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Differentiability Properties of the Autonomous Composition Operator in Sobolev Spaces

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Abstract. In this paper, we study the autonomous composition operator, which takes a pair of functions (f, g) into its composite function $f \circ g$. We assume that f and g belong to Sobolev spaces defined on open subsets of $\mathbb{R}^n$, and we concentrate on the case in which the space for g is a Banach algebra. We give a sufficient condition in order that the composition maps bounded sets to bounded sets, and we exploit the density of the polynomial functions in the space for f in order to prove that for suitable Sobolev exponents of the spaces for f and g, the composition is continuous and differentiable with continuity up to order r, with $r \geq 1$. Then we show the optimality of such conditions by means of theorems of 'inverse' type.

Keywords: *Superposition operators, nonlinear operators, Sobolev spaces*

AMS subject classification: Primary 47 H 30, secondary 46 E 35

1. Introduction

In this paper we study continuity and r-th order differentiability properties of the autonomous composition operator defined by

$$(f, g) \longmapsto f \circ g \qquad \left(f \in W^{m_1+r,p_1}(\Omega_1),\, g \in (W^{m,p}(\Omega))^n\right) \tag{1}$$

where Ω and Ω_1 are open subsets of $\mathbb{R}^n$, $g(\Omega) \subseteq \Omega_1$, and where $W^{m_1+r,p_1}(\Omega_1)$ and $W^{m,p}(\Omega)$ denote Sobolev spaces of exponents $m_1 + r$, p_1 and m, p, respectively. We note that, in general, the composition of an equivalence class of functions of $W^{m_1+r,p_1}(\Omega_1)$ with an element of $(W^{m,p}(\Omega))^n$ which maps Ω into Ω_1 does not make sense. Indeed, the representatives of the elements of $W^{m_1+r,p_1}(\Omega_1)$ are defined only up to a set of measure zero. Accordingly, we will be able to consider the composition in (1) only for g's such that the g-preimage of a set of measure zero has measure zero. Even by taking $r = 0$, and by composing $f \in W^{m_1,p_1}(\Omega_1)$ with a smooth g, we cannot expect that, in general, $f \circ g$ could be more regular than a function of $W^{m_1,p_1}(\Omega)$. Thus, as a range space, we choose $W^{m_1,p_1}(\Omega)$. Similarly, in order to have $f \circ g \in W^{m_1,p_1}(\Omega)$, we must require, in general, that g is at least as regular as f. Thus we choose $m_1 \leq m$ and $p_1 \leq p$. To ensure that $W^{m,p}(\Omega)$ is a Banach algebra, we assume $mp > n$. To ensure

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that $f \circ g \in W^{m_1,p_1}(\Omega)$ when $f \in W^{m_1,p_1}(\Omega_1)$ and $g \in (W^{m,p}(\Omega))^n$, and that the set of g's we consider is open in $(W^{m,p}(\Omega))^n$, we assume that $(m-1)p > n$.

The main finding of this paper is that the composition operator in (1) is of class C^r from an appropriate subset of $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$ to $W^{m_1,p_1}(\Omega)$. To prove such statement, we exploit the abstract results of Lanza [16], which are in the wake of those of Lanza [15] for Schauder spaces, and some estimates of the Sobolev norm of $f \circ g$ of the type of those contained in Lanza [13]. Then we show, with the only exception of the case $(m_1, p_1) = (0, 1)$, that such statement is optimal, in the sense that if f is a real-valued function defined on $\mathbb{R}^n$ and if $g \mapsto f \circ g$ were to be of class C^r from the set of g's for which we have considered (1) to $W^{m_1,p_1}(\Omega)$, then $f \in W^{m_1+r,p_1}_{\mathrm{loc}}(\mathbb{R}^n)$.

The composition operator normally arises in problems of nonlinear analysis, and has been studied by several authors. For extensive references, we refer to the monograph of Appell and Zabrejko [3] and to that of Runst and Sickel [26]. In the Sobolev space setting we mention, in particular, the papers of Marcus and Mizel [17 - 22], Adams [1], Szigeti [28, 29], Valent [31 - 33], Gol'dshtein and Reshetnyak [11], Drábek and Runst [10], Musina [23], Bourdaud and Meyer [7], Bourdaud [4, 5], Bourdaud and Kateb [6], and Sickel [27]. However, as far as considering the differentiability of the composition operator when both the functions f and g belong to a Sobolev space, the author is only aware of the paper of Brokate and Colonius [8], who have proved a first order differentiability statement for the composition operator from a suitable subset of $W^{1,\infty} \times W^{1,\infty}$ to L^p, with a finite p and with f and g depending on a single real variable.

2. Preliminaries and notation

We denote the norm on a (real) normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$ or, in case of no ambiguity, more simply by $\|\cdot\|$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We equip the product space $\mathcal{X} \times \mathcal{Y}$ with the norm $\|\cdot\|_{\mathcal{X}\times\mathcal{Y}} = \|\cdot\|_{\mathcal{X}} + \|\cdot\|_{\mathcal{Y}}$, while we use the Euclidean norm for $\mathbb{R}^n$. We say that $\mathcal{X}$ is imbedded into $\mathcal{Y}$ provided that there exists a continuous linear injective map of $\mathcal{X}$ into $\mathcal{Y}$. By $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ we denote the normed space of the continuous linear maps of $\mathcal{X}$ to $\mathcal{Y}$ equipped with the topology of uniform convergence on the unit sphere of $\mathcal{X}$. For any non-zero natural number s, $\mathcal{L}^{(s)}(\mathcal{X}, \mathcal{Y})$ denotes the normed space of continuous s-linear maps of $\mathcal{X}^s$ to $\mathcal{Y}$. For all standard definitions and theorems of Calculus in normed spaces, we refer the reader to Cartan [9].

Further, $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper, n is an element of $\mathbb{N} \setminus \{0\}$. Let $r \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{O}$ be an open subset of $\mathcal{X}$. Then $C^r(\mathcal{O}, \mathcal{Y})$ denotes the space of r-times continuously differentiable maps of $\mathcal{O}$ to $\mathcal{Y}$. Let f be a function. The f-preimage of a set D is denoted $f^{\leftarrow}(D)$. The inverse function of an invertible function f is denoted $f^{(-1)}$ as opposed to the reciprocal of a real-valued function g or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. For all $R > 0$ and $x \in \mathbb{R}^n$, $|x|$ denotes the Eucledian modulus of x in $\mathbb{R}^n$, and $B(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$ or the matrix product.

Let Ω be an open subset of $\mathbb{R}^n$, $\mathrm{diam}\,[\Omega]$ its diameter and $\mathrm{cl}\,\Omega$ its closure. The space of m-times continuously differentiable real-valued functions on Ω is denoted by

$C^m(\Omega)$. We denote by $C^\infty(\Omega)$ the vector space $\cap_{m\in\mathbb{N}} C^m(\Omega)$ and by $C_c^\infty(\Omega)$ the space of functions in $C^\infty(\Omega)$ with compact support. Let $f \in (C^m(\Omega))^n$. Then we denote by Df the gradient matrix

$$Df = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\dots,n}.$$

Further, if $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ and $|\beta| := \beta_1 + \dots + \beta_n$, then we set

$$D^\beta f := \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}.$$

The subspace of $C^m(\Omega)$ of functions f such that f and its derivatives $D^\beta f$ of order $|\beta| \leq m$ can be extended with continuity to $\operatorname{cl}\Omega$ is denoted $C^m(\operatorname{cl}\Omega)$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\operatorname{cl}\Omega)$ equipped with the norm

$$\|f\|_{C^m(\operatorname{cl}\Omega)} := \sum_{|\beta|\leq m} \sup_{\operatorname{cl}\Omega} |D^\beta f|$$

is a Banach space. Let $B \subseteq \mathbb{R}^n$. We denote by χ_B the characteristic function of B, *i.e.* $\chi_B(\xi) = 1$ if $\xi \in B$ and $\chi_B(\xi) = 0$ if $\xi \in \mathbb{R}^n \setminus B$. We say that a function ψ of $[0, +\infty)$ to itself is increasing, provided that $\psi(\rho_1) \leq \psi(\rho_2)$ whenever $0 \leq \rho_1 < \rho_2$.

Let $1 \leq p < +\infty$ and $m \in \mathbb{N}$. We denote by $W^{m,p}(\Omega)$ the Sobolev space of the (equivalence classes of) real-valued functions in $L^p(\Omega)$, which have all distributional derivatives up to order m in $L^p(\Omega)$. We introduce in $W^{m,p}(\Omega)$ its usual norm

$$\|u\|_{W^{m,p}(\Omega)} := \sum_{|\beta|\leq m,\, \beta\in\mathbb{N}^n} \|D^\beta u\|_{L^p(\Omega)}. \tag{2}$$

As usual,

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(\xi)|^p \, d\xi\right)^{\frac{1}{p}}.$$

Further, $W^{m,p}_{\mathrm{loc}}(\Omega)$ denotes the space of the (equivalence classes of) functions f in Ω, such that $f \in W^{m,p}(V)$, for all open and relatively compact subsets V of Ω.

We say that an open subset Ω of $\mathbb{R}^n$ has the *cone property*, if there exist $h > 0$ and $\alpha > 0$ such that, for all points $\xi \in \partial\Omega$, there exists an open cone of heigth h, opening α and vertex ξ contained in Ω. We say that an open subset Ω of $\mathbb{R}^n$ is of *class* $C^{0,1}$ (or that Ω is bounded and has the strong Lipschitz property) if Ω is bounded and if, locally around each point of $\partial\Omega$, $\partial\Omega$ is a graph of a Lipschitz function and Ω lies above the graph. For further details, we refer to Adams [2: p. 66]. It is well-known that if Ω is of class $C^{0,1}$, then Ω has the cone property (cf., e.g., Adams [2: p. 66]).

We collect in the following theorem three well-known results on Sobolev spaces (cf., e.g., Adams [2: Theorem 5.4/p. 97 and Theorem 5.23/p. 115], Valent [33: Theorem 2.2/p. 26] and Reshetnyak [24: Corollary 1/p. 28].) For a standard definition of Banach algebra we refer, for example, to Lanza [16: Definition 2.1].

Theorem 2.1. *Let $m, m_1 \in \mathbb{N}$, $1 \leq p < +\infty$, and $mp > n$. Then the following statements hold:*

(i) *If Ω is an open subset of $\mathbb{R}^n$ of class $C^{0,1}$, then $\left(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)}\right)$ is continuously imbedded into $\left(C^0(\mathrm{cl}\,\Omega), \|\cdot\|_{C^0(\mathrm{cl}\,\Omega)}\right)$.*

(ii) *If Ω is an open subset of $\mathbb{R}^n$ with finite measure and with the cone property, then $W^{m,p}(\Omega)$ equipped with a suitable positive multiple of the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ is a commutative Banach algebra with unity. Furthermore, for all $p_1 \in [1,p]$ and $0 \leq m_1 \leq m$, the pointwise multiplication is bilinear and continuous from $W^{m_1,p_1}(\Omega) \times W^{m,p}(\Omega)$ into $W^{m_1,p_1}(\Omega)$.*

(iii) *Let Ω be an open subset of $\mathbb{R}^n$. Each $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$ has a continuous representative $\tilde{u}$. The function $\tilde{u}$ is differentiable in the ordinary sense almost everywhere in Ω and the ordinary partial derivative $\frac{\partial \tilde{u}}{\partial \xi_j}$ of $\tilde{u}$ is a representative of the D_j-distributional derivative $D_j u$ of u, for all $j \in \{1, \ldots, n\}$.*

Remark 2.2. We note that in statement (i) the inclusion is to be understood in the sense that each equivalence class of functions of $W^{m,p}(\Omega)$ contains exactly one representative which admits a continuous extension to $\mathrm{cl}\,\Omega$. Concerning the first part of statement (ii), we note that we have $\|uv\|_{W^{m,p}(\Omega)} \leq c\|u\|_{W^{m,p}(\Omega)}\|v\|_{W^{m,p}(\Omega)}$ for some constant $c > 0$ depending on Ω, m and p, and that such constant may well be greater than 1. However, we could obtain $c = 1$ by simply replacing the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ with the equivalent norm $\|\cdot\|'_{W^{m,p}(\Omega)} = c\|\cdot\|_{W^{m,p}(\Omega)}$.

Remark 2.3. If in statement (i) we further assume that $(m-1)p > n$, then $W^{m,p}(\Omega)$ is imbedded into $C^1(\mathrm{cl}\,\Omega)$. If this is the case, we will identify an element of $W^{m,p}(\Omega)$ with its representative of class $C^1(\mathrm{cl}\,\Omega)$.

We now note that the representatives of f are defined almost everywhere and that, accordingly, the composition of f with an equivalence class of functions g of Ω to Ω_1 may not make sense. Thus we introduce the following

Definition 2.4. Let Ω and Ω_1 be open subsets of $\mathbb{R}^n$. Let $f \in L^1_{\mathrm{loc}}(\Omega_1)$ and $g \in (L^1_{\mathrm{loc}}(\Omega))^n$. Assume that $g(\Omega) \subseteq \Omega_1$, *i.e.* for each fixed representative $\tilde{g}$ of g, we have $\tilde{g}(\xi) \in \Omega_1$ for almost all $\xi \in \Omega$. We say that the composition of the equivalence classes f and g is *well-defined*, provided that for all representatives f_1, f_2 of f, and g_1, g_2 of g, we have

$$f_1 \circ g_1 = f_2 \circ g_2 \qquad \text{a.e. in } \Omega \tag{3}$$

(note that any of the two hand-sides of equation (3) may be undefined on some subset of measure zero of Ω, *i.e.* whenever $g_i(\xi) \notin \Omega_1$ for $\xi \in \Omega$). In case the composition of f and g is well-defined, we denote by $f \circ g$ the equivalence class of those functions which are almost everywhere equal to any of the composite functions in (3).

Concerning Definition 2.4, it is perhaps worth to note that if $g \in \left(L^1_{\mathrm{loc}}(\Omega)\right)^n$ and if $\tilde{g}(\xi) \in \Omega_1$, for almost all $\xi \in \Omega$, for at least one representative $\tilde{g}$ of g, then the same holds for all representatives of g, so that we can conclude that $g(\Omega) \subseteq \Omega_1$. Also, it is not difficult to realize that the following holds (cf. Lanza [16: Lemma 3.23]).

Lemma 2.5. *Let Ω and Ω_1 be open subsets of $\mathbb{R}^n$. Let $g \in (L^1_{\text{loc}}(\Omega))^n$ and $g(\Omega) \subseteq \Omega_1$. Then the following statements hold.*

(i) *The composition $f \circ g$ is well-defined for all $f \in L^1_{\text{loc}}(\Omega_1)$ if and only if, for all subsets N of $\mathbb{R}^n$ of measure zero, the preimage $\tilde{g}^{\leftarrow}(N)$ has measure zero for all representatives $\tilde{g}$ of g.*

(ii) *A representative $\tilde{g}$ of g has the property that for all subsets N of $\mathbb{R}^n$ of measure zero the preimage $\tilde{g}^{\leftarrow}(N)$ has measure zero, if and only if all the representatives of g have the same property.*

Thus, we will consider only g's with the property of statement (ii) of Lemma 2.5. If $g \in (W^{m,p}(\Omega))^n$ $(mp > n)$, the formula of change of variables in multiple integrals (cf. Marcus and Mizel [19: Corollary 2/p. 791 and Theorem 2/ p. 792]) implies that if $\tilde{g}$ is the continuous representative of g and if $M := \{\xi \in \Omega : |\det D\tilde{g}(\xi)| = 0\}$, then $\int_{\tilde{g}(M)} d\eta \leq \int_M |\det D\tilde{g}(\xi)|\, d\xi$. Thus we must consider g's with $|\det D\tilde{g}(\xi)| \neq 0$ for almost all $\xi \in \Omega$, otherwise $\tilde{g}$ would not satisfy the property of statement (ii) of Lemma 2.5. Now, in the specific case in which the continuous representative of g is injective and $|\det Dg|^{-1} \in L^\gamma(\Omega)$, for some $0 < \gamma \leq +\infty$, it can be shown (cf. Lanza [13: Theorem 3.2]) that if $f \in W^{m,p}(\Omega_1)$, and if $g \in (W^{m,p}(\Omega))^n$, and if the the real number t defined by

$$t := \begin{cases} pn\big\{m[n-(m-1)p] + n + n\gamma^{-1}\big\}^{-1} & \text{if } (m-1)p < n, \\ p\big\{1+\gamma^{-1}\big\}^{-1} & \text{if } (m-1)p \geq n, \end{cases}$$

satisfies $t > 1$, then $f \circ g \in W^{m,t}(\Omega)$. Here $\gamma^{-1} := 0$ if $\gamma = +\infty$. Since, as announced in the introduction, we require $t = p$ when $m = m_1$ and $p = p_1$, we take $(m-1)p \geq n$ and $\gamma = +\infty$. To satisfy condition $\gamma = +\infty$, we assume the existence of some constant $c > 0$ such that $|\det Dg| > c$ a.e. in Ω. Since we are interested in the differentiability of the composition operator, we require that the set

$$\Big\{g \in (W^{m,p}(\Omega))^n : |\det Dg| > c \ \text{ a.e. in } \Omega\Big\}$$

be open in the space $(W^{m,p}(\Omega))^n$. Such requirement suggests that we should assume the map $g \mapsto |\det Dg|$ to be continuous from $(W^{m,p}(\Omega))^n$ to $L^\infty(\Omega)$. To ensure such continuity, we take $(m-1)p > n$. Such considerations indicate that it is for us natural to consider the composition of

$$(f, g) \in W^{m_1,p_1}(\Omega_1) \times \Big\{g \in (W^{m,p}(\Omega))^n : g(\Omega) \subseteq \Omega_1 \text{ and } |\det Dg| > 0 \text{ a.e. in } \Omega\Big\}$$

with $(m-1)p > n$. Then we have the following result, which is in the spirit of Lanza [13: Theorem 3.2]. We note that related results can be found in Gol'dshtein and Reshetnyak [11: Chapter 5].

Theorem 2.6. *Let $m, m_1 \in \mathbb{N}$, $0 \leq m_1 \leq m$, $1 \leq p_1 \leq p < +\infty$, $p(m-1) > n$, $0 \leq c < +\infty$, and $0 < \lambda \leq +\infty$. Let Ω and Ω_1 be open subsets of $\mathbb{R}^n$, Ω with finite* *measure and the cone property. Let*

$$\mathcal{G}_{m,p,c,\lambda}(\Omega,\Omega_1) := \left\{ g \in (W^{m,p}(\Omega))^n \;\middle|\; \begin{array}{l} g(\Omega) \subseteq \Omega_1, \text{ the unique representative} \\ \tilde{g} \in (C^1(\Omega))^n \text{ of } g \text{ satisfies } |\det D\tilde{g}(\xi)| > 0 \\ \forall \xi \in \Omega, \text{ and } \Gamma(\tilde{g},\eta) \leq \lambda \ \forall \eta \in \mathbb{R}^n \end{array} \right\}, \tag{4}$$

where $\Gamma(\tilde{g},\eta)$ denotes the (possibly equal to $+\infty$) number of elements of the preimage $\tilde{g}^{\leftarrow}(\{\eta\})$.

Then the following assertions hold:

(i) *For all $0 \leq c < +\infty$ and $0 < \lambda \leq +\infty$, the composition $f \circ g$ is well-defined for all $(f,g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\lambda}(\Omega,\Omega_1)$ and belongs to $W^{m_1,p_1}(\Omega)$.*

(ii) *For each fixed value of $(c,\lambda) \in (0,+\infty)^2$, there exists an increasing function ψ of $[0,+\infty)$ to itself such that*

$$\|f \circ g\|_{W^{m_1,p_1}(\Omega)} \leq \|f\|_{W^{m_1,p_1}(\Omega_1)} \psi\big(\|g\|_{(W^{m,p}(\Omega))^n}\big) \tag{5}$$

for all $(f,g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\lambda}(\Omega,\Omega_1)$.

(iii) *If $m_1 > 0$ and $j \in \{1,\dots,n\}$, then the j-th distributional derivative $D_j(f \circ g)$ of $f \circ g$ coincides with $\sum_{l=1}^{n} [(D_l f) \circ g] D_j g_l$, where $g := (g_1,\dots,g_n)$.*

Proof. With our assumptions on Ω, m, p and n, the existence of a unique representative $\tilde{g} \in (C^1(\Omega))^n$ of g is guaranteed by the Sobolev Imbedding Theorem (cf. Adams [2: Theorem 5.4/p. 97]). By assumption $|\det D\tilde{g}| > 0$ in Ω, $\tilde{g}$ is a local diffeomorphism of Ω onto $\tilde{g}(\Omega)$, and thus $\tilde{g}^{\leftarrow}(N)$ has measure zero whenever N is a subset of measure zero of $\mathbb{R}^n$. Indeed, each compact subset of Ω can be covered by a finite number of balls, say B, on which g is a diffeomorphism onto its image, $\tilde{g}_{|B}^{\leftarrow}(N)$ has measure zero and Ω is a countable union of compact subsets of Ω. Thus, by Lemma 2.5/(i), the composition is well-defined. Since each measurable set of $\mathbb{R}^n$ is the union of a Borel set and of a subset of measure zero of $\mathbb{R}^n$, we can also assert that $\tilde{f} \circ \tilde{g}$ is measurable, for all representatives $\tilde{f}$ of f. Then all representatives of $f \circ g$ are measurable. It can be easily shown that ψ as in the statement exists provided that the composition maps bounded sequences of $W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\lambda}(\Omega,\Omega_1)$ to bounded sequences of $W^{m_1,p_1}(\Omega)$ (cf. Lanza [16: Proposition 3.11]). Thus to prove the theorem, it suffices to fix $m \geq 2$ and to show by (finite) induction on $m_1 \in \{0,\dots,m\}$ that whenever $(m-1)p > n$, the composition maps bounded sequences of $W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\lambda}(\Omega,\Omega_1)$ to bounded sequences of $W^{m_1,p_1}(\Omega)$ and that the chain rule holds when $m_1 > 0$.

Let $m_1 = 1$. By the Sobolev Imbedding Theorem (cf. Adams [2: Theorem 5.4/p. 97]), $W^{m-1,p}(\Omega)$ is imbedded into $L^\infty(\Omega)$. Then by using the Hölder inequality and the rule of change of variables in multiple integrals (cf. Marcus and Mizel [19: Theorem

2/p. 792]), we deduce the validity of the inequalities

$$\begin{aligned}\int_\Omega |f\circ\tilde g(\xi)|^{p_1}\,d\xi &\le \frac{1}{c}\int_\Omega |f\circ\tilde g(\xi)|^{p_1}|\mathrm{det}D\tilde g(\xi)|\,d\xi\\ &= \frac{1}{c}\int_{\tilde g(\Omega)} |f(\xi)|^{p_1}\Gamma(\tilde g,\eta)\,d\eta\\ &\le \frac{\lambda}{c}\|f\|^{p_1}_{L^{p_1}(\Omega_1)}\end{aligned} \tag{6$_a$}$$

and

$$\begin{aligned}&\int_\Omega \left|[(D_l f)\circ\tilde g(\xi)]D_j\tilde g_l(\xi)\right|^{p_1}d\xi\\ &\le \frac{c'}{c}\|D_j g_l\|^{p_1}_{W^{m-1,p}(\Omega)}\int_\Omega |(D_l f)\circ\tilde g(\xi)|^{p_1}|\mathrm{det}D\tilde g(\xi)|\,d\xi\\ &= \frac{c'}{c}\|D_j g_l\|^{p_1}_{W^{m-1,p}(\Omega)}\int_{\tilde g(\Omega)}|D_l f(\eta)|^{p_1}\Gamma(\tilde g,\eta)\,d\eta\\ &\le \frac{\lambda}{c}c'\|f\|^{p_1}_{W^{1,p_1}(\Omega_1)}\|g\|^{p_1}_{(W^{m,p}(\Omega))^n}\end{aligned} \tag{6$_b$}$$

for some constant $c' > 0$. Thus, to conclude the proof of the case $m_1 = 1$, it suffices to show that the chain rule holds. Since $\tilde g$ is a local diffeomorphism, for all $P \in \Omega$, there exists $\rho > 0$ such that $\mathrm{cl}\,B(P,\rho) \subseteq \Omega$ and $\tilde g_{|B(P,\rho)}$ is a diffeomorphism onto its image. Since each $f \in W^{1,p_1}(\Omega_1)$ can be approximated by a sequence $\{f_s\}_{s\in\mathbb{N}}$ in $W^{1,p_1}(\Omega_1)\cap C^\infty(\Omega_1)$ (cf., e.g., Adams [2: Theorem 3.16/p. 52]), the validity of the chain rule in $B(P,\rho)$ follows from the equalities

$$\begin{aligned}&\int_{B(P,\rho)} f(\tilde g(\xi))D_j\phi(\xi)\,d\xi\\ &= \lim_{s\to+\infty}\int_{\tilde g(B(P,\rho))} f_s(\eta)\left[(D_j\phi)\circ\tilde g^{(-1)}_{|B(P,\rho)}(\eta)\right]\left|(\mathrm{det}D\tilde g)\circ\tilde g^{(-1)}_{|B(P,\rho)}(\eta)\right|^{-1}d\eta\\ &= \lim_{s\to+\infty}\int_{B(P,\rho)} f_s(\tilde g(\xi))D_j\phi(\xi)\,d\xi\\ &= -\lim_{s\to+\infty}\int_{B(P,\rho)}\sum_{l=1}^n\left[(D_l f_s)\circ g(\xi)\right]D_j g_l(\xi)\phi(\xi)\,d\xi\\ &= -\int_{B(P,\rho)}\sum_{l=1}^n\left[(D_l f)\circ g(\xi)\right]D_j g_l(\xi)\phi(\xi)\,d\xi\end{aligned} \tag{7}$$

for all $\phi \in C^\infty_c(B(P,\rho))$, which hold by the rule of change of variables in multiple integrals, by the validity of the chain rule when $f_s \in C^\infty(\Omega_1)$ (cf., e.g., Reshetnyak [24: Theorem 2.8/p. 21]), by the condition $|\mathrm{det}D\tilde g|^{-1} < c^{-1}$ in Ω, by the membership of $D_l g_j$ in $W^{m-1,p}(\Omega)$ (and thus, as remarked above, in $L^\infty(\Omega)$), by the membership $\phi \in C^\infty_c(B(P,\rho))$, and by the Hölder inequality. Since the chain rule holds in $B(P,\rho)$, a standard argument based on the partition of unity implies the validity of the chain rule in Ω.

We now assume that the claim holds for $1 \leq m_1 < m$ and prove it for $m_1 + 1$. Since $W^{m_1+1,p_1}(\Omega_1)$ is imbedded into $W^{m_1,p_1}(\Omega_1)$, then by inductive assumption the composition maps bounded sequences of $W^{m_1+1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\lambda}(\Omega, \Omega_1)$ into bounded sequences of $W^{m_1,p_1}(\Omega)$ and the chain rule holds. By inductive assumption, the map $(f,g) \mapsto (D_l f) \circ g$ maps bounded sequences of $W^{m_1+1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\lambda}(\Omega, \Omega_1)$ into bounded sequences of $W^{m_1,p_1}(\Omega)$. Since $(m-1)p > n$, $m_1 \leq m-1$ and $1 \leq p_1 \leq p$, Theorem 2.1/(ii) ensures that the pointwise multiplication is continuous from $W^{m_1,p_1}(\Omega) \times W^{m-1,p}(\Omega)$ to $W^{m_1,p_1}(\Omega)$, and thus we conclude that the map $(f,g) \mapsto [(D_l f) \circ g] D_j g_l$ maps bounded sequences of $W^{m_1+1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\lambda}(\Omega, \Omega_1)$ to bounded sequences of $W^{m_1,p_1}(\Omega)$ and the proof of the case $m_1 + 1$ is complete. The case $m_1 = 0$ follows by the inequality $\|f \circ g\|_{L^{p_1}(\Omega)} \leq \left(\frac{\lambda}{c}\right)^{\frac{1}{p_1}} \|f\|_{L^{p_1}(\Omega_1)}$ proved for the case $m_1 = 1$ ∎

We point out that our proof of the existence of ψ as in (5) heavily relies on the assumptions $c > 0$ and $\lambda < +\infty$. Accordingly, one could not deduce from our proof the existence of ψ as in (5) for $(f,g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0,+\infty}(\Omega, \Omega_1)$.

We now state an abstract result that we need to prove our differentiability theorem for the composition. The following includes the content of Lanza [16: Remark 2.5 and Theorem 2.7].

Proposition 2.7. *Let $\mathcal{P}(\mathbb{R}^n)$ be the space of polynomials in n real variables with real coefficients and let $\|\cdot\|_{\mathcal{Y}}$ be a norm on $\mathcal{P}(\mathbb{R}^n)$. Let, for all $r \in \mathbb{N}$, $\mathcal{Y}_r$ be the completion of $(\mathcal{P}(\mathbb{R}^n), \|\cdot\|_{\mathcal{Y}_r})$, where $\|\cdot\|_{\mathcal{Y}_r}$ is the norm on $\mathcal{P}(\mathbb{R}^n)$ defined by*

$$\|p\|_{\mathcal{Y}_r} := \sum_{|\beta| \leq r,\, \beta \in \mathbb{N}^n} \|D^\beta p\|_{\mathcal{Y}}. \tag{8}$$

(Sometimes, we write $\mathcal{Y}$ to denote the space $\mathcal{Y}_0$.) Let $s,t \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$ with $t - |\beta| = s$. Then there exists one and only one linear and continuous operator of $\mathcal{Y}_t$ to $\mathcal{Y}_s$ which coincides with the ordinary partial derivation of multiindex β on the elements of $\mathcal{P}(\mathbb{R}^n)$. By abuse of notation, we shall denote such operator by D^β, just as the usual partial derivative of multiindex β. We have

$$D^\beta y = \lim_{j \to \infty} D^\beta p_j \quad \text{in } \mathcal{Y}_s, \quad \text{whenever} \quad \lim_{j \to \infty} p_j = y \quad \text{in } \mathcal{Y}_t. \tag{9}$$

By analogy with the usual derivations, Dy denotes the matrix $(D_1 y, \ldots, D_n y)$.

We now note that the following assertion holds.

Proposition 2.8. *Let $m_1, r \in \mathbb{N}$ and $1 \leq p_1 < +\infty$. Let Ω_1 be an open subset of $\mathbb{R}^n$ of class $C^{0,1}$. Let $\|\cdot\|_{\mathcal{Y}} := \|\cdot\|_{W^{m_1,p_1}(\Omega_1)}$. Then $W^{m_1+r,p_1}(\Omega_1)$ is a completion of $(\mathcal{P}(\mathbb{R}^n), \|\cdot\|_{\mathcal{Y}_r})$, with $\|\cdot\|_{\mathcal{Y}_r}$ as in (8). Thus, up to a linear homeomorphism, the space $W^{m_1+r,p_1}(\Omega_1)$ coincides with the space $\mathcal{Y}_r$. Furthermore, for all $\beta \in \mathbb{N}^n$ with $|\beta| \leq r$, the operator D^β from $W^{m_1+r,p_1}(\Omega_1)$ into $W^{m_1+r-|\beta|,p_1}(\Omega_1)$ defined in Proposition 2.7 coincides with the distributional derivative of multiindex β.*

Proof. Since Ω_1 is of class $C^{0,1}$, it is well-known (cf., e.g., Adams [2: p. 67 and Theorem 3.18/p. 54]) that the set of restrictions of the functions of $C_c^\infty(\mathbb{R}^n)$ to $\operatorname{cl}\Omega_1$ is dense in $W^{m_1+r,p_1}(\Omega_1)$. By the Weierstrass Approximation Theorem, the functions of $C_c^\infty(\mathbb{R}^n)$ can be approximated, uniformly with their derivatives up to order $m_1 + r$ on compact subsets of $\mathbb{R}^n$, with elements of $\mathcal{P}(\mathbb{R}^n)$ (cf., e.g., Rohlin and Fuchs [25: p. 185]). Accordingly, $W^{m_1+r,p_1}(\Omega_1)$ is a completion of $(\mathcal{P}(\mathbb{R}^n), \|\cdot\|_{\mathcal{Y}_r})$. Indeed, the norm $\|\cdot\|_{\mathcal{Y}_r}$, with $\|\cdot\|_{\mathcal{Y}} := \|\cdot\|_{W^{m_1,p_1}(\Omega_1)}$, is clearly equivalent to the norm $\|\cdot\|_{W^{m_1+r,p_1}(\Omega_1)}$ on $\mathcal{P}(\mathbb{R}^n)$. Furthermore, if $F \in W^{m_1+r,p_1}(\Omega_1)$ is the limit of a sequence $\{p_j\}$ of polynomials in $W^{m_1+r,p_1}(\Omega_1)$, and if $\beta \in \mathbb{N}^n$ with $|\beta| \le r$, then clearly the sequence $\{D^\beta p_j\}$ converges to the D^β-distributional derivative of F in $W^{m_1+r-|\beta|,p_1}(\Omega_1)$, and thus in $\mathcal{Y}_{r-|\beta|}$. Accordingly, the operator D^β introduced in Proposition 2.7 coincides with the distributional derivative of multiindex β ∎

Finally, we need the following abstract result, which has been proved in Lanza [16: Theorems 3.1 and 4.1, and Proposition 4.17]. In order to write the formulas in a coincise way, we put a '$\hat{\ }$' symbol on a term which we wish to suppress. So, for example, $\xi_1 \cdots \hat{\xi}_j \cdots \xi_s$ denotes $\prod_{l=1,\dots,s,\ l\neq j} \xi_l$.

Theorem 2.9. *Let $r \in \mathbb{N}$. Let $\|\cdot\|_{\mathcal{Y}}$ be a norm on $\mathcal{P}(\mathbb{R}^n)$, and let $\mathcal{Y}_r$ be the completion of $\mathcal{P}(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{\mathcal{Y}_r}$ defined in (8). (As above, we write $\mathcal{Y}$ to denote the space $\mathcal{Y}_0$.) Let $\mathcal{X}$ be a real commutative Banach algebra with unity and $\tilde{\mathcal{X}}$ a real Banach space. Assume that there exists a continuous linear and injective map $\mathcal{J}$ of $\mathcal{X}$ into $\tilde{\mathcal{X}}$ and let $(\cdot) * (\cdot)$ be a continuous and bilinear map of $\tilde{\mathcal{X}} \times \mathcal{X}$ into $\tilde{\mathcal{X}}$ with '$*$' satisfying the condition*

$$\mathcal{J}[x_1] * x_2 = \mathcal{J}[x_1 x_2] \qquad \text{for all } x_1, x_2 \in \mathcal{X}. \tag{10}$$

Let $\mathcal{A}$ be a subset of $\mathcal{X}^n$. Assume that there exists an increasing function ψ of $[0,+\infty)$ to itself such that

$$\left\|\mathcal{J}\left[p(x_1,\dots,x_n)\right]\right\|_{\tilde{\mathcal{X}}} \le \|p\|_{\mathcal{Y}}\, \psi(\|(x_1,\dots,x_n)\|_{\mathcal{X}^n}) \tag{11}$$

for all $(p,(x_1,\dots,x_n)) \in \mathcal{P}(\mathbb{R}^n) \times \mathcal{A}$. Then there exists a unique map $\tilde{A}$ of $\mathcal{Y} \times \mathcal{A}$ to $\tilde{\mathcal{X}}$ such that the following two conditions hold:

$$\tilde{A}[p,x] = \mathcal{J}[p(x)] \qquad \text{for all } (p,x) \in \mathcal{P}(\mathbb{R}^n) \times \mathcal{A}, \tag{12}$$

and

$$\begin{array}{l}\text{for all fixed } x := (x_1,\dots,x_n) \in \mathcal{A}, \\ \text{the map } y \mapsto \tilde{A}[y,x] \text{ is continuous from } \mathcal{Y} \text{ into } \tilde{\mathcal{X}}.\end{array} \tag{13}$$

Furthermore, the map $\tilde{A}[\cdot,x]$ of (13) is linear, and $\tilde{A}$ is continuous from $\mathcal{Y} \times \mathcal{A}$ into $\tilde{\mathcal{X}}$, and if $y \in \mathcal{Y}$ with $y = \lim_{j\to\infty} p_j$ in $\mathcal{Y}$ for $p_j \in \mathcal{P}(\mathbb{R}^n)$ and $x \in \mathcal{A}$, then

$$\tilde{A}[y,x] = \lim_{j\to\infty} \mathcal{J}[p_j(x)] \qquad \text{in } \tilde{\mathcal{X}}, \tag{14}$$

and

$$\|\tilde{A}[y,x]\|_{\tilde{\mathcal{X}}} \le \|y\|_{\mathcal{Y}}\, \psi(\|x\|_{\mathcal{X}^n}). \tag{15}$$

If we further assume that $\mathcal{A}$ is open, then $\tilde{A}$ is of class C^r from $\mathcal{Y}_r \times \mathcal{A}$ to $\tilde{\mathcal{X}}$, for all $r \ge 1$. If $r \ge 1$ and $s \in \{1,\dots,r\}$, then the differential $d^s\tilde{A}$ of order s of $\tilde{A}$ at

$(y^{\#}, x^{\#}) \in \mathcal{Y}_r \times \mathcal{A}$, *which can be identified with an element of* $\mathcal{L}^{(s)}(\mathcal{Y}_r \times \mathcal{X}^n, \tilde{\mathcal{X}})$, *is delivered by the formula*

$$
\begin{aligned}
& d^s \tilde{A}[y^{\#}, x^{\#}]((v_{[1]}, w_{[1]}), \ldots, (v_{[s]}, w_{[s]})) \\
& = \sum_{j=1}^{s} \sum_{l_1, \ldots, \widehat{l_j}, \ldots, l_s = 1}^{n} \tilde{A}[D_{l_s} \cdots \widehat{D_{l_j}} \cdots D_{l_1} v_{[j]}, x^{\#}] * (w_{s,l_s} \cdots \widehat{w_{j,l_j}} \cdots w_{1,l_1}) \\
& \quad + \sum_{l_1, \ldots, l_s = 1}^{n} \tilde{A}[D_{l_s} \cdots D_{l_1} y^{\#}, x^{\#}] * (w_{s,l_s} \cdots w_{1,l_1})
\end{aligned}
\tag{16}
$$

for all $(v_{[j]}, w_{[j]} := (w_{j,1}, \ldots, w_{j,n})) \in \mathcal{Y}_r \times \mathcal{X}^n$ $(j = 1, \ldots, s)$, *where the symbols* $l_1, \ldots, l_s$ *denote summation indexes ranging from* 1 *to* n. *In particular, if* $s = 1$, *we have the map*

$$
(v, w) \mapsto \tilde{A}[v, x^{\#}] + \sum_{l=1}^{n} \tilde{A}[D_l y^{\#}, x^{\#}] * w_l \tag{17}
$$

for all $(v, w) := (v, (w_1, \ldots, w_n)) \in \mathcal{Y}_r \times \mathcal{X}^n$. *(The symbol* D_l *has been defined in Proposition* 2.7.*)*

3. Continuity and differentiability theorems for the composition operator in Sobolev spaces

If Ω and Ω_1 are open subsets of $\mathbb{R}^n$ with Ω of class $C^{0,1}$, $1 \le p < +\infty$ and $m \in \mathbb{N}$ with $(m-1)p > n$, then we introduce the notation

$$
\mathcal{G}_{m,p,c}(\Omega, \Omega_1) := \left\{ g \in (W^{m,p}(\Omega))^n \,\middle|\, \begin{array}{l} g(\Omega) \subseteq \Omega_1, \text{ the unique represen-} \\ \text{tative } \tilde{g} \in (C^1(\mathrm{cl}\,\Omega))^n \text{ of } g \\ \text{satisfies } |\det D\tilde{g}(\xi)| > c \ \forall \xi \in \mathrm{cl}\,\Omega \end{array} \right\}.
$$

As we have indicated in the discussion preceding Theorem 2.6, $\mathcal{G}_{m,p,0}(\Omega, \Omega_1)$ is a natural set for our g's. In order to study the composition on $W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1)$ with $0 \le m_1 \le m$ and $1 \le p_1 \le p$, and to apply Theorem 2.9, we need the existence of a function ψ as in (11) for $(f, g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1)$, while Theorem 2.6 guarantees the existence of ψ only for $(f, g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c,\frac{1}{c}}(\Omega, \Omega_1)$, with $c > 0$. To circumvent this difficulty, we need the following technical proposition, which is known at least in part.

Proposition 3.1. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$. *Then the following statements hold.*

(i) *Let* $g \in (C^1(\mathrm{cl}\,\Omega))^n$ *be such that* $\det Dg(\xi) \neq 0$ *for all* $\xi \in \mathrm{cl}\,\Omega$. *Assume that there exist* $R > 0$ *such that* $\mathrm{cl}\,\Omega \subseteq B(0, R)$ *and* $\gamma \in (C^1(\mathrm{cl}\,B(0,R)))^n$ *such that* $\gamma_{|\mathrm{cl}\,\Omega} = g$. *Then, for all* $\eta \in \mathbb{R}^n$, *the number of elements* $\Gamma(g, \eta)$ *of the set* $g^{\leftarrow}(\{\eta\})$ *is finite and*

$$
C_g := \sup_{\eta \in \mathbb{R}^n} \Gamma(g, \eta) < +\infty. \tag{19}
$$

(ii) *Let the sequence* $\{g_l\}_{l\in\mathbb{N}}$ *converge to* g *in* $(C^1(\mathrm{cl}\,\Omega))^n$ *and let* $\det Dg(\xi) \neq 0 \neq \det Dg_l(\xi)$ *for all* $\xi \in \mathrm{cl}\,\Omega$ *and* $l \in \mathbb{N}$. *Assume that there exist* $R > 0$ *such that* $\mathrm{cl}\,\Omega \subseteq B(0,R)$ *and a sequence* $\{\gamma_l\}_{l\in\mathbb{N}}$ *converging to some* γ *in* $\left(C^1(\mathrm{cl}\,B(0,R))\right)^n$ *such that* $\gamma_{|\mathrm{cl}\,\Omega} = g$ *and* $\gamma_{l|\mathrm{cl}\,\Omega} = g_l$ *for all* $l \in \mathbb{N}$. *Then* $\limsup_{l\to+\infty} C_{g_l} \leq C_g$.

(iii) *Let* Ω *have the* C^1*-extension property, i.e. there exist* $R > 0$ *such that* $\mathrm{cl}\,\Omega \subseteq B(0,R)$ *and a linear and continuous operator* E *of* $C^1(\mathrm{cl}\,\Omega)$ *into* $C^1(\mathrm{cl}\,B(0,R))$ *such that* $(Ef)_{|\mathrm{cl}\,\Omega} = f$ *for all* $f \in C^1(\mathrm{cl}\,\Omega)$. *Then, for all* $g \in (C^1(\mathrm{cl}\,\Omega))^n$ *such that* $\det Dg(\xi) \neq 0$ *for all* $\xi \in \mathrm{cl}\,\Omega$, *there exists an open neighborhood* $\mathcal{W}_g$ *in the Banach space* $(C^1(\mathrm{cl}\,\Omega))^n$, *with*

$$\mathcal{W}_g \subseteq \left\{h \in (C^1(\mathrm{cl}\,\Omega))^n : \det Dh(\xi) \neq 0 \ \ \textit{for all} \ \ \xi \in \mathrm{cl}\,\Omega\right\},$$

such that

$$\sup_{h\in\mathcal{W}_g} C_h \leq C_g. \tag{20}$$

In particular, the integer-valued map $g \mapsto C_g$ *is upper semicontinuous on*

$$\left\{h \in (C^1(\mathrm{cl}\,\Omega))^n : \det Dh(\xi) \neq 0 \ \ \textit{for all} \ \ \xi \in \mathrm{cl}\,\Omega\right\}.$$

Proof. Let $\eta \in g(\mathrm{cl}\,\Omega)$. By applying the Inverse Function Theorem to the map γ around the points of $g^{\leftarrow}(\{\eta\})$, we see that the set $g^{\leftarrow}(\{\eta\})$ is discrete. Since $g^{\leftarrow}(\{\eta\})$ is clearly compact, we conclude that $\Gamma(g,\eta) < +\infty$ (which is a known fact).

To prove at once both statements (i) and (ii), it suffices to show the following claim: If $\{g_l\}_{l\in\mathbb{N}}$ and g are as in statement (ii), and if $\{\eta_l\}_{l\in\mathbb{N}}$ is a sequence converging to $\eta \in \mathbb{R}^n$, then $\limsup_{l\to\infty} \Gamma(g_l,\eta_l) \leq \Gamma(g,\eta)$. Indeed, if the claim were true, then by taking $g_l = g$ and $\gamma_l = \gamma$, there could be no bounded sequence $\{\eta_l\}_{l\in\mathbb{N}}$ with $\{\Gamma(g,\eta_l)\}_{l\in\mathbb{N}}$ converging to infinity, and since $\Gamma(g,\cdot)$ is zero outside of $g(\mathrm{cl}\,\Omega)$, statement (i) would follow. Similarly, there could exist no sequence $\{g_l\}_{l\in\mathbb{N}}$ converging to g as in statement (ii) with $C_{g_l} > C_g$ for all $l \in \mathbb{N}$, otherwise, we would have $\Gamma(g_l,\eta_l) \geq C_g + 1$ for some bounded sequence $\{\eta_l\}_{l\in\mathbb{N}}$, and by taking a convergent subsequence of $\{\eta_l\}_{l\in\mathbb{N}}$, our claim would yield a contradiction. Thus also statement (ii) would follow.

We now turn to prove our claim. If $\limsup_{l\to\infty} \Gamma(g_l,\eta_l) > \Gamma(g,\eta)$, then by possibly selecting a subsequence, we can assume that for each l there exist at least $t := \Gamma(g,\eta)+1$ distinct points $\xi_{l,1},\ldots,\xi_{l,t}$ of $\mathrm{cl}\,\Omega$ such that $g_l^{\leftarrow}(\{\eta_l\}) \supseteq \{\xi_{l,1},\ldots,\xi_{l,t}\}$. Since $(\mathrm{cl}\,\Omega)^t$ is compact, there exists a subsequence $\{(\xi_{l_k,1},\ldots,\xi_{l_k,t})\}_{k\in\mathbb{N}}$ of $\{(\xi_{l,1},\ldots,\xi_{l,t})\}_{l\in\mathbb{N}}$ converging to some $(\bar{\xi}_1,\ldots,\bar{\xi}_t) \in (\mathrm{cl}\,\Omega)^t$. Then by the inequality

$$\left|g(\bar{\xi}_j) - \eta_{l_k}\right| \leq \left|g(\bar{\xi}_j) - g(\xi_{l_k,j})\right| + \left|g(\xi_{l_k,j}) - g_{l_k}(\xi_{l_k,j})\right| \tag{21}$$

and by taking the limit as $k \to \infty$, we obtain $\eta = g(\bar{\xi}_j)$ for all $j \in \{1,\ldots,t\}$. Since $\Gamma(g,\eta) < t$, at least two of the points $\bar{\xi}_j$ must coincide. There is no loss of generality in assuming that $\bar{\xi}_1 = \bar{\xi}_2 =: \bar{\xi}$. By the Inverse Function Theorem, there exists $\rho > 0$ such that $\mathrm{cl}\,B(\bar{\xi},\rho) \subseteq B(0,R)$ and that $\gamma_{|\mathrm{cl}\,B(\bar{\xi},\rho)}$ be injective and satisfy $\det D\gamma(\xi) \neq 0$ for all $\xi \in \mathrm{cl}\,B(\bar{\xi},\rho)$. Since

$$\lim_{k\to\infty} \gamma_{l_k|\mathrm{cl}\,B(\bar{\xi},\rho)} = \gamma_{|\mathrm{cl}\,B(\bar{\xi},\rho)} \qquad \text{in} \ \ \left(C^1(\mathrm{cl}\,B(\bar{\xi},\rho))\right)^n,$$

there exists k_0 such that $\gamma_{l_k|\mathrm{cl}\,B(\bar{\xi},\rho)}$ is injective for all $k \geq k_0$ (cf., e.g., Lanza [14: Corollary 4.30]). Since

$$\lim_{k\to+\infty} \xi_{l_k,1} = \bar{\xi} = \lim_{k\to+\infty} \xi_{l_k,2},$$

we can assume that $\xi_{l_k,1} \in \mathrm{cl}\,B(\bar{\xi},\rho)$ and $\xi_{l_k,2} \in \mathrm{cl}\,B(\bar{\xi},\rho)$ for all $k \geq k_0$. Since for all k we have $\xi_{l_k,1} \neq \xi_{l_k,2}$ and $g_{l_k}(\xi_{l_k,1}) = g_{l_k}(\xi_{l_k,2})$, then we have a contradiction.

Statement (iii) is a trivial consequence of statement (ii) and of the continuity of the map $g \mapsto |\mathrm{det}Dg|$ from $(C^1(\mathrm{cl}\,\Omega))^n$ to $C^0(\mathrm{cl}\,\Omega)$ ∎

As an application of Theorem 2.9, case $r = 0$, we now deduce the following fact, a variant of which has already been proved in Lanza [13: Theorem 3.2]. Related continuity results for the composition operator can be found in Marcus and Mizel [22], Valent [32, 33], Drábek and Runst [10], Musina [23], Sickel [27], Runst and Sickel [26].

Theorem 3.2. *Let $m, m_1 \in \mathbb{N}$ with $0 \leq m_1 \leq m$, $1 \leq p_1 \leq p < +\infty$, and $(m-1)p > n$. Let Ω and Ω_1 be open subsets of $\mathbb{R}^n$. Let Ω be of class of class $C^{0,1}$. Then the following statements hold.*

(i) *The composition $f \circ g$ is well-defined for all $(f,g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega,\Omega_1)$ and belongs to $W^{m_1,p_1}(\Omega)$. For each value of $c > 0$, there exists an increasing function ψ of $[0,+\infty)$ to itself such that*

$$\|f \circ g\|_{W^{m_1,p_1}(\Omega)} \leq \|f\|_{W^{m_1,p_1}(\Omega_1)}\, \psi\big(\|g\|_{(W^{m,p}(\Omega))^n}\big), \tag{22}$$

for all $(f,g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c}(\Omega,\Omega_1)$.

(ii) *If the set of restrictions to Ω_1 of the polynomials is dense in $W^{m_1,p_1}(\Omega_1)$, then the composition is continuous from $W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega,\Omega_1)$ to $W^{m_1,p_1}(\Omega)$.*

Proof. The obvious inclusion $\mathcal{G}_{m,p,0}(\Omega,\Omega_1) \subseteq \mathcal{G}_{m,p,0,+\infty}(\Omega,\Omega_1)$ and Theorem 2.6 imply that the composition of $(f,g) \in W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega,\Omega_1)$ is well-defined and belongs to $W^{m_1,p_1}(\Omega)$. Let $R > 0$ be such that $B(0,R) \supseteq \mathrm{cl}\,\Omega$. Since Ω is of class $C^{0,1}$, there exists a linear and continuous extension operator $\tilde{E}$ of $(W^{m,p}(\Omega))^n$ into $\big(W^{m,p}(B(0,R))\big)^n$ (cf., e.g., Jones [12: Theorem A/p. 72]). Now let $\{(f_l, g_l)\}_{l\in\mathbb{N}}$ be a bounded sequence of $W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c}(\Omega,\Omega_1)$, with $c > 0$, and assume by contradiction that, by possibly selecting a subsequence, $\lim_{l\to+\infty} \|f_l \circ g_l\|_{W^{m_1,p_1}(\Omega)} = +\infty$. Since $(m-1)p > n$, then $W^{m,p}(B(0,R))$ is compactly imbedded into $C^1(\mathrm{cl}\,B(0,R))$ (cf., e.g., Adams [2: Theorem 5.4, Part II/p. 98] together with Lanza [15: Lemma 2.4]), and thus there exists a subsequence $\{\tilde{E}g_{l_k}\}_{k\in\mathbb{N}}$ of $\{\tilde{E}g_l\}_{l\in\mathbb{N}}$ and $\gamma \in \big(C^1(\mathrm{cl}\,B(0,R))\big)^n$ such that $\lim_{k\to+\infty} \tilde{E}g_{l_k} = \gamma$ in $(C^1(\mathrm{cl}\,B(0,R)))^n$. Obviously, $|\mathrm{det}D\gamma| \geq c$ in $\mathrm{cl}\,\Omega$. Let $\tilde{g}_{l_k}$ be the representative of class C^1 of g_{l_k}. Then by Proposition 3.1 there exists $k_0 \in \mathbb{N}$ such that $C_{\tilde{g}_{l_k}} \leq C_{\gamma_{|\mathrm{cl}\,\Omega}} < +\infty$ for all $k \geq k_0$. Then by Theorem 2.6, we have $\sup_{k_0 \leq k \in \mathbb{N}} \|f_{l_k} \circ g_{l_k}\|_{W^{m_1,p_1}(\Omega)} < +\infty$, which is a contradiction. Since the composition maps bounded sequences of $W^{m_1,p_1}(\Omega_1) \times \mathcal{G}_{m,p,c}(\Omega,\Omega_1)$ to bounded sequences of $W^{m_1,p_1}(\Omega)$, it can be easily seen that ψ as in statement (i) exists (cf. Lanza [16: Proposition 3.11]). Clearly, ψ may well depend on c. Since $W^{m,p}(\Omega)$ is imbedded into $C^1(\mathrm{cl}\,\Omega)$ and the map $g \mapsto |\mathrm{det}Dg|$ is continuous from $(C^1(\mathrm{cl}\,\Omega))^n$ into $C^0(\mathrm{cl}\,\Omega)$, then $\mathcal{G}_{m,p,c}(\Omega,\Omega_1)$ is open in $\mathcal{G}_{m,p,0}(\Omega,\Omega_1)$ for all $c > 0$, and $\cup_{c>0}\mathcal{G}_{m,p,c}(\Omega,\Omega_1) =$

$\mathcal{G}_{m,p,0}(\Omega,\Omega_1)$. Then the continuity of the composition on $W^{m_1,p_1}(\Omega_1)\times\mathcal{G}_{m,p,0}(\Omega,\Omega_1)$ follows from that on $W^{m_1,p_1}(\Omega_1)\times\mathcal{G}_{m,p,c}(\Omega,\Omega_1)$ for all $c>0$, which in turn follows from statement (i) and Theorem 2.9, case $r=0$ ∎

Now, we note that, in general, $\mathcal{G}_{m,p,c}(\Omega,\Omega_1)$ is not open in $(W^{m,p}(\Omega))^n$. Indeed, a map g^* may be close to some $g\in\mathcal{G}_{m,p,c}(\Omega,\Omega_1)$ in the norm of $(W^{m,p}(\Omega))^n$, but the condition $g^*(\Omega)\subseteq\Omega_1$ may well be violated. Thus, in order to study differentiability properties of the composition, we introduce a suitable open subset of $\mathcal{G}_{m,p,c}(\Omega,\Omega_1)$ by means of the following statement.

Proposition 3.3. *Let $m\in\mathbb{N}\setminus\{0\}$ and $1\le p<+\infty$, with $(m-1)p>n$. Let Ω and Ω_1 be open subsets of $\mathbb{R}^n$. Let Ω be of class $C^{0,1}$. Then, for all $0\le c<+\infty$, the set*

$$\mathcal{K}_{m,p,c}(\Omega,\Omega_1):=\left\{g\in\mathcal{G}_{m,p,c}(\Omega,\Omega_1)\;\middle|\;\begin{array}{l}\text{the unique representative } \tilde g \text{ in}\\ (C^1(\mathrm{cl}\,\Omega))^n \text{ of } g \text{ satisfies } \tilde g(\mathrm{cl}\,\Omega)\subseteq\Omega_1\end{array}\right\}\tag{23}$$

is open in $(W^{m,p}(\Omega))^n$. Furthermore,

$$\mathcal{K}_{m,p,0}(\Omega,\Omega_1)=\bigcup_{c>0}\mathcal{K}_{m,p,c}(\Omega,\Omega_1).\tag{24}$$

Proof. Since in the proof of Theorem 3.2 we have already pointed out that

$$\mathcal{G}_{m,p,0}(\Omega,\Omega_1)=\bigcup_{c>0}\mathcal{G}_{m,p,c}(\Omega,\Omega_1),$$

equality (24) holds, and thus it suffices to show that the set $\mathcal{K}_{m,p,c}(\Omega,\Omega_1)$ is open in $(W^{m,p}(\Omega))^n$. Since $(W^{m,p}(\Omega))^n$ is imbedded into $(C^1(\mathrm{cl}\,\Omega))^n$, it suffices to show that

$$\left\{g\in(C^1(\mathrm{cl}\,\Omega))^n:\ |\mathrm{det}Dg|>c\ \text{ and }\ g(\mathrm{cl}\,\Omega)\subseteq\Omega_1\right\}$$

is open in $(C^1(\mathrm{cl}\,\Omega))^n$. Now if $g,g_1\in(C^1(\mathrm{cl}\,\Omega))^n$ with $g(\mathrm{cl}\,\Omega)\subseteq\Omega_1$, and if $\sup_{\mathrm{cl}\,\Omega}|g-g_1|$ is smaller than the distance of $g(\mathrm{cl}\,\Omega)$ to $\mathbb{R}^n\setminus\Omega_1$, then $g_1(\mathrm{cl}\,\Omega)\subseteq\Omega_1$. Since the map $g\mapsto|\mathrm{det}Dg|$ is continuous from $(C^1(\mathrm{cl}\,\Omega))^n$ to $C^0(\mathrm{cl}\,\Omega)$, the proof is complete ∎

We now state our main differentiability theorem. We note that previous reults on the differentiability of the composition operator in Sobolev spaces were given in Valent [31, 33], who considered the first order differentiability in the variable (f,g), with f of class C^{m_1+1} and $g\in W^{m,p}$ in order to have $f\circ g\in W^{m_1,p}$, and the differentiability of order $r\ge1$ of the map $g\mapsto f\circ g$ from $W^{m,p}$ into $W^{m_1,p}$ for a fixed f of the class C^{m_1+r}, and by Sickel [27], Runst and Sickel [26], who considered the infinite differentiability of the map $g\mapsto f\circ g$ in $W^{m,p}$ with an f of class C^∞. A first order differentiability theorem when both f and g belong to a Sobolev class was given, as mentioned in the introduction, by Brokate and Colonius [8]. The methods and results of those authors are different from those of this paper.

Theorem 3.4. *Let $m, m_1 \in \mathbb{N}$ with $0 \le m_1 \le m$, $1 \le p_1 \le p < +\infty$ with $(m-1)p > n$, and $r \in \mathbb{N}$. Let Ω and Ω_1 be open subsets of $\mathbb{R}^n$. Let Ω be of class $C^{0,1}$. Then the composition is well-defined and of class C^r from the open subset*

$$W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,0}(\Omega, \Omega_1) \tag{25}$$

of $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$ to $W^{m_1,p_1}(\Omega)$. If $r \ge 1$ and $s \in \{1, \dots, r\}$, then the differential of order s of the composition at $(f^\#, g^\#) \in W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,0}(\Omega, \Omega_1)$, which can be identified with an s-linear function of $\mathcal{L}^{(s)}(W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n, W^{m_1,p_1}(\Omega))$, is delivered by the map

$$\begin{aligned} &((v_{[1]}, w_{[1]}), \dots, (v_{[s]}, w_{[s]})) \\ &\longmapsto \sum_{j=1}^{s} \sum_{l_1, \dots, \widehat{l_j}, \dots, l_s = 1}^{n} [(D_{l_s} \cdots \widehat{D_{l_j}} \cdots D_{l_1} v_{[j]}) \circ g^\#] w_{s,l_s} \cdots \widehat{w_{j,l_j}} \cdots w_{1,l_1} \\ &\quad + \sum_{l_1, \dots, l_s = 1}^{n} [(D_{l_s} \cdots D_{l_1} f^\#) \circ g^\#] w_{s,l_s} \cdots w_{1,l_1} \end{aligned} \tag{26}$$

for all $(v_{[j]}, w_{[j]} := (w_{j,1}, \dots, w_{j,n})) \in W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$ $(j = 1, \dots, s)$, where the symbols $l_1, \dots, l_s$ denote summation indexes ranging from 1 to n. In particular, if $s = 1$, we have the map

$$(v, w) \longmapsto v \circ g^\# + \sum_{l=1}^{n} [(D_l f^\#) \circ g^\#] w_l \tag{27}$$

for all $(v, w := (w_1, \dots, w_n)) \in W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$.

Proof. By equality (24), the set in (25) can be written as

$$W^{m_1+r,p_1}(\Omega_1) \times \left\{ \bigcup_{c>0} \mathcal{K}_{m,p,c}(\Omega, \Omega_1) \right\}, \tag{28}$$

and by Proposition 3.3, the set $W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,c}(\Omega, \Omega_1)$ is open in $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$. For the sake of brevity we understand that, for a given $g \in (W^{m,p}(\Omega))^n$, the inclusion $g(\mathrm{cl}\, \Omega) \subseteq \Omega_1$ means that the unique representative $\tilde{g} \in (C^1(\mathrm{cl}\, \Omega))^n$ of g satisfies $\tilde{g}(\mathrm{cl}\, \Omega) \subseteq \Omega_1$. Now, let $(f^\#, g^\#) \in W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,c}(\Omega, \Omega_1)$. By Theorem 3.2/(i), the composition $f^\# \circ g^\#$ is well-defined. Since $g^\#$ has a unique continuous representative in $\mathrm{cl}\, \Omega$, there exists an open and relatively compact subset V of Ω_1 such that $g^\#(\mathrm{cl}\, \Omega) \subseteq V \subseteq \mathrm{cl}\, V \subseteq \Omega_1$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $\phi = 1$ on $\mathrm{cl}\, V$ and $\phi = 0$ on $\mathbb{R}^n \setminus \Omega_1$. Let $R > 0$ be such that the support of ϕ is contained in the ball $B(0, R)$. As it is well-known, $\phi f \in W^{m_1+r,p_1}(B(0,R))$, for all $f \in W^{m_1+r,p_1}(\Omega_1)$. Furthermore, $(\phi f) \circ g = f \circ g$ for all $(f, g) \in W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,c}(\Omega, V)$. By Proposition 3.3, $W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,c}(\Omega, V)$ is an open neighborhood of $(f^\#, g^\#)$ in $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$ contained in $W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,c}(\Omega, \Omega_1)$.

Thus it suffices to show that the map

$$(f,g) \mapsto (\phi f) \circ g$$

is of class C^r from $W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,c}(\Omega, V)$ to $W^{m_1,p_1}(\Omega)$. By writing the Leibnitz rule for the derivatives of ϕf, it is immediate to recognize that the map $f \mapsto \phi f$ is linear and continuous from $W^{m_1+r,p_1}(\Omega_1)$ to $W^{m_1+r,p_1}(B(0,R))$. Then it suffices to show that the composition is of class C^r from $W^{m_1+r,p_1}(B(0,R)) \times \mathcal{K}_{m,p,c}(\Omega, V)$ to $W^{m_1,p_1}(\Omega)$. Now, by Proposition 2.8, the space $W^{m_1+r,p_1}(B(0,R))$ coincides with the completion of $(\mathcal{P}(\mathbb{R}^n), \|\cdot\|_{\mathcal{Y}_r})$, with $\|\cdot\|_{\mathcal{Y}} := \|\cdot\|_{W^{m_1,p_1}(B(0,R))}$ and with $\|\cdot\|_{\mathcal{Y}_r}$ as in (8). Furthermore, by Theorem 3.2/(i), there exists an increasing function ψ of $[0,+\infty)$ to itself such that

$$\|f \circ g\|_{W^{m_1,p_1}(\Omega)} \leq \|f\|_{W^{m_1,p_1}(B(0,R))} \psi\big(\|g\|_{(W^{m,p}(\Omega))^n}\big) \tag{29}$$

for all $(f,g) \in W^{m_1,p_1}(B(0,R)) \times \mathcal{K}_{m,p,c}(\Omega, V)$, and by Theorem 2.1/(ii), $W^{m,p}(\Omega)$ is a commutative Banach algebra with unity, and the pointwise product is bilinear and continuous from $W^{m_1,p_1}(\Omega) \times W^{m,p}(\Omega)$ into $W^{m_1,p_1}(\Omega)$. Then by Theorem 2.9, $\tilde{A}[f,g]$ coincides with the composition $f \circ g$, for all (f,g) as in (29), and by the same Theorem 2.9, we can conclude that the composition is of class C^r on $W^{m_1+r,p_1}(B(0,R)) \times \mathcal{K}_{m,p,c}(\Omega, V)$. Furthermore, if $r \geq 1$, then the differential of the composition at $(\phi f^\#, g^\#)$ is delivered by

$$(u,w) \longmapsto u \circ g^\# + \sum_{l=1}^{n} [D_l(\phi f^\#) \circ g^\#] w_l \tag{30}$$

for all $(u, w := (w_1, \ldots, w_n)) \in W^{m_1+r,p_1}(B(0,R)) \times (W^{m,p}(\Omega))^n$. We note that in Proposition 2.8 we have shown that D_l in (30) actually coincides with the D_l-distributional derivative. Since $D_l(\phi f^\#) \circ g^\# = (D_l f^\#) \circ g^\#$ and $(\phi v) \circ g^\# = v \circ g^\#$ for all $v \in W^{m_1+r,p_1}(\Omega_1)$, we obtain the formula (27) by (30) and by the chain rule. Formula (26) can be obtained similarly ∎

We observe that sometimes in applications the condition $g(\mathrm{cl}\,\Omega) \subseteq \Omega_1$ may not be satisfied, although $g(\Omega) \subseteq \Omega_1$ and the composition $f \circ g$ is well-defined. The role of such condition was to ensure that the domain of the composition be open in $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$. Indeed, in general $W^{m_1+r,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1)$ is not open in $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$. Now that we have studied the case in which the domain is open, we are ready to consider the case in which $g \in \mathcal{G}_{m,p,0}(\Omega, \Omega_1)$.

Theorem 3.5. *Let $m, m_1 \in \mathbb{N}$ with $0 \leq m_1 \leq m$, $1 \leq p_1 \leq p < +\infty$ with $(m-1)p > n$, and $r \in \mathbb{N}$. Let Ω and Ω_1 be open subsets of $\mathbb{R}^n$. Let Ω be of class $C^{0,1}$. Let Ω_1 have the W^{m_1+r,p_1}-extension property, i.e. there exists a linear and continuous operator E from $W^{m_1+r,p_1}(\Omega_1)$ to $W^{m_1+r,p_1}(\mathbb{R}^n)$ such that $Ef_{|\Omega_1} = f$ for all $f \in W^{m_1+r,p_1}(\Omega_1)$.*

Then there exists an open neighborhood $\mathcal{W}$ of $W^{m_1+r,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1)$ in the Banach space $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$ and an operator $\hat{\circ}$ of class C^r from $\mathcal{W}$ to $W^{m_1,p_1}(\Omega)$ such that

$$f \hat{\circ} g = f \circ g \qquad \text{for all } (f,g) \in W^{m_1+r,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1). \tag{31}$$

If $r \geq 1$ and $s \in \{1, \ldots, r\}$, then the differential of order s of the operator $\hat{o}$ at $(f^{\#}, g^{\#}) \in W^{m_1+r,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1)$, which can be identified with an s-linear function of the space $\mathcal{L}^{(s)}(W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n, W^{m_1,p_1}(\Omega))$, is delivered by the map

$$
\begin{aligned}
&((v_{[1]}, w_{[1]}), \ldots, (v_{[s]}, w_{[s]})) \\
&\quad \longmapsto \sum_{j=1}^{s} \sum_{l_1, \ldots, \widehat{l_j}, \ldots, l_s = 1}^{n} \left[(D_{l_s} \cdots \widehat{D_{l_j}} \cdots D_{l_1} v_{[j]}) \circ g^{\#}\right] w_{s,l_s} \cdots \widehat{w_{j,l_j}} \cdots w_{1,l_1} \\
&\qquad + \sum_{l_1, \ldots, l_s = 1}^{n} \left[(D_{l_s} \cdots D_{l_1} f^{\#}) \circ g^{\#}\right] w_{s,l_s} \cdots w_{1,l_1}
\end{aligned} \tag{32}
$$

for all $(v_{[j]}, w_{[j]} := (w_{j,1}, \ldots, w_{j,n})) \in W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$ $(j = 1, \ldots, s)$, where the symbols $l_1, \ldots, l_s$ denote summation indexes ranging from 1 to n.

In particular, if $s = 1$, we have the map

$$
(v, w) \longmapsto v \circ g^{\#} + \sum_{l=1}^{n} \left[(D_l f^{\#}) \circ g^{\#}\right] w_l \tag{33}
$$

for all $(v, w := (w_1, \ldots, w_n)) \in W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$.

Proof. Let $\Lambda : W^{m_1+r,p_1}(\Omega_1) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1) \longmapsto W^{m_1+r,p_1}(\mathbb{R}^n) \times \mathcal{G}_{m,p,0}(\Omega, \Omega_1)$ be defined by $\Lambda[(f,g)] := (Ef, g)$. The operator Λ is clearly the restriction of the operator $\tilde{\Lambda}$ of $\mathcal{W} := W^{m_1+r,p_1}(\Omega_1) \times \mathcal{K}_{m,p,0}(\Omega, \mathbb{R}^n)$ to $W^{m_1+r,p_1}(\mathbb{R}^n) \times \mathcal{K}_{m,p,0}(\Omega, \mathbb{R}^n)$ defined by $\tilde{\Lambda}[(f,g)] := (Ef, g)$. Moreover, the domain of $\tilde{\Lambda}$ contains the domain of Λ and is open in the space $W^{m_1+r,p_1}(\Omega_1) \times (W^{m,p}(\Omega))^n$, by Proposition 3.3. We define $f\hat{o}g = T \circ \tilde{\Lambda}[(f,g)]$, where we have denoted by T the composition of Theorem 3.4 in case $\Omega_1 = \mathbb{R}^n$. Since $\tilde{\Lambda}$ is linear and continuous, $\tilde{\Lambda}$ is of class C^∞ and thus the statement follows by Theorem 3.4 ∎

As shown in Jones [12], extension operators as in the statement of Theorem 3.5 exist for a general class of domains. We now have the following 'inverse' result.

Theorem 3.6. *Let Ω be a non-empty open subset of $\mathbb{R}^n$. Let $p, p_1 \in [1 + \infty)$, $r \in \mathbb{N}$ and $m, m_1 \in \mathbb{N}$ with $mp > n$. Let f be a function of $\mathbb{R}^n$ to $\mathbb{R}$. Let $\mathcal{A}$ be a subset of $(W^{m,p}(\Omega))^n$ containing the equivalence classes of the restrictions to Ω of the affine invertible functions of $\mathbb{R}^n$ into itself (a function G of $\mathbb{R}^n$ to itself is said to be affine if there exists an element $c \in \mathbb{R}^n$ such that $G - c \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$). Then the following statements hold.*

(i) *If $g \mapsto f \circ g$ maps $\mathcal{A}$ to $W^{m_1,p_1}(\Omega)$, then f is a representative of an element of $W^{m_1,p_1}_{\text{loc}}(\mathbb{R}^n)$.*

(ii) *Assume that if $g \in \mathcal{A}$, then for all subsets N of $\mathbb{R}^n$ of measure zero, the preimage $\tilde{g}^{\leftarrow}(N)$ has measure zero in Ω, for all representatives $\tilde{g}$ of g. If $\mathcal{A}$ is open and if the map $g \mapsto f \circ g$ is of class C^r from $\mathcal{A}$ to $W^{m,p}(\Omega)$, then f is a representative of an element of $W^{m+r,p}_{\text{loc}}(\mathbb{R}^n)$.*

Proof. We first prove statement (i). It clearly suffices to show that f is a representative of an element of $W^{m_1,p_1}(B(0,R))$, for all $R > 0$. Let G be an affine and invertible map of $\mathbb{R}^n$ such that $G(\Omega) \supseteq B(0,R)$. By assumption, $G_{|\Omega}$ is the representative of an element in $\mathcal{A}$ and thus $f \circ G_{|\Omega}$ defines an element of $W^{m_1,p_1}(\Omega)$ by the hypothesis of statement (i). By the inclusion $G^{(-1)}(B(0,R)) \subseteq \Omega$, the function $f \circ G_{|G^{(-1)}(B(0,R))}$ defines an element of $W^{m_1,p_1}(G^{(-1)}(B(0,R)))$. Since $G^{(-1)}$ is affine and invertible, and maps $B(0,R)$ onto $G^{(-1)}(B(0,R))$, and $f \circ G_{|G^{(-1)}(B(0,R))}$ defines an element of $W^{m_1,p_1}(G^{(-1)}(B(0,R)))$, then a well known result on the change of variables by means of smooth diffeomorphisms (cf., e.g., Adams [2: Section 3.34, Theorem 3.35/p. 63]), implies that

$$\left(f \circ G_{|G^{(-1)}(B(0,R))}\right) \circ \left(G^{(-1)}_{|B(0,R)}\right) = f_{|B(0,R)}$$

defines an element of $W^{m_1,p_1}(B(0,R))$.

We now prove statement (ii) by induction on $r \in \mathbb{N}$. If $r = 0$, then we can conclude by statement (i). Let statement (ii) hold for $r \in \mathbb{N}$ and assume that the map $g \mapsto f \circ g$ is of class C^{r+1} from $\mathcal{A}$ into $W^{m,p}(\Omega)$. Since $g \mapsto f \circ g$ is of class C^{r+1}, then the same map is of class C^0 and accordingly f defines an element of $W^{m,p}_{\text{loc}}(\mathbb{R}^n)$ by case $r = 0$. By Theorem 2.1/(iii), the element of $W^{m,p}_{\text{loc}}(\mathbb{R}^n)$ defined by f admits a continuous representative $\tilde{f}$, and $\tilde{f}$ is differentiable in the ordinary sense outside of some subset S of measure zero of $\mathbb{R}^n$. Thus the ordinary partial derivatives $\frac{\partial \tilde{f}}{\partial \eta_l}$ $(l \in \{1,\dots,n\})$ exist in $\mathbb{R}^n \setminus S$. Since $f = \tilde{f}$ a.e. in $\mathbb{R}^n$ and since the $\tilde{g}$-preimage of sets of measure zero has measure zero, for all representatives $\tilde{g}$ of $g \in \mathcal{A}$, we have $\tilde{f} \circ g = f \circ g$ for all $g \in \mathcal{A}$. In particular, the map $g \mapsto \tilde{f} \circ g$ is of class C^{r+1} from $\mathcal{A}$ to $W^{m,p}(\Omega)$.

Let $T_{\tilde{f}}[g] := \tilde{f} \circ g$. We now compute $dT_{\tilde{f}}[g]$. Let g be an arbitrary element of $\mathcal{A}$, $h := (h_1,\dots,h_n) \in (W^{m,p}(\Omega))^n$ and

$$\|h\|_{(W^{m,p}(\Omega))^n} = \sum_{l=1}^{n} \|h_l\|_{W^{m,p}(\Omega)} \neq 0.$$

Let $\tilde{g}$, $\tilde{h}$ and $\tilde{h}_l$ be representatives of g, h and h_l, repectively. By Theorem 2.1/(iii), $\tilde{f}$ is differentiable at $\tilde{g}(\xi)$ for all $\xi \in \Omega \setminus \tilde{g}^{\leftarrow}(S)$. Since S has measure zero, the set $\tilde{g}^{\leftarrow}(S)$ has measure zero by our hypothesis on the elements of $\mathcal{A}$. Then we have

$$\lim_{t \to 0} \frac{\tilde{f}(\tilde{g}(\cdot) + t\tilde{h}(\cdot)) - \tilde{f}(\tilde{g}(\cdot))}{t} = \sum_{l=1}^{n} F_l(\tilde{g}(\cdot))\, \tilde{h}_l(\cdot) \qquad \text{a.e. in } \Omega, \tag{34}$$

where F_l denotes the function of $\mathbb{R}^n$ to $\mathbb{R}$ defined by $F_l(\eta) = \frac{\partial \tilde{f}}{\partial \eta_l}(\eta)$ if $\eta \in \mathbb{R}^n \setminus S$ and $F_l(\eta) = 0$ if $\eta \in S$. Since $\mathcal{A}$ is open, for any fixed element $g \in \mathcal{A}$ we have $g + th \in \mathcal{A}$ for $|t|$ sufficiently small. Since $T_{\tilde{f}}$ is differentiable at $g \in \mathcal{A}$, we have

$$\lim_{t \to 0} \frac{\tilde{f}(g(\cdot) + th(\cdot)) - \tilde{f}(g(\cdot))}{t} = dT_{\tilde{f}}[g](h) \qquad \text{in } W^{m,p}(\Omega). \tag{35}$$

Now let $\{t_n\}$ be an arbitrary sequence of non-zero real numbers converging to zero. Since the limiting relation in (35) holds also in $L^p(\Omega)$, the sequence $\{t_n^{-1}[\tilde{f}(\tilde{g}(\cdot) + t_n\tilde{h}(\cdot)) -$

$\tilde{f}(\tilde{g}(\cdot))]\}$ has a subsequence converging almost everywhere in Ω to a representative of $dT_{\tilde{f}}[g](h)$. Then, by (34), we have

$$dT_{\tilde{f}}[g](h) = \sum_{l=1}^{n} F_l(g(\cdot))h_l(\cdot). \tag{36}$$

We now show that Ω must have finite measure. By assumption, the restriction to Ω of the identity map in $\mathbb{R}^n$ belongs to $\mathcal{A} \subseteq (W^{m,p}(\Omega))^n$. Then the inequality

$$\chi_\Omega(\xi) \le \sup\left\{\chi_{[-1,1]^n}(\xi), \chi_\Omega(\xi)\sum_{l=1}^{n}|\xi_l|^p\right\} \qquad \forall\, \xi := (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \tag{37}$$

implies that the measure of Ω is finite. Now let $\tilde{\delta}^j$ be the function of Ω to $\mathbb{R}^n$ with the j-th component equal to 1, and with the remaining components equal to zero. Since the measure of Ω is finite, $\tilde{\delta}^j$ defines an element δ^j of $(W^{m,p}(\Omega))^n$. Since the map $T_{\tilde{f}}$ is of class C^{r+1}, then $dT_{\tilde{f}}$ is a map of of class C^r from $\mathcal{A}$ into the space $\mathcal{L}((W^{m,p}(\Omega))^n, W^{m,p}(\Omega))$. Since the 'evaluation' map $A \mapsto A[\delta^j]$ $(j = 1, \dots, n)$ is linear and continuous from $\mathcal{L}((W^{m,p}(\Omega))^n, W^{m,p}(\Omega))$ into $W^{m,p}(\Omega)$, we conclude that the map $g \mapsto dT_{\tilde{f}}[g](\delta^j) = F_j(g)$ is of class C^r from $\mathcal{A}$ into $W^{m,p}(\Omega)$. Then by inductive assumption, F_j defines an element of $W^{m+r,p}_{\mathrm{loc}}(\mathbb{R}^n)$. By Theorem 2.1/(iii), F_j is a representative of the D_j-distributional derivative of the element of $W^{m,p}_{\mathrm{loc}}(\mathbb{R}^n)$ defined by $\tilde{f}$ (or by f). Thus we can conclude that f defines an element of $W^{m+r+1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ ∎

In part from Theorem 3.6, we deduce the following result.

Proposition 3.7. *Let Ω be a non-empty open subset of $\mathbb{R}^n$ of class $C^{0,1}$. Let $r \in \mathbb{N}$. Let $p, p_1 \in [1, +\infty)$ and $m, m_1 \in \mathbb{N}$ with $(m-1)p > n$. Let f be a function of $\mathbb{R}^n$ to $\mathbb{R}$. Then the following statements hold:*

(i) *If $g \mapsto f \circ g$ maps $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$ to $W^{m_1,p_1}(\Omega)$, then f is a representative of an element in $W^{m_1,p_1}_{\mathrm{loc}}(\mathbb{R}^n)$.*

(ii) *Let $m_1 > 0$. If the map $g \mapsto f \circ g$ is of class C^r from $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$ to $W^{m_1,p_1}(\Omega)$, then f is a representative of an element in $W^{m_1+r,p_1}_{\mathrm{loc}}(\mathbb{R}^n)$.*

(iii) *Let $m_1 = 0$ and $p_1 > 1$. If the map $g \mapsto f \circ g$ is of class C^r from $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$ to $W^{m_1,p_1}(\Omega)$, then f is a representative of an element in $W^{m_1+r,p_1}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. By Proposition 3.3, the set $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$, which equals $\mathcal{K}_{m,p,0}(\Omega, \mathbb{R}^n)$, is open in $(W^{m,p}(\Omega))^n$. Obviously, $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$ contains the elements of $(W^{m,p}(\Omega))^n$ defined by the restrictions to Ω of invertible affine maps. Furthermore, if $g \in \mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$, and if $\tilde{g}$ is the unique representative of class $(C^1(\mathrm{cl}\,\Omega))^n$ of g, then $\tilde{g}_{|\Omega}$ is a local diffeomorphism of Ω onto $\tilde{g}(\Omega)$, and thus, as we have already pointed out at the beginning of the proof of Theorem 2.6, $\tilde{g}_{|\Omega}^{(-1)}(N)$ has measure zero, whenever N is a subset of measure zero of $\mathbb{R}^n$. Then we can conclude the proof of statement (i) by Lemma 2.5/(ii) and Theorem 3.6.

We now turn to prove by induction statements (ii) and (iii) at one time. As in the previous proof, we proceed by induction on r. If $r = 0$, then we can conclude by statement (i). Let the statement (ii) or (iii) hold for $r \in \mathbb{N}$ and assume that the map $T_f[\cdot]$ defined by $T_f[g] = f \circ g$ is of class C^{r+1} from $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$ into $W^{m_1,p_1}(\Omega)$. By case $r = 0$, f defines an element of $W^{m_1,p_1}_{\text{loc}}(\mathbb{R}^n)$. In the case (ii), we have $f \in W^{m_1,p_1}_{\text{loc}}(\mathbb{R}^n) \subseteq W^{1,p_1}_{\text{loc}}(\mathbb{R}^n)$. We now prove that $f \in W^{1,p_1}_{\text{loc}}(\mathbb{R}^n)$ holds also if $m_1 = 0$ and $p_1 > 0$. Let $\delta^j \in (W^{m,p}(\Omega))^n$ be as in the proof of Theorem 3.6. By assumption, T_f is differentiable at all invertible affine transformations G. Then

$$\lim_{t\to 0} \left\| \frac{f(G(\cdot) + t\delta^j) - f(G(\cdot))}{t} - dT_f[G](\delta^j) \right\|_{L^{p_1}(\Omega)} = 0. \tag{38}$$

By changing the variable in the integral which defines the norm in the limiting relation of (38) by means of the affine transformation $G^{(-1)}$ (cf., e.g., Adams [2: Section 3.34/p. 63]), we obtain

$$\lim_{t\to 0} \frac{f(\text{id}_{G(\Omega)} + t\delta^j) - f(\text{id}_{G(\Omega)})}{t} = \{dT_f[G](\delta^j)\} \circ G^{(-1)}$$

in $L^{p_1}(G(\Omega))$ $(j = 1, \dots, n)$, where $\text{id}_{G(\Omega)}$ denotes the identity map in $G(\Omega)$. Since $\Omega \neq \emptyset$, for all open and relatively compact subset ω of $\mathbb{R}^n$, there exists G as above such that $G(\Omega) \supseteq \omega$. Then the well-known difference quotient method and the assumption $p_1 > 1$ imply that $f \in W^{1,p_1}_{\text{loc}}(\mathbb{R}^n)$ (cf., e.g., Troianiello [30: Theorem 1.21/p. 43]). Thus, in both cases (ii) and (iii), we have $f \in W^{1,p_1}_{\text{loc}}(\mathbb{R}^n)$. Now let $g^{\#} \in \mathcal{K}_{m,p,0}(\Omega, \mathbb{R}^n)$. Assume that the ball $B(0, R_{g^{\#}})$ contains $\tilde{g}^{\#}(\text{cl}\Omega)$, where $\tilde{g}^{\#}$ is the representative of class $(C^1(\text{cl}\Omega))^n$ of $g^{\#}$. By Proposition 3.3, $\mathcal{K}_{m,p,0}(\Omega, B(0, R_{g^{\#}}))$ is an open neighborhood of $g^{\#}$ contained in $\mathcal{K}_{m,p,0}(\Omega, \mathbb{R}^n)$, and by Theorem 3.4 the map $g \mapsto f \circ g$ is of class C^1 from $\mathcal{K}_{m,p,0}(\Omega, B(0, R_{g^{\#}}))$ to $L^{\min\{p,p_1\}}(\Omega)$. Then the map $g \mapsto f \circ g$ is of class C^1 from $\mathcal{K}_{m,p,0}(\Omega, \mathbb{R}^n)$, which coincides with $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$, to $L^{\min\{p,p_1\}}(\Omega)$, with differential delivered by (27), with $v = 0$. By inductive assumption, both in cases (ii) and (iii), T_f is of class C^1 from $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$ to $W^{m_1,p_1}(\Omega)$, and thus to $L^{\min\{p,p_1\}}(\Omega)$. Then

$$dT_f[g](w) = \sum_{l=1}^{n} [(D_l f) \circ g] w_l \qquad \text{for all } w := (w_1, \dots, w_n) \in (W^{m,p}(\Omega))^n.$$

By computing $dT_f[g]$ on δ^j as in the proof of Theorem 3.6, we conclude that $g \mapsto (D_j f) \circ g$ is of class C^r from $\mathcal{G}_{m,p,0}(\Omega, \mathbb{R}^n)$ to $W^{m_1,p_1}(\Omega)$, for all $j \in \{1, \dots, n\}$. Then, by inductive assumption, each of the representatives of $D_j f$ defines an element of class $W^{m_1+r,p_1}_{\text{loc}}(\mathbb{R}^n)$, and accordingly $f \in W^{(m_1+r)+1,p_1}_{\text{loc}}(\mathbb{R}^n)$ ∎

Remark 3.8. Concerning Theorem 3.6/(i), we mention that Marcus and Mizel [22], Bourdaud and Meyer [7], Bourdaud [4, 5], Bourdaud and Kateb [6], and Sickel [27] have investigated the problem of characterizing the f's of one real variable such that $g \mapsto f \circ g$ maps $W^{m,p}(\mathbb{R}^n)$ to itself. Their results cover large classes of values of the exponents m and p and their approach is different from that of ours. If $n = 1$, then the statement (i) of Theorem 3.6 becomes a variant of the corresponding results of Bourdaud [4] and of Sickel [27].

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