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Liouville Theorems for Fuchsian-Type Operators on the Heisenberg Group

M. R. Lancia and M. V. Marchi

Abstract. We prove some Liouville theorems for a degenerate elliptic operator whose principal part is given in divergence form with respect to the Heisenberg vector fields and lower terms satisfy Fuchsian-type conditions with respect to the intrinsic norm.

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1. Introduction

In this paper we study the asymptotic behavior of the local solutions of the equation

$$Lu = 0 \tag{1.1}$$

where the operator L is given by

$$Lu = -\sum_{i,j=1}^{2n} X_j^*\big(a_{ij}(x)X_iu + d_j(x)u\big) + \sum_{i=1}^{2n} b_i(x)X_iu + c(x)u \tag{1.2}$$

and X_j are the Heisenberg vector fields in $\mathbb{R}^{2n+1}$, X_j^* is the L^2-adjoint of X_j. The operator L is assumed to be uniformly subelliptic and weakly Fuchsian with respect to the intrinsic dilations. More precisely we will assume the following:

(A) a_{ij} are measurable functions on $\mathbb{R}^{2n+1}$, and there exist positive constants $\mu \le M$ such that

$$\mu|\xi|^2 \le \sum_{i,j=1}^{2n} a_{ij}(x)\xi_i\xi_j \le M|\xi|^2 \tag{1.3}$$

for all $x \in \mathbb{R}^{2n+1}$ and $\xi \in \mathbb{R}^{2n}$.

M. R. Lancia: Università di Roma "La Sapienza", Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Via A. Scarpa 10, 00161 Roma, Italia
M. V. Marchi: Università di Roma "La Sapienza", Dipartimento di Matematica, Piazzale A. Moro 2, 00185 Roma, Italia

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(B) There exist $q > 2n+2$, $\Lambda > 0$, $0 < a < 1 < b$ and a sequence $R_k \nearrow +\infty$ such that, for every $k \geq 1$,

$$\rho b_i \in L^q(A_k), \quad \rho d_i \in L^q(A_k) \quad (1 \leq i \leq n), \qquad \rho^2 c \in L^{\frac{q}{2}}(A_k) \tag{1.4}$$

and

$$\begin{gathered} \|\rho b_i\|_{L^q(A_k)} \leq \Lambda |A_k|^{\frac{1}{q}}, \quad \|\rho d_i\|_{L^q(A_k)} \leq \Lambda |A_k|^{\frac{1}{q}} \qquad (1 \leq i \leq 2n) \\ \|\rho c\|_{L^{\frac{q}{2}}(A_k)} \leq \Lambda |A_k|^{\frac{2}{q}} \end{gathered} \tag{1.5}$$

where ρ is the norm defined in (2.3), intrinsically associated with the vector fields X_j, A_k are the open annuli:

$$A_k = \left\{ x \in \mathbb{R}^{2n+1} : aR_k < \rho(x) < bR_k \right\}$$

and $|A_k|$ denotes the Lebesgue measure of A_k.

We will also assume that L satisfies the Maximum Principle (see Section 3).

The topics include the following results.

Proposition 4.4. *For every two positive local solutions u and v of L in $\mathbb{R}^{2n+1} \setminus B_R$, there exists*

$$\lim_{\rho(x) \to +\infty} \frac{u(x)}{v(x)}. \tag{1.6}$$

where B_R denotes the intrinsic ball defined in (2.8).

Theorem 4.5 (Liouville Theorem). *There exists a unique, up to a constant, positive local solution of the equation* (1.1) *in* $\mathbb{R}^{2n+1}$.

When L is a Fuchsian operator (see condition (4.11)), we can prove also the following statement.

Theorem 4.7 (Liouville Property). *Every bounded local solution of the equation* (1.1) *in* $\mathbb{R}^{2n+1}$ *is of constant sign.*

Our method to study the asymptotic behavior of the solutions is to show, as in [9, 10, 14, 15], that the positive solutions satisfy a uniform Harnack inequality. The local Harnack inequality for the operator defined in (1.2) has been proved by M. Biroli and U. Mosco in [3], under the assumptions a_{ij} symmetric and $b_i \equiv d_i \equiv c \equiv 0$, in the more general context of the Dirichlet forms. In [13] there is given a local Harnack inequality for the operator (1.2), where X_j are Hörmander vector fields.

In order to have a uniform Harnack inequality we need some control on the growth of the lower terms (see, for example, [1, 5, 14, 15]). In the classical case this control is given by Fuchsian assumptions, related to the homogeneity of the differential operators $\frac{\partial}{\partial x_i}$ with respect to the usual dilations. In our case the assumptions on the growth of the lower terms reflect the non-isotropic character of the fields X_j. The Liouville theorems for the Heisenberg Laplacian has been proved by B. Gaveau [8]. For the nonlinear case see also [2] and [7].

The plan of the paper is the following. In Section 2 we recall the main properties of the Heisenberg group and of the associated Sobolev spaces. In Section 3 we recall some results about the operator L in bounded domains, we discuss the Maximum Principle and the existence and uniqueness for the Dirichlet problem, and we prove an existence theorem for the equation (1.1) in $\mathbb{R}^{2n+1}$. In Section 4, by assuming L to be a weakly Fuchsian operator, we prove the existence of the limit (1.6) and Liouville's theorems.

Finally we recall that all the previous results still hold for the operator

$$\widetilde{L}u = Lu + g(x)X_0u \tag{1.7}$$

where L, defined in (1.2), is assumed to satisfy assumptions (A) and (B) and, in addition, the following one:

(C) X_0 is a bounded smooth vector field in $\mathbb{R}^{2n+2}$, i.e.

$$X_0 = \sum_{i=1}^{2n} \sigma_i X_i + \sigma_{2n+1}T \qquad (\sigma_i \in C^\infty(\mathbb{R}^{2n+1}),\ i = 1, \ldots, 2n+1)$$

with bounded coefficients, i.e.

$$\sigma_i \in L^\infty(R^{2n+1}) \qquad (i = 1, \ldots, 2n+1) \tag{1.8}$$

such that, for some $1 \leq h \leq n$ and every $k \geq 1$,

$$\rho X_h(\sigma_{2n+1}) \in L^q(A_k) \qquad \text{and} \qquad \rho X_{h+n}(\sigma_{2n+1}) \in L^q(A_k) \tag{1.9}$$

with

$$\|\rho X_h(\sigma_{2n+1})\|_{L^q(A_k)} \leq \Lambda |A_k|^{\frac{1}{q}} \qquad \text{and} \qquad \|\rho X_{h+n}(\sigma_{2n+1})\|_{L^q(A_k)} \leq \Lambda |A_k|^{\frac{1}{q}}.$$

(D) g is a bounded measurable function such that

$$\rho g \in L^q(A_k), \qquad \rho X_h g \in L^q(A_k), \qquad \rho X_{h+n} g \in L^q(A_k) \tag{1.10}$$

with

$$\begin{aligned} \|\rho g\|_{L^q(A_k)} &\leq \Lambda |A_k|^{\frac{1}{q}} \\ \|\rho X_h g\|_{L^q(A_k)} &\leq \Lambda |A_k|^{\frac{1}{q}} \\ \|\rho X_{h+n} g\|_{L^q(A_k)} &\leq \Lambda |A_k|^{\frac{1}{q}}. \end{aligned} \tag{1.11}$$

2. The Heisenberg group

In this Section we recall the main properties of the Heisenberg group that we will use later. For more details see [17].

The space $\mathbb{R}^{2n+1}$, whose elements we denote by $x = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$, equipped with the multiplication law

$$x \cdot x' = \left(x_1 + x_1', \ldots, x_n + x_n', y_1 + y_1', \ldots, y_n + y_n', t + t' + \sum_{i=1}^{n} (x_i' y_i - x_i y_i') \right) \quad (2.1)$$

is a group whose identity is the origin and where the inverse is given by

$$x^{-1} = (-x_1, \ldots, -x_n, -y_1, \ldots, -y_n, -t).$$

The space $\mathbb{R}^{2n+1}$ with the structure (2.1) is the Heisenberg group, denoted by H^n. The non-isotropic dilations

$$\delta \circ x = (\delta x_1, \delta x_n, \delta y_1, \delta y_n, \delta^2 t) \qquad (\delta \in \mathbb{R},\, x \in H^n) \quad (2.2)$$

are automorphisms of H^n. The non-negative function

$$\rho(x) = \left(\sum_{i=1}^{n} (x_i^2 + y_i^2)^2 + t^2 \right)^{\frac{1}{4}} \quad (2.3)$$

is a norm for the Heisenberg group, in particular it is homogeneous of degree 1 with respect to the dilations (2.2), i.e.

$$\rho(\delta \circ x) = |\delta| \rho(x) \quad (2.4)$$

for every $x \in H^n$ and $\rho \in \mathbb{R}$. Moreover, there exist positive constants c_1 and c_2 such that

$$c_1 |x| < \rho(x) < c_2 |x|^{\frac{1}{2}} \quad (2.5)$$

for every $x \in H^n$, where $|x|$ denotes the Euclidean norm in $\mathbb{R}^{2n+1}$.

By (2.4) and (2.5) it follows that the function d defined by

$$d(x, x') = \rho(x'^{-1} \cdot x) \quad (2.6)$$

is a distance in H^n, topologically equivalent to the Euclidean one and left invariant with respect to the law (2.1). By using the distance d we define the intrinsic balls, spheres and neighborhoods of infinity

$$B_R(x) = \{x' \in H^n : d(x, x') < R\}$$
$$S_R(x) = \{x' \in H^n : d(x, x') = R\}$$
$$D_R(x) = \mathbb{R}^{2n+1} \setminus B_R(x).$$

In the following, for brevity, we will set

$$B_R = B_R(0), \qquad S_R = S_R(0), \qquad D_R = D_R(0). \tag{2.8}$$

The Lebesgue measure $dx = dx_1 \cdots dx_n \cdot dy_1 \cdots dy_n \cdot dt$ is invariant with respect to the translations (2.1), so that for any $x \in H^n$ and $R > 0$ we have

$$|B_R(x)| = |B_R|. \tag{2.9}$$

Since the Jacobian of the dilations (2.2) is given by

$$J_\delta = \delta^{2n+2} \tag{2.10}$$

from (2.9) and (2.10) it follows that for every $x \in H^n$ and $R > 0$

$$|B_R(x)| = R^{2n+2}|B_1|. \tag{2.11}$$

In our context it is convenient to look for a basis of vector fields invariant with respect to the translations (2.1). Such a basis is given by

$$\begin{aligned} X_i &= \begin{cases} \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} & \text{for } i = 1, \dots, n \\ \frac{\partial}{\partial y_{i-n}} + 2x_{i-n} \frac{\partial}{\partial t} & \text{for } i = n+1, \dots, 2n \end{cases} \\ T &= \frac{\partial}{\partial t}. \end{aligned} \tag{2.12}$$

For the vector fields (2.12) we have the commutative law

$$[X_i, X_{i+n}] = -4T \qquad \text{for every } \ i = 1, \dots, n \tag{2.13}$$

while the other commutators vanish. We recall that a commutator of two vector fields V_1 and V_2 is the new vector field given by

$$[V_1, V_2] = V_1 V_2 - V_2 V_1. \tag{2.14}$$

Therefore, $X_1, \dots, X_{2n}$ are a basis for the Lie algebra of the vector fields invariant with respect to (2.1). Moreover, they are homogeneous of degree 1 with respect to the dilations (2.2), whereas, by (2.13), T is homogeneous of degree 2, i.e.

$$X_i(u(\delta \circ x)) = \delta((X_i u)(\delta \circ x)) \qquad (i = 1, \dots, 2n) \tag{2.15}$$

and

$$T(u(\delta \circ x)) = \delta^2((Tu)(\delta \circ x)). \tag{2.16}$$

Given an open set $\Omega \subseteq \mathbb{R}^{2n+1}$, we denote by $S^2(\Omega)$ the Sobolev-type space of the functions $u \in L^2(\Omega)$, such that the distribution derivatives $X_i u \in L^2(\Omega)$ for $i = 1, \dots, 2n$. The norm in $S^2(\Omega)$ is given by

$$|||u|||^2 = \int_\Omega \left(|u|^2 + \sum_{i=1}^{2n} |X_i u|^2 \right) dx. \tag{2.17}$$

The closure of $C_0^\infty(\Omega)$ in the above norm is denoted by $\mathring{S}^2(\Omega)$. By $S^2_{\text{loc}}(\Omega)$ we mean the set of functions $u \in S^2(\Omega')$ for every $\Omega' \subset\subset \Omega$.

We recall the following properties of the above Sobolev spaces.

Poincaré Inequality (see [11]). For every $B_R(x) \subset \mathbb{R}^{2n+1}$ and $u \in \mathring{S}^2(B_R(x))$ we have

$$\int_{B_R(x)} |u|^2 dx \leq cR^2 \int_{B_R(x)} \sum_{j=1}^{2n} |X_j u|^2 dx. \tag{2.18}$$

Sobolev Inequality (see [12]). There exists $S > 0$ such that for every $u \in S^2(\mathbb{R}^{2n+1})$ we have

$$\left(\int_{\mathbb{R}^{2n+1}} |u|^{2^*} dx \right)^{\frac{2}{2^*}} \leq S \int_{\mathbb{R}^{2n+1}} \sum_{j=1}^{2n} |X_j u|^2 dx \tag{2.19}$$

where $2^* = \frac{2n+2}{n}$.

Compact Embedding (see [6]). For every bounded domain Ω the Sobolev space $\mathring{S}^2(\Omega)$ is compactly embedded into $L^p(\Omega)$, for every $p < 2^*$.

3. Dirichlet problems and maximum principles

Let Ω be an open subset of $\mathbb{R}^{2n+1}$. We consider the differential operator

$$Lu = -\sum_{i,j=1}^{2n} X_j^* \big(a_{ij}(x) X_i u + d_j(x) u\big) + \sum_{i=1}^{2n} b_i(x) X_i u + c(x) u \tag{3.1}$$

whose associated bilinear form is given by

$$a(u, v) = \int_\Omega \left(\sum_{i,j} \big(a_{ij} X_i u X_j v + b_i (X_i u) v + d_j u X_j v\big) + cuv \right) dx \tag{3.2}$$

for $u \in S^2_{\text{loc}}(\Omega)$ and $v \in \mathring{S}^2(\Omega)$.

Let $f_j \in L^2(\Omega)$ $(j = 1, \dots, 2n)$ and $f \in L^{2^{*'}}(\Omega)$.

Definition 3.1. We say that the function $u \in S^2_{\text{loc}}(\Omega)$ is a *local solution* of the equation

$$Lu = -\sum_{j=1}^{2n} X_j^* f_j + f \tag{3.3}$$

in Ω, if for every $\varphi \in \mathring{S}^2(\Omega)$ we have

$$a(u, \varphi) = \int_\Omega \left(\sum_j f_j X_j \varphi + f \varphi \right) dx. \tag{3.4}$$

When $u \in S^2(\Omega)$ or $u \in \mathring{S}^2(\Omega)$), we say that u is a *solution* or a *solution vanishing on the boundary*, respectively, of the equation (3.3). For brevity, when $f_j \equiv f \equiv 0$ $(i = 1, \dots, 2n)$, we say that u is a *local solution* or a *solution*, respectively, of (the operator) L.

We start recalling some results concerning the operator L when Ω is a bounded domain in $\mathbb{R}^{2n+1}$. In the following we will assume that L satisfies condition (A) on Ω, $b_i \in L^q(\Omega)$ and $d_i \in L^q(\Omega)$ $(i = 1, \dots, 2n)$, $c \in L^{\frac{q}{2}}(\Omega)$ with $q > 2n + 2$.

Proposition 3.2. *Let T be a linear bounded operator on $\mathring{S}^2(\Omega)$, $\Omega \subset \mathbb{R}$ and $\sigma_0 \in \mathbb{R}$. Then, whenever σ_0 is sufficiently large or* $\operatorname{diam}\Omega$ *is sufficiently small, the equation*

$$Lu + \sigma_0 u = T \tag{3.5}$$

admits exactly one solution $u_{\sigma_0,T} \in \mathring{S}^2(\Omega)$. In particular this holds for the equation

$$Lu + \sigma_0 u = -\sum_{j=1}^{2n} X_j^* f_j + f \tag{3.6}$$

where $f_j \in L^2(\Omega)$ $(j = 1, \dots, 2n)$ and $f \in L^{2^{*'}}(\Omega)$. Furthermore,the map $L_{\sigma_0}^{-1} : T \mapsto u_{\sigma_0,T}$ is continuous.

Proof. See [16] ∎

Proposition 3.3 (Caccioppoli Inequality). *Let $\Omega' \subset\subset \Omega$. Then there exists a positive constant k such that, for every local solution or non-negative subsolution u of L in Ω, we have*

$$\sum_{j=1}^{2n} \|X_j u\|_{L^2(\Omega')} \leq k\|u\|_{L^2(\Omega)}. \tag{3.7}$$

Proof. See [16: Lemma 5.2] or [13: Lemma 4.4] ∎

Proposition 3.4 (Local Harnack Inequality). *For any compact $K \subset \Omega$ there exists a positive constant $C = C(M, \mu, K, \Omega, \|b_i\|_{L^q}, \|d_i\|_{L^q}, \|c\|_{L^{\frac{q}{2}}})$ such that, for any positive local solution u of L in Ω,*

$$Cu(x) < u(y) \tag{3.8}$$

holds for every $x, y \in K$.

Proof. For every $B_R(x) \subset \Omega$ with R sufficiently small, we can prove the following:

Step 1: For every $p > 0$ there exists a positive constant c_p such that

$$\sup_{B_R(x)} u \leq c_p \left(\fint_{B_{2R}(x)} |u|^p dx \right)^{\frac{1}{p}}$$

(see [13: Theorem 4.2]).

Step 2: From Step 1 it follows that for every $q < 0$ there exists a positive constant c_q such that

$$\inf_{B_R(x)} u \geq c_q \left(\fint_{B_{2R}(x)} |u|^q dx \right)^{\frac{1}{q}}$$

(see [13: Proposition 4.8]).

Step 3: There exist $\delta \in (0,1)$ and $A \geq 1$ such that

$$\fint_{B_{2R}(x)} |u|^\delta dx \cdot \fint_{B_{2R}(x)} |u|^{-\delta} dx \leq A$$

(see [13: Proposition 5.2]).

The proof follows from Steps 1-3 and from compactness arguments ∎

Remark 3.5. We can prove that Step 1 still holds for non-negative solutions and positive subsolutions and Step 3 for positive supersolutions (see [16: Section 8]).

Proposition 3.6 (Hölder continuity). *For any compact $K \subset \Omega$ there exist positive constants δ, c, c' and $0 < \lambda < 1$ such that, for any local solution u of L in Ω,*

$$|u(x) - u(y)| \leq c'\|u\|_{L^2} d(x,y)^\lambda \leq c\|u\|_{L^2}|x-y|^{\frac{\lambda}{2}} \tag{3.9}$$

holds for every $x, y \in K$ with $d(x,y) < \delta$.

Proof. See [13: Theorem 5.6] ∎

Remark 3.7. The previous results still hold under the weaker assumption $b_i \in L^{2n+2}(\Omega)$ $(i = 1, \ldots, 2n)$, but in this case the constants involved in the inequalities depend on b_i and not only on $\|b_i\|_{L^q}$. In the next section, in order to prove a uniform local Harnack inequality, we need the constant C of inequality (3.8) to depend only on $\|b_i\|_{L^q}$, therefore we assume $b_i \in L^q$ from now on.

Following [10: Theorem 8.3], we will prove the existence for the Dirichlet problem via the Maximum Principle. In order to formulate it we need the following notion of supremum on the boundary for a function $u \in S^2_{\text{loc}}(\Omega)$:

$$\sup_{\partial\Omega} u = \inf\{k \in \mathbb{R} : u \leq k\}$$

x where by $u \leq k$ on $\partial\Omega$ and $u \geq k$ on $\partial\Omega$ we mean $(u-k)^+ \in \mathring{S}^2(\Omega)$ and $(u-k)^- \in \mathring{S}^2(\Omega)$, respectively. If $u \leq k$ and $u \geq k$ on $\partial\Omega$, we say that $u = k$ on $\partial\Omega$.

Weak Maximum Principle. Let u be a local solution of L in a bounded domain Ω. Then

$$\sup_\Omega u \leq \max\left(0, \sup_{\partial\Omega} u\right) \qquad \text{and} \qquad \min\left(0, \inf_{\partial\Omega} u\right) \leq \inf_\Omega u$$

where $\inf_{\partial\Omega} u = -\sup_{\partial\Omega}(-u)$.

In the following we will need also the

Strong Maximum Principle. Let u be a non-negative local solution of L on a domain Ω. Then either $u \equiv 0$ or $u > 0$ on Ω.

From now on we will assume that the operator L satisfies the Weak Maximum Principle.

Proposition 3.8. *Let $u \in \mathring{S}^2(\Omega)$ be a solution of L. Then $u \equiv 0$.*

From Proposition 3.8 and the Fredholm alternative there follows an existence and uniqueness theorem for the Dirichlet problem in bounded domains, analogous to [10: Theorem 8.3]. We give here the simplified version that we will use later.

Proposition 3.9 (Existence Theorem). *For every constant k there exists exactly one solution $u \in S^2(\Omega)$ of*

$$\left.\begin{aligned} Lu &= 0 \quad on\ \Omega \\ u &= k \quad on\ \partial\Omega. \end{aligned}\right\} \tag{3.10}$$

Proof. The problem (3.10) is equivalent to the problem

$$Lw = k\left(\sum_{j=1}^{2n} X_j^* d_j - c\right) \qquad (w \in \mathring{S}^2(\Omega)) \tag{3.11}$$

where $w = u - k$. Let σ_0 be sufficiently large so that the equation

$$Lv + \sigma_0 v = k\left(\sum_{j=1}^{2n} X_j^* d_j - c\right) \tag{3.12}$$

admits exactly one solution $v \in \mathring{S}^2(\Omega)$. Then the equation (3.11) is equivalent to the equations

$$(Lw + \sigma_0 w) - \sigma_0 w = k\left(\sum_{j=1}^{2n} X_j^* d_j - c\right) \tag{3.13}$$

$$w - \sigma_0 L_{\sigma_0}^{-1} I w = k L_{\sigma_0}^{-1}\left(\sum_{j=1}^{2n} X_j^* d_j - c\right) \tag{3.14}$$

where I is the compact embedding of $\mathring{S}^2(\Omega)$ into $L^2(\Omega)$ and $L^{-1}\sigma_0$ is the map defined in Proposition 3.2. By the Compact Embedding Property of Sobolev spaces and Proposition 3.2, $\sigma_0 L_{\sigma_0}^{-1} I$ is a compact operator from $\mathring{S}^2(\Omega)$ into itself so that, by Proposition 3.8 and the Fredholm alternative, there exists exactly one solution of (3.14) and so of (3.10) ∎

Theorem 3.10. *Let L satisfy assumption* (A) *in $\mathbb{R}^{2n+1}$, $b_i \in L^q_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ and $d_i \in L^q_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ $(i = 1, \ldots, 2n)$, $c \in L^{\frac{q}{2}}_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ with $q > 2n+2$. Let us assume that L satisfies also the Strong Maximum Principle. Then there exists at least a positive local solution $u \in S^2_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ of L.*

Proof. Let $R_k \nearrow +\infty$. By Proposition (3.9) there exists a sequence $\{v_k\}$ of positive solutions of the problems

$$\left.\begin{aligned} Lv_k &= 0 \quad \text{on } B_{R_k} \\ v_k &= 1 \quad \text{on } \partial B_{R_k}. \end{aligned}\right\}$$

Let us consider the sequence of solutions $u_k = \frac{v_k}{v_k(0)}$. Fixed $N \geq 1$, the sequence $\{u_k\}$ converges in $\overline{B}_{R_N}$ to a continuous positive function $u \in S^2(\overline{B}_{R_N})$. Actually, by the local Harnack inequality, there exists $c_N > 0$ such that for every $k > N$

$$c_N u_k(x) \leq u_k(0) = 1. \tag{3.15}$$

Hence

$$\|u_k\|_{L^2(\overline{B}_{R_N})} \leq \|u_k\|_{L^\infty(\overline{B}_{R_N})} |B_{R_N}|^{\frac{1}{2}} \leq \frac{|B_{R_N}|^{\frac{1}{2}}}{c_N}. \tag{3.16}$$

By (3.16) and (3.9) the sequence $\{u_k\}$ is bounded in $C(\overline{B}_{R_N})$ and uniformly equicontinuous. So by Ascoli's Theorem it converges (up to a subsequence) to a continuous function u, that is a positive local solution of L in $\mathbb{R}^{2n+2}$. Actually, by the Caccioppoli inequality and (3.16), the sequence $\{\|X_i u_k\|_{L^2(B_{R_N})}\}$ is bounded, so $\{u_k\}$ converges in every $\overline{B}_{R_N}$ to u also in the norm (2.17) ∎

We give now two sufficient conditions for the Maximum Principles to hold.

Proposition 3.11. *Let $-\sum_{i,j=1}^{2n} X_j^* d_j + c$ be positive in the sense of distibutions on Ω, i.e.*

$$\int_\Omega \left(\sum_j d_j X_j \varphi + c\varphi \right) dx \geq 0 \qquad \text{for every } \ 0 \leq \varphi \in \mathring{S}^2(\Omega). \tag{3.17}$$

Then the Weak Maximum Principle holds.

Proof. For sake of completeness we give the proof, that is analogous to that of [10: Theorem 8.1]. Let u be a local solution of L. Let us suppose that there exist constants k such that

$$\max\left(0, \sup_{\partial\Omega} u\right) < k < \sup_\Omega u. \tag{3.18}$$

Set $v_k = (u-k)^+$. Since $v_k \in \mathring{S}^2(\Omega)$, we have

$$\begin{aligned} 0 &= a(u, v_k) \\ &= \int_\Omega \left(\sum_j \Big(a_{i,j} X_i u X_j v_k + (b_i - d_i)(X_i u) v_k + d_i X_i(u v_k) \Big) + c u v_k \right) dx. \end{aligned} \tag{3.19}$$

From (3.17), (3.19) and $uv_k \geq 0$ it follows

$$\int_\Omega \sum_{i,j} a_{ij} X_i u X_j v_k \, dx \leq \int_\Omega \sum_i (d_i - b_i)(X_i u) v_k \, dx. \tag{3.20}$$

Hence, from the assumptions on the coefficients of L and from the Sobolev inequality, it follows

$$\begin{aligned}
\mu\sum_j \|X_j v_k\|_{L^2}^2 &\le \int_\Omega \sum_{i,j} a_{ij} X_i v_k X_j v_k \, dx \\
&= \int_\Omega \sum_{i,j} a_{ij} X_i u X_j v_k \, dx \\
&\le \int_\Omega \sum_i (d_i - b_1)(X_i u) v_k \, dx \\
&= \int_\Omega \sum_i (d_i - b_i)(X_i v_k) v_k \, dx \\
&\le K \sum_i |\operatorname{supp} X_i v_k|^{\frac{q-2n-2}{q(2n+2)}} \sum_i \|b_i - d_i\|_{L^q} \sum_i \|X_i v_k\|_{L^2}^2 .
\end{aligned} \tag{3.21}$$

Therefore for every k that satisfies (3.18) we have

$$|\operatorname{supp} X_i v_k| \ge \mu^{\frac{q(2n+2)}{q-2n-2}} \left(\sum_i \|b_i - d_i\|_{L^q} \right)^{\frac{q(2n+2)}{q-2n-2}} . \tag{3.22}$$

So u attains its supremum on a set of positive measure where $X_i u \neq 0$. This contradicts the existence of k satisfying (3.18) ∎

Proposition 3.12. *Let L satisfy assumption* (3.17). *Then the Strong Maximum Principle holds.*

Proof. Let u be a non-negative local solution of L. By assumption (3.17), $(u+\varepsilon)$ is a positive local supersolution of L and $(u+\varepsilon)^q$ $(q<0)$ a positive local subsolution of $\widetilde{L}$ where

$$\widetilde{L} = -\sum_{i,j=1}^{2n} X_j^*\big(a_{ij}(x)X_i u + q d_j(x) u\big) + \sum_{i=1}^{2n} \big(b_i(x) + (q-1)d_i(x)\big)X_i u + qc(x)u$$

so that Steps 2 and 3 of Proposition 3.4 still hold for $(u+\varepsilon)$. By the monotone convergence theorem they also hold for u. Therefore, by inequality (3.8) applied to u, it follows that either $u \equiv 0$ or $u > 0$ ∎

Proposition 3.13. *Let there exists a positive local solution w of L in Ω, positive on $\partial\Omega$ and such that $X_i w \in L^q(\Omega)$ with $q > 2n+2$, for every $i = 1, \dots, 2n$. Then L satisfies the Weak and Strong Maximum Principles.*

Proof. Let u be a local solution of L non-negative on $\partial\Omega$. Set $v = \frac{u}{w}$. For every $\varphi \in \mathring{S}^2(\Omega)$ we have

$$0 = a(u,\varphi) = a(vw,\varphi) = a(w, v\varphi) + \tilde{a}(v,\varphi) = \tilde{a}(v,\varphi)$$

where

$$\tilde{a}(v,\varphi) = \int_\Omega \left(\sum_{i,j} w a_{ij} X_i v X_j \varphi + \left(\sum_i (wb_i - wd_i) - \sum_{i,j} a_{ij} X_j w \right) (X_i v) \varphi \right) dx \,.$$

By the assumptions on w, the coefficients of $\tilde{a}$ satisfy the assumptions of Proposition 3.11. Therefore from $v \geq 0$ on $\partial\Omega$ it follows that v, and so u, are non-negative on Ω and also that they are either positive or identically equal to zero ∎

For a thorough discussion of the Maximum Principle see also [4] and the included references.

4. Liouville Theorems

In this section we prove Liouville's theorems assuming L to be a Fuchsian-type operator.

Definition 4.1. We say that the operator defined in (1.2) is a *Fuchsian operator in the weak sense*, if there exist $0 < a < 1 < b$ and $R_k \nearrow +\infty$ such that L satisfies assumptions (A) and (B).

From now on we will assume that L is a Fuchsian operator in the weak sense.

Proposition 4.2 (Uniform local Harnack inequality). *There exists a positive constant C depending on μ, M, Λ, a, b, a' and b' such that, for every $k \geq 1$ and every positive solution u of L in A_k, we have*

$$Cu(x) < u(y) \qquad \textit{for all } \; x, y \in A'_k \tag{4.1}$$

where A'_k is the annulus

$$A'_k = \left\{ x \in \mathbb{R}^{2n+1} : a' R_k < \rho(x) < b' R_k \right\} \tag{4.2}$$

and $a < a' < 1 < b' < b$.

Proof. Let u be a positive solution of L in A_k. Let us define in the annulus

$$A = \left\{ \xi \in \mathbb{R}^{2n+1} : a < \rho(\xi) < b \right\}$$

the function

$$w(\xi) = w_k(\xi) = u(R_k \circ \xi).$$

By the homogeneity of X_j with respect to the dilations (2.2) and by (2.10), we have for every $\varphi \in \mathring{S}^2(A_k)$

$$
\begin{aligned}
0 &= \int_{A_k} \Bigg(\sum_{i,j} \Big(a_{ij}(x) X_i u(x) X_j \varphi(x) + b_i(x)(X_i u(x))\varphi(x) \\
&\quad + d_j(x) u(x) X_j \varphi(x) \Big) + c(x) u(x) \varphi(x) \Bigg) dx \\
&= \int_A \Bigg(\sum_{i,j} \Big(\frac{1}{R_k^2} a_{ij}(R_k \circ \xi) X_i w(\xi) X_j \psi(\xi) \\
&\quad + \frac{1}{R_k} b_i(R_k \circ \xi)(X_i w(\xi))\psi(\xi) + \frac{1}{R_k} d_j(R_k \circ \xi) w(\xi) X_j \psi(\xi)\Big) \\
&\quad + c(R_k \circ \xi) w(\xi) \psi(\xi) \Bigg) R_k^{2n+2} d\xi \\
&= R_k^{2n} \int_A \Bigg(\sum_{i,j} \Big(a_{ij}^k X_i w X_j \psi + b_i^k (X_i w)\psi + d_j^k w X_j \psi \Big) + c^k w \psi \Bigg) d\xi
\end{aligned} \tag{4.3}
$$

where $a_{ij}^k(\xi) = a_{ij}(R_k \circ \xi)$, $b_i^k(\xi) = R_k b_i(R_k \circ \xi)$, $d_i^k(\xi) = R_k d_i(R_k \circ \xi)$ $(i = 1 \ldots, 2n)$, $c^k(\xi) = R_k^2 c(R_k \circ \xi)$ and $\psi(\xi) = \varphi(R_k \circ \xi)$. Since for every $\psi \in \mathring{S}^2(A)$ we have $\psi(\xi) = \varphi(R_k \circ \xi)$ where $\varphi \in \mathring{S}^2(A_k)$ is given by $\varphi(x) = \psi(\frac{1}{R_k} \circ x)$, from (4.3) it follows that w is a positive solution of L_k in A, where

$$
L_k w = - \sum_{i,j=1}^{2n} X_j^* \big(a_{ij}^k(\xi) X_i w + d_j^k(\xi) w\big) + \sum_{i=1}^{2n} b_i^k(\xi) X_i w + c^k(\xi) w. \tag{4.4}
$$

By the homogeneity of ρ with respect to the dilations (2.2), by (2.10), (1.5) and (2.11) it follows for every $i = 1, \ldots, 2n$ that

$$
\begin{aligned}
\int_A (b_i^k)^q d\xi &= \int_A R_k^q b_i(R_k \circ \xi)^q d\xi = \int_A \left(\frac{\rho(R_k \circ \xi)}{\rho(\xi)} \cdot b_i(R_k \circ \xi) \right)^q d\xi \\
&\le \frac{1}{a^q} \int_A \big(\rho(R_k \circ \xi) b_i(R_k \circ \xi)\big)^q d\xi = \frac{1}{a^q R^{2n+2}} \int_{A_k} (\rho(x) b_i(x))^q dx \\
&\le \frac{1}{a^q} \Lambda^q \frac{|A_k|}{R_k^{2n+2}} \le \frac{b^{2n+2} - a^{2n+2}}{a^q} \Lambda^q |B_1|.
\end{aligned} \tag{4.5}
$$

In the same way we can prove for every $k \ge 1$ and every $j = 1 \ldots, 2n$ that

$$
\|d_j^k\|_{L^q(A)}^q \le \frac{b^{2n+2} - a^{2n+2}}{a^q} \Lambda^q |B_1| \tag{4.6}
$$

and

$$
\|c^k\|_{L^{\frac{q}{2}}(A)}^{\frac{q}{2}} \le \frac{b^{2n+2} - a^{2n+2}}{a^q} \Lambda^{\frac{q}{2}} |B_1|. \tag{4.7}
$$

Let $x, y \in A'_k$. Then $\xi = \frac{1}{R_k} \circ x$ and $\eta = \frac{1}{R_k} \circ y$ belong to A'. By the local Harnack inequality (3.8) there exists C, depending only on a, b, a', b' and on μ, M, Λ, but not on k, such that

$$
C w_k(\xi) < w_k(\eta) \, . \tag{4.8}
$$

By the definition of w, from (4.8) it follows (4.1) ∎

Corollary 4.3. *Let u and v be two positive local solutions of L in A_k. Then we have*

$$C^2\frac{u(x)}{v(x)} < \frac{u(y)}{v(y)} \qquad \textit{for all } x, y \in A'_k \tag{4.9}$$

where C is the same constant appearing in the inequality (4.1).

Proof. From $Cu(x) < u(y)$ and $Cv(y) < v(x)$ it follows (4.9) ∎

From now on we will assume that L satisfies the Weak and Strong Maximum Principles.

Proposition 4.4. *Let u and v be two positive local solutions of L in D_R for some $R > 0$. Then there exists*

$$\lim_{\rho(x)\to+\infty} \frac{u(x)}{v(x)}. \tag{4.10}$$

Proof. Let $a_k = \min_{S_{R_k}} \frac{u(x)}{v(x)}$ and $b_k = \max_{S_{R_k}} \frac{u(x)}{v(x)}$. Set $\alpha_k = \min(a_{k-1}, a_{k+1})$ and $\beta_k = \max(b_{k-1}, b_{k+1})$. The functions $u - \alpha_k v$ and $\beta_k v - u$ are solutions of L in

$$G_k = \{x \in \mathbb{R}^{2n+1} : R_{k-1} < \rho(x) < R_{k+1}\},$$

non-negative on ∂G_k. Hence by the Maximum Principles they are positive in G_k or they are identically equal to zero. Therefore either $u(x) = \alpha_k v(x)$ on S_{R_k} for some R_k and so on D_R, or the sequence $\{a_k\}$ is definitively strictly monotone. Actually, if $a_k \leq a_{k-1}$, by the Strong Maximum Principle saying that $a_k > \alpha_k$, it follows $a_{k+1} < a_k$. In the same way we can prove that either b_k is constant or it is definitively monotone.

Let $a = \lim_{k\to+\infty} a_k$ and $b = \lim_{k\to+\infty} b_k$. Since $a_k \leq b_k$ if $a = +\infty$ we are done. If $a < +\infty$, by the Harnack inequality it still follows that $a = b$. Actually, let $x_k, y_k \in S_{R_k}$ be such that $\frac{u(x_k)}{v(x_k)} = b_k - \varepsilon_k$ and $\frac{u(y_k)}{v(y_k)} = a_k + \eta_k$ with $\varepsilon_k \searrow 0$ and $\eta_k \searrow 0$. Then by inequality (4.9) applied to the positive solutions $u - \alpha_k v$ and v it follows

$$C^2(b_k - \varepsilon_k - \alpha_k) \leq (a_k + \eta_k - \alpha_k).$$

By passing to the limit we have $0 \leq C^2(b - a) \leq 0$. Since $\alpha_k \leq \frac{u(x)}{v(x)} \leq \beta_k$ holds for every $x \in G_k$, the existence of the limit (4.10) follows from $a = b$ ∎

As a by-product of this limit theorem we obtain a Liouville theorem for the operator L.

Theorem 4.5 (Liouville Theorem). *There exists a unique, up to a constant, positive local solution of L in $\mathbb{R}^{2n+1}$.*

Proof. The existence of at least one local solution is given by Theorem 3.10. To prove the uniqueness it is sufficient to observe that if the solutions u and v in Proposition 4.4 are defined in $\mathbb{R}^{2n+1}$, then by the Weak Maximum Principle the sequences $\{a_k\}$ and $\{b_k\}$ must be non-increasing and non-decreasing, respectively. This, toghether with $a = b$, yields $u(x) \equiv av(x)$ on $\mathbb{R}^{2n+1}$ ∎

From now on we will assume that L is a Fuchsian operator, i.e. that it is a Fuchsian operator in the weak sense and there exists $R > 0$ such that

$$D_R \subset \cup A'_k. \tag{4.11}$$

Lemma 4.6. *Let u be a positive local solution of L in $\mathbb{R}^{2n+1}$. Then* $\inf u > 0$.

Proof. Let us assume that $\inf_{\mathbb{R}^{2n+1}} u = 0$. Then, by the Strong Maximum Principle, a minimizing sequence $\{x_i\}$ must diverge, i.e. $\rho(x_i) \to +\infty$. Let $R_i = \rho(x_i)$, $a_i = \min_{S_{R_i}} u$ and $b_i = \max_{S_{R_i}} u$. By (4.11), for every i there exists k_i such that $S_{R_i} \subset A'_{k_i}$. Since $\{a_i\}$ converges to zero, by the uniform Harnack inequality $\{b_i\}$ also converges to zero. Therefore, by the Weak Maximum Principle, for every k_i and every $\varepsilon > 0$ we have $\sup_{B_{R_{k_i}}} u < \varepsilon$, i.e. $u \equiv 0$ on $\mathbb{R}^{2n+1}$ ∎

Theorem 4.7 (Liouville Property). *Let v be a local solution bounded below of L in $\mathbb{R}^{2n+1}$. Then either $v \equiv 0$ or v is of constant sign. The same result holds if v is bounded above.*

Proof. Let $v > k$. If $k \geq 0$, we are done. If $k < 0$, by Theorem 3.10 and Lemma 4.6, there exists a positive local solution $u > -k$ in $\mathbb{R}^{2n+1}$. Then $v + u$ is a positive local solution, too. Therefore, by the Liouville Theorem, there exists $\lambda > 0$ such that $v = (\lambda - 1)u$. The second part of the proposition follows by observing that if $v < k$, then $-v$ is of constant sign ∎

Remark 4.8. All the previous results still hold when we consider the operator

$$\widetilde{L}u = Lu + g(x)X_0 u$$

where L, X_0 and g satisfy assumptions (A), (B), (C) and (D) of Section 1. Actually, by (2.13),

$$X_0 = \sum_{i=1}^{2n} \sigma_i X_i - \frac{1}{4}\sigma_{2n+1}[X_h, X_{h+1}],$$

so that

$$Lu = -\sum_{i,j=1}^{2n} X_j^*(\widetilde{a}_{ij}(x)X_i u + d_j(x)u) + \sum_{i=1}^{2n} \widetilde{b}_i(x)X_i u + c(x)u$$

were $\widetilde{a}_{ij} = a_{ij}$, when $(i,j) \neq (h, h+n), (h+n, h)$ while

$$\widetilde{a}_{h,h+n} = a_{h,h+n} - \frac{1}{4}g\sigma_{2n+1} \qquad \text{and} \qquad \widetilde{a}_{h+n,h} = a_{h+n,h} + \frac{1}{4}g\sigma_{2n+1}$$

and $\widetilde{b}_i = b_i + g\sigma_i$, when $i \neq h, h+n$ while

$$\widetilde{b}_h = b_h + g\sigma_h - \frac{1}{4}gX_{h+n}(\sigma_{2n+1}) - \frac{1}{4}\sigma_{2n+1}X_{h+n}g$$

and

$$\widetilde{b}_{h+n} = b_{h+n} + g\sigma_{h+n} + \frac{1}{4}gX_h(\sigma_{2n+1}) + \frac{1}{4}\sigma_{2n+1}X_n g.$$

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