0,\infty)$ is a continuous function. There has been much work on equation (1.3). We refer to Chuanxi and Ladas [3], Zhang and Gopalsamy [12], Yu [10] and the newest monograph [4]. In this paper we develope the thoughts of [2], wanting to establish the equivalence of the oscillation of equation (1.3) and that of some linear ordinary differential equation. If the equivalence could be established, we can give oscillation criteria of equation (1.3) by using oscillation theorems of linear ordinary differential equations. It is well known that there have been many profound results in the oscillation theory of linear ordinary differential equations (see, for example, Kiguradze and Chanturia [6]).

In Section 2, we establish the equivalence of the oscillation of equation (1.3) and that of some linear ordinary differential equation. In Section 3, we apply the results of Section 2 to equation (1.3) and obtain a sharp criterion for its oscillation, which improves the newest result in [10]. We also establish a class of comparison theorems of integral type. We find the important fact that the deviation σ has no effection on the oscillation of equation (1.3). .

As is customary, a solution of equation (1.3) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise it is said to be non-oscillatory. If every solution of equation (1.3) is oscillatory, equation (1.3) itself is said to be oscillatory. Otherwise equation (1.3) is said to be non-oscillatory.

2. Oscillation equivalence of neutral delay and ordinary differential equations

First we need the following two lemmas.

Lemma 2.1. *A necessary and sufficient condition for equation* (1.3) *to oscillate is that the neutral differential inequality*

$$(x(t) - x(t-r))^{(n)} + p(t)\,x(t-\sigma) \leq 0 \tag{2.1}$$

has no eventually positive solution, where $r > 0$ and $\sigma \in \mathbb{R}$ are constants, $n \geq 1$ is an odd integer and $p : [0,\infty) \to [0,\infty)$ is a continuous function.

In the case $\sigma \geq 0$ Lemma 2.1 is an immediate corrollary of [4: Theorem 5.5.1]. The proof in the case $\sigma < 0$ is quite similar to that in the case $\sigma \geq 0$; only a slight modification is needed and we omit it.

Lemma 2.2. *Let $n \geq 2$ be even and let the function $q : [0, \infty) \to [0, \infty)$ be continuous. Then a necessary and sufficient condition for the equation*

$$x^{(n)}(t) + q(t)\,x(t) = 0 \tag{2.2}$$

to oscillate is that the differential inequality

$$x^{(n)}(t) + q(t)\,x(t) \leq 0 \tag{2.3}$$

has no eventually positive solution.

The proof is quite easy and similar to that of Lemma 2.1 and we omit it.

The main result of this section is the following

Theorem 2.3. *A necessary and sufficient condition for equation* (1.3) *to oscillate is that the ordinary differential equation*

$$x^{(n+1)}(t) + \frac{p(t)}{r}\,x(t) = 0 \tag{2.4}$$

is oscillatory.

Proof. Without loss of generality, we assume $r = 1$ and $n = 3$. That is, we will prove that the oscillation of the two equations

$$(x(t) - x(t-1))''' + p(t)x(t - \sigma) = 0 \tag{2.5}$$

and

$$x''''(t) + p(t)x(t) = 0 \tag{2.6}$$

is equivalent.

Sufficiency, i.e. the oscillation of equation (2.6) implies that of equation (2.5). Suppose to the contrary that equation (2.5) has an eventually positive solution x and set $y(t) = x(t) - x(t-1)$. Then $y'''(t) \leq 0$. By [4: Lemma 5.1.4] we know that $y(t) > 0$ eventually. From [4: Lemma 5.1.2] we see that there are only two possibilities for the solution $x(t)$:

Case (I): $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$ eventually.

Case (II): $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ eventually.

If x is a (I)-type solution, then $\lim_{t\to\infty} y(t) = k \geq 0$, and so y is bounded. Choose T sufficiently large such that $x(t) > 0, y(t) > 0$ and $y'(t) < 0$ on $[T-1, \infty)$, and let $m = \min_{[T-1,T]} x(t)$. Then $m > 0$. If $t \in [T, T+1]$, we have

$$x(t) = y(t) + x(t-1) \geq \int_t^{t+1} y(s)\,ds + m$$

and by induction we get

$$x(t) \geq \int_{t-n}^{t+1} y(s)\,ds + m$$

on $[T+n, T+n+1]$. Hence

$$x(t) \geq \int_{T+1}^{t} y(s)\,ds + m$$

on $[T+1, \infty)$. If $\sigma \geq 0$, we have

$$x(t-\sigma) \geq \int_{T+1}^{t-\sigma} y(s)\,ds + m \geq \int_{T+1+\sigma}^{t} y(s)\,ds + m$$

for $t \in [T+1+\sigma, +\infty)$. If $\sigma < 0$, we have

$$x(t-\sigma) \geq \int_{T+1}^{t-\sigma} y(s)\,ds + m \geq \int_{T+1}^{t} y(s)\,ds + m$$

for $t \in [T+1, +\infty)$. So, whether $\sigma \geq 0$ or $\sigma < 0$, we can always find a $T^* \geq T+1$ such that

$$x(t-\sigma) \geq \int_{T^*}^{t} y(s)\,ds + m \tag{2.7}$$

for $t \in [T^*, \infty)$. Substituting (2.7) into (2.5), we get

$$y'''(t) + p(t)\left(\int_{T^*}^{t} y(s)\,ds + m\right) \leq 0.$$

Let $z(t) = \int_{T^*}^{t} y(s)\,ds$. Then $z(t) > 0$ eventually, and $z''''(t) + p(t)z(t) \leq 0$. From Lemma 2.2 we see that equation (2.6) is non-oscillatory, which is a contradiction.

Now let x be a (II)-type solution, choose T sufficiently large such that $x(t) > 0$, $y(t) > 0$, $y'(t) > 0$ and $y''(t) > 0$ on $[T-1, \infty)$, and let $m = \min_{[T-1,T]} x(t)$. Then $m > 0$. If $t \in [T, T+1]$, we have

$$x(t) = y(t) + x(t-1) \geq \int_{t-1}^{t} y(s)\,ds + m.$$

By induction we have

$$x(t) \geq \int_{t-n-1}^{t} y(s)\,ds + m$$

on $[T+n, T+n+1]$. Hence

$$x(t) \geq \int_{T}^{t} y(s)\,ds + m$$

on $[T+1,\infty)$. In the following discussion,we will distinguish four cases.

1°. If $\sigma \le 0$, we have

$$x(t-\sigma) \ge \int_T^{t-\sigma} y(s)\,ds + m \ge \int_T^{t} y(s)\,ds + m.$$

Letting $z(t) = \int_T^t y(s)\,ds$ and repeating the same argument as about (I)-type solutions, we get a contradiction.

2°. If $\sigma > 0$ and $\lim_{t\to\infty} y''(t) = k > 0$, then

$$y'(t) = kt + o(t), \qquad y(t) = \frac{k}{2}\,t^2 + o(t^2), \qquad \int_T^t y(s)\,ds = \frac{k}{6}\,t^3 + o(t^3)$$

as $t \to \infty$. As long as t is sufficiently large, we have $\int_{t-\sigma}^t y(s)\,ds \le \sigma k t^2$ and then

$$x(t-\sigma) \ge \int_T^{t-\sigma} y(s)\,ds + m = \int_T^t y(s)\,ds - \int_{t-\sigma}^{t} y(s)\,ds + m \ge \int_T^t y(s)\,ds - \sigma k t^2 + m.$$

Let $z(t) = \int_T^t y(s)\,ds - \sigma k t^2$. Then $z(t) > 0$ eventually. Repeating the same argument as about (I)-type solutions, we get a contradiction.

3°. If $\sigma > 0$, $\lim_{t\to\infty} y''(t) = 0$ and $\lim_{t\to\infty} y'(t) = \infty$, then

$$y'(t) = o(t), \quad y(t) = o(t^2),\ t = o(y(t)), \quad \int_T^t y(s)\,ds = o(t^3), t^2 = o\left(\int_T^t y(s)\,ds\right)$$

as $t \to \infty$. So we have

$$\int_{t-\sigma}^t y(s)\,ds < t^2$$

eventually. Let $z(t) = \int_T^t y(s)\,ds - t^2$. Then $z(t) > 0$ and $x(t-\sigma) \ge z(t)$ eventually. Repeating the same argument as about (I)-type solutions, we get a contradiction.

4°. If $\sigma > 0$, $\lim_{t\to\infty} y''(t) = 0$ and $\lim_{t\to\infty} y'(t) = k > 0$, then

$$y(t) = kt + o(t) \qquad \text{and} \qquad \int_T^t y(s)\,ds = \frac{k}{2}\,t^2 + o(t^2)$$

as $t \to \infty$. So we have

$$\int_{t-\sigma}^t y(s)\,ds < 2\sigma k t$$

eventually. Let

$$z(t) = \int_T^t y(s)\,ds - 2\sigma k t.$$

Then $z(t) > 0$ and $x(t-\sigma) \geq z(t)$ eventually. Repeating the same argument as about (I)-type solutions, we get a contradiction. The proof of sufficiency is complete.

Necessity, i.e. the oscillatority of equation (2.5) implies that of equation (2.6). Suppose to the contrary that equation (2.6) has an eventually positive solution y. Then $y''''(t) \leq 0$. From [4: Lemma 5.1.1] we see that there are only the following two possibilities for y:

Case (I): $y'(t) > 0$, $y''(t) < 0$, $y'''(t) > 0$ eventually.

Case (II): $y'(t) > 0$, $y''(t) > 0$, $y'''(t) > 0$ eventually.

If y is a (I)-type solution, choose T sufficiently large such that $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ and $y'''(t) > 0$ on $[T-1-|\sigma|,\infty)$. Because $y'(t) = o(y(t))$ as $t \to \infty$, we can also assume $y(t) > M > 0$ and $y'(t) < M(2(1+|\sigma|))^{-1}$ on $[T-1-|\sigma|,\infty)$, where M is a positive constant. Set

$$H(t) = \begin{cases} y'(t) & \text{if } t \geq T \\ (t-T+1)y'(T) & \text{if } T-1 \leq t \leq T \\ 0 & \text{if } t \leq T-1. \end{cases}$$

Clearly, $H \in C(\mathbb{R},\mathbb{R}^+)$. Define

$$z(t) = \sum_{i=0}^{\infty} H(t-i).$$

Then $z \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $z(t) - z(t-1) = H(t)$. Obviously, $z(t) - z(t-1) = y'(t)$ on $[T,\infty)$. Let $m = \max_{[T-1,T]} z(t)$. Then $m = y'(T) > 0$. If $t \in [T, T+1]$, we have

$$z(t) = y'(t) + z(t-1) \leq \int_{t-1}^t y'(s)\,ds + m.$$

By induction we have

$$z(t) \leq \int_{t-n}^t y'(s)\,ds + m$$

on $[T+n-1, T+n]$. Hence $z(t) \leq \int_T^t y'(s)\,ds + m$ on $[T,\infty)$. If $\sigma \geq 0$, we have

$$z(t-\sigma) \leq \int_T^{t-\sigma} y'(s)\,ds + m \leq \int_T^t y'(s)\,ds + m$$

for $t \in [T+\sigma, +\infty)$. If $\sigma < 0$, we have

$$z(t-\sigma) \le \int_T^{t-\sigma} y'(s)\,ds + m \le \int_T^{t} y'(s)\,ds + m + \int_t^{t-\sigma} y'(s)\,ds$$

for $t \in [T, +\infty)$. So, whether $\sigma \ge 0$ or $\sigma < 0$, we have

$$\begin{aligned} z(t-\sigma) &\le \int_T^t y'(s)\,ds + m + |\sigma|\frac{M}{2(1+|\sigma|)} \\ &\le y(t) - y(T) + (1+|\sigma|)\frac{M}{2(1+|\sigma|)} \\ &\le y(t) - M + \frac{M}{2} \\ &\le y(t). \end{aligned}$$

Substituting the above inequality and $z(t) - z(t-1) = y'(t)$ into (2.6), we get

$$(z(t) - z(t-1))''' + p(t)z(t-\sigma) \le 0.$$

From Lemma 2.1 we see that equation (2.5) is non-oscillatory, which is a contradiction.

If y is a (II)-type solution, then $\lim_{t\to\infty} y'''(t) = k \ge 0$. Choose T sufficiently large such that $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$ on $[T-1, \infty)$. Define

$$H(t) = \begin{cases} y'(t) & \text{if } t \ge T \\ (t-T+1)y'(T) & \text{if } T-1 \le t \le T \\ 0 & \text{if } t \le T-1 \end{cases}$$

and

$$z(t) = \sum_{i=0}^{\infty} H(t-i).$$

Then $H, z \in C(\mathbb{R}^+, \mathbb{R}^+)$ and satisfy $z(t) - z(t-1) = y'(t)$ on $[T, \infty)$. Let $M = \max_{[T-1,T]} z(t)$ and $m = \max_{[T-1,T]} y(t)$. If $t \in [T, T+1]$, we have

$$z(t) = y'(t) + z(t-1) \le \int_t^{t+1} y'(s)\,ds + M \le y(t+1) - y(t) + M \le y(t+1) + M$$

and

$$z(t) \ge \int_{t-1}^{t} y'(s)\,ds + z(t-1) \ge y(t) - y(t-1) \ge y(t) - m.$$

By induction we have $z(t) \le y(t+1) + M$ and $z(t) \ge y(t) - m$ on $[T, \infty)$. Hence

$$\begin{aligned} z(t-\sigma) &\le y(t+1-\sigma) + M \\ &= y(t) - y(t) + y(t+1-\sigma) + M \\ &\le y(t) + |1-\sigma|\, y'(\max(t, t+1-\sigma)) + M. \end{aligned}$$

In the following discussion, we will distinguish three cases.

1°. Let $k > 0$. Then

$$y''(t) = kt + o(t), \qquad y'(t) = \frac{k}{2}\, t^2 + o(t^2), \qquad y(t) = \frac{k}{6}\, t^3 + o(t^3)$$

as $t \to \infty$. Obviously, $z(t) = \frac{k}{6}\, t^3 + o(t^3)$ as $t \to \infty$. As long as t is sufficiently large, we have

$$z(t-\sigma) \le y(t) + |1-\sigma| k t^2 + M.$$

Let

$$\bar{z}(t) = z(t) - |1-\sigma| k (t+\sigma)^2 - M.$$

Then $\bar{z}(t) > 0$ eventually, and $y(t) \ge \bar{z}(t-\sigma)$. Obviously, we have

$$y''''(t) = (z(t) - z(t-1))''' = (\bar{z}(t) - \bar{z}(t-1))'''.$$

Substituting these expressions into equation (2.6), we get

$$(\bar{z}(t) - \bar{z}(t-1))''' + p(t)\bar{z}(t-\sigma) \le 0.$$

From Lemma 2.1 we see that equation (2.5) is non-oscillatory, which is a contradiction.

2°. Let $k = 0$ and $\lim_{t\to\infty} y''(t) = l$. Then

$$y'(t) = lt + o(t) \qquad \text{and} \qquad y(t) = \frac{l}{2}\, t^2 + o(t^2)$$

as $t \to \infty$. Obviously, $z(t) = \frac{l}{2}\, t^2 + o(t^2)$ as $t \to \infty$. As long as t is sufficiently large, we have

$$z(t-\sigma) \le y(t) + 2|1-\sigma| l t + M.$$

Let

$$\bar{z}(t) = z(t) - 2|1-\sigma| l (t+\sigma) - M.$$

Then $\bar{z}(t) > 0$ eventually, and $y(t) \ge \bar{z}(t-\sigma)$. Repeating the same argument as in the case 1°, we get a contradiction.

3°. If $k = 0$ and $\lim_{t\to\infty} y''(t) = \infty$, then

$$y''(t) = o(t), \qquad y'(t) = o(t^2),\ t = o(y'(t)), \qquad y(t) = o(t^3),\ t^2 = o(y(t))$$

as $t \to \infty$. Obviously, $z(t) = o(t^3)$ and $t^2 = o(z(t))$ as $t \to \infty$. So we have

$$z(t-\sigma) \le y(t) + |1-\sigma| t^2 + M$$

eventually. Let

$$\bar{z}(t) = z(t) - |1-\sigma|(t+\sigma)^2 - M.$$

Then $\bar{z}(t) > 0$ eventually and $y(t) \ge \bar{z}(t-\sigma)$. Repeating the same argument as in the case 1°, we get a contradiction. The proof is complete ∎

Remark 2.4. From Theorem 2.3 we know that the deviation σ in equation (1.3) has no influence on its oscillation.

Theorem 2.5. *Let $n \geq 2$ be even, let the function $p : [0,\infty) \to [0,\infty)$ be continuous and $\sigma \in \mathbb{R}$ constant. Then the oscillation of the two equations*

$$x^{(n)}(t) + p(t)\,x(t-\sigma) = 0 \tag{2.8}$$

and

$$x^{(n)}(t) + p(t)\,x(t) = 0 \tag{2.9}$$

is equivalent.

The proof is quite similar to that of Theorem 2.4 and we omit it.

3. Applications

First we introduce some notations. Let M_n denote the maximum of $P_n(x) = x(1-x)\cdots(n-1-x)$ on $(0,1)$. The following lemma is known.

Lemma 3.1 (see [6]). *Let $n \geq 2$ be even and let the function $p : [0,\infty) \to [0,\infty)$ be continuous. Then either of the conditions*

$$\liminf_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\,ds > M_n \tag{3.1}$$

and

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\,ds > (n-1)! \tag{3.2}$$

is sufficient for the equation

$$x^{(n)}(t) + p(t)\,x(t) = 0 \tag{3.3}$$

to oscillate.

Combining Lemma 3.1 and Theorem 2.3, we get the following

Theorem 3.2. *Let $n \geq 1$ be odd. Then either of the conditions*

$$\liminf_{t\to\infty} t \int_t^\infty s^{n-1} p(s)\,ds > rM_{n+1} \tag{3.4}$$

and

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-1} p(s)\,ds > rn! \tag{3.5}$$

is sufficient for equation (1.3) *to oscillate.*

Similarly we have

Theorem 3.3. *Let $n \geq 2$ be even, let the function $p: [0,\infty) \to [0,\infty)$ be continuous and $\sigma \in R$ constant. Then either condition* (3.1) *or* (3.2) *is sufficient for equation* (2.8) *to oscillate.*

Remark 3.4. Theorem 3.2 improves the main result Theorem 3.1 in [10].

Remark 3.5. It is known (see [4: Theorem 5.2.8]) that $\int^\infty s^n p(s)\,ds < \infty$ is a necessary and sufficient condition for equation (1.3) to have a bounded positive solution.

Example 3.6. Consider the equation

$$(x(t) - x(t-1))''' + \frac{1}{t^\alpha} x(t-1) = 0 \qquad (t > 1). \tag{3.6}$$

According to the result in [10], if $\alpha \leq \frac{7}{2}$, then every solution of equation (3.6) is oscillatory. From Theorem 3.2, if $\alpha \leq 4$, then every solution of equation (3.6) is oscillatory. Condition $\alpha \leq 4$ is not only sufficient, but also necessary for equation (3.6) to oscillate. In fact, from Remark 3.5, we see if $\alpha > 4$, then equation (3.6) has a bounded positive solution.

A more simple criterion is the following one which is also better than [10: Theorem 3.1].

Corollary 3.7. *If*

$$\int^\infty s^{n-\epsilon} p(s)\,ds = \infty \qquad (\epsilon \in (0,n)), \tag{3.7}$$

then equation (1.3) *is oscillatory.*

Now we compare equation (1.3) with the following one:

$$(x(t) - x(t-r))^{(n)} + q(t)x(t-\sigma_1) = 0 \tag{3.8}$$

where $r > 0$ and $\sigma_1 \in \mathbb{R}$ are constants, $n \geq 1$ is an odd integer and $q: [0,\infty) \to [0,\infty)$ is a continuous function. From Theorem 2.3 we know that the oscillation of the equation (3.8) and the equation

$$x^{(n+1)}(t) + \frac{q(t)}{r} x(t) = 0 \tag{3.9}$$

is equivalent. Using [6: Theorem 1.4], we get the following comparison result.

Theorem 3.8. *If*

$$\int_t^\infty s^{n-1} p(s)\,ds \geq \int_t^\infty s^{n-1} q(s)\,ds \qquad (t \in \mathbb{R}^+), \tag{3.10}$$

then the oscillation of equation (3.8) *implies that of equation* (1.3).

Remark 3.9. Condittion (3.10) improves the simple comparison condition $p(t) \geq q(t)$ in [4: Theorem 5.1.2], and does not require $\sigma = \sigma_1$.

Similarly we have

Theorem 3.10. *If $n \geq 2$ is even, $p(t) > 0, q(t) > 0$ and*

$$\int_t^\infty s^{n-2} p(s)\, ds \geq \int_t^\infty s^{n-2} q(s)\, ds \qquad (t \in \mathbb{R}^+), \tag{3.11}$$

then the oscillation of the equation

$$x^{(n)}(t) + q(t)\, x(t - \sigma_1) = 0 \tag{3.12}$$

implies that of the equation

$$x^{(n)}(t) + p(t)\, x(t - \sigma) = 0 \tag{3.13}$$

and no relation of σ and σ_1 is required.

Remark 3.11. If $n \geq 1$ is odd, note that the equation

$$x^{(n+1)}(t) + \frac{M_{n+1}}{t^{n+1}}\, x(t) = 0$$

always has a positive solution $x(t) = t^\alpha$, where α is the smallest root of the equation $P_{n+1}(x) = M_{n+1}$. By Theorems 3.8 and 2.3, we know that if $p(t)t^{n+1} \leq rM_{n+1}$ eventually, then equation (1.3) is non-oscillatory.

To conclude our paper, we consider the equation

$$(x(t) - q(t)\, x(t - r))^{(n)} + p(t)\, x(t - \delta) = 0 \tag{3.14}$$

where $r > 0$ and $\delta \in \mathbb{R}$ are constants, $n \geq 1$ is an odd integer and $p, q : [0, \infty) \to [0, \infty)$ are continuous functions.

Theorem 3.12. *Let the conditions*

$$q(t - \delta)\, p(t) \geq p(t - r) \quad \textit{eventually} \tag{3.15}$$

$$q(t^* + ir) \leq 1 \quad \textit{for some } t^* \quad (i \in \mathbb{N}) \tag{3.16}$$

hold. Then either condition (3.4) *or condition* (3.5) *is sufficient for equation* (3.14) *to be oscillatory.*

Proof. Assume to the contrary that equation (3.14) has a positive solution x and define $y(t) = x(t) - q(t)x(t - r)$. By [10: Lemma 2.2] we see that $y(t) > 0$ eventually. By the same method as in the proof of [10: Theorem 3.2] we get

$$(y(t) - y(t - r))^{(n)} + p(t)\, y(t - \delta) \leq 0. \tag{3.17}$$

Then, by Lemma 2.1, we conclude that the equation

$$(x(t) - x(t - r))^{(n)} + p(t)\, x(t - \delta) = 0 \tag{3.18}$$

has a positive solution, which contradicts Theorem 3.2 ∎

Remark 3.13. Theorem 3.12 improves [10: Theorem 3.2].

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