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On the Mixed Problem for Quasilinear Partial Differential-Functional Equations of the First Order

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Abstract. We consider the mixed problem for the quasilinear partial differential-functional equation of the first order

$$D_x z(x,y) = \sum_{i=1}^{n} f_i(x,y,z_{(x,y)}) D_{y_i} z(x,y) + G(x,y,z_{(x,y)})$$
$$z(x,y) = \phi(x,y) \qquad ((x,y) \in [-\tau,a] \times [-b,b+h] \setminus (0,a] \times [-b,b))$$

where $z_{(x,y)} : [-\tau,0] \times [0,h] \to \mathbb{R}$ is a function defined by $z_{(x,y)}(t,s) = z(x+t,y+s)$ for $(t,s) \in [-\tau,0] \times [0,h]$. Using the method of characteristics and the fixed-point method we prove, under suitable assumptions, a theorem on the local existence and uniqueness of solutions of the problem.

Keywords: *Partial differential-functional equations, classical solutions, local existence, bicharacteristics, fixed-point theorem*

AMS subject classification: 35 F 30, 35 L 60, 35 R 10

1. Introduction

If X,Y are any metric spaces, then we denote by $C(X;Y)$ the class of all continuous functions from X to Y. Let $B = [-\tau,0] \times [0,h]$, where $h = (h_1,\ldots,h_n) \in \mathbb{R}_+^n$ and $\tau \in \mathbb{R}_+$, with $\mathbb{R}_+ = [0,+\infty)$. For a given function

$$z : [-\tau,a] \times [-b,b+h] \to \mathbb{R}$$

where $a > 0$ and $b = (b_1,\ldots,b_n)$, with $b_i > 0$ $(i = 1,\ldots,n)$, and a point $(x,y) = (x,y_1,\ldots,y_n) \in [0,a] \times [-b,b]$, we define the function $z_{(x,y)} : B \to \mathbb{R}$ by the formula

$$z_{(x,y)}(t,s) = z(x+t,y+s) \qquad ((t,s) \in B).$$

Define

$$\partial_0 E_{\bar a} = [0,\bar a] \times [-b,b+h] \setminus [0,\bar a] \times [-b,b)$$
$$E_{\bar a} = [0,\bar a] \times [-b,b]$$
$$E^*_{\bar a} = [-\tau,\bar a] \times [-b,b+h]$$

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for any $\bar{a} \in [0, a]$.

For given functions

$$\begin{aligned} \phi &: E_0^* \cup \partial_0 E_a \to \mathbb{R} \\ G &: E_a \times C(B; \mathbb{R}) \to \mathbb{R} \\ f = (f_1, \dots, f_n) &: E_a \times C(B; \mathbb{R}) \to \mathbb{R}^n \end{aligned}$$

we consider the following mixed problem:

$$D_x z(x, y) = \sum_{i=1}^{n} f_i(x, y, z_{(x,y)}) D_{y_i} z(x, y) + G(x, y, z_{(x,y)}) \tag{1}$$

$$z(x, y) = \phi(x, y) \qquad \big((x, y) \in E_0^* \cup \partial_0 E_a\big). \tag{2}$$

In this paper we consider classical solutions of problem (1),(2) local with respect to the first variable. In other words, a function $z \in C^1(E_{\bar{a}}^*; \mathbb{R})$ is said to be a *solution* of problem (1),(2) if it satisfies equation (1) on $E_{\bar{a}}$ and fulfils initial-boundary condition (2) on $E_0^* \cup \partial_0 E_{\bar{a}}$, for a certain $\bar{a} \in (0, a]$.

Note that in equation (1) the given functions f and G are functional operators on $C(B; \mathbb{R})$ with respect to the last variable. This model of functional dependence contains as a particular case equations with a deviated argument, and if $\tau = h = 0$ equations without any functional dependence. In non-functional setting generalized (in the "almost everywhere" sense) solutions of quasilinear systems with Cauchy and boundary conditions have been discussed in [1, 6, 7], while continuous solutions (i.e. solutions satisfying integral systems arising from differential equations by integrating along characteristics) of mixed problems have been discussed in [1, 15].

As a particular case of (1) we may also obtain some differential-integral equations and equations with operators of the Volterra type (cf. [16]). Classical solutions of quasilinear systems with such operators were investigated in [8, 9]. From the literature concerning other problems for first order partial differential-functional equations where classical solutions are considered we refer here to the papers [12, 13]. Differential-integral problems are often used as mathematical models of various problems in nonlinear optics [4, 5] and may be used to describe the growth of a population of cells [10]. Differential problems for equations with a deviated argument arise in the theory of the distribution of wealth [11].

In this paper we prove a theorem on the local existence and uniqueness of solutions of the mixed problem (1),(2). Our result is analogous to that of [14] for generalized solutions of weak-coupled systems in two independent variables. We use the well known method of bicharacteristics (cf. [2, 3, 8, 14]) and the Banach fixed point theorem.

2. Bicharacteristics

If $\|\cdot\|_0$ denotes the supremum norm in $C(X;Y)$, where X is a domain in $\mathbb{R}^{1+n}$ and Y is an Euclidean space, then the norm in $C^1(X;Y)$ is defined by $\|w\|_1 = \|w\|_0 + \|D_{(x,y)}w\|_0$, where $D_{(x,y)}w$ denotes the Jacobi matrix of w. For any $w \in C(X;Y)$ let

$$\|w\|_L = \sup\left\{|w(x,y) - w(\bar{x},\bar{y})| \cdot \left[|x-\bar{x}| + |y-\bar{y}|\right]^{-1} : \ (x,y),(\bar{x},\bar{y}) \in X\right\}.$$

If we put $\|w\|_{0,L} = \|w\|_0 + \|w\|_L$ and $\|w\|_{1,L} = \|w\|_1 + \|D_{(x,y)}w\|_L$, then we denote by $C^{i,L}(X;Y)$ $(i = 0,1)$ the space of all functions $z \in C^i(X;Y)$ such that $\|z\|_{i,L} < +\infty$ with the norm $\|\cdot\|_{i,L}$.

Assumption (H$_1$). Suppose that $\phi \in C^1(E_0^* \cup \partial_0 E_a; \mathbb{R})$ and that

$$\|\phi\|_0 \le \Lambda_0, \quad \|D_x\phi\|_0 \le \Lambda_1, \quad \|D_y\phi\|_0 \le \Lambda_1, \quad \|D_x\phi\|_L \le \Lambda_2, \quad \|D_y\phi\|_L \le \Lambda_2,$$

where $\Lambda_0, \Lambda_1, \Lambda_2$ are given non-negative constants.

Supposed that Assumption (H$_1$) is satisfied and given non-negative Q_0, Q_1, Q_2 such that $Q_i \ge \Lambda_i$ $(i = 0,1,2)$ we will denote by $C_{\bar{a}}^{1,L}(Q)$, where $\bar{a} \in (0,a]$, the set of all functions $z \in C(E_{\bar{a}};\mathbb{R})$ such that

(i) $z(x,y) = \phi(x,y)$ on $E_0^* \cup \partial_0 E_{\bar{a}}$

(ii) $\|z\|_0 \le Q_0$, $\|D_x z\|_0 \le Q_1$, $\|D_y z\|_0 \le Q_1$, $\|D_x z\|_L \le Q_2$, $\|D_y z\|_L \le Q_2$.

Assumption (H$_2$). Suppose the following:

1° $f = (f_1, \ldots, f_n) \in C(E_a \times C(B;\mathbb{R});\mathbb{R}^n)$ is a function of the variables (x,y,w), and the derivatives $D_y f$ and $D_w f$ exist on $E_a \times C^1(B;\mathbb{R})$.

2° There exist non-decreasing functions $L_0, L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $(x,y),(\bar{x},\bar{y}) \in E_a$ we have

$$|f(x,y,w)| \le L_0(q) \qquad (w \in C(B;\mathbb{R}),\ \|w\|_0 \le q)$$

$$|f(x,y,w) - f(\bar{x},y,w)| \le L_1(q)|x-\bar{x}| \qquad (w \in C^{0,L}(B;\mathbb{R}),\ \|w\|_{0,L} \le q)$$

$$|D_y f(x,y,w)|, \|D_w f(x,y,w)\| \le L_1(q) \qquad (w \in C^1(B;\mathbb{R}),\ \|w\|_1 \le q)$$

and

$$|D_y f(x,y,w) - D_y f(\bar{x},\bar{y},\bar{w})| \le L_2(q)[|x-\bar{x}| + |y-\bar{y}| + \|w-\bar{w}\|_0]$$

$$\|D_w f(x,y,w) - D_w f(\bar{x},\bar{y},\bar{w})\| \le L_2(q)[|x-\bar{x}| + |y-\bar{y}| + \|w-\bar{w}\|_0]$$

where $w, \bar{w} \in C^{1,L}(B;\mathbb{R})$ with $\|w\|_{1,L}, \|\bar{w}\|_{1,L} \le q$.

3° For every $q \in \mathbb{R}_+$ there is $\delta(q) > 0$ such that $f_i(x,y,w) \ge \delta(q)$ $(i = 1,\ldots,n)$ for $(x,y,w) \in E_a \times C(B;\mathbb{R})$ with $\|w\|_0 \le q$.

For a fixed $z \in C^{1,L}_{\bar{a}}(Q)$, where $\bar{a} \in (0, a]$, and for any $(x, y) \in E_a$, we consider the Cauchy problem

$$\left.\begin{aligned} \frac{d}{dt}\rho(t) &= -f(t, \rho(t), z_{(t,\rho(t))}) \\ \rho(x) &= y. \end{aligned}\right\} \tag{3}$$

If Assumption (H_2) is satisfied, then there exists a unique solution of problem (3) which we denote by

$$g[z](\cdot, x, y) = \big(g_1[z](\cdot, x, y), \ldots, g_n[z](\cdot, x, y)\big).$$

Let $\lambda[z](x, y)$ be the left end of the maximal interval on which the solution $g[z](\cdot, x, y)$ is defined. Then

$$\big(\lambda[z](x, y), g[z](\lambda[z](x, y), x, y)\big) \in (E_0^* \cup \partial_0 E_{\bar{a}}) \cap E_{\bar{a}}$$

because of condition 3° of Assumption (H_2) and we may define the following two sets:

$$E_{\bar{a}0}[z] = \Big\{(x, y) \in E_{\bar{a}} : \lambda[z](x, y) = 0\Big\}$$

$$E_{\bar{a}b}[z] = \Big\{(x, y) \in E_{\bar{a}} : g_i[z](\lambda[z](x, y), x, y) = b_i \quad \text{for some } \ 1 \le i \le n\Big\}.$$

Furthermore, we define the constants

$$\Gamma_{\bar{a}} = L_1^* \bar{a} \exp\{L_1^*[1 + Q_1]\bar{a}\}$$

$$\Gamma_{1\bar{a}} = (1 + L_0^*) \exp\{L_1^*[1 + Q_1]\bar{a}\}$$

$$\Gamma_{2\bar{a}} = \big\{L_1^*[1 + Q_1](1 + \Gamma_{1\bar{a}}) + \big[L_2^*[1 + Q_1]^2 + L_1^*\big]\Gamma_{1\bar{a}}^2 \bar{a}\big\} \exp\{L_1^*[1 + Q_1]\bar{a}\}$$

where $L_i^* = L_i(\sum_{j=0}^{i} Q_j)$, for $i = 0, 1, 2$.

Lemma 1. *Suppose that Assumption (H_2) is satisfied, $z, \bar{z} \in C^{1,L}_{\bar{a}}(Q)$, and (x, y), $(\bar{x}, \bar{y}) \in E_{\bar{a}}$. If the intervals*

$$K_1 = \big[\max\{\lambda[z](x, y), \lambda[z](\bar{x}, \bar{y})\},\ \min\{x, \bar{x}\}\big]$$

$$K_2 = \big[\max\{\lambda[z](x, y), \lambda[\bar{z}](x, y)\},\ x\big]$$

are non-empty, then we have the estimates

$$\big|D_x g[z](t, x, y)\big| \le \Gamma_{1\bar{a}}, \quad \big|D_y g[z](t, x, y)\big| \le \Gamma_{1\bar{a}} \qquad \text{if } t \in [\lambda[z](x, y), x] \tag{4}$$

$$\big|D_x g[z](t, x, y) - D_x g[z](t, \bar{x}, \bar{y})\big| \le \Gamma_{2\bar{a}}\big[|x - \bar{x}| + |y - \bar{y}|\big] \quad \text{if } t \in K_1 \tag{5}$$

$$\big|D_y g[z](t, x, y) - D_y g[z](t, \bar{x}, \bar{y})\big| \le \Gamma_{2\bar{a}}\big[|x - \bar{x}| + |y - \bar{y}|\big] \quad \text{if } t \in K_1 \tag{6}$$

$$\big|g[z](t, x, y) - g[\bar{z}](t, x, y)\big| \le \Gamma_{\bar{a}}\|z - \bar{z}\|_0 \qquad \text{if } t \in K_2. \tag{7}$$

Proof. Let $g = g[z]$ and $\bar{g} = g[\bar{z}]$. It follows from classical theorems on differentiation of solutions with respect to initial data that the derivatives $D_x g$ and $D_y g$ exist and fulfil the integral equations

$$D_x g(t,x,y) = f(x,y,z_{(x,y)}) - \int_x^t \Big[D_y f(P_\tau) + D_w f(P_\tau) \circ (D_y z)_{(\tau, g(\tau,x,y))}\Big] D_x g(\tau,x,y)\, d\tau$$

$$D_y g(t,x,y) = I - \int_x^t \Big[D_y f(P_\tau) + D_w f(P_\tau) \circ (D_y z)_{(\tau, g(\tau,x,y))}\Big] D_y g(\tau,x,y)\, d\tau$$

for $t \in \big[\lambda[z](x,y),\, x\big]$ and $(x,y) \in E_{\bar{a}}$, where I denotes the identity matrix and $P_\tau = \big(\tau, g(\tau,x,y), z_{(\tau,g(\tau,x,y))}\big)$. Hence, by Assumption ($H_2$), we have

$$|D_x g(t,x,y)| \le L_0^* + \left| \int_x^t L_1^*[1+Q_1]\, |D_x g(\tau,x,y)|\, d\tau \right|$$

$$|D_y g(t,x,y)| \le 1 + \left| \int_x^t L_1^*[1+Q_1]\, |D_y g(\tau,x,y)|\, d\tau \right|$$

from which (4) follows by the Gronwall lemma. Analogously, by Assumption (H_2) and (4), we get

$$\begin{aligned} &\big|D_x g(t,x,y) - D_x g(t,\bar{x},\bar{y})\big| \\ &\quad \le L_1^*[1+Q_1]\,\big[|x-\bar{x}| + |y-\bar{y}|\big] + \left| \int_x^{\bar{x}} L_1^*[1+Q_1]\Gamma_{1\bar{a}}\, d\tau \right| \\ &\qquad + \left| \int_x^t \big\{L_2^*[1+Q_1]^2 + L_1^*\big\}\, \Gamma_{1\bar{a}}^2 \big[|x-\bar{x}| + |y-\bar{y}|\big]\, d\tau \right| \\ &\qquad + \left| \int_x^t L_1^*[1+Q_1]\, \big|D_x g(\tau,x,y) - D_x g(\tau,\bar{x},\bar{y})\big|\, d\tau \right| \end{aligned}$$

and

$$\begin{aligned} &\big|D_y g(t,x,y) - D_y g(t,\bar{x},\bar{y})\big| \\ &\quad \le \left| \int_x^{\bar{x}} L_1^*[1+Q_1]\Gamma_{1\bar{a}}\, d\tau \right| \\ &\qquad + \left| \int_x^t \big\{L_2^*[1+Q_1]^2 + L_1^*\big\}\Gamma_{1\bar{a}}^2 \big[|x-\bar{x}| + |y-\bar{y}|\big]\, d\tau \right| \\ &\qquad + \left| \int_x^t L_1^*[1+Q_1]\big|D_y g(\tau,x,y) - D_y g(\tau,\bar{x},\bar{y})\big|\, d\tau \right| \end{aligned}$$

for $t \in K_1$, from which (5) and (6) follow by the Gronwall lemma. In the same way we may get for $t \in K_2$ the estimate

$$\begin{aligned} &\big|g(t,x,y) - \bar{g}(t,x,y)\big| \\ &\quad \le \left| \int_x^t L_1^* \|z - \bar{z}\|_{E_a}\, d\tau \right| + \left| \int_x^t L_1^*[1+Q_1]\big|g(\tau,x,y) - \bar{g}(\tau,x,y)\big|\, d\tau \right| \end{aligned}$$

from which using again the Gronwall lemma we get (7) which completes the proof of Lemma 1 ∎

Lemma 2. *If Assumption* (H_2) *is satisfied and* $z \in C^{1,L}_{\bar{a}}(Q)$, *then* $\lambda[z]$ *is piecewise of class* C^1 *on* $E_{\bar{a}b}[z]$ *and*

$$\big|\lambda[z](x,y) - \lambda[z](\bar{x},\bar{y})\big| \le \frac{1}{\delta^*}\Gamma_{1\bar{a}}\big[|x-\bar{x}| + |y-\bar{y}|\big] \tag{8}$$

for $(x,y) \in E_{\bar{a}b}[z]$, *where* $\delta^* = \delta(Q_0)$.

Proof. In the proof of this lemma, for simplicity, we will write λ and g instead of $\lambda[z]$ and $g[z]$, respectively. Note that λ is defined by the relation

$$g_i(\lambda(x,y),x,y) = b_i \qquad ((x,y) \in E_{\bar{a}b}[z])$$

for some $1 \le i \le n$. Thus, since g_i is of class C^1 and $\frac{dg_i}{dt} \neq 0$, we see by the theorem on implicit differentiation that λ is locally of class C^1, and its partial derivatives are given by the formulas

$$D_x\lambda(x,y) = \frac{D_x g_i(\lambda(x,y),x,y)}{f_i\big(\lambda(x,y), g(\lambda(x,y),x,y), \phi_{(\lambda(x,y),g(\lambda(x,y),x,y))}\big)} \tag{9}$$

$$D_y\lambda(x,y) = \frac{D_y g_i(\lambda(x,y),x,y)}{f_i\big(\lambda(x,y), g(\lambda(x,y),x,y), \phi_{(\lambda(x,y),g(\lambda(x,y),x,y))}\big)}. \tag{10}$$

From the above relations we get

$$|D_x\lambda(x,y)| \le \frac{1}{\delta^*}\Gamma_{1\bar{a}} \qquad \text{and} \qquad |D_y\lambda(x,y)| \le \frac{1}{\delta^*}\Gamma_{1\bar{a}}$$

which gives (8) ∎

Remark 1. Note that from the proof of Lemma 2 it follows that $\lambda[z]$ is of class C^1 on each of the sets $\{(x,y) \in E_{\bar{a}b}[z] : g_i[z](\lambda[z](x,y),x,y) = b_i\}$ $(1 \le i \le n)$.

Lemma 3. *If Assumption* (H_2) *is satisfied and* $z, \bar{z} \in C^{1,L}_{\bar{a}}(Q)$, *then we have*

$$\big|\lambda[z](x,y) - \lambda[\bar{z}](x,y)\big| \le \frac{1}{\delta^*}\Gamma_{\bar{a}}\|z - \bar{z}\|_0 \tag{11}$$

on $E_{\bar{a}}$.

Proof. Since (11) is obviously satisfied if $(x,y) \in E_{\bar{a}0}[z] \cap E_{\bar{a}0}[\bar{z}]$, without loss of generality we may assume that $\lambda[\bar{z}](x,y) \le \lambda[z](x,y)$ and $(x,y) \in E_{\bar{a}b}[z]$. Let $1 \le i \le n$ be such that $g_i[z]\big(\lambda[z](x,y),x,y\big) = b_i$. Then we have

$$\begin{aligned}
&g_i[z]\big(\lambda[z](x,y),x,y\big) - g_i[\bar{z}]\big(\lambda[z](x,y),x,y\big)\\
&\quad\ge g_i[\bar{z}]\big(\lambda[\bar{z}](x,y),x,y\big) - g_i[\bar{z}]\big(\lambda[z](x,y),x,y\big)\\
&\quad= \int_{\lambda[\bar{z}](x,y)}^{\lambda[z](x,y)} f_i\big(\tau, g[\bar{z}](\tau,x,y), z_{(\tau,g[\bar{z}](\tau,x,y))}\big)\,d\tau\\
&\quad\ge \delta^*\big[\lambda[z](x,y) - \lambda[\bar{z}](x,y)\big].
\end{aligned}$$

The above estimate together with

$$0 \le g_i[z]\big(\lambda[z](x,y),x,y\big) - g_i[\bar{z}]\big(\lambda[z](x,y),x,y\big) \le \Gamma_{\bar{a}}\|z - \bar{z}\|_0$$

gives (11) ∎

Remark 2. Note that condition 3° of Assumption (H_2) is essential in the proof of Lemma 3. In Lemma 2 it suffices to assume that $f_i(x,y,w) \geq \delta(q)$ for $(x,y) \in E_a$ such that $y_i = b_i$ for some $1 \leq i \leq n$ and $f_i(x,y,w) \geq 0$ on $E_a \times C(B;\mathrm{R})$ while in Lemma 1 only the latter condition is necessary.

3. The main result

Now we prove a theorem on existence and uniqueness of solutions of the mixed problem (1),(2).

Assumption (H$_3$). Suppose the following:

1° $G \in C\big(E_a \times C(B;\mathbb{R});\mathbb{R}\big)$ is a function of the variables (x,y,w), and the derivatives D_yG and D_wG exist on $E_a \times C^1(B;\mathbb{R})$.

2° There exist non-decreasing functions $M_0, M_1, M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that G fulfils conditions analogous to those given in 2° of Assumption (H$_2$), with L_i replaced by M_i, respectively.

3° The consistency condition

$$D_x\phi(x,y) - \sum_{i=1}^{n} f_i(x,y,\phi_{(x,y)})D_y\phi(x,y) = G(x,y,\phi_{(x,y)}) \tag{12}$$

holds true on $(E_0^* \cup \partial_0 E_a) \cap E_a$.

We define the operator W on $C_{\tilde{a}}^{1,L}(Q)$ by the formula

$$(Wz)(x,y) = \begin{cases} \phi\big(\lambda[z](x,y), g[z](\lambda[z](x,y),x,y)\big) & \\ \quad + \displaystyle\int_{\lambda[z](x,y)}^{x} G\big(t, g[z](t,x,y), z_{(t,g[z](t,x,y))}\big)dt & \text{for } (x,y) \in E_{\tilde{a}} \\ \phi(x,y) & \text{for } (x,y) \in E_0^* \cup \partial_0 E_{\tilde{a}}. \end{cases} \tag{13}$$

Remark 3. The right-hand side of (13) arises in the following way. We consider (1) along bicharacteristics

$$\begin{aligned} D_x z\big(t, g[z](t,x,y)\big) - \sum_{i=1}^{n} f_i\big(t, g[z](t,x,y), z_{(t,g[z](t,x,y))}\big) D_{y_i} z\big(t, g[z](t,x,y)\big) \\ = G\big(t, g[z](t,x,y), z_{(t,g[z](t,x,y))}\big) \end{aligned}$$

from which by (3) we get

$$\frac{d}{dt} z\big(t, g[z](t,x,y)\big) = G\big(t, g[z](t,x,y), z_{(t,g[z](t,x,y))}\big).$$

Integrating this equation with respect to t on the interval $[\lambda[z](x,y),\, x]$ we get the righ-hand side of (13).

Assumption (H$_4$). Suppose that

$$
\begin{aligned}
&Q_0 > \Lambda_0\\
&Q_1 > \Lambda_1(1+L_0^*) + M_0^*\\
&Q_2 > \Lambda_2\left[\frac{1}{\delta^*}(1+L_0^*)+1\right](1+L_0)^2 + \Lambda_1\left[\frac{1}{\delta^*}L_1^*(1+L_0)^2 + L_1^*[1+Q_1](2+L_0^*)\right]\\
&\qquad + M_1^*[1+Q_1] + \left[1+\frac{1}{\delta^*}(1+L_0)\right]M_1^*[1+Q_1](1+L_0)
\end{aligned}
$$

where $M_i^* = M_i(\sum_{j=0}^{i} Q_i)$, for $i=0,1,2$.

Define the constants

$$
\begin{aligned}
S_{0\bar a} &= \Lambda_0^* + \bar a M_0^*\\
S_{1\bar a} &= \Lambda_1\Gamma_{1\bar a} + M_0^* + \bar a M_1^*[1+Q_1]\Gamma_{1\bar a}\\
S_{2\bar a} &= \Lambda_2\left[\frac{1}{\delta^*}(1+L_0^*)+1\right]\Gamma_{1\bar a}^2 + \Lambda_1\left[\frac{1}{\delta^*}L_1^*\Gamma_{1\bar a}^2 + \Gamma_{2\bar a}\right]\\
&\quad + M_1^*[1+Q_1] + \left[1+\frac{1}{\delta^*}\Gamma_{1\bar a}\right]M_1^*[1+Q_1]\Gamma_{1\bar a}\\
&\quad + \bar a\Big[M_2^*(1+Q_1)^2\Gamma_{1\bar a}^2 + M_1^* Q_2\Gamma_{1\bar a} + M_1^*[1+Q_1]\Gamma_{2\bar a}\Big].
\end{aligned}
$$

Remark 4. Note that since

$$
\lim_{\bar a\to 0+}\Gamma_{1\bar a} = 1+L_0^* \qquad\text{and}\qquad \lim_{\bar a\to 0+}\Gamma_{2\bar a} = L_1^*[1+Q_1](2+L_0^*)
$$

we may by Assumption (H$_4$) choose $\bar a \in (0,a]$ sufficiently small in order that $S_{i\bar a} \le Q_i$, for $i=0,1,2$.

Theorem 1. *If Assumptions $(H_1)-(H_3)$ are satisfied, then for $\bar a\in(0,a]$ sufficiently small the operator W defined by* (13) *maps $C_{\bar a}^{1,L}(Q)$ into itself.*

Proof. Let $z\in C_{\bar a}^{1,L}(Q)$. As in the proof of Lemma 2, for simplicity, we will write λ and g instead of $\lambda[z]$ and $g[z]$, respectively. From (13) it follows that

$$
\begin{aligned}
D_x(Wz)(x,y) &= D_y\phi(0,g(0,x,y))D_x g(0,x,y) + G(x,y,z_{(x,y)})\\
&\quad + \int_0^x \big[D_yG(P_t) + D_wG(P_t)\circ D_y z\big]D_x g(t,x,y)\,dt
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
D_y(Wz)(x,y) &= D_y\phi(0,g(0,x,y))D_y g(0,x,y)\\
&\quad + \int_0^x \big[D_yG(P_t) + D_wG(P_t)\circ D_y z\big]D_y g(t,x,y)\,dt
\end{aligned}
\tag{15}
$$

on $E_{\bar{a}0}[z]$, where $P_t = (t, g(t,x,y), z_{(t,g(t,x,y))})$. Suppose that $(x,y) \in E_{\bar{a}b}[z]$, which means that $g_j(\lambda(x,y),x,y) = b_j$ for some $1 \le j \le n$. From (13) and (3) we have then

$$\begin{aligned} D_x(Wz)(x,y) &= D_x\phi(\lambda(x,y), g(\lambda(x,y),x,y))D_x\lambda(x,y) \\ &+ \sum_{i=1,i\neq j}^{n} D_{y_i}\phi(\lambda(x,y), g(\lambda(x,y),x,y)) \\ &\times \Big[-f_i(P_{\lambda(x,y)})D_x\lambda(x,y) + D_x g_i(\lambda(x,y),x,y)\Big] \\ &+ G(x,y,z_{(x,y)}) - G(P_{\lambda(x,y)})D_x\lambda(x,y) \\ &+ \int_{\lambda(x,y)}^{x} \Big[D_yG(P_t) + D_wG(P_t)\circ(D_yz)_{(t,g(t,x,y))}\Big] D_xg(t,x,y)\,dt. \end{aligned}$$

Using consistency condition (12) and (9) we may transform the above relation into the form

$$\begin{aligned} &D_x(Wz)(x,y) \\ &= D_{y_j}\phi(\lambda(x,y), g(\lambda(x,y),x,y)) f_j(P_{\lambda(x,y)})D_x\lambda(x,y) \\ &\quad + \sum_{i=1,i\neq j}^{n} D_{y_i}\phi(\lambda(x,y), g(\lambda(x,y),x,y)) D_x g_i(\lambda(x,y),x,y) \\ &\quad + G(x,y,z_{(x,y)}) \\ &\quad + \int_{\lambda(x,y)}^{x} \Big[D_yG(P_t) + D_wG(P_t)\circ(D_yz)_{(t,g(t,x,y))}\Big] D_xg(t,x,y)\,dt \\ &= D_y\phi(\lambda(x,y), g(\lambda(x,y),x,y)) D_x g(\lambda(x,y),x,y) + G(x,y,z_{(x,y)}) \\ &\quad + \int_{\lambda(x,y)}^{x} \Big[D_yG(P_t) + D_wG(P_t)\circ(D_yz)_{(t,g(t,x,y))}\Big] D_xg(t,x,y)\,dt. \end{aligned} \tag{16}$$

Analogously, by consistency condition (12) and (10), we get

$$\begin{aligned} &D_y(Wz)(x,y) \\ &= D_y\phi(\lambda(x,y), g(\lambda(x,y),x,y)) D_y g(\lambda(x,y),x,y) \\ &\quad + \int_{\lambda(x,y)}^{x} \Big[D_yG(P_t) + D_wG(P_t)\circ(D_yz)_{(t,g(t,x,y))}\Big] D_yg(t,x,y)\,dt. \end{aligned} \tag{17}$$

Note that the right-hand sides of (16) and (17) do not depend on $1 \le j \le n$, which means that Wz is of class C^1 on $E_{\bar{a}b}[z]$.

It is obvious that Wz is continuous on $E_{\bar{a}}^*$ and that

$$D_y(Wz)(0,y) = D_y\phi(0,0,y) = D_y\phi(0,y)$$

for $y \in [-b,b]$. Moreover, the relation

$$D_x(Wz)(0,y) = D_y\phi(0,y)D_xg(0,0,y) + G(0,y,\phi_{(0,y)}) = D_x\phi(0,y)$$

for $y \in [-b, b]$ follows from (14) and from the consistency condition (12). Analogously, (16) and (17) give

$$D_y(Wz)(x,y) = D_y\phi(x,y)D_yg(x,x,y) = D_y\phi(x,y)$$

and

$$D_x(Wz)(x,y) = D_y\phi(x,y)D_xg(x,x,y) + G(x,y,\phi_{(x,y)}) = D_x\phi(x,y)$$

for $(x,y) \in E_{\bar{a}}$ such that $y_i = b_i$ for some $1 \le i \le n$. In order to get $Wz \in C^1(E^*_{\bar{a}};\mathbb{R})$ it remains to prove that formulas (14),(15) and (16),(17) define the same values for $(x,y) \in E_{\bar{a}0}[z] \cap E_{\bar{a}b}[z]$, but this is obvious since $\lambda(x,y) = 0$ in this case.

Now we prove that

$$|(Wz)(x,y)| \le Q_0, \quad |D_x(Wz)(x,y)| \le Q_1, \quad |D_y(Wz)(x,y)| \le Q_1 \tag{18}$$

on $E^*_{\bar{a}}$. From (13), (16) and (17) we have

$$|(Wz)(x,y)| \le \Lambda_0 + \int_{\lambda(x,y)}^{x} M_0^* dt \le S_{0\bar{a}}$$

$$|D_x(Wz)(x,y)| \le \Lambda_1\Gamma_{1\bar{a}} + M_0^* + \int_{\lambda(x,y)}^{x} M_1^*[1+Q_1]\Gamma_{1\bar{a}}\, dt \le S_{1\bar{a}}$$

$$|D_y(Wz)(x,y)| \le \Lambda_1\Gamma_{1\bar{a}} + \int_{\lambda(x,y)}^{x} M_1^*[1+Q_1]\Gamma_{1\bar{a}}\, dt \le S_{1\bar{a}}$$

on $E_{\bar{a}b}[z]$. Note that since the integral $\int_{\lambda(x,y)}^{x}$ is estimated by $\int_0^{\bar{a}}$ the above estimates will still be valid on $E_{\bar{a}0}[z]$. Taking $\bar{a}$ sufficiently small in order that $S_{0\bar{a}} \le Q_0$ and $S_{1\bar{a}} \le Q_1$ we get (18) for all $(x,y) \in E_{\bar{a}}$. Since $\Lambda_0 < Q_0$ and $\Lambda_1 < Q_1$ we see that (18) hold true for all $(x,y) \in E^*_{\bar{a}}$.

Finally, we prove that

$$\begin{aligned} \left|D_x(Wz)(x,y) - D_x(Wz)(\bar{x},\bar{y})\right| &\le Q_2\left[|x-\bar{x}| + |y-\bar{y}|\right] \\ \left|D_y(Wz)(x,y) - D_y(Wz)(\bar{x},\bar{y})\right| &\le Q_2\left[|x-\bar{x}| + |y-\bar{y}|\right] \end{aligned} \tag{19}$$

on $E^*_{\bar{a}}$. For $(x,y),(\bar{x},\bar{y}) \in E_{\bar{a}b}[z]$ we have

$$\begin{aligned} &\left|D_x(Wz)(x,y) - D_x(Wz)(\bar{x},\bar{y})\right| \\ &\le \Big|D_y\phi\big(\lambda(x,y), g(\lambda(x,y),x,y)\big)D_xg(\lambda(x,y),x,y) \\ &\quad - D_y\phi\big(\lambda(\bar{x},\bar{y}), g(\lambda(\bar{x},\bar{y}),\bar{x},\bar{y})\big)D_xg(\lambda(\bar{x},\bar{y}),\bar{x},\bar{y})\Big| \\ &\quad + \left|G(x,y,z_{(x,y)}) - G(\bar{x},\bar{y},z_{(\bar{x},\bar{y})})\right| \\ &\quad + \left|\int_x^{\bar{x}} \left[D_yG(\bar{P}_t) + D_wG(\bar{P}_t)\circ D_yz(t,g(t,\bar{x},\bar{y}))\right]D_xg(t,\bar{x},\bar{y})\,dt\right| \end{aligned}$$

$$+\left|\int_{\lambda(x,y)}^{\lambda(\bar{x},\bar{y})}\Big[D_yG(\bar{P}_t)+D_wG(\bar{P}_t)\circ D_yz(t,g(t,\bar{x},\bar{y}))\Big]D_xg(t,\bar{x},\bar{y})\,dt\right|$$
$$+\left|\int_{\lambda(x,y)}^{x}\Big\{\Big[D_yG(\bar{P}_t)+D_wG(\bar{P}_t)\circ D_yz(t,g(t,\bar{x},\bar{y}))\Big]D_g(t,\bar{x},\bar{y})\right.$$
$$\left.-\Big[D_yG(\bar{P}_t)+D_wG(\bar{P}_t)\circ D_yz(t,g(t,\bar{x},\bar{y}))\Big]D_g(t,\bar{x},\bar{y})\Big\}dt\right|$$
$$\le\Big\{\Lambda_2\left[\frac{1}{\delta^*}(1+L_0^*)+1\right]\Gamma_{1\bar{a}}^2+\Lambda_1\left[\frac{1}{\delta^*}L_1\Gamma_{1\bar{a}}^2+\Gamma_{2\bar{a}}\right]$$
$$+M_1^*[1+Q_1]+\left[1+\frac{1}{\delta^*}\Gamma_{1\bar{a}}\right]M_1^*[1+Q_1]\Gamma_{1\bar{a}}$$
$$+\int_0^x\Big[M_2^*[1+Q_1]^2\Gamma_{1\bar{a}}^2+M_1^*Q_2\Gamma_{1\bar{a}}+M_1^*[1+Q_1]\Gamma_{2\bar{a}}\Big]dt\Big\}\big[|x-\bar{x}|+|y-\bar{y}|\big]$$

where $\bar{P}_t=\big(t,g(t,\bar{x},\bar{y}),z_{(t,g(t,\bar{x},\bar{y}))}\big)$. Analogously we get the estimate

$$\big|D_y(Wz)(x,y)-D_y(Wz)(\bar{x},\bar{y})\big|$$
$$\le\Big\{\Lambda_2\left[\frac{1}{\delta^*}(1+L_0^*)+1\right]\Gamma_{1\bar{a}}^2+\Lambda_1\left[\frac{1}{\delta^*}L_1\Gamma_{1\bar{a}}^2+\Gamma_{2\bar{a}}\right]\frac{1}{\delta^*}\Gamma_{1\bar{a}}M_1^*[1+Q_1]\Gamma_{1\bar{a}}$$
$$+\int_0^x\Big[M_2^*[1+Q_1]^2\Gamma_{1\bar{a}}^2+M_1^*Q_2\Gamma_{1\bar{a}}+M_1^*[1+Q_1]\Gamma_{2\bar{a}}\Big]dt\Big\}\big[|x-\bar{x}|+|y-\bar{y}|\big].$$

The above estimates hold true also in the case $(x,y),(\bar{x},\bar{y})\in E_{\bar{a}0}[z]$, or $(x,y)\in E_{\bar{a}0}[z]$ and $(\bar{x},\bar{y})\in E_{\bar{a}b}[z]$. Taking $\bar{a}$ sufficiently small in order that $S_{2\bar{a}}\le Q_2$ and making use of the relation $\Lambda_2<Q_2$ we get (19), which completes the proof of Theorem 1 ∎

Theorem 2. *If Assumptions* $(H_1)-(H_4)$ *are satisfied, then for sufficiently small* $\bar{a}\in(0,a]$ *the problem* (1), (2) *has a unique solution on* $E_{\bar{a}}$ *in the class* $C_{\bar{a}}^{1,L}(Q)$.

Proof. We prove that for sufficiently small $\bar{a}\in(0,a]$ the operator $W:C_{\bar{a}}^{1,L}(Q)\to C_{\bar{a}}^{1,L}(Q)$ is a contraction. Indeed, if $z,\bar{z}\in C_{\bar{a}}^{1,L}(Q)$, $g=g[z],\bar{g}=g[\bar{z}]$, $\lambda=\lambda[z]$ and $\bar{\lambda}=\lambda[\bar{z}]$, then we have

$$\big|Wz(x,y)-W\bar{z}(x,y)\big|$$
$$\le\Big|\phi\big(\lambda(x,y),g(\lambda(x,y),x,y)\big)-\phi\big(\bar{\lambda}(x,y),\bar{g}(\bar{\lambda}(x,y),x,y)\big)\Big|$$
$$+\left|\int_{\lambda(x,y)}^{\bar{\lambda}(x,y)}G\big(t,\bar{g}(t,x,y),\bar{z}_{(t,\bar{g}(t,x,y))}\big)\,dt\right|$$
$$+\int_{\lambda(x,y)}^{x}\Big|G\big(t,g(t,x,y),z_{(t,g(t,x,y))}\big)-G\big(t,g(t,x,y),\bar{z}_{(t,g(t,x,y))}\big)\Big|dt$$
$$\le\Lambda_1\Big[(1+L_0^*)\big|\lambda(x,y)-\bar{\lambda}(x,y)\big|+\big|g(\lambda(x,y),x,y)-\bar{g}(\lambda(x,y),x,y)\big|\Big]$$
$$+M_0^*\big|\lambda(x,y)-\bar{\lambda}(x,y)\big|$$
$$+\int_0^x M_1^*\Big\{[1+Q_1]\big|g(t,x,y)-\bar{g}(t,x,y)\big|+\big\|z_{(t,g(t,x,y))}-\bar{z}_{(t,g(t,x,y))}\big\|_0\Big\}dt$$

from which by (7), (11) and the obvious relation $(Wz)(x,y) = (W\bar{z})(x,y)$ on $E_0^* \cup \partial_0 E_{\bar{a}}$ we obtain

$$\|Wz - W\bar{z}\|_0 \leq S_{\bar{a}}\|z - \bar{z}\|_0$$

where

$$S_{\bar{a}} = \Lambda_1 \left[\frac{1}{\delta_*}(1 + L_0^*) + 1\right] \Gamma_{\bar{a}} + M_0^* \frac{1}{\delta_*}\Gamma_{\bar{a}} + \bar{a}M_1^*\{\Gamma_{\bar{a}}[1 + Q_1] + 1\}.$$

Since $\lim_{\bar{a}\to 0^+} S_{\bar{a}} = 0$ we may choose $\bar{a} \in (0,a]$ sufficiently small in order that $S_{\bar{a}} < 1$. Consequently W is a contraction, and by the Banach theorem there exists a unique fixed-point of W. Denoting this fixed point by z^* we prove that it is a solution of equation (1).

For any $(x,y) \in E_{\bar{a}0}[z^*]$ we have

$$z^*(x,y) = \phi(0, g(0,x,y)) + \int_0^x G\big(t, g(t,x,y), z^*_{(t,g(t,x,y))}\big)dt. \tag{20}$$

For a fixed x we consider the transformation $y \mapsto g(0,x,y) = \xi$. Using this transformation and the group property (20) takes the form

$$z^*(x, g(x,0,\xi)) = \phi(0,\xi) + \int_0^x G\big(t, g(t,0,\xi), z^*_{(t,g(t,0,\xi))}\big)dt.$$

Differentiating this equation with respect to x we get

$$D_x z^*(x, g(x,0,\xi)) + \sum_{i=1}^n D_{y_i} z^*(x, g(x,0,\xi)) \frac{dg_i}{dt}(x,0,\xi)$$
$$= G\big(x, g(x,0,\xi), z^*_{(x,g(x,0,\xi))}\big).$$

Making use of the inverse transformation $\xi \mapsto g(x,0,\xi) = y$ and (3), we get (1).

For any $(x,y) \in E_{\bar{a}b}[z^*]$ we have

$$z^*(x,y) = \phi\big(\lambda(x,y), g(\lambda(x,y),x,y)\big) + \int_{\lambda(x,y)}^x G\big(t, g(t,x,y), z^*_{(t,g(t,x,y))}\big)dt. \tag{21}$$

For simplicity of notation suppose that $g_i(\lambda(x,y),x,y) = b_i$ for $i = n$, and write

$$\xi' = (\xi_1, \dots, \xi_{n-1}) \qquad \text{and} \qquad g' = (g_1, \dots, g_{n-1}).$$

Fixing x and using the transformation

$$y \mapsto \big(g'(\lambda(x,y),x,y), \lambda(x,y)\big) = (\xi', \eta)$$

we see that (21) takes the form

$$z^*\big(x, g(x,\eta,\xi',b_n)\big) = \phi(\eta,\xi',b_n) + \int_\eta^x G\big(t, g(t,\eta,\xi',b_n), z^*_{(t,g(t,\eta,\xi',b_n))}\big)dt.$$

Differentiating the above equation with respect to x we get

$$D_x z^*\big(x, g(x,\eta,\xi',b_n)\big) + \sum_{i=1}^n D_{y_i} z^*\big(x, g(x,\eta,\xi',b_n)\big) \frac{dg_i}{dt}(x,\eta,\xi',b_n)$$
$$= G\big(x, g(x,\eta,\xi',b_n), z^*_{(x,g(x,\eta,\xi',b_n))}\big).$$

Making use of the inverse transformation $(\xi',\eta) \mapsto g(x,\eta,\xi',b_n) = y$ and (3) we get (1). Since $z^* \in C_{\bar{a}}^{1,L}(Q)$ obviously fulfils the mixed condition (2) this completes the proof of Theorem 2 ∎

4. Some noteworthy particular cases

Given $\hat{f}_i, \hat{G} : E_a \times \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, n)$ let us consider the differential-integral equation with deviated argument

$$\begin{aligned} D_x z(x,y) = \sum_{i=1}^{n} \hat{f}_i\big(x,y,z(x,y), z\big(\alpha(x,y),\beta(x,y)\big)\big) D_{y_i} z(x,y) \\ + \hat{G}\big(x,y,z(x,y), z\big(\alpha(x,y),\beta(x,y)\big)\big) \end{aligned} \tag{22}$$

where $\alpha : E_{\bar{a}} \to \mathbb{R}$ and $\beta : E_{\bar{a}} \to \mathbb{R}^n$. We give sufficient conditions for the existence and uniqueness of solutions of the problem (22),(2).

Assumption (H$_5$). Suppose the following:

1° $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n) \in C(E_a \times \mathbb{R} \times \mathbb{R}; \mathbb{R}^n)$ and $\hat{G} \in C(E_a \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ are functions of the variables (x, y, z, p), and the derivatives $D_y\hat{f}$, $D_z\hat{f}$, $D_p\hat{f}$, $D_y\hat{G}$, $D_z\hat{G}$ and $D_p\hat{G}$ exist on $E_a \times \mathbb{R} \times \mathbb{R}$.

2° There exist non-decreasing functions $\hat{L}_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 0, 1, 2)$ such that

$$\begin{aligned} &\big|\hat{f}(x,y,z,p)\big| \le \hat{L}_0(q), \quad \big|\hat{f}(x,y,z,p) - \hat{f}(\bar{x},y,z,p)\big| \le \hat{L}_1(q)|x-\bar{x}| \\ &\big|D_y\hat{f}(x,y,z,p)\big| \le \hat{L}_1(q), \quad \big|D_z\hat{f}(x,y,z,p)\big| \le \hat{L}_1(q), \quad \big|D_p\hat{f}(x,y,z,p)\big| \le \hat{L}_1(q) \end{aligned}$$

and

$$\begin{aligned} \big|D_y\hat{f}(x,y,z,p) - D_y\hat{f}(\bar{x},\bar{y},\bar{z},\bar{p})\big| \le \hat{L}_2(q)\big[|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}| + |p-\bar{p}|\big] \\ \big|D_z\hat{f}(x,y,z,p) - D_z\hat{f}(\bar{x},\bar{y},\bar{z},\bar{p})\big| \le \hat{L}_2(q)\big[|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}| + |p-\bar{p}|\big] \\ \big|D_p\hat{f}(x,y,z,p) - D_p\hat{f}(\bar{x},\bar{y},\bar{z},\bar{p})\big| \le \hat{L}_2(q)\big[|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}| + |p-\bar{p}|\big] \end{aligned}$$

for $(x,y), (\bar{x},y), (\bar{x},\bar{y}) \in E_a$ and $z, \bar{z}, p, \bar{p} \in \mathbb{R}$ with $|z|, |\bar{z}|, |p|, |\bar{p}| \le q$.

3° There exist non-decreasing functions $\hat{M}_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 0, 1, 2)$ such that $\hat{G}$ fulfils conditions analogous to those given in 2°, with $\hat{L}_i$ replaced by $\hat{M}_i$, respectively.

4° For every $q \in \mathbb{R}_+$ there is $\delta(q) > 0$ such that $f_i(x,y,z,p) \ge \delta(q)$ $(i = 1, \dots, n)$ for $(x,y,z,p) \in E_a \times \mathbb{R} \times \mathbb{R}$ with $|z|, |p| \le q$.

5° The consistency condition

$$\begin{aligned} D_x\phi(x,y) - \sum_{i=1}^{n} \hat{f}_i\big(x,y,\phi(x,y), \phi\big(\alpha(x,y),\beta(x,y)\big)\big) D_{y_i}\phi(x,y) \\ = \hat{G}\big(x,y,\phi(x,y), \phi\big(\alpha(x,y),\beta(x,y)\big)\big) \end{aligned}$$

holds true on $(E_0^* \cup \partial_0 E_a) \cap E_a$.

Assumption (H$_6$). Suppose the following:

1° $\alpha \in C(E_a;\mathbb{R})$ and $\beta \in C(E_a;\mathbb{R}^n)$ are functions of the variables (x,y) such that $(\alpha(x,y)-x, beta(x,y)-y) \in B$ for $(x,y) \in E_a$.

2° The derivatives $D_y\alpha$ and $D_y\beta$ exist on E_a, and there are constants $\hat{N}_i, \hat{P}_i \in \mathbb{R}_+$ $(i=1,2)$ such that

$$|\alpha(x,y)-\alpha(\bar{x},y)| \le \hat{N}_1|x-\bar{x}| \qquad \text{and} \qquad |\beta(x,y)-\beta(\bar{x},y)| \le \hat{P}_1|x-\bar{x}|$$

on E_a and

$$\|D_y\alpha\|_0 \le \hat{N}_1, \quad \|D_y\beta\|_0 \le \hat{P}_1, \quad \|D_y\alpha\|_L \le \hat{N}_2, \quad \|D_y\beta\|_L \le \hat{P}_2.$$

Theorem 3. *If Assumptions (H_1), (H_5) and (H_6) are satisfied, then there are $Q_i \in \mathbb{R}_+$ with $Q_i > \Lambda_i$ $(i=0,1,2)$ such that for sufficiently small $\bar{a} \in (0,a]$ the problem* (22),(2) *has a unique solution on $E_{\bar{a}}$ in the class $C_{\bar{a}}^{1,L}(Q)$.*

Proof. If we define the function $f=(f_1,\dots,f_n)$ by

$$f(x,y,w) = \hat{f}\big(x,y,w(0,0),w\big(\alpha(x,y)-x,\beta(x,y)-y\big)\big)$$

for $(x,y,w) \in E_{\bar{a}} \times C(B;\mathbb{R})$, then the relations

$$\begin{aligned} D_yf(x,y,w) &= D_y\hat{f}\big(x,y,w(0,0),w\big(\alpha(x,y)-x,\beta(x,y)-y\big)\big) \\ &\quad + D_p\hat{f}\big(x,y,w(0,0),w\big(\alpha(x,y)-x,\beta(x,y)-y\big)\big) \\ &\quad \times \Big[D_xw\big(\alpha(x,y)-x,\beta(x,y)-y\big)D_y\alpha(x,y) \\ &\quad + D_yw\big(\alpha(x,y)-x,\beta(x,y)-y\big)\big(D_y\beta(x,y)-1\big)\Big] \end{aligned}$$

and

$$\begin{aligned} D_wf(x,y,w)\circ h &= D_z\hat{f}\big(x,y,w(0,0),w\big(\alpha(x,y)-x,\beta(x,y)-y\big)\big)h(0,0) \\ &\quad + D_p\hat{f}\big(x,y,w(0,0),w\big(\alpha(x,y)-x,\beta(x,y)-y\big)\big) \\ &\quad \times h\big(\alpha(x,y)-x,\beta(x,y)-y\big), \end{aligned}$$

where $(x,y,w) \in E_{\bar{a}} \times C^1(B;\mathbb{R})$ and $h \in C^1(B;\mathbb{R})$, imply that f fulfils Assumption (H_2) with the functions

$$\begin{aligned} L_0(q) &= \hat{L}_0(q) \\ L_1(q) &= \hat{L}_1(q)\big[2+q(\hat{N}_1+\hat{P}_1+1)\big] \\ L_2(q) &= \hat{L}_2(q)\big\{1+\big[1+q(\hat{N}_1+\hat{P}_1+1)\big]^2\big\} \\ &\quad + \hat{L}_1(q)\big[q(1+\hat{N}_1+\hat{P}_1)+q(1+\hat{N}_2+\hat{P}_2)\big]. \end{aligned}$$

Analogously, the function G defined by

$$G(x,y,w) = \hat{G}\big(x,y,w(0,0),w\big(\alpha(x,y)-x,\beta(x,y)-y\big)\big)$$

for $(x, y, w) \in E_{\tilde{a}} \times C(B; \mathbb{R})$ fulfils Assumption (H_3) with the functions

$$
\begin{aligned}
M_0(q) &= \hat{M}_0(q) \\
M_1(q) &= \hat{M}_1(q)\left[2 + q(\hat{N}_1 + \hat{P}_1 + 1)\right] \\
M_2(q) &= \hat{M}_2(q)\left\{1 + \left[1 + q(\hat{N}_1 + \hat{P}_1 + 1)\right]^2\right\} \\
&\quad + \hat{M}_1(q)\left[q(1 + \hat{N}_1 + \hat{P}_1) + q(1 + \hat{N}_2 + \hat{P}_2)\right].
\end{aligned}
$$

Then we choose $Q_i > \Lambda_i$ $(i = 0, 1, 2)$ such that Assumption (H_4) holds true, and our claim follows by Theorem 2 ∎

Remark 5. The equation with a deviated argument considered by Eichorn and Gleissner [11] is a special case of (22).

Remark 6. With $\hat{f}$ and $\hat{G}$ as in equation (22) consider the differential-integral equation

$$
\begin{aligned}
D_x z(x,y) = &\sum_{i=1}^{n} \hat{f}_i \left(x, y, z(x,y), \int_B z(x+t, y+s)\, dt ds\right) D_{y_i} z(x,y) \\
&+ \hat{G}\left(x, y, z(x,y), \int_B z(x+t, y+s)\, dt ds\right).
\end{aligned}
\tag{23}
$$

If we define the functions f and G by

$$
\begin{aligned}
f(x,y,w) &= \hat{f}\left(x, y, w(0,0), \int_B w(t,s)\, dt ds\right) \\
G(x,y,w) &= \hat{G}\left(x, y, w(0,0), \int_B w(t,s)\, dt ds\right)
\end{aligned}
$$

for $(x, y, w) \in E_{\tilde{a}} \times C(B; \mathbb{R})$, then it is also easy to formulate assumptions on $\hat{f}$ and $\hat{G}$ in order to get an existence and uniqueness theorem for problem (23),(2) as a particular case of problem (1),(2).

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