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'Restricted' Closed Graph Theorems

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Dedicated to Professor W. Meyer-König on the occasion of his 85th birthday

Abstract. The paper deals with 'restricted' closed graph theorems in the following sense: Let a suitable class $\mathcal{E}$ of Mackey spaces as domain spaces, a class $\mathcal{F}$ of locally convex spaces as range spaces and for each $F \in \mathcal{F}$ a suitable class $C_{F,\mathfrak{P}}$ of linear maps $T : E \longrightarrow F$ $(E \in \mathcal{E})$ with closed graph defined by a (general) property $\mathfrak{P}$ be given. In demand is a sufficient or even a sufficient as well as necessary condition to $F \in \mathcal{F}$ such that each $T \in C_{F,\mathfrak{P}}$ is continuous. Note that in the situation of the well-known closed graph theorems of Pták and Kalton, $\mathcal{E}$ is the class of barrelled spaces and Mackey spaces with sequentially complete weak dual space, respectively, $\mathcal{F}$ is the class of all locally convex spaces and for each $F \in \mathcal{F}$ the class $C_{F,\mathfrak{P}}$ is defined to be the set of all linear maps $T : E \longrightarrow F$ $(E \in \mathcal{E})$ with closed graph; a sufficient condition for F in the asked sense is B_r-completeness (Pták [10]) and to be a separable B_r-complete space (Kalton [8]), respectively; a sufficient as well as necessary condition for F is A-completeness (Zhu and Zhao [20]) and to be an L_r-space (Qiu [11], see [12] too), respectively. Based on the ideas developed in [4] and [2] in the present paper the notions of Θ-completeness and of $(\Theta, \mathcal{F}, \mathfrak{P})$-spaces are introduced and a very general 'restricted' closed graph theorem of the described type (see Theorem 2.4) is proved. Furthermore, it is shown that this main result contains both new interesting special cases and a series of known closed graph theorems, for example those due to Pták and Kalton and more generally that of Zhu and Zhao as well as that of Qiu. In addition, the paper deals with further aspects which are related to $(\Theta, \mathcal{F}, \mathfrak{P})$-spaces and motivated by the φ-topologies introduced by Ruckle [13]. At last, the general considerations serve to extend the well-known inclusion theorems of Bennett and Kalton [1, Theorem 5 and 4] which establish a connection between functional analysis (weak sequential completeness and barrelledness) and summability.

Keywords: *Closed graph theorems, $(\Theta, \mathcal{F}, \mathfrak{P})$-spaces, B_r- and A-complete spaces, barrelled spaces, L_r-, L_φ- and A_φ-spaces, matrix maps, inclusion theorems for sequence spaces*

AMS subject classification: 46A45, 46A30, 40H05

1. Introduction

Let E and F be (separated) locally convex spaces. Then the following well-known closed graph theorems hold:

1° (Pták, see [10] and [17, Theorem 12.5.7]). If E is barrelled and F is B_r-complete then every linear map $T : E \longrightarrow F$ with closed graph is continuous.

2° (Kalton, see [8, Theorem 2.4] and [17, Theorem 12.5.13]). If E is a Mackey space with weakly sequentially complete (topological) dual E' and F is a separable B_r-complete space then every linear map $T : E \longrightarrow F$ with closed graph is continuous.

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In both cases the maximal class of range spaces F has been identified: in the first case by Zhu and Zhao [20] and in the second by Qiu [11], its elements are called *A-complete spaces* and *L_r-spaces*, respectively.

If E and F are sequence spaces (more generally, sequence spaces over a locally convex space) with K–topologies and if E has sequentially complete weak dual, then each matrix map $A : E \longrightarrow F$ has closed graph. Therefore 1° and 2° are true in the case of matrix maps between K–spaces. The maximal class of range spaces in the situation of the closed graph theorem of Kalton and matrix maps between K–spaces, namely the class of all L_φ–spaces, has been determined and examined by the authors in [4] and in a joint paper with Große–Erdmann [2]. Note that K–spaces being L_r–spaces are L_φ–spaces and the class of all L_φ–spaces contains all separable B_r–complete K–spaces and all domains E_A of operator–valued matrices A with respect to an L_φ–space E. Those domains are not necessarily L_r–spaces, that is, L_φ–spaces are not L_r–spaces in general.

In this paper we continue the idea of 'restricted' closed graph theorems: We fix a class $\mathcal{F}$ of locally convex spaces and a property $\mathfrak{P}$ which defines for each $F \in \mathcal{F}$ a family $\mathcal{M}_{F,\mathfrak{P}}$ of subspaces M of F^* (algebraic dual) such that $M \cap F'$ is total. Then we consider for a given class $\mathcal{E}$ of domain spaces, for example, the class of all barrelled spaces or the class of all Mackey spaces with sequentially complete weak dual, the class $C_{F,\mathfrak{P}}$ of all linear maps $T : E \longrightarrow F$, $(E \in \mathcal{E})$ such that there exists an $M \in \mathcal{M}_{F,\mathfrak{P}}$ with

$$M \cap F' \subset \Delta_T := \left\{ f \in F' \;\middle|\; f \circ T \in E' \right\}.$$

Each member of $C_{F,\mathfrak{P}}$ has closed graph and –as mentioned in the case of matrix maps– we may ask for the maximal class of range spaces such that each T of this class is continuous. Introducing in 2.1 the notion of Θ–complete spaces and considering the class $\mathcal{E}$ of all Mackey spaces with Θ–complete weak dual we'll prove in Theorem 2.4 that the maximal class is the family of all $(\Theta, \mathcal{F}, \mathfrak{P})$–spaces which will be defined in 2.3.

In Section 3 we apply the 'restricted' closed graph theorem 2.4 to special classes $\mathcal{E}$ and $\mathcal{F}$ and properties $\mathfrak{P}$ and get in this way several well–known as well as some new closed graph theorems. Furthermore, in Section 4 we study aspects which are related to $(\Theta, \mathcal{F}, \mathfrak{P})$–spaces and are motivated by the φ–topologies due to Ruckle [13]. In Section 5 we apply the main results of the paper to extend an inclusion theorem due to Bennett and Kalton which emphasizes once more the close relation between the theory of topological sequence spaces and Summability.

The terminology from the theory of locally convex spaces and Summability is standard, we refer to Wilansky [17], [18]. Throughout the whole paper locally convex spaces are separated.

Let X be a locally convex space. Then we use the notations

$$\omega(X) := \left\{ x = (x_k) \;\middle|\; x_k \in X \ (k \in \mathbf{N}) \right\},$$
$$\varphi(X) := \left\{ x \in \omega(X) \;\middle|\; \exists\, k_0 \in \mathbf{N} \ \forall\, k > k_0 : x_k = 0 \right\}.$$

In the case of $X := \mathbf{K}$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) we write ω and φ instead of $\omega(X)$ and $\varphi(X)$, respectively. Obviously, $\varphi(X)$ equals the linear span of $\{e_k(a) \mid a \in X,\ k \in \mathbf{N}\}$ where $e_k : X \longrightarrow \omega(X)$ is the k–th embedding map defined by $e_k(a) := (0, \dots, a, 0, \dots)$. A subspace E of $\omega(X)$ is called *sequence space (over X)*. Furthermore, a locally convex space E is called a *K(X)–space* if E is a sequence space over X and if the coordinate functions

$$\pi_k : E \longrightarrow X,\ x = (x_i) \longrightarrow x_k \qquad (k \in \mathbf{N})$$

are continuous. If, in addition, E is a Fréchet (Banach) space, then E is called an FK(X)-space (BK(X)-space). In the case of $X := \mathbf{K}$ we use the notations K-, FK- and BK-space.

For any K(X)-space E the space $\varphi(X')$ is a $\sigma(E', E)$-dense subspace of E' and the *β-dual*

$$E^\beta := \left\{ u = (u_k) \in \omega(X') \;\middle|\; \forall\,(x_k) \in E : \sum_k u_k(x_k) \ \text{converges} \right\}$$

is a subspace of E^* where we identify $\varphi(X')$ and E^β as subspaces of E^* in a canonical way. Note that $E^\beta \subset E'$ holds if E is barrelled.

For each $x \in \omega(X)$ and $n \in \mathbf{N}$ the n^{th} section $x^{[n]}$ of x is defined by

$$x^{[n]} := (x_1, x_2, \ldots, x_n, 0, \ldots) = \sum_{k=1}^{n} e^k(x_k).$$

If E is a K(X)-space containing $\varphi(X)$ then we put

$$W_E := \left\{ x \in E \mid x^{[n]} \longrightarrow x \quad (\sigma(E, E')) \right\},$$
$$S_E := \left\{ x \in E \mid x^{[n]} \longrightarrow x \ \text{in } E \right\}.$$

A K(X)-space E containing $\varphi(X)$ with $E = S_E$ and $E = W_E$ is called an AK-space and SAK-space, respectively.

2. A general version of 'restricted' closed graph theorems

We enter now into the topic of 'restricted' closed graph theorems and introduce for this purpose several notions which seem to be theoretical, but the applications and examples in the next section will prove that this access is very useful.

For each $H = (H, \tau_H)$ in a class of locally convex spaces let $\Theta = \Theta[H]$ be a fixed set of τ_H-bounded nets in H which is given by a general property and satisfies the following (technical) conditions:

(i) $\forall\, h \in H \quad \exists\,(h_\alpha) \in \Theta : h_\alpha \longrightarrow h$.

(ii) $\forall\, a \in \mathbf{K} \quad \forall\,(h_\alpha) \in \Theta : (a h_\alpha) \in \Theta$.

(iii) $\forall\, f, g \in H \quad \forall\,(f_\alpha)_{\alpha\in\mathcal{A}}, (g_\beta)_{\beta\in\mathcal{B}} \in \Theta : (f_\alpha \longrightarrow f \text{ and } g_\beta \longrightarrow g)$
$\Longrightarrow \exists\,(h_\gamma) \in \Theta : h_\gamma \in (\{f_\alpha \mid \alpha \in \mathcal{A}\} + \{g_\beta \mid \beta \in \mathcal{B}\}) \text{ and } h_\gamma \longrightarrow f + g$.

For example, if $\Theta =: \Theta_{bs}$ is the set of all bounded sequences then the conditions (i)–(iii) are obviously fulfilled (take $h_n := f_n + g_n$ in (iii)); if $\Theta =: \Theta_{bn}$ is the set of all bounded nets then the conditions (i)–(iii) are fulfilled too: (i) and (ii) are trivial; for (iii) we may assume $\mathcal{A} = \mathcal{U} = \mathcal{B}$, where $\mathcal{U}$ is any given neighbourhood base of zero, and put $h_U := f_U + g_U \quad (U \in \mathcal{U})$.

Definition 2.1 . A locally convex space H is *Θ-complete* if each Cauchy net contained in Θ is convergent in H.

For example, if $\Theta = \Theta_{bs}$ then H is Θ-complete if and only if H is sequentially complete.

Let F be a linear space and let S and M be subspaces of the algebraic dual F^* of F. For a given class of nets $\Theta = \Theta[(F^*, \sigma(F^*, F))]$ we put

$$\overline{S}^{\Theta} := \left\{ g \in F^* \mid \exists\,(g_\alpha)_{\alpha\in\mathcal{A}} \text{ in } S : (g_\alpha) \in \Theta \text{ and } g_\alpha \longrightarrow g\,(\sigma(F^*, F)) \right\},$$
$$\overline{M}^{\Theta} := \bigcap \left\{ S \mid M \subset S < F^* \text{ and } S = \overline{S}^{\Theta} \right\}.$$

Note that $S \subset \overbracket{S}^{\Theta}$ on account of (i), and $\overbracket{S}^{\Theta}$ is a subspace of F^{*} because of (ii) and (iii). If $S = \overbracket{S}^{\Theta}$ then S is called Θw^{*}*-closed*; furthermore, $\overbrace{S}^{\Theta}$ is called the Θw^{*}*-closure* of S. In the special case that $\Theta = \Theta_{bs}$ is the set of bounded sequences, we have (see [4]) $\overbracket{S} = \overbracket{S}^{\Theta}$ and $\overbrace{S} = \overbrace{S}^{\Theta}$; in this case the terms Θw*–closed and Θw*–closure coincide with *sequentially closed* and *sequential closure*, respectively.

In the next step we aim for a 'restricted' closed graph theorem in the following sense: We consider locally convex spaces E, F and linear maps $T : E \longrightarrow F$ having closed graph which is equivalent to the statement that $\Delta_T := \{f \in F' \mid f \circ T \in E'\}$ is $\sigma(F', F)$–dense in F'. Recall that T is weakly continuous, thus continuous in the case of a Mackey space E, if and only if $F' = \Delta_T$. Now, we fix a family $\mathcal{M}_F$ of subspaces M of F^{*} such that $M \cap F'$ is total for each $M \in \mathcal{M}_F$, and restrict our interest to linear maps $T : E \longrightarrow F$ with $M \cap F' \subset \Delta_T$ for at least one $M \in \mathcal{M}_F$; since $M \cap F'$ is total, thus $\sigma(F', F)$–dense in F', such maps have closed graph. In this situation we will get a 'restricted' closed graph theorem if we assume that the Θw*–closure of $M \cap F'$ $(M \in \mathcal{M}_F)$ contains F' and that E is a Mackey space with Θ–complete weak dual. To that end we need the following

General assumption: Throughout the remaining paper we consider only classes Θ such that for all locally convex spaces E, F and all linear maps $T : E \longrightarrow F$ the implication

$$(f_\gamma) \in \Theta = \Theta[(F^{*}, \sigma(F^{*}, F))] \quad \Longrightarrow \quad (f_\gamma \circ T) \in \Theta = \Theta[(E^{*}, \sigma(E^{*}, E))] \tag{2.1}$$

holds for each net (f_γ) in F^{*}. Note that $T^{*}(B)$ is $\sigma(E^{*}, E)$–bounded for any $\sigma(F^{*}, F)$–bounded subset B of F^{*} and each linear map $T : E \longrightarrow F$.

Theorem 2.2. *Let Θ satisfy (2.1), let $F = (F, \tau_F)$ be a locally convex space and $\mathcal{M}_F$ be a family of subspaces M of F^{*} such that $M \cap F'$ is total for each $M \in \mathcal{M}_F$. Then the following statements are equivalent:*

(a) $\forall\, M \in \mathcal{M}_F : F' \subset \overbrace{M \cap F'}^{\Theta}$.

(b) *If E is a Mackey space and $(E', \sigma(E', E))$ is Θ–complete then every linear map $T : E \longrightarrow F$ such that there exists an $M \in \mathcal{M}_F$ with $M \cap F' \subset \Delta_T$ is continuous.*

Note that '(a)⇒(b)' is a 'restricted' closed graph theorem whereas '(b)⇒(a)' tells us that the condition $F' \subset \overbrace{M \cap F'}^{\Theta}$ $(M \in \mathcal{M}_F)$ is best possible for the range space if we restrict our interest to domain spaces and linear maps as considered in (b).

Proof of 2.2. ***(a) ⇒ (b)*** : Let E, T and M be given according to (b). First of all, we prove the Θw*–closedness of Δ_T in F'. Then, on account of $F' \subset \overbrace{M \cap F'}^{\Theta} \cap F' \subset \overbrace{\Delta_T}^{\Theta} \cap F' = \overbracket{\Delta_T}^{\Theta} \cap F' = \Delta_T$, we have $\Delta_T = F'$ which implies the (weak) continuity of T.
Let (f_α) be a net in Δ_T with $(f_\alpha) \in \Theta$ and $f_\alpha \longrightarrow f$ in $(F', \sigma(F', F))$. Consequently, $(f_\alpha \circ T) \in \Theta = \Theta[(E', \sigma(E', E))]$ and it is a Cauchy net in $(E', \sigma(E', E))$. Thus, because of the Θ–completeness of $(E', \sigma(E', E))$, it is convergent to a suitable g in $(E', \sigma(E', E))$. On the other hand

$$f_\alpha \circ T(x) = f_\alpha(T(x)) \longrightarrow f(T(x)) = f \circ T(x) \quad \text{for each } x \in E$$

which implies $f \circ T = g \in E'$, that means $f \in \Delta_T$. Therefore, $\overbracket{\Delta_T}^{\Theta} \cap F' = \Delta_T$.

(b) ⇒ (a) : Let $M \in \mathcal{M}_F$ be given and $G := \overbrace{M}^{\Theta}$. Obviously, G is Θw*–closed in F^{*}, thus

$(F, \tau(F,G))$ has a Θ–complete dual $(G, \sigma(G,F))$. Therefore by statement (b), the inclusion map $i : (F,G) \longrightarrow (F, \tau_F)$ is (weakly) continuous, that is $F' \subset \overline{M \cap F'}^{\Theta}$ ∎

The general form of Theorem 2.2 suggests to distinguish the class of locally convex spaces fulfilling statement (a) in dependence of Θ and a property $\mathfrak{P}$ which defines the family $\mathcal{M}_F$. More precisely, we have the situation of the following definition:

Definition 2.3. Let Θ be determined according to (2.1), $\mathcal{F}$ a class of locally convex spaces and let $\mathfrak{P}$ be a general property defining for each $F \in \mathcal{F}$ a family $\mathcal{M}_{F,\mathfrak{P}}$ of subspaces M of F^* such that $M \cap F'$ is total. An $F \in \mathcal{F}$ is called $(\Theta, \mathcal{F}, \mathfrak{P})$*-space* if $F' \subset \overline{M \cap F'}^{\Theta}$ holds for each $M \in \mathcal{M}_F$. (If we define $\mathcal{M}_{F,\mathfrak{P}}$ explicitely for each $F \in \mathcal{F}$ then we use vicariously the notation $\mathfrak{P}$ for all those definitions.)

For examples of $(\Theta, \mathcal{F}, \mathfrak{P})$–spaces we refer to the next section. However, we should already mention that we are interested in big families $\mathcal{F}$ and $\mathcal{M}_{F,\mathfrak{P}}$ as well as in small families: For example, $\mathcal{F}$ may be the set of all locally convex spaces and $\mathfrak{P}$ may be the 'property' *all subspaces of the algebraic dual such that the intersection with the topological dual is total* which we denote by $\mathfrak{P}_r$. In this case we use the notion Θ_r*-space* instead of $(\Theta, \mathcal{F}, \mathfrak{P})$–space and remark that, if E is a Mackey space and $\mathfrak{P} = \mathfrak{P}_r$, a linear map $T : E \longrightarrow F$ has closed graph if and only if there exists an $M \in \mathcal{M}_{F,\mathfrak{P}_r}$ with $M \subset \Delta_T$. On the other hand, we are interested in the case $\mathcal{F} := \{F\}$, $\mathcal{M}_{F,\mathfrak{P}} := \{M\}$ where F is any fixed locally convex space and M is a suitable subspace of F^* such that $M \cap F'$ is total. Another example of interest is given if $\mathcal{F}$ is the set of all K(Y)–spaces over a fixed locally convex space Y and $M_{F,\mathfrak{P}} := \{\varphi(Y')\}$ for any $F \in \mathcal{F}$. Note furthermore, the property to be a $(\Theta, \mathcal{F}, \mathfrak{P})$–space does not depend on a concrete topology τ_F of $F \in \mathcal{F}$ but it depends only on the dual pair (F, F') : If (F, τ_F) is a $(\Theta, \mathcal{F}, \mathfrak{P})$–space, then the same is true for (F, τ) where τ is any topology of the dual pair (F, F') with $F' := (F, \tau_F)'$.

As an immediate consequence of Theorem 2.2 we get a 'restricted' closed graph theorem for $(\Theta, \mathcal{F}, \mathfrak{P})$–spaces as range space and Mackey spaces with Θ–complete weak dual as domain spaces:

Theorem 2.4. *Under the assumptions and notations in Definition 2.3 the following statements are equivalent for any* $F \in \mathcal{F}$:

(a) F *is a* $(\Theta, \mathcal{F}, \mathfrak{P})$*-space.*

(b) *If* E *is a Mackey space and* $(E', \sigma(E',E))$ *is* Θ*–complete then each linear map* $T : E \longrightarrow F$ *such that there exists an* $M \in \mathcal{M}_{F,\mathfrak{P}}$ *with* $M \cap F' \subset \Delta_T$ *is continuous.*

Theorem 2.4 tells us that the class of all $(\Theta, \mathcal{F}, \mathfrak{P})$–spaces is maximal in $\mathcal{F}$ in the sense, that the 'restricted' closed graph theorem is true in the case of Mackey spaces with Θ–complete weak dual as domain spaces and the class of linear maps with closed graph according to statement (b).

Closing the section we should mention that Qiu [12, Theorem 1] stated a general closed graph theorem containing also important special cases like Kalton's and Ptàk's closed graph theorem, respectively. More generally as in the present paper he considers on the dual spaces so–called Hellinger–Toeplitz topologies (instead of the weak topology), however he restricted the completeness considerations on the dual of the domain space to sequential completeness and quasi–completeness, respectively, which are special cases of Θ–completeness. Because the authors of the present paper do not know any application of Qiu's theorem in the case of non–weak Hellinger–Toeplitz topologies we don't try to generalize Theorem 2.4 with the aim to cover

Qiu's theorem. However, in the next section we will point out that -except [12, Theorem 7] which is trivial as well as [12, Theorem 8] and its corollary- all applications of Qiu's theorem done in [12] are also applications of Theorem 2.4. If we define $\overleftarrow{S}^{\Theta}$ and $\overrightarrow{S}^{\Theta}$ in the case of Hellinger–Toeplitz topologies α and if we consider $\Theta := \Theta_{bn}$ and $\mathfrak{P} := \mathfrak{P}_r$, then Theorem 2.2 remains true if we replace in (b) the topology $\sigma(E', E)$ by the corresponding Hellinger–Toeplitz topology $\alpha(E', E)$; then Qiu's Theorem 8 is contained in this result.

3. Known and new 'restricted' closed graph theorems

In this section we apply Theorem 2.4 with the aim to get known and new ('restricted') closed graph theorems.

3.1 $L_{\mathcal{F},\mathfrak{P}}$–spaces, Kalton's closed graph theorem

Let $\Theta =: \Theta_{bs}$ be the set of all bounded sequences. Then a locally convex space H is Θ–complete if and only if it is sequentially complete, and the Θw^*–closure is identical with the sequential closure. In this case we use the notion $L_{\mathcal{F},\mathfrak{P}}$*-space* instead of $(\Theta, \mathcal{F}, \mathfrak{P})$-space. Furthermore, we use instead of $L_{\mathcal{F},\mathfrak{P}}$-space the following notions:

- L_r*-space* if $\mathcal{F}$ is the class of all locally convex spaces and $\mathfrak{P} = \mathfrak{P}_r$. The consideration of L_r-spaces was first done by J. Qiu (see [11] and [12]).
- L_φ*-space* if Y is a fixed locally convex space, $\mathcal{F}$ is the class of all K(Y)-spaces and $M_{F,\mathfrak{P}} := \{\varphi(Y')\}$.
- L_β*-space* if Y is a fixed locally convex space, $\mathcal{F}$ is the class of all K(Y)-spaces and $M_{F,\mathfrak{P}} := \{F^\beta\}$.

Based on the idea of L_r-spaces, the notions of L_φ-spaces and L_β-spaces were introduced by the authors in [4] (see [2] as well). In the case of any K(Y)-space we have (see [4])

$$F \text{ is an } L_r\text{-space} \implies F \text{ is an } L_\beta\text{-space} \iff F \text{ is an } L_\varphi\text{-space}$$

where the converse of the first implication is not true in general.

In the case of L_r-spaces we get Qiu's closed graph theorem as a corollary of Theorem 2.4 .

Theorem 3.1 (Qiu [11, Theorems 1 and 3] and [12, Theorem 2], see also [4]) . *A locally convex space F is an L_r-space if and only if any linear map $T : E \longrightarrow F$ with closed graph is continuous where E is an arbitrary Mackey space with sequentially complete dual $(E', \sigma(E', E))$.*

Additional remark: Each separable Fréchet space is an L_r-space, thus Kalton's closed graph theorem cited in 2° is included in the implication '⇒'.

Proof. Apply 2.4 to the situation in the definition of L_r-spaces and note $\mathfrak{P} = \mathfrak{P}_r$ ∎

In the case of L_φ-spaces we get as a corollary of Theorem 2.4 a closed graph theorem for matrix maps; thereby, the equivalence '(a)⇔(c)' is due to the authors and K.-G. Große-Erdmann [2, Theorem 5.1]:

Theorem 3.2 . *If Y is an arbitrary locally convex space and F is a $K(Y)$-space then the following statements are equivalent:*

(a) *F is an L_φ-space.*

(b) *If E is a Mackey space and $(E', \sigma(E', E))$ is sequentially complete then each linear map $T : E \longrightarrow F$ with $\varphi(Y') \subset \Delta_T$ is continuous.*

(c) *If X is any locally convex space, E is a Mackey $K(X)$-space and $(E', \sigma(E', E))$ is sequentially complete then every (weak operator-valued) matrix map $A : E \longrightarrow F$ is continuous.*

Proof. Noting that $\varphi(Y')$ is a total subset of F', thus $\sigma(F', F)$-dense in F', we get '(a)⇔(b)' as a corollary of Theorem 2.2. The second part of the proof of 2.2 shows that '(a)⇔(b)' remains true if we consider especially K(X)-spaces E instead of general locally convex spaces. In this case the linear maps $T : E \longrightarrow F$ with $\varphi(Y') \subset \Delta_T$ are exactly the (weak) operator-valued matrix maps ∎

3.2 $A_{\mathcal{F},\mathfrak{P}}$–spaces, Pták's closed graph theorem

Let $\Theta =: \Theta_{bn}$ be the set of all bounded nets. In this case we denote $(\Theta, \mathcal{F}, \mathfrak{P})$-spaces as *$A_{\mathcal{F},\mathfrak{P}}$-spaces* and for special $\mathcal{F}$ and $\mathfrak{P}$ introduce instead of $A_{\mathcal{F},\mathfrak{P}}$-space the following notions:

- *A_φ-space* if Y is a fixed locally convex space and $\mathcal{F}$ is the class of all K(Y)-spaces and $M_{F,\mathfrak{P}} := \{\varphi(Y')\}$.
- *A_β-space* if Y is a fixed locally convex space and $\mathcal{F}$ is the class of all K(Y)-spaces and $M_{F,\mathfrak{P}} := \{F^\beta\}$.

Remarks 3.3 . (a) If $\mathcal{F}$ and $\mathfrak{P}$ are fixed, then each $L_{\mathcal{F},\mathfrak{P}}$-space is also an $A_{\mathcal{F},\mathfrak{P}}$-space; the converse implication does not hold in general since the BK-space m is an A_φ-space (see Remark (c) below) and not an L_φ-space (see [4, Example 3.13 (a)]). Moreover, in Example 4.6 we will give a big class of such examples.

(b) Each barrelled subWCG-$A_{\mathcal{F},\mathfrak{P}}$-space is an $L_{\mathcal{F},\mathfrak{P}}$-space (see [4, Theorem 3.3] and [7]).

(c) Each FK(X)-space is an A_φ-space, each separable FK(X)-space is an L_φ-space.

The last statement is [4, Remark 3.6(c)] and the first one follows from the fact that FK(X)-spaces are barrelled, thus B_r-complete, therefore A-complete. Recall that a locally convex space F is called *B_r-complete* (Ptàk [10]) and *A-complete*[1] (Zhu and Zhao [20]) if each $\sigma(F', F)$-dense subspace S of F' which is aw^*-closed and bw^*-closed, respectively, is closed in $(F', \sigma(F', F))$; by definition, a subspace S of F' is bw^*-closed if $S \cap U^\circ$ is $\sigma(F', F)$-closed for any τ_F-barrel U in F; it is aw^*-closed if $S \cap U^\circ$ is $\sigma(F', F)$-closed for any $U \in B$, whereby B is any neighbourhood base of zero in (F, τ_F). Consequently, F is A-complete if and only if there does not exist a strict $\sigma(F', F)$-dense $bw*$-closed subspace of F'.

The notions bw^*- and Θw*-closedness are connected as follows:

Proposition 3.4 (see [20] and [17, Example 12.2.2]). *Let $\Theta =: \Theta_{bn}$, (F, τ_F) be a locally convex space and let S be a subspace of F'. Then S is bw^*-closed if and only if S is Θw^*-closed.*

Proof. $\Leftarrow$: Let (g_α) be a $\sigma(F^*, F)$-bounded net in S with $g_\alpha \longrightarrow g$ in $(F^*, \sigma(F^*, F))$. Then there exists a barrel U in (F, τ_F) such that $g_\alpha \in U^\circ$ for each α. Since $S \cap U^\circ$ is $\sigma(F^*, F)$-closed we get $g \in S \cap U^\circ \subset S$.

[1] The definition may be confusing since **A**-complete spaces are defined on the basis of bw*-closedness and $\boldsymbol{B}_r$-complete spaces on the basis of aw*-closedness. However, these notions are historically justified.

$\Rightarrow$: Let U be a τ_F-barrel and (g_α) be any net in $S \cap U^\circ$ with $g_\alpha \longrightarrow g(\sigma(F^*, F))$. Then $g \in U^\circ$ and (g_α) is $\sigma(F^*, F)$-bounded. Thus $g \in \overline{S}^\Theta = S$ because S is bw*-closed ∎

As a consequence we get A-complete spaces as a special case of $(\Theta, \mathcal{F}, \mathfrak{P})$-spaces, thus we can apply Theorem 2.4 to them. To that we characterize barrelled spaces in terms of Θ-completeness and deduce Ptàk's closed graph theorem from Theorem 2.4.

Proposition 3.5 . *A locally convex space F is A-complete if and only if it is a $(\Theta, \mathcal{F}, \mathfrak{P})$-space where $\Theta = \Theta_{bn}$, $\mathcal{F}$ is the set of all locally convex spaces and $\mathfrak{P} = \mathfrak{P}_r$.*

Proposition 3.6 . *In the case of a dual pair (F, S) the following statements are equivalent:*

(a) (F, S) *is barrelled,* that is, $(F, \tau(F, S))$ is a barrelled space.

(b) $(S, \sigma(S, F))$ *is boundedly complete,* that is, any $\sigma(S, F)$-bounded closed subset R of S is $\sigma(S, F)|_R$-complete.

(c) *Each bounded Cauchy net in $(S, \sigma(S, F))$ converges.*

(d) *S is bw*-closed, that is Θw^*-closed were $\Theta = \Theta_{bn}$.*

Proof. The equivalence '(a)$\Leftrightarrow$(b)' is known (see [17, Theorem 9.3.13]). The proofs of the other implications are straightforward ∎

Theorem 3.7 (Zhu and Zhao [20], see also Qiu [12, Theorem 6]) . *A locally convex space F is A-complete if and only if for any barrelled space E each linear map $T : E \longrightarrow F$ with closed graph is continuous.*

Additional remark: Each B_r-complete space is A-complete, thus the implication '$\Rightarrow$' contains Ptàk's closed graph theorem cited in 1°.

Drawing a parallel between the case of A_φ- and A_β-spaces to that of L_φ- and L_β-spaces we get

Proposition 3.8 . *Let (E, τ_E) be a $K(X)$-space. Then (E, τ_E) is an A_β-space if and only if it is an A_φ-space.*

Proof. The proof is quite similar to that of the corresponding result in the case of L_φ- and L_β-spaces (see [4, Remark 3.6(b)] and note $\varphi(X') \subset E'$ and $E^\beta \subset \overline{\varphi(X')} \subset \overline{\varphi(X')}^\Theta$ where $\Theta = \Theta_{bn}$.) ∎

Theorem 3.9 . *If Y is an arbitrary locally convex space and F is a $K(Y)$-space then the following statements are equivalent:*

(a) *F is an A_φ-space.*

(b) *If E is a barrelled space then each linear map $T : E \longrightarrow F$ with $\varphi(Y') \subset \Delta_T$ is continuous.*

(c) *If X is any locally convex space and E is a barrelled $K(X)$-space then every (operator-valued) matrix map $A : E \longrightarrow F$ is continuous.*

As the following example shows, Theorem 3.9 is not a corollary of Theorem 3.7.

Example 3.10 (see [3] for the product of L_φ–spaces). Let $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of infinite dimensional BK–spaces with card $\mathcal{A} \geq 2^{\aleph_0}$. The topological product $F := \prod_{\alpha\in\mathcal{A}} F_\alpha$ is a barrelled space which is not B_r–complete (see [15, Theorem 6]). Therefore, since the notions of B_r– and A–completeness coincide in the case of barrelled spaces, F *is not A-complete*. Now we are going to prove that F *is an A_φ-space*. For that purpose we consider F as a sequence space over $X := \prod_{\alpha\in\mathcal{A}} \mathbf{K}$ endowed with the product topology. Thus F is a K(X)–space and

$$\varphi(X') = \varphi\left(\bigoplus_{\alpha\in\mathcal{A}} \mathbf{K}\right) = \bigoplus_{\alpha\in\mathcal{A}} \varphi \subset \bigoplus_{\alpha\in\mathcal{A}} F_\alpha{}' = F' .$$

Now we assume M to be a bw*–closed subspace of F' containing $\varphi(X')$. Thus, if pr_α denotes the projection from F' to $F_\alpha{}'$, we have

$$\varphi = \mathrm{pr}_\alpha \varphi(X') \subset \mathrm{pr}_\alpha M =: M_\alpha \subset \mathrm{pr}_\alpha F' = F_\alpha{}' .$$

Therefore, using the continuity of the injections

$$g_\alpha : (F_\alpha{}', \sigma(F_\alpha{}', F_\alpha)) \longrightarrow (F', \sigma(F', F)),\ h \longrightarrow (f_\beta) \quad \text{with } f_\alpha = h \text{ and } f_\beta = 0 \text{ if } \alpha \neq \beta ,$$

we get the bw*–closedness of M_α, hence $M_\alpha = F_\alpha{}'$ for each $\alpha \in \mathcal{A}$ which implies

$$M = \bigoplus_{\alpha\in\mathcal{A}} M_\alpha = \bigoplus_{\alpha\in\mathcal{A}} F_\alpha{}' = F' .$$

Consequently, F is an A_φ–space.

Remark. Analogous to the case of L_φ–spaces in [3] we may prove that the product of A_φ–spaces is an A_φ–space.

3.3 ω_r– and α_r–complete spaces, Marquina's closed graph theorems

By definition, let $(g_\gamma) \in \Theta =: \Theta_{bt}$ if and only if there exists a bounded countable subset N in H with[2] $g_\gamma \in \operatorname{clabsconv} N$ for all γ.

Proposition 3.11. *If $\Theta = \Theta_{bt}$ and E is any Mackey space then $(E', \sigma(E', E))$ is Θ-complete if and only if $E = (E, \tau_E)$ is ω-barrelled,* that is, each $\sigma(E', E)$–bounded countable set in E' is τ_E–equicontinuous.

A locally convex space F is called *ω_r-complete* if it is a $(\Theta, \mathcal{F}, \mathfrak{P})$–space where $\mathcal{F}$ is the class of all locally convex spaces and $\mathfrak{P} = \mathfrak{P}_r$.

From 3.11 and Theorem 2.4 we get the following closed graph theorem:

Theorem 3.12. *A locally convex space F is ω_r-complete if and only if for any ω-barrelled Mackey space E each linear map $T : E \longrightarrow F$ with closed graph is continuous.*

Remark 3.13. Marquina (see [9, Corollary 1.1]) proved the sufficient part of Theorem 3.12 under the assumption that F is a weakly compactly generated Banach space. Thus, with '$\Leftarrow$' we may conclude that each weakly compactly generated Banach space is ω_r–complete; conversely, if we know that a weakly compactly generated Banach space is ω_r–complete then Marquina's closed graph theorem [9, Corollary 1.1] is a corollary of '$\Rightarrow$'.

[2] clabsconv := closure of the absolutely convex hull

Aiming to get a more general closed graph theorem as Theorem 3.12 which contains a further closed graph theorem due to Marquina (see [9, Theorem 1]) we define: $(g_\gamma) \in \Theta = \Theta_{b\alpha}$ if and only if there exists a bounded subset B, $\operatorname{card} B \leq \alpha$, with $g_\gamma \in \operatorname{clabsconv} B$ for all γ.
Then:

For any Mackey space E the weak dual $(E', \sigma(E', E))$ is Θ-complete if and only if E is α-barrelled.

That is, each $\sigma(E', E)$-bounded set B in E', $\operatorname{card} B \leq \alpha$, is $\tau(E, E')$-equicontinuous where α is an infinite cardinal number. (If we consider countable sets B in the definition of α-barrelledness then we get exactly ω-barrelledness.) A locally convex space F is called *α_r-complete* if it is a $(\Theta, \mathcal{F}, \mathfrak{P})$-space where $\mathcal{F}$ is the class of all locally convex spaces and $\mathfrak{P} = \mathfrak{P}_r$.

Analogous to Theorem 3.12 we get the following closed graph theorem and the corresponding remark.

Theorem 3.14 . *A locally convex space F is α_r-complete if and only if for each α-barrelled Mackey space E any linear map $T : E \longrightarrow F$ with closed graph is continuous.*

Remark 3.15 . Marquina (see [9, Theorem 1]) proved the sufficient part of Theorem 3.14 under the assumption that F is an α-WCG-space. Thus, with '$\Leftarrow$' we may conclude that each α-WCG-space is α_r-complete; conversely, if we know that each weakly compactly generated Banach space is α_r-complete then Marquina's closed graph theorem [9, Theorem 1] is a corollary of '$\Rightarrow$'.

3.4 Sequentially barrelled Mackey spaces as domain space

By definition, let the sequence be in $(g_n) \in \Theta =: \Theta_{c0}$ if and only if there exists a sequence (h_k) converging to zero with $g_n \in \operatorname{clabsconv}\{h_k \mid k \in \mathbf{N}\}$ for all $n \in \mathbf{N}$. The following Proposition gives us a useful characterization of sequential completeness of Mackey spaces by the Θ-completeness of the weak dual.

Proposition 3.16 . *Let $\Theta = \Theta_{c0}$ and E be a Mackey space. Then E is sequentially barrelled*[3]*, if and only if $(E', \sigma(E', E))$ is Θ-complete.*

Proof. Let E be sequentially barrelled and $(g_n) \in \Theta = \Theta[(E', \sigma(E', E))]$ be a $\sigma(E', E)$-Cauchy sequence in E'. Then there exists $g \in E^*$ with $g_n \longrightarrow g$ in $(E^*, \sigma(E^*, E))$. According to the definition of Θ we may choose a sequence (h_k) converging to zero in $(E', \sigma(E', E))$ with $g_n \in B := \operatorname{clabsconv}\{h_k \mid k \in \mathbf{N}\}$. Since B is τ_E-equicontinuous there exists a τ_E-neighbourhood U of zero with $B \subset U^0$. Therefore, for each $\varepsilon > 0$ we get $|g_n(x)| \leq \varepsilon$ $(x \in \varepsilon U,\ n \in \mathbf{N})$ and consequently $|g(x)| \leq \varepsilon$ $(x \in \varepsilon U)$. The last inequality implies $g \in E'$, thus the Θ-completeness of $(E', \sigma(E', E))$ is proved.

Conversely, let $(E', \sigma(E', E))$ be Θ-complete and (h_k) be a sequence converging to zero in $(E', \sigma(E', E))$ and let $B := \operatorname{clabsconv}\left\{h_k \mid k \in \mathbf{N}\right\}$. Then $(B, \sigma(E', E)|_B)$ is metrizable (see [8, Theorem 1.4, Corollary]). Obviously, B is $\sigma(E', E)$-precompact and, since $(E', \sigma(E', E))$ is Θ-complete, B is $\sigma(E', E)|_B$-complete. This implies the $\sigma(E', E)|_B$-compactness and therefore the τ_E-equicontinuity of B. In particular, (h_k) is τ_E-equicontinuous ∎

Applying 3.16 and 2.4 to the situation in question ($\Theta = \Theta_{c0}$ and $\mathfrak{P} = \mathfrak{P}_r$) we get the following 'restricted' closed graph theorem:

[3] E is sequentially barrelled (see [16]) if each null sequence in $(E', \sigma(E', E))$ is $\tau(E', E)$-equicontinuous.

Theorem 3.17. *A locally convex space F is a Θ_r-space if and only if for all sequentially barrelled Mackey spaces E any linear map $T : E \longrightarrow F$ with closed graph is continuous.*

3.5 H–spaces

Generalizing K(X)–spaces we may consider H–spaces: Let (H, τ_H) be a locally convex space. Then a locally convex space (F, τ_F) is called *H-space* if $F \subset H$ and $\tau_F \supset \tau_H|_F$ (cf. [18, p. 55]). In particular, each K(X)–space is an H–space with $H := \omega(X)$ endowed with the topology τ_H of coordinatewise convergence. Obviously, $H'|_F$ is total and *the inclusion $\Delta_T \supset H'$ holds if and only if $T : (E, \sigma(E, E')) \longrightarrow (F, \sigma(F, H'))$ is continuous.* As an immediate consequence of Theorem 2.2 we get

Theorem 3.18. *Let Θ be determined in accordance with (2.1), $\mathcal{F}_H$ be the set of all H-spaces for a fixed locally convex space H and $\mathcal{M}_{F,\mathfrak{P}} := \{H'|_F\}$. Then an H-space $F = (F, \tau_F)$ is a $(\Theta, \mathcal{F}_H, \mathfrak{P})$-space if and only if for any Mackey space E with Θ-complete dual $(E', \sigma(E', E))$ each $\tau(E, E')$-τ_H-continuous linear map $T : E \longrightarrow F$ is $\tau(E, E')$-τ_F-continuous.*

Analogous to the considerations in Theorems 3.2 and 3.9 the following special results hold:

(a) *An H-space F is an $L_{\mathcal{F}_H,\mathfrak{P}}$-space if and only if for any Mackey space E with sequentially $\sigma(E', E)$-complete dual each $\tau(E, E')$-τ_H-continuous linear map $T : E \longrightarrow F$ is $\tau(E, E')$-τ_F-continuous.*

(b) *An H-space F is an $A_{\mathcal{F}_H,\mathfrak{P}}$-space if and only if for any barrelled space E each $\tau(E, E')$-τ_H-continuous linear map $T : E \longrightarrow F$ is $\tau(E, E')$-τ_F-continuous.*

4. A_φ– and L_φ– spaces, M–topologies

Ruckle [13], [14] calls a locally convex topology τ_F on a sequence space F (over $\mathbb{K}$) *φ-topology* if τ_F coincides with a polar topology τ_S where $\mathcal{S}$ is a family of $\sigma(\varphi, F)$–bounded subsets of φ and φ is embedded in F^* in the usual way.

Definition 4.1. Let F be a linear space and M be a total subspace of F^*. A locally convex topology τ on F is called *M-topology* if τ coincides with a polar topology τ_S where $\mathcal{S}$ is a family of $\sigma(M, F)$–bounded subsets of $M \cap F'$. If E is any sequence space over a locally convex space X then for convenience we use the notation *φ-topology* and *β-topology* in the case of a $\varphi(X')$–topology and E^β–topology, respectively.

By definition we obtain that a locally convex topology τ on a linear space F is an M–topology if there exists a family $\mathcal{S}$ of $\sigma(M, F)$–bounded subsets of $M \cap F'$ such that $\{S^\circ \mid S \in \mathcal{S}\}$ is a τ–neighbourhood base of zero. If $\tau = \tau_S$ is an M–topology then, without loss of generality, we may assume that $\mathcal{S}$ is saturated (see [17, Definition 8.5.9]) and $S \in \mathcal{S}$ implies $S = S^{\circ\circ} \cap M$.

Proposition 4.2. *Let (F, τ_F) be a locally convex space and M be a subspace of F^* such that $M \cap F'$ is total.*

(a) *The following statements are equivalent:*

 (i) *τ_F is an M-topology.*

 (ii) *τ_F is generated by a family of seminorms p_γ $(\gamma \in \Gamma)$ having the property*

$$\forall \gamma \in \Gamma \quad \exists S_\gamma \subset M \cap F' \quad \forall x \in F : p_\gamma(x) = \sup_{f \in S_\gamma} |f(x)|. \tag{4.1}$$

(iii) *There exists a τ_F-neighbourhood base $\mathcal{B}$ of zero such that*

$$\forall\, U \in \mathcal{B} : \; M \cap U^\circ \;\text{ is } \sigma(F', F)\text{-dense in } U^\circ . \tag{4.2}$$

(b) *If τ_F is an M-topology and $\Theta = \Theta_{bn}$ then $F' \subset \overline{M \cap F'}^{\Theta}$; therefore (F, τ_F) is an $\mathcal{A}_{\mathcal{F},\mathfrak{P}}$-space where $\mathcal{F} = \{F\}$ and $M_{F,\mathfrak{P}} = \{M\}$.*

Proof. (a) The stated equivalence is an immediate consequence of Definition 4.1 and the subsequent remark.
(b) Let τ_F be an M-topology on F and $\mathcal{B}$ be a neighbourhood base of zero satisfying (4.2) and let $f \in F'$ be given. Then there exists $U \in \mathcal{B}$ with $f \in U^\circ$. Thus we may choose a net $(f_\alpha)_{\alpha \in \mathcal{A}}$ in $M \cap U^\circ$ with $f_\alpha \longrightarrow f \;(\sigma(F', F))$. Since (f_α) is τ_F-equicontinuous, thus $\sigma(F^*, F)$-bounded, we get $F' \subset \overline{M \cap F'}^{\Theta}$ ∎

Remarks 4.3 . Let (F, τ_F) and M be chosen as in 4.2.
(a) If F' is the smallest aw*-closed subspace of F' which contains $M \cap F'$ then there exists an M-topology τ on F with $(F, \tau)' = F' := (F, \tau_F)'$.
(b) If (F, τ_F) is separable and τ_F is an M-topology, then $\overline{M \cap F'} = F'$. (Recall that a K-space (F, τ_F) containing φ has *φ-sequentially dense dual if $\overline{\varphi} = F'$*, see [4, Definition 3.5].) [A proof of this statement may be based on [17, Theorem 9.5.3].]

For a given subspace M of F', in general, F-spaces do not carry an M-topology in contradiction to the statement in [14, p. 69] in the case of FK-spaces and $M := \varphi$ as the following example proves.

Example 4.4 . P. Erdös and G. Piranian [5, Theorem 1] gave an example of a regular real-valued (row infinite) matrix A such that there exists no row finite regular matrix B with $c_A \subset c_B$. Thus, as stated in [4, Example 3.12], the FK-space c_A does not have φ-sequentially dense dual. Consequently, since c_A is separable, on account of 4.3(b) the FK-topology of c_A cannot be a φ-topology on c_A.

Proposition 4.5 . *Let (E, τ_E) be a $K(X)$-space. If τ_E is a β-topology (φ-topology) then (E, τ_E) is an A_φ-space.*

Proof. The statement is an immediate corollary of Proposition 4.2(b) and 3.8 ∎

Example 4.6 . Let X be a locally convex space, let $\mathcal{P}$ be a family of seminorms generating the topology of X and let $B_{\alpha k} \in X'$ $(\alpha \in \mathcal{A},\, k \in \mathbf{N})$, where $\mathcal{A}$ is a directed set. In the case of the (generalized) matrix $B := (B_{\alpha k})_{\alpha \in \mathcal{A}, k \in \mathbf{N}}$ we introduce analogously to the case $X := \mathbf{K}$ the following notations:

$$\omega_B := \left\{ x \in \omega(X) \;\middle|\; \sum_k B_{\alpha k}(x_k) \text{ converges for each } \alpha \in \mathcal{A} \right\},$$

$$m_B := \left\{ x \in \omega_B \;\middle|\; \|x\|_B := \sup_\alpha \left| \sum_k B_{\alpha k}(x_k) \right| < \infty \right\},$$

$$c_B := \left\{ x \in \omega_B \;\middle|\; \lim_B x := \lim_\alpha \sum_k B_{\alpha k}(x_k) \text{ exists} \right\}.$$

The family of the seminorms $\| \|_B$, s_α $(\alpha \in \mathcal{A})$ and r_{pk} $(p \in \mathcal{P},\ k \in \mathbf{N})$ with

$$s_\alpha(x) := \sup_r \left| \sum_{k=1}^{r} B_{\alpha k}(x_k) \right| \quad \text{and} \quad r_{pk}(x) := p(x_k)$$

generates a locally convex topology τ_B on m_B and its members are seminorms with the property (4.1) in the case of $M := m_B^\beta$. By Proposition 4.2(a) τ_B is a β-topology, *therefore* (m_B, τ_B) *is an* A_φ*-space* (see Proposition 4.5(b)). However, (m_B, τ_B) is not an L_φ-space in general (see [4, Example 3.13(a)]).

Now, in the situation of Example 4.6, we assume that B has bounded and convergent (generalized) columns $(B_{\alpha k})_\alpha$ $(k \in \mathbf{N})$, that is, for each $k \in \mathbf{N}$ and $a \in X$ the net $(B_{\alpha k}(a))_\alpha$ is bounded and $\lim_B e^k(a) = \lim_\alpha B_{\alpha k}(a) =: B_k(a)$ exists. Then $\varphi(X) \subset m_B \cap c_B$. Analogously to the classical case $X := \mathbf{K}$ and $\mathcal{A} := \mathbf{N}$ we define

$$\Lambda_B^\perp := \left\{ x \in m_B \cap c_B \;\middle|\; \sum_k B_k(x_k) \text{ exists and } \lim_B x = \sum_k B_k(x_k) \right\}.$$

Obviously, $\Lambda_B^\perp$ includes the space $W_B := W_{m_B \cap c_B}$ since $\lim_B |_{m_B \cap c_B} \in (m_B \cap c_B)'$ where $m_B \cap c_B$ carries the topology $\tau_B|_{m_B \cap c_B}$.

Parallel to [4, Theorem 3.7] we state:

Proposition 4.7 (see [4, Theorem 3.7]). *If E is a $K(X)$-space with $E^\beta \subset E'$ then we consider the following statements:*

(a) *There exists a β-topology τ with $(E, \tau)' = E'$.*

(b) $\overline{E^\beta \cap E'}^{\Theta} \supset E'$ *where* $\Theta = \Theta_{bn}$.

(c) *For each $f \in E'$ there exists a (generalized) matrix $B = (B_{\alpha k})$ with $B_{\alpha k} \in X'$ $(\alpha \in \mathcal{A},\ k \in \mathbf{N})$ such that $E \subset m_B \cap c_B$ and $f = \lim_B |_E$ holds.*

Then we have (a)$\Rightarrow$(b)$\Leftrightarrow$(c) and, if $E^\beta \not\subset E'$, (a)$\Rightarrow$(b)$\Rightarrow$(c).

Proof. The implication (a)$\Rightarrow$(b) follows (also in the case $E^\beta \not\subset E'$) from Proposition 4.2(b) and the proofs of the remaining statements are straightforward ∎

We supply the above results on L_φ- and A_φ-spaces with some statements on the domain of operator-valued matrices. To that end let X and Y together with a family $\mathcal{P}$ and $\mathcal{Q}$ of seminorms, respectively, be locally convex spaces. For each matrix $A = (A_{nk})$ with $A_{nk} \in B(X, Y)$ $(n, k \in \mathbf{N})$ the sequence space

$$\omega(Y)_A := \left\{ x \in \omega(X) \;\middle|\; \sum_k A_{nk}(x_k) \text{ converges for each } n \in \mathbf{N} \right\},$$

is an K(X)-AK-space with the topology generated by the family of seminorms r_{pk} $(p \in \mathcal{P}, k \in \mathbf{N})$ and s_{qn} $(q \in \mathcal{Q}, n \in \mathbf{N})$ where $r_{pk}(x) := p(x_k)$ $(p \in \mathcal{P},\ k \in \mathbf{N})$ and

$$s_{qn}(x) := \sup_r q\left(\sum_{k=1}^{r} A_{nk}(x_k) \right) \qquad (q \in \mathcal{Q}, n \in \mathbf{N}).$$

Now, if (E, τ_E) is a K(Y)-space and τ_E is generated by a family of seminorms Q_γ $(\gamma \in \Gamma)$ then the *domain of* A *with respect to* E defined by

$$E_A := \left\{ x \in \omega(Y)_A \;\middle|\; Ax := \left(\sum_k A_{nk}(x_k)\right)_n \in E \right\}$$

together with the family of seminorms r_{pk} $(p \in \mathcal{P}, k \in \mathbf{N})$ and s_{qn} $(q \in \mathcal{Q}, n \in \mathbf{N})$ and $Q_{A,\gamma}$ $(\gamma \in \Gamma)$ is a K(X)-space with topology τ_{E_A} where $Q_{A,\gamma}(x) := Q_\gamma(Ax)$ $(x \in E_A\,,\ \gamma \in \Gamma)$.

Theorem 4.8 . *Let $A = (A_{nk})$ be a matrix with $A_{nk} \in B(X,Y)$. If E is any A_φ-$K(Y)$-space (L_φ-$K(Y)$-space) then E_A is an A_φ-$K(X)$-space (L_φ-$K(X)$-space).*

Proof. In the case of an L_φ-space E we refer to [2, Theorem 5.5] and remark that this theorem remains true if we consider more general locally convex spaces X and Y since $\widetilde{\omega(Y)_A}$ is an L_φ-space (thus an A_φ-space). Furthermore, noting $\widetilde{\varphi(X')} = \widetilde{E_A^\beta}$, we may replace $\widetilde{E_A^\beta}$ by $\widetilde{\varphi(X')}$ in the proof of [2, Theorem 5.5]. Now, it is easy to adapt that proof to the case of an A_φ-space E ∎

5. An inclusion theorem for K–spaces

By means of Theorems 3.2 and 3.9 we may prove for a given vector-valued sequence space E two inclusion theorems which extend essentially in the case of $X := \mathbb{K}$ inclusion theorems due to Bennett and Kalton (see [1, Theorem 5 and 4]). The first one, which is based on Theorem 3.2'(a)⇒(c)' and the consideration of L_φ-spaces, contains a characterization of the weak sequential completeness of the β-dual E^β and is already done by the authors in [4, Theorem 4.4]; for an extension of that result to so-called T-matrix and weak matrix maps see [19]. The second one is based on Theorem 3.9'(a)⇒(c)' and the consideration of A_φ-spaces and gives us a characterization of the barrelledness of $(E, \tau(E, E^\beta))$. Both shed a new light on the Bennett-Kalton theorems and give -in the situation of the theorems- a connection between *weak sequential completeness* and *barrelledness* on one hand and *Summability* on the other hand.

Theorem 5.1 . *Let X be a locally convex space. For any sequence space E over X containing $\varphi(X)$ the following statements are equivalent:*

(a) *$(E, \tau(E, E^\beta))$ is barrelled.*

(b) *Any matrix map $B : (E, \tau(E, E^\beta)) \longrightarrow (F, \tau_F)$ is continuous whenever F is an A_φ-$K(Y)$-space and Y is a locally convex space.*

(c) *The implication $E \subset F \implies E \subset W_F$ holds whenever F is an A_φ-$K(X)$-space.*

(d) *The implication $E \subset F \implies E \subset S_F$ holds whenever F is an A_φ-$K(X)$-space.*

(e) *The implication $E \subset m_B \implies E \subset W_B := W_{m_B \cap c_B}$ holds for any matrix $B = (B_{\alpha k})_{\alpha \in \mathcal{A}, k \in \mathbf{N}}$ with $B_{\alpha k} \in X'$, a directed set $\mathcal{A}$ and $\varphi(X) \subset c_B$.*

(f) *The implication $E \subset m_B \implies E \subset S_B := S_{m_B \cap c_B}$ holds for any matrix $B = (B_{\alpha k})_{\alpha \in \mathcal{A}, k \in \mathbf{N}}$ with $B_{\alpha k} \in X'$, a directed set $\mathcal{A}$ and $\varphi(X) \subset c_B$.*

(g) *The implication $E \subset m_B \implies E \subset \Lambda_B^\perp$ holds for any matrix $B = (B_{\alpha k})_{\alpha \in \mathcal{A}, k \in \mathbf{N}}$ with $B_{\alpha k} \in X'$, a directed set $\mathcal{A}$ and $\varphi(X) \subset c_B$.*

(h) *In* $(E^\beta, \sigma(E^\beta, E))$ *each bounded Cauchy net is convergent.*

The case $X := \mathbf{K}$: *Principally we may consider matrices in the usual sense, that is $\mathcal{A} := \mathbf{N}$ with the natural order. Furthermore, the theorem remains true if we replace 'A_φ-K-space' by 'FK-space' and if we extend the theorem by the following statements:*

(b′) *For each matrix $B = (b_{nk})_{n,k\in\mathbf{N}}$ with $b_{nk} \in \mathbf{K}$ and $E \subset m_B$ the matrix maps*

$$B : (E, \tau(E, E^\beta)) \longrightarrow m\,,\; x \longrightarrow Bx \quad \text{and} \quad j : (E, \tau(E, E^\beta)) \longrightarrow \omega_B\,,\; x \longrightarrow x$$

are continuous where m and ω_B carry the FK-topology.

(h′) *Each $\sigma(E^\beta, E)$-bounded subset of E^β is relatively sequentially $\sigma(E^\beta, E)$-compact.*

Proof. '(a)⇒(b)' is an immediate consequence of Theorem 3.9'(a)⇒(c)'.

(b)⇒(c): If F is an A_φ-space containing E then the inclusion map $i : (E, \tau(E, E^\beta)) \longrightarrow (F, \tau_F)$ as a matrix map is continuous, thus weakly continuous. Since $(E, \sigma(E, E^\beta))$ is an AK-space each $x = i(x) \in E \subset F$ has the AK-property in $(F, \sigma(F, F'))$, that is $E \subset W_F$.

The implications '(c)⇒(e)' and '(d)⇒(f)' are valid, since m_B is an A_φ-space (see Example 4.6). On account of $S_B \subset W_B \subset \Lambda_B^\perp$ we get '(e)⇒(g)' and '(f)⇒(g)'.

(g)⇒(h): Let $(B^{(\alpha)})_{\alpha\in\mathcal{A}}$ be a bounded Cauchy net in $(E^\beta, \sigma(E^\beta, E))$, then $\sup_\alpha \left|\sum_k B_k^{(\alpha)}(x_k)\right| < \infty$ and the limit $\lim_\alpha \sum_k B_k^{(\alpha)}(x_k)$ exists for each $x \in E$, that is $E \subset m_B \cap c_B$ if we consider the (generalized) matrix $B = (B_k^{(\alpha)})_{\alpha,k}$. On account of (g) we get $E \subset \Lambda_B^\perp$, thus $\Phi := \left(\lim_\alpha B_k^{(\alpha)}\right)_k \in E^\beta$ and $B^{(\alpha)} \longrightarrow \Phi\, (\sigma(E^\beta, E))$.

'(h)⇒(a)' is an immediate corollary of Proposition 3.6.

(c)⇒(d): Let (c) be true and let F be an A_φ-space with $F \supset E$. Since we have already proved the equivalence '(c)⇔(a)' we may assume that $(E, \tau(E, E^\beta))$ is barrelled. Since $E = W_E$ and a barrelled SAK-space is an AK-space we get $E = S_E$. On account of '(c)⇔(b)' the inclusion map $i : (E, \tau(E, E^\beta)) \longrightarrow (F, \tau_F)$ is continuous and thus $E = S_E \subset S_F$.

The case $X := \mathbf{K}$: We give a proof of $(g) \Rightarrow (h')$ and $(h') \Rightarrow (a)$, the remarks on replacing the 'class of A_φ-K-spaces' by the 'class of FK-spaces' are obviously true.

$(g) \Rightarrow (h')$: Let D be a $\sigma(E^\beta, E)$-bounded subset of E^β and let $(a^{(r)})$ be a sequence in D. Obviously $(a^{(r)})$ is bounded in (ω, τ_ω), thus we may choose a coordinatewise convergent subsequence $(b^{(n)})$ of $(a^{(r)})$. We put $b_k := \lim_n b_k^{(n)}$ $(k \in \mathbf{N})$. Because of the $\sigma(E^\beta, E)$-boundedness of $\left\{b^{(n)} \mid n \in \mathbf{N}\right\}$ we get $E \subset m_B$, where $B := (b_k^{(n)})_{n,k}$. Using (g) we get $E \subset \Lambda_B^\perp$, thus

$$b := (b_k) \in E^\beta \quad \text{and} \quad b^{(n)} \longrightarrow b\, (\sigma(E^\beta, E)).$$

Altogether we proved that D is relatively sequentially compact in $(E^\beta, \sigma(E^\beta, E))$.

$(h') \Rightarrow (a)$: Using the fact (see [6, Theorem 8]) that in the K-space $(E^\beta, \sigma(E^\beta, E))$ each relatively sequentially compact subset is also relatively compact, the implication $(h') \Rightarrow (a)$ is trivially true since (h') tells us $\tau(E, E^\beta) = \beta(E, E^\beta)$ ∎

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