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Nonlinear Geometric Optics for Shock Waves Part I: Scalar Case

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Abstract. In this paper we study the nonlinear geometric optics of the shock wave for a scalar conservation law in one space variable. The existence of the oscillatory shock wave and its leading terms are obtained. Meanwhile, we rigorously justify the asymptotic properties of the shock wave as well as the shock front.

Keywords: *Nonlinear geometric optics, scalar conservation law, shock wave*

AMS subject classification: 35 B 05, 35 L 65, 35 L 67

1. Introduction

The study of small amplitude, rapidly oscillatory waves for nonlinear hyperbolic equations is widespread and very important in the field of applied mathematics. There is a rich literature devoted to the formal analysis and rigorous justification of weakly nonlinear geometric optics for rapidly oscillatory waves. Most of the rigorous justification is given in the setting of smooth solutions. See articles of J. L. Joly, G. Métivier and J. Rauch [8, 9] and references therein for Cauchy problems, and those of J. Chikhi [2] and M. Williams [16, 17] for mixed value problems with fixed boundaries. The asymptotic analysis of nonlinear hyperbolic problems together with applications had also been investigated by Y. He and T. B. Moodie in [7].

With the publication of the important work of R. DiPerna and A. Majda [5] which contains the first rigorous result in the setting of bounded variation solutions, much attention has been paid to the rigorous study of the formal analysis beyond the formation of shocks in recent years. In this aspect, we mention the interesting work of C. Cheverry [1], S. Schochet [14] for the initial value problem, and that of M. Sablé-Tougeron [13] for the boundary value problem. To my knowledge, all of these rigorous justifications for weak solutions are developed in the setting of bounded variation solutions, in which some shock waves might appear, but it does not contain any detailed information on the propagation of wave fronts.

The present paper is the first attempt to rigorously justify the nonlinear geometric optics of shock waves when a shock wave is perturbed by rapidly oscillatory initial data, which was considered by A. Majda and M. Artola in [12] for the formal analysis. We

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study the simpliest case, the scalar conservation law in one space variable. When the piecewise constant initial data, which produces a plane shock, is perturbed by small amplitude, high frequency oscillatory data, both of the rapidly oscillatory shock wave and the asymptotic principal term satisfy free boundary value problems, in which the unknown state function and shock front are nonlinearily coupled. As A. Majda in [11], we transform this free boundary into a fixed one by using a transformation depending on the unknown shock front.

For the scalar conservation law, this coupled problem can be decoupled into two problems for the state function and shock front respectively, which makes us possible to use the theory of J. L. Joly et al. in [8] to study our problems. In the case of systems, it is impossible to make the problem decoupled. This case will be discussed in the forthcoming Part II.

The remainder of this paper is arranged as follows. In Section 2, we deduce the problems of the oscillatory shock wave and leading profiles by using the method of multiple scales, and state the main results. The problems of the oscillatory shock wave as well as the leading profiles are studied in Sections 3 and 4. Finally, Section 5 is devoted to the proof of the asymptotic property of the shock wave, which gives the nonlinear geometric optics.

2. Statement of problems and main results

For the scalar conservation law in one space variable

$$\partial_t u + \partial_x f(u) = 0 \tag{2.1}$$

we assume that the piecewise constant state

$$u_0 = \begin{cases} u_0^+ & \text{for } x > \sigma t \\ u_0^- & \text{for } x < \sigma t \end{cases} \tag{2.2}$$

is a shock wave with the speed $\sigma \in (-\infty, \infty)$, and the smooth flux function f is genuinely nonlinear when

$$|u - u_0| < \eta$$

for a fixed small constant $\eta > 0$, i.e. $f''(u_0) \neq 0$.

In the following discussion, we will always use u^+ and u^- to denote the right and left values of u, respectively, on both sides of the shock front, and denote by

$$[u] = u^+ - u^-$$

the jump of u across the front. From the well known Rankine-Hugoniot and Lax's entropy conditions, we know that the plane shock (2.2) satisfies

$$\sigma[u_0] = [f(u_0)] \tag{2.3}$$

and

$$f'(u_0^+) < \sigma < f'(u_0^-). \tag{2.4}$$

Let us study the following Cauchy problem of (2.1) with the initial data being a small perturbation of the piecewise constant (2.2):

$$\left.\begin{aligned} &\partial_t U^\varepsilon + \partial_x f(U^\varepsilon) = 0 \qquad (t > 0,\, x \in \mathbb{R}) \\ &U^\varepsilon(0, x) = \begin{cases} u_0^+ + \varepsilon u_{+,0}^\varepsilon(x) & \text{for } x > 0 \\ u_0^- + \varepsilon u_{-,0}^\varepsilon(x) & \text{for } x < 0 \end{cases} \end{aligned}\right\} \tag{2.5}$$

where $\varepsilon > 0$ is small enough, and $u_{\pm,0}^\varepsilon \in C^1$.

As A. Corli and M. Sablé-Tougeron in [4], the initial value problem (2.5) determines a shock locally around the origin for any fixed small $\varepsilon \in (0, \varepsilon_0]$.

Before giving assumptions on the problem (2.5), we introduce some notations. Given a small closed neighborhood $\omega \subset \{t = 0\}$ of the origin, suppose Ω is the closure of a determinacy domain of ω for the Cauchy problem (2.5) when $|U^\varepsilon - u_0| < \eta$. Set

$$\begin{aligned} \Omega^+ &= \Omega \cap \{x \geq 0\} & & \omega^+ = \omega \cap \{x \geq 0\} \\ & & \text{and} & \\ \Omega_T^+ &= \Omega^+ \cap \{t \leq T\} & & \omega^- = \omega \cap \{x \leq 0\}. \end{aligned}$$

The space $C^k(\Omega)$ is the usual one of functions whose derivatives of order less or equal k are continuous in Ω. Equip this space with the family of norms

$$\|u\|_{\varepsilon,k,\Omega} = \sum_{|\alpha| \leq k} \varepsilon^{|\alpha|} \|\partial^\alpha u\|_{L^\infty(\Omega)}.$$

A family $\{u^\varepsilon\}_\varepsilon \subset C^k(\Omega)$ is *bounded* in $C_\varepsilon^k(\Omega)$ if the norms $\|u^\varepsilon\|_{\varepsilon,k,\Omega}$ are bounded, and ϕ^ε is *bounded* in $\widetilde{C}_\varepsilon^k([0,T])$ if $\phi^\varepsilon \in C^k[0,T]$ and $\|d_t \phi^\varepsilon\|_{\varepsilon,k-1,[0,T]}$ are bounded for $k \geq 1$.

Set $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$, and let $C_p^0(\mathbb{R} \times \mathbb{T}^1)$ be the space of continuous functions $u = u(\tau, \theta)$ periodic in $\theta \in \mathbb{T}^1$ and almost periodic in $\tau \in \mathbb{R}$. Denote by

$$C^0(\Omega : \mathbb{R} \times \mathbb{T}^1) = C^0(\Omega : C_p^0(\mathbb{R} \times \mathbb{T}^1))$$

the space of continuous functions from Ω into $C_p^0(\mathbb{R} \times \mathbb{T}^1)$. For $k \in \mathbb{N}$, define the space $C^k(\Omega : \mathbb{R} \times \mathbb{T}^1)$ of those functions $U \in C^0(\Omega : \mathbb{R} \times \mathbb{T}^1)$ whose derivatives $\partial_{(t,x;\tau,\theta)}^\alpha U$ belong to $C^0(\Omega : \mathbb{R} \times \mathbb{T}^1)$ for any $|\alpha| \leq k$. Let $C^k(\omega : \mathbb{T}^1)$ be the set of $u \in C^k(\Omega : \mathbb{R} \times \mathbb{T}^1)$ independent of (t, τ).

For the problem (2.5), we assume that there are $U_0^\pm(x, \theta) \in C^1(\omega^\pm : \mathbb{T}^1)$ such that

$$\left\| u_{\pm,0}^\varepsilon(x) - U_0^\pm(x, \tfrac{x}{\varepsilon}) \right\|_{\varepsilon,1,\omega^\pm} = o(1) \tag{2.6}$$

when $\varepsilon \to 0$, which immediately implies the boundedness of $u_{\pm,0}^\varepsilon$ in $C_\varepsilon^1(\omega^\pm)$, where we have used the notation "$\pm$" to mean two cases according to the upper and lower signs, and it will be used in this whole paper.

The aim of this paper is to study the local existence of the shock wave U^ε for the problem (2.5), and asymptotic expansions of U^ε and its shock front $\{x = \Psi_\varepsilon(t)\}$ with respect to ε.

Now, let us simplify the problem (2.5), and deduce the problem of leading terms of U^ε and the shock curve $\{x = \Psi_\varepsilon(t)\}$. Let

$$U^\varepsilon(t,x) = \begin{cases} u_0^+ + \varepsilon u_+^\varepsilon(t,x) & \text{for } x > \sigma t + \varepsilon\phi^\varepsilon(t) \\ u_0^- + \varepsilon u_-^\varepsilon(t,x) & \text{for } x < \sigma t + \varepsilon\phi^\varepsilon(t) \end{cases} \tag{2.7}$$

be the shock wave solution of the problem (2.5), i.e. $(u_\pm^\varepsilon, \phi^\varepsilon)$ satisfy

$$\left.\begin{aligned} &\partial_t(u_0^+ + \varepsilon u_+^\varepsilon) + \partial_x f(u_0^+ + \varepsilon u_+^\varepsilon) = 0 \qquad \text{for } x > \sigma t + \varepsilon\phi^\varepsilon(t) \\ &\partial_t(u_0^- + \varepsilon u_-^\varepsilon) + \partial_x f(u_0^- + \varepsilon u_-^\varepsilon) = 0 \qquad \text{for } x < \sigma t + \varepsilon\phi^\varepsilon(t) \end{aligned}\right\}$$

which is equivalent to

$$\left.\begin{aligned} &\partial_t u_+^\varepsilon + f'(u_0^+ + \varepsilon u_+^\varepsilon)\partial_x u_+^\varepsilon = 0 \qquad \text{for } x > \sigma t + \varepsilon\phi^\varepsilon(t) \\ &\partial_t u_-^\varepsilon + f'(u_0^- + \varepsilon u_-^\varepsilon)\partial_x u_-^\varepsilon = 0 \qquad \text{for } x < \sigma t + \varepsilon\phi^\varepsilon(t) \end{aligned}\right\} \tag{2.8}$$

and satisfy the Rankine-Hugoniot condition

$$\left(\sigma + \varepsilon\frac{d\phi^\varepsilon}{dt}\right)(\varepsilon[u^\varepsilon] + [u_0]) = [f(u_0 + \varepsilon u^\varepsilon)] \tag{2.9}$$

on $\{x = \sigma t + \varepsilon\phi^\varepsilon(t)\}$.

At this stage, both of functions $u_\pm^\varepsilon$ and ϕ^ε are unknown, hence (2.8) - (2.9) is a free boundary value problem. In order to transform this problem into a fixed boundary problem, we perform the tranformation

$$\left.\begin{aligned} &\tilde{t} = t \\ &\tilde{x} = x - \sigma t - \varepsilon\phi^\varepsilon(t) \end{aligned}\right\} \tag{2.10}$$

in (2.8), and obtain that $\tilde{u}_\pm^\varepsilon(\tilde{t},\tilde{x}) = u_\pm^\varepsilon(t,x)$ satisfies

$$\left.\begin{aligned} &\partial_{\tilde{t}}\tilde{u}_+^\varepsilon + \Big(f'(u_0^+ + \varepsilon u_+^\varepsilon) - (\sigma + \varepsilon d_t\phi^\varepsilon)\Big)\partial_{\tilde{x}}\tilde{u}_+^\varepsilon = 0 \quad \text{for } \tilde{x} > 0 \\ &\partial_{\tilde{t}}\tilde{u}_-^\varepsilon + \Big(f'(u_0^- + \varepsilon u_-^\varepsilon) - (\sigma + \varepsilon d_t\phi^\varepsilon)\Big)\partial_{\tilde{x}}\tilde{u}_-^\varepsilon = 0 \quad \text{for } \tilde{x} < 0. \end{aligned}\right\} \tag{2.11}$$

By using the transformation

$$\left.\begin{aligned} &\bar{t} = \tilde{t} \\ &\bar{x} = -\tilde{x} \end{aligned}\right\}$$

in the second line of (2.11), we know that $(u^\varepsilon_\pm, \phi^\varepsilon)$ satisfy the following coupled problem with the fixed boundary $\{x = 0\}$:

$$\left.\begin{aligned} \partial_t u^\varepsilon_\pm \pm \Big(f'(u_0^\pm + \varepsilon u^\varepsilon_\pm) - (\sigma + \varepsilon d_t\phi^\varepsilon)\Big)\partial_x u^\varepsilon_\pm = 0 \qquad (t, x > 0)\\ (\sigma + \varepsilon d_t\phi^\varepsilon)(\varepsilon[u^\varepsilon] + [u_0]) = [f(u_0 + \varepsilon u^\varepsilon)] \qquad (x = 0)\\ \phi^\varepsilon(0) = 0\\ u^\varepsilon_\pm(0, x) = u^\varepsilon_{\pm,0}(x) \end{aligned}\right\} \tag{2.12}$$

where we have dropped the tilde and bar of notations for simplicity.

Suppose that the solutions $(u^\varepsilon_\pm, \phi^\varepsilon)$ of the problem (2.12) have the forms

$$\left.\begin{aligned} u^\varepsilon_\pm(t,x) &= U^\pm(t,x;\tfrac{t}{\varepsilon},\tfrac{x}{\varepsilon}) + \varepsilon V^\pm(t,x;\tfrac{t}{\varepsilon},\tfrac{x}{\varepsilon}) + O(\varepsilon^2)\\ \phi^\varepsilon(t) &= \phi(t,\tfrac{t}{\varepsilon}) + \varepsilon\varphi(t,\tfrac{t}{\varepsilon}) + O(\varepsilon^2) \end{aligned}\right\} \tag{2.13}$$

where $U^\pm(t,x;\tau,\theta), V^\pm(t,x;\tau,\theta), \phi(t,\tau)$ and $\varphi(t,\tau)$ are almost periodic in $\tau \in \mathbb{R}$ and periodic in $\theta \in \mathbb{T}^1$ (in fact, we will see that they are also periodic in τ with a different period). Let us formally deduce the problem of $(U^\pm, \phi)$ from (2.12).

Set $\tau = \frac{t}{\varepsilon}$ and $\theta = \frac{x}{\varepsilon}$. Plugging the formal expressions (2.13) into the equation of (2.12), expanding $f'(u_0^\pm + \varepsilon u^\varepsilon_\pm)$ by Taylor's formula and grouping each power of ε, it follows that the term of "ε^{-1}" is

$$\frac{\partial U^\pm}{\partial\tau} \pm \left(f'(u_0^\pm) - \sigma - \frac{\partial\phi}{\partial\tau}\right)\frac{\partial U^\pm}{\partial\theta} = 0 \tag{2.14}$$

and the term of "ε^0" is

$$\begin{aligned} \frac{\partial U^\pm}{\partial t} + \frac{\partial V^\pm}{\partial\tau} &\pm \left(f'(u_0^\pm) - \sigma - \frac{\partial\phi}{\partial\tau}\right)\left(\frac{\partial U^\pm}{\partial x} + \frac{\partial V^\pm}{\partial\theta}\right)\\ &\pm \left(f''(u_0^\pm)U^\pm - \left(\frac{\partial\phi}{\partial t} + \frac{\partial\varphi}{\partial\tau}\right)\right)\frac{\partial U^\pm}{\partial\theta} = 0. \end{aligned} \tag{2.15}$$

Similarly, the boundary condition of (2.12) implies that on $\{x = \theta = 0\}$ we have

$$\left(\sigma + \frac{\partial\phi}{\partial\tau}\right)[u_0] = [f(u_0)] \tag{2.16}$$

and

$$\left(\frac{\partial\phi}{\partial t} + \frac{\partial\varphi}{\partial\tau}\right)[u_0] + \left(\sigma + \frac{\partial\phi}{\partial\tau}\right)(U^+ - U^-) - f'(u_0^+)U^+ + f'(u_0^-)U^- = 0. \tag{2.17}$$

Employing the Rankine-Hugoniot condition (2.3) and the obvious fact $[u_0] \neq 0$ for (2.16), it follows

$$\frac{\partial\phi}{\partial\tau} = 0, \tag{2.18}$$

i.e. the leading term ϕ of $\phi^\varepsilon(t)$ is independent of $\tau = \frac{t}{\varepsilon}$, which means that ϕ has not any oscillation.

To simplify the problem of $U^\pm$, for any continuous function $u(t,x;\tau,\theta)$ periodic in $\theta \in \mathbb{T}^1$ and almost periodic in $\tau \in \mathbb{R}$, define the mean value operator $\mathbb{E}_\pm$ by

$$\mathbb{E}_\pm u(t,x;\tau,\theta) = \lim_{\rho\to\infty} \frac{1}{2\rho} \int_{-\rho}^{\rho} u\Big(t,x;\tau+s,\theta \pm \big(f'(u_0^\pm)-\sigma\big)s\Big)\,ds \tag{2.19}$$

and let

$$(\mathbf{m}_\tau u)(t,x,\theta) = \lim_{\rho\to\infty} \frac{1}{2\rho} \int_{-\rho}^{\rho} u(t,x;\tau,\theta)\,d\tau \tag{2.20}$$

be the mean value operator in the τ-variable. Then, from the equations (2.14), (2.15) and (2.17), by using the result (2.18) and the assumption (2.6) we obtain that the leading profiles of $(u_\pm^\varepsilon, \phi^\varepsilon)$ satisfy the problem

$$\left.\begin{aligned} \mathbb{E}_\pm U^\pm &= U^\pm \\ \mathbb{E}_\pm\Big(\partial_t U^\pm \pm \big(f'(u_0^\pm)-\sigma\big)\partial_x U^\pm \pm \big(f''(u_0^\pm)U^\pm - \chi\big)\partial_\theta U^\pm\Big) &= 0 \\ \chi[u_0] - \big[\big(f'(u_0)-\sigma\big)U\big] &= 0 \quad (x=\theta=0) \\ U^\pm|_{t=\tau=0} &= U_0^\pm(x,\theta) \end{aligned}\right\} \tag{2.21}$$

and

$$\left.\begin{aligned} d_t\phi[u_0] - \mathbf{m}_\tau[(f'(u_0)-\sigma)U] = 0 \quad (x=\theta=0) \\ \phi(0) = 0 \end{aligned}\right\} \tag{2.22}$$

where $\chi(t,\tau) = d_t\phi + \partial_\tau\varphi$.

Let us state the main result of this paper as follows.

Theorem 2.1. *Suppose that the initial data $u_{\pm,0}^\varepsilon \in C^1(\omega^\pm)$ satisfy the asymptotic property* (2.6).

(1) *There are $T>0$ and $\varepsilon_0>0$ such that the problem* (2.12) *has unique bounded solutions $u_\pm^\varepsilon \in C_\varepsilon^1(\Omega_T^\pm)$ and $\phi^\varepsilon \in \widetilde{C}_\varepsilon^2[0,T]$ for any $\varepsilon \in (0,\varepsilon_0]$.*

(2) *There are unique solutions $U^\pm \in C^1(\Omega_T^\pm : \mathbb{R}\times\mathbb{T}^1)$, $\chi \in C^1([0,T]:\mathbb{R})$ and $\phi \in C^2[0,T]$ to the problems* (2.21) *and* (2.22).

(3) *For the above solutions $(u_\pm^\varepsilon, \phi^\varepsilon)$, we have the asymptotic properties*

$$\big\|u_\pm^\varepsilon(t,x) - U^\pm(t,x;\tfrac{t}{\varepsilon},\tfrac{x}{\varepsilon})\big\|_{\varepsilon,1,\Omega_T^\pm} = o(1) \tag{2.23}$$

and

$$\left.\begin{aligned} \big\|d_t\phi^\varepsilon(t) - \chi(t,\tfrac{t}{\varepsilon})\big\|_{\varepsilon,1,[0,T]} = o(1) \\ \big\|\phi^\varepsilon(t) - \phi(t)\big\|_{L^\infty[0,T]} = o(1) \end{aligned}\right\} \tag{2.24}$$

when $\varepsilon \to 0$.

Remark 2.1. From the results (2.24), it is easy to obtain the asymptotics of the shock front $\{x = \sigma t + \varepsilon\phi^\varepsilon(t)\}$.

3. Existence of exact solutions

This section is devoted to the proof of Theorem 2.1/(1), which implies the existence and uniqueness of the exact solutions $(u^\varepsilon_\pm, \phi^\varepsilon)$ to the problem (2.12).

Set $v^\varepsilon = (u^\varepsilon_+,\ u^\varepsilon_-)^T$,

$$A(\varepsilon v^\varepsilon) = \begin{pmatrix} f'(u_0^+ + \varepsilon u^\varepsilon_+) & 0 \\ 0 & -f'(u_0^- + \varepsilon u^\varepsilon_-) \end{pmatrix}$$

and

$$B(\varepsilon v^\varepsilon) = \frac{[f(u_0 + \varepsilon u^\varepsilon)]}{[u_0] + \varepsilon[u^\varepsilon]} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, from (2.12) we obtain that when

$$|\varepsilon[u^\varepsilon]| \le \frac{1}{2}|u_0^+ - u_0^-|, \tag{3.1}$$

the problem (2.12) is equivalent to the systems

$$\left.\begin{aligned} \partial_t v^\varepsilon + (A(\varepsilon v^\varepsilon) + B(\varepsilon v^\varepsilon))\partial_x v^\varepsilon = 0 \qquad (t, x > 0) \\ v^\varepsilon(0, x) = \left(u^\varepsilon_{+,0}(x),\ u^\varepsilon_{-,0}(x)\right)^T \end{aligned}\right\} \tag{3.2}$$

and

$$\left.\begin{aligned} & d_t\phi^\varepsilon = \varepsilon^{-1}\Big(\big[f(u_0 + \varepsilon u^\varepsilon)\big]\big([u_0] + \varepsilon[u^\varepsilon]\big)^{-1} - \sigma\Big) \\ & \phi^\varepsilon(0) = 0. \end{aligned}\right\} \tag{3.3}$$

Obviously, the eigenvalues of $A(\varepsilon v^\varepsilon) + B(\varepsilon v^\varepsilon)$ are

$$\begin{aligned} \lambda_1 &= f'(u_0^+ + \varepsilon u^\varepsilon_+) - \frac{[f(u_0 + \varepsilon u^\varepsilon)]}{[u_0] + \varepsilon[u^\varepsilon]} \\ \lambda_2 &= \frac{[f(u_0 + \varepsilon u^\varepsilon)]}{[u_0] + \varepsilon[u^\varepsilon]} - f'(u_0^- + \varepsilon u^\varepsilon_-). \end{aligned} \tag{3.4}$$

By simple computation and using the Rankine-Hugoniot condition (2.3), λ_1 and λ_2 can be rewritten as

$$\lambda_1 = f'(u_0^+) - \sigma + f''(u_0^+ + \tau_1\varepsilon u^\varepsilon_+)(\varepsilon u^\varepsilon_+) - \frac{\varepsilon[(f'(u_0 + \tau_2\varepsilon u^\varepsilon) - \sigma)u^\varepsilon]}{[u_0] + \varepsilon[u^\varepsilon]} \tag{3.5}$$

$$\lambda_2 = -f'(u_0^-) + \sigma - f''(u_0^- + \tau_3\varepsilon u^\varepsilon_-)(\varepsilon u^\varepsilon_-) + \frac{\varepsilon[(f'(u_0 + \tau_2\varepsilon u^\varepsilon) - \sigma)u^\varepsilon]}{[u_0] + \varepsilon[u^\varepsilon]} \tag{3.6}$$

with $\tau_i \in (0,1)$ $(i = 1, 2, 3)$.

Employing the Lax entropy condition (2.4) for (3.5) and (3.6), we obtain that there is $\eta > 0$ such that when

$$\|\varepsilon v^\varepsilon\|_{L^\infty(\Omega_+)} \le \eta \tag{3.7}$$

we have

$$\lambda_1, \lambda_2 < 0. \tag{3.8}$$

Hence, even though the problem (3.2) is defined in the quarter space $\{t, x > 0\}$, it does not need any boundary condition on $\{x = 0\}$ when (3.7) holds. The problem (2.12) is decoupled into problems (3.2) and (3.3).

Though the term $B(\varepsilon v^\varepsilon)$ of the problem (3.2) depends only upon the value of v^ε on the boundary $\{x = 0\}$, it is not difficult to see that the classical method of Cauchy problems for one space dimensional quasilinear hyperbolic systems is still valid. By applying the theory of P. Hartmen and A. Wintner in [6], and J. L. Joly et al. in [8: Subsection 6.2] in the problem (3.2), we immediately obtain the following

Theorem 3.1. *There are $T > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the iteration scheme*

$$\left.\begin{aligned} \partial_t v_{n+1}^\varepsilon + \big(A(\varepsilon v_n^\varepsilon) + B(\varepsilon v_n^\varepsilon)\big)\partial_x v_{n+1}^\varepsilon = 0 \quad (t, x > 0) \\ v_{n+1}^\varepsilon(0, x) = \big(u_{+,0}^\varepsilon(x),\ u_{-,0}^\varepsilon(x)\big)^T \end{aligned}\right\} \tag{3.9}$$

with $v_0^\varepsilon(t, x) \equiv 0$ defines a sequence $\{v_n^\varepsilon\}_n \subset C^1(\Omega_T^+)$ such that:

(1) *There is $M > 0$ such that, for all n and $\varepsilon \in (0, \varepsilon_0]$, we have*

$$\|v_n^\varepsilon\|_{\varepsilon,1,\Omega_T^+} \le M \qquad \text{and} \qquad \varepsilon\, \|v_n^\varepsilon\|_{L^\infty(\Omega_T^+)} \le \eta \tag{3.10}$$

with η given in (3.7).

(2) *For each fixed $\varepsilon \in (0, \varepsilon_0]$, the sequence $\{v_n^\varepsilon\}_n$ converges in $C^1(\Omega_T^+)$ to the solution v^ε of the problem* (3.2).

(3) *Moreover, as $n \to \infty$, $\|v_n^\varepsilon - v^\varepsilon\|_{L^\infty(\Omega_T^+)} \to 0$ uniformly in $\varepsilon \in (0, \varepsilon_0]$.*

By using Taylor's formula for $f(u_0 + \varepsilon u^\varepsilon)$, the problem (3.3) can be reformulated as

$$\left.\begin{aligned} &d_t\phi^\varepsilon = \big([u_0] + \varepsilon[u^\varepsilon]\big)^{-1}\left(\int_0^1 [f'(u_0 + \tau\varepsilon u^\varepsilon)u^\varepsilon]d\tau - \sigma[u^\varepsilon]\right) \\ &\phi^\varepsilon(0) = 0. \end{aligned}\right\} \tag{3.11}$$

With the function $v^\varepsilon = (u_+^\varepsilon, u_-^\varepsilon)^T \in C_\varepsilon^1(\Omega_T^+)$ determined by Theorem 3.1, we can easily solve the problem (3.11), and obtain the following

Theorem 3.2. *There is a unique solution $\phi^\varepsilon \in C^2[0, T]$ to the problem* (3.11) *with ϕ^ε bounded in $\widetilde{C}_\varepsilon^2[0, T]$.*

4. Existence of profiles

In this section, we study the problems (2.21) and (2.22) of leading profiles, and obtain the proof of Theorem 2.1/(2).

From the definition (2.19) of $\mathbb{E}_\pm$, we know that $\mathbb{E}_\pm U^\pm = U^\pm$ means that $U^\pm(t,x;\tau,\theta)$ can be regarded as a function of $\big(t,x;\theta \mp (f'(u_0^\pm)-\sigma)\tau\big)$. Denote

$$U^\pm(t,x;\tau,\theta) = \underline{U}^\pm\Big(t,x;\theta \mp (f'(u_0^\pm)-\sigma)\tau\Big)$$

$$V(t,x;\theta) = \Big(\underline{U}^+(t,x;\theta), \underline{U}^-(t,x;\theta)\Big)^T.$$

Then from (2.21) and (2.22), we know that (V,χ,ϕ) satisfy the following systems:

$$\left.\begin{aligned} &\partial_t V + A\partial_x V + V^T B \partial_\theta V + C\partial_\theta V = 0 \quad (t, x > 0) \\ &V(0,x;\theta) = \big(U_0^+(x,\theta), U_0^-(x,\theta)\big)^T \end{aligned}\right\} \tag{4.1}$$

$$\begin{aligned} \chi(t,\tau) = \frac{1}{[u_0]}\Big\{ &\big(f'(u_0^+)-\sigma\big)\underline{U}^+\big(t,0;(\sigma - f'(u_0^+))\tau\big) \\ &- \big(f'(u_0^-)-\sigma\big)\underline{U}^-\big(t,0;(f'(u_0^-)-\sigma)\tau\big)\Big\} \end{aligned} \tag{4.2}$$

and

$$\left.\begin{aligned} &d_t\phi = \big[(f'(u_0)-\sigma)\mathbf{m}_\theta \underline{U}\big][u_0]^{-1} \\ &\phi(0) = 0 \end{aligned}\right\} \tag{4.3}$$

where

$$A = \begin{pmatrix} f'(u_0^+)-\sigma & 0 \\ 0 & \sigma - f'(u_0^-) \end{pmatrix}$$

$$B = \begin{pmatrix} f''(u_0^+) & 0 \\ 0 & -f''(u_0^-) \end{pmatrix}$$

$$C = \frac{\big[(f'(u_0)-\sigma)\mathbf{m}_\theta \underline{U}\big]}{[u_0]}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\mathbf{m}_\theta$ is the mean value operator in θ-variable

$$(\mathbf{m}_\theta u)(t,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t,x;\theta)\,d\theta$$

for any $u(t,x;\theta)$ periodic in $\theta \in \mathbb{T}^1$.

As in Section 3, the eigenvalues of A are

$$\lambda_1 = f'(u_0^+) - \sigma < 0 \qquad \text{and} \qquad \lambda_2 = \sigma - f'(u_0^-) < 0$$

which implies that the equation in (4.1) does not need any boundary condition on $\{x = 0\}$. Thus, the problem (2.21) is decoupled into problems (4.1) and (4.2).

By taking the mean value operator $\mathbf{m}_\theta$ in (4.1), we obtain that $(\mathbf{m}_\theta V)(t,x)$ satisfies

$$\left.\begin{aligned}(\partial_t + A\partial_x)\mathbf{m}_\theta V = 0 \quad (t, x > 0)\\ (\mathbf{m}_\theta V)(0,x) = (\mathbf{m}_\theta U_0^+, \mathbf{m}_\theta U_0^-)^T\end{aligned}\right\}$$

which immediately gives rise to

$$\mathbf{m}_\theta V(t,x) = \Big(\mathbf{m}_\theta U_0^+\big(x - (f'(u_0^+) - \sigma)t\big), \mathbf{m}_\theta U_0^-\big(x + (f'(u_0^-) - \sigma)t\big)\Big)^T. \tag{4.4}$$

From the problem (4.1), we know that two elements of V satisfy two decoupled scalar quasilinear equations with

$$C = \frac{[(f'(u_0) - \sigma)\mathbf{m}_\theta \underline{U}]}{[u_0]}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

being determined by (4.4), which can be easily studied by classical methods (cf. J. L. Joly et al. [8: Subsection 6.3]). Hence, we have the following

Theorem 4.1.

(1) *Given the initial data $U_0^\pm \in C^1(\omega^+ : \mathbb{T}^1)$, periodic in $\theta \in \mathbb{T}^1$ with $\omega^+ = \omega \cap \{x \geq 0\}$. There is $T > 0$ such that the iteration scheme*

$$\left.\begin{aligned}&\partial_t V_{n+1} + A\partial_x V_{n+1} + V_n^T \cdot B \cdot \partial_\theta V_{n+1} + C\partial_\theta V_{n+1} = 0 \quad (t, x > 0)\\ &V_{n+1}(0,x;\theta) = \big(U_0^+(x,\theta), U_0^-(x,\theta)\big)^T\end{aligned}\right\} \tag{4.5}$$

with $V_0(t,x,\theta) \equiv 0$ defines a sequence $\{V_n\}_n \subset C^1(\Omega_T^+ : \mathbb{T}^1)$ periodic in θ, such that the limit V of V_n in $C^1(\Omega_T^+ : \mathbb{T}^1)$ is the unique solution to the problem (4.1).

(2) *There are unique solutions $\chi \in C^1([0,T] : \mathbb{R})$ and $\phi \in C^2[0,T]$ to the problems (4.2) and (4.3) with $(T,0) \in \Omega$.*

Remark 4.1. From Theorem 4.1 and the definition of V, we know that the solution

$$U^\pm(t,x;\tau,\theta) = \underline{U}^\pm\Big(t,x;\theta \mp (f'(u_0^\pm) - \sigma)\tau\Big)$$

of the problem (2.21) is periodic in τ with the period being $2\pi\big(f'(u_0^\pm) - \sigma\big)^{-1}$.

5. Asymptotic properties

This section is devoted to the study of the asymptotic property of exact solutions $(u_\pm^\varepsilon, \phi^\varepsilon)$ to the problem (2.12), which completes the proof of Theorem 2.1.

As it is discussed in Section 3, even though the problem (3.2) is defined in the quarter space $\{t, x > 0\}$, intrinsically, it is an initial value problem when (3.7) holds. Under the assumption (2.6) for the initial data, by using the result of J. L. Joly et al. [8: Theorem 2.9.1], we have that the solution v^ε to the problem (3.2) obtained by Theorem 3.1 has the asymptotic property

$$v^\varepsilon(t,x) - W\left(t, x; \frac{x - (f'(u_0^+) - \sigma)t}{\varepsilon}, \frac{x + (f'(u_0^-) - \sigma)t}{\varepsilon}\right) = o(1) \qquad (5.1)$$

in $C_\varepsilon^1(\Omega_T^+)$, where

$$W(t,x;\theta_1,\theta_2) = \big(W_1(t,x,\theta_1), W(t,x,\theta_2)\big)^T \in C^1(\Omega_T^+ : \mathbb{T}^1 \times \mathbb{T}^1)$$

satisfy the problem

$$\left.\begin{aligned} &(\partial_t + (f'(u_0^+) - \sigma)\partial_x)W_1 + f''(u_0^+)W_1\partial_{\theta_1}W_1 - a(t)\partial_{\theta_1}W_1 = 0 \\ &(\partial_t - (f'(u_0^-) - \sigma)\partial_x)W_2 - f''(u_0^-)W_2\partial_{\theta_2}W_2 + a(t)\partial_{\theta_2}W_2 = 0 \\ &\qquad\qquad\qquad\qquad W_1(0,x,\theta_1) = U_0^+(x,\theta_1) \\ &\qquad\qquad\qquad\qquad W_2(0,x,\theta_2) = U_0^-(x,\theta_2) \end{aligned}\right\} \qquad (5.2)$$

with

$$a(t) = [u_0]^{-1}\Big((f'(u_0^+) - \sigma)(\mathbf{m}_{\theta_1}W_1)(t,0) - (f'(u_0^-) - \sigma)(\mathbf{m}_{\theta_2}W_2)(t,0)\Big).$$

Obviously, the problem (5.2) is the same as (4.1). From the uniqueness of the problem (4.1), the conclusion (5.1) implies the following

Theorem 5.1. *Suppose* $v^\varepsilon \in C_\varepsilon^1(\Omega_T^+)$, *and*

$$V(t,x,\theta) = (\underline{U}^+(t,x,\theta), \quad \underline{U}^-(t,x,\theta))^T \in C^1(\Omega_T^+ : \mathbb{T}^1)$$

are the unique solutions of the problems (3.2) *and* (4.1), *respectively. Then we have*

$$\begin{aligned} v^\varepsilon(t,x) - \Bigg(&\underline{U}^+\left(t, x; \frac{x - (f'(u_0^+) - \sigma)t}{\varepsilon}\right), \\ &\underline{U}^-\left(t, x; \frac{x + (f'(u_0^-) - \sigma)t}{\varepsilon}\right)\Bigg)^T = o(1) \end{aligned} \qquad (5.3)$$

in $C_\varepsilon^1(\Omega_T^+)$ *when* $\varepsilon \to 0$.

As a simple consequence of Theorem 5.1, from problems (3.11) and (4.2), we immediately obtain

Corollary 5.1. *Suppose that $\phi^\varepsilon \in \widetilde{C}^2_\varepsilon[0,T]$ and $\chi \in C^1([0,T] : \mathbb{R})$ are the unique solutions of the problems* (3.11) *and* (4.2), *respectively. Then we have*

$$d_t\phi^\varepsilon(t) - \chi(t, \tfrac{t}{\varepsilon}) = o(1) \quad \text{in } C^1_\varepsilon[0,T] \qquad \text{when } \varepsilon \to 0. \tag{5.4}$$

Before studying the relationship between ϕ^ε and its leading term ϕ, at first we give a result which can be obtained in the same way as in M. Slemrod [15: Section 6]:

Lemma 5.1. *For any continuous function $f = f(\theta)$ almost periodic in $\theta \in \mathbb{R}$, we have*

$$f(\tfrac{x}{\varepsilon}) \longrightarrow \mathbf{m}(f) = \lim_{\rho\to\infty} \frac{1}{2\rho} \int_{-\rho}^{\rho} f(\theta)\,d\theta \qquad \text{in } L^\infty_{loc} - weak^* \tag{5.5}$$

when $\varepsilon \to 0$, which means that, for any function $\varphi \in L^1(\mathbb{R})$ with compact support,

$$\lim_{\varepsilon\to 0} \int f(\tfrac{x}{\varepsilon})\varphi(x)\,dx = \mathbf{m}(f) \int \varphi(x)\,dx. \tag{5.6}$$

The result of ϕ^ε and ϕ is stated as follows:

Theorem 5.2. *Suppose $\phi^\varepsilon \in \widetilde{C}^2_\varepsilon[0,T]$ and $\phi \in C^2[0,T]$ are the unique solutions of the problems* (3.3) *and* (4.3), *respectively. Then we have*

$$\phi^\varepsilon(t) - \phi(t) = o(1) \qquad \text{in } \; L^\infty[0,T] \tag{5.7}$$

when $\varepsilon \to 0$.

Proof. By comparing the problem (4.2) of χ and (4.3) of ϕ, it follows

$$d_t\phi(t) = (\mathbf{m}_\tau\chi)(t) = \lim_{\rho\to\infty} \frac{1}{2\rho} \int_{-\rho}^{\rho} \chi(t,\tau)\,d\tau \tag{5.8}$$

which implies

$$\chi(t, \tfrac{t}{\varepsilon}) \longrightarrow d_t\phi(t) \qquad \text{in } L^\infty[0,T] - weak^* \tag{5.9}$$

when $\varepsilon \to 0$ by using Lemma 5.1. Hence, by combining (5.4) with (5.9) it gives rise to $d_t\phi^\varepsilon \longrightarrow d_t\phi$ in $L^\infty[0,T] - weak^*$, which immediately implies $\|\phi^\varepsilon - \phi\|_{L^\infty[0,T]} \to 0$ as $\varepsilon \to 0$ ∎

Note in the proof. After the finish of this work together with the system case, the author was informed that a similar problem had been investigated by A. Corli in [3].

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