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About Strongly Fejér Monotone Mappings and Their Relaxations

D. Schott

Abstract. We consider the general class of strongly Fejér monotone mappings and some of their basic properties. These properties are useful for a convergence theory of corresponding iterative methods which are widely used to solve convex problems. Especially, we study the relation between these mappings and their relaxations.

Keywords: *Set-valued mappings, Fejér monotone mappings, non-expansive operators, relaxations, convex sets*

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1. Strongly Fejér monotone mappings

Let H be a (real) Hilbert space. We consider a non-empty, convex and closed subset Q of H and (set-valued) mappings $g : Q \to \mathbb{P}(Q)$, where $\mathbb{P}(Q)$ consists of all non-empty subsets of Q. For g we introduce sets of *weak* and *strong fixed points*, namely

$$F_-(g) := \{x \in Q : x \in g(x)\} \qquad \text{and} \qquad F_+(g) := \{x \in Q : \{x\} = g(x)\}.$$

Obviously, $F_+(g) \subseteq F_-(g)$. As usual, operators (i.e. single-valued mappings) $g : Q \to Q$ are included as embeddings. Here both kinds of fixed point sets coincide with the set $F(g) = \{x \in Q : x = g(x)\}$. We exclude the uninteresting special case $g = I$ (I the identity).

Now we turn to the basic concepts.

Definition 1.1. For a mapping $g : Q \to \mathbb{P}(Q)$ and a number $\lambda \in \mathbb{R}$ we call the modified mapping

$$g_\lambda := (1 - \lambda) I + \lambda g \tag{1}$$

a *relaxation* of g with the parameter λ.

Although the above concept is usually used only for values of λ in $(0, 2)$, we omit this restriction here.

D. Schott: Hochschule Wismar, Fachbereich Elektrotechnik und Informatik, Philipp-Müller-Straße, PF 1210, D – 23952 Wismar

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Definition 1.2. Let M be a non-empty (proper) subset of Q and $\alpha > 0$. A mapping $g : Q \to \mathbb{P}(Q)$ is said to be *α-strongly M-Fejér monotone* (in notation: $g \in \mathbb{F}^\alpha(M)$) if

$$\|y - x\|^2 - \|z - x\|^2 \geq \alpha \|y - z\|^2 \qquad \text{for all } x \in M, y \in Q, z \in g(y) \tag{2}$$

and

$$y \notin g(y) \qquad \text{for all } y \in Q \backslash M \tag{3}$$

hold. Besides, g is called *α-strongly Fejér monotone* (in notation: $g \in \mathbb{F}^\alpha$) if g is α-strongly M-Fejér monotone for some M. Moreover, g is called *strongly M-Fejér monotone* (in notation: $g \in \mathbb{F}_s(M)$) if g is α-strongly M-Fejér monotone for some $\alpha > 0$. Finally, M can also be omitted here (in notation: $g \in \mathbb{F}_s$).

Remark 1.3. General *M-Fejér monotone* mappings $g \in \mathbb{F}(M)$ satisfy (2) with the limit value $\alpha = 0$. Here we have the relation $M \subseteq F_+(g)$. If also (3) is added, then g is said to be *regularly M-Fejér monotone* (in notation: $g \in \mathbb{F}_r(M)$). By the way, $\mathbb{F}_r(M)$ can be regarded as the limit class $\mathbb{F}^0(M)$ of $\mathbb{F}^\alpha(M)$. In the regular case M is uniquely determined by g. Namely, we get the convex and closed set

$$M = F_+(g) = F_-(g).$$

All the more this holds for strongly Fejér monotone mappings. The inequality in condition (2) is fulfilled automatically for $y \in M$ if $M = F_+(g)$. Besides, (3) can be omitted for $M = F_-(g)$. For the above mentioned facts consult, for instance, [7, 8].

Remark 1.4. Obviously, the sets of strongly Fejér monotone mappings satisfy

$$\mathbb{F}^\beta(M) \subseteq \mathbb{F}^\alpha(M) \subseteq \mathbb{F}_s(M) \subseteq \mathbb{F}_r(M) \qquad \text{for } \beta > \alpha > 0.$$

Even the strict inclusions are fulfilled (see the remarks after Theorem 5.5).

Lemma 1.5. *Let A be an index set of numbers $\alpha \geq 0$ with $\alpha^* = \sup A$. Then we have the equivalence*

$$g \in \mathbb{F}^\alpha \ \textit{for all}\ \alpha \in A \qquad \Longleftrightarrow \qquad g \in \mathbb{F}^{\alpha^*}.$$

Proof. Both parts of the equivalence imply $M = F_-(g) \neq \emptyset$, that is (3). Choose for g fixed elements $x \in M$, $y \in Q$ and $z \in g(y)$. If the inequality in (2) holds for each $\alpha \in A$, then also for α^*. So the right-hand implication follows. The left-hand implication is a direct consequence of Remark 1.4 ∎

Naturally, often it is useful to know the best possible parameter α for strongly Fejér monotone mappings.

Definition 1.6. Let be $g \in \mathbb{F}_s$. Then the number

$$\alpha^* = \alpha_F^*(g) := \sup\{\alpha : g \in \mathbb{F}^\alpha\}$$

is said to be the *F-index* of g. (Mappings $g \in \mathbb{F}_r \setminus \mathbb{F}_s$ obtain the F-index 0.)

Remark 1.7. The F-index $\alpha_F^*(g)$ of $g \in \mathbb{F}_r$ is determined by

$$\alpha_F^*(g) = \inf\left\{ \frac{\|y-x\|^2 - \|z-x\|^2}{\|y-z\|^2} \,\middle|\, x \in M,\ y \in Q \setminus M,\ z \in g(y) \right\},$$

where $M = F_-(g)$. Because of Lemma 1.5 we get

$$\alpha_F^*(g) = \max\left\{\alpha : g \in \mathbb{F}^\alpha\right\}.$$

Moreover, the set

$$\mathbb{F}_*^\alpha := \mathbb{F}^\alpha \setminus \cup_{\beta>\alpha}\mathbb{F}^\beta$$

contains all mappings g with index $\alpha_F^*(g) = \alpha$. Later we will show that $\mathbb{F}_*^\alpha \neq \emptyset$ (see Theorem 5.5).

The geometrical background of the introduced concepts is given in [9]. Iterative methods generated by Fejér monotone mappings are widely used to solve convex problems (see, e.g., [3, 6, 7, 10]). The following convergence result illustrates the importance of strongly Fejér monotone mappings g since they fulfil at least the first two conditions.

Theorem 1.8. *Under the assumptions*

a) $g \in \mathbb{F}_r(M)$

b) g *asymptotically regular*

c) $g' = I - g$ *demiclosed* (*I the identity*)

the iterative method (x_k) *defined by*

$$x_{k+1} \in g(x_k)$$

converges weakly to an element x^* *in* M.

Additional conditions ensure also strong or even geometric convergence of the method.

2. Classes of strongly Fejér monotone mappings

We start with mappings $g : Q \to \mathbb{P}(Q)$ which are slight generalizations of contractive operators (operators with Lipschitz norm $q \in [0,1)$).

Definition 2.1. A mapping $g : Q \to \mathbb{P}(Q)$ is said to be *Fejér q-contractive* for a number $q \in [0,1)$ (in notation: $g \in \mathbb{F}_c^q$) if g has a (weak) fixed point x (i.e. $x \in F_-(g)$) such that

$$\|z - x\| \leq q\,\|y - x\| \qquad \text{for all } y \in Q,\ z \in g(y) \tag{4}$$

holds.

Theorem 2.2. *A Fejér q-contractive mapping $g :\to \mathbb{P}(Q)$ has exactly one weak fixed point x which is even a strong one* (*i.e.* $F_-(g) = F_+(g) = \{x\}$). *Besides, g is α-strongly M-Fejér monotone with $M = \{x\}$ and $\alpha = \alpha(q) := \frac{1-q}{1+q}$.*

Proof. Let x be a weak fixed point of $g \in \mathbb{F}_c^q$. Choosing $y = x$ in (4) yields $z = x$ such that x is a strong fixed point. Let x' be a weak fixed point of g, too. Choosing $y = x'$ in (4) supplies $\|x' - x\| \leq q\|x' - x\|$ which is only possible for $x' = x$, since $q \in [0,1)$. Hence $M = \{x\}$ contains all weak fixed points of g. Moreover, in view of (4) and the triangle inequality we get the estimates

$$\|y - z\| \leq \|y - x\| + \|z - x\| \leq (1+q)\,\|y - x\|$$

and

$$\begin{aligned}\|y - x\|^2 - \|z - x\|^2 &\geq (1 - q^2)\,\|y - x\|^2 \\ &\geq \frac{1 - q^2}{(1+q)^2}\,\|y - z\|^2 = \frac{1-q}{1+q}\,\|y - z\|^2.\end{aligned}$$

Consequently, g is α-strongly M-Fejér monotone with the given α and M ∎

The function $\alpha = \alpha(q)$ is strictly monotone decreasing with $\alpha(0) = 1$ and the left limit $\alpha(1-0) = 0$. So α produces only the range $(0,1]$. But observe the relation $\alpha_F^*(g) \geq \alpha(q)$ for $g \in \mathbb{F}_c^q$. There are mappings $g \in \mathbb{F}_c^q$ with $\alpha_F^*(g) = \alpha(q)$, but indeed, there are also such mappings with $\alpha_F^*(g) > 1 \geq \alpha(q)$ for each $q > 0$. By the way, the F-index can become arbitrarily great if q tends to 1 (see [9]).

The reversion of Theorem 2.2 is not true. A mapping g which is α-strongly M-Fejér monotone with $\alpha = \alpha(q)$ and $M = \{x\}$ need not to be Fejér q-contractive (see [9]).

Now we turn to special classes of non-expansive operators $g : Q \to Q$.

Definition 2.3. An operator $g : Q \to Q$ is said to be *α-strongly non-expansive* for $\alpha > 0$ (in notation: $g \in \mathbb{L}^\alpha$) if

$$\|y - x\|^2 - \|g(y) - g(x)\|^2 \geq \alpha\,\|g'(y) - g'(x)\|^2 \qquad \text{for all } x, y \in Q, \tag{5}$$

where g' denotes the complement $I - g$ of g. Besides, g is called *strongly non-expansive* (in notation: $g \in \mathbb{L}_s$) if g is α-strongly non-expansive for some $\alpha > 0$.

Remark 2.4. The limit case $\alpha = 0$ in (5) characterizes *non-expansive* operators g (operators with Lipschitz norm less or equal to 1). If the fixed point property is added (i.e. $F(g) \neq \emptyset$), then we speak of *regularly non-expansive* operators g (in notation: $g \in \mathbb{L}_r$) which are also regularly M-Fejér monotone with $M = F(g)$ (see Remark 1.3 and, for the proof, [8]). Observe that $F(g)$ is convex and closed.

Sometimes it is useful to specify the fixed point set $M = F(g)$ of operators $g \in \mathbb{L}^\alpha$ with $F(g) \neq \emptyset$. Then we write $g \in \mathbb{L}^\alpha(M)$. This appears in accordance with the notation $\mathbb{F}^\alpha(M)$ in Definition 1.2. At all, it will turn out that the aspects we are interested in contain a great analogy between both classes of mappings. Therefore results for the second case are often omitted or only outlined in the following.

Fejér q-contractive operators need not to be α-strongly non-expansive and vice versa (see [9]).

Evidently, the sets of strongly non-expansive operators fulfil the relations

$$\mathbb{L}^\beta \subseteq \mathbb{L}^\alpha \subseteq \mathbb{L}_s \subseteq \mathbb{L}^0 \qquad \text{for } \beta > \alpha > 0.$$

Lemma 2.5. *Let A be an index set of numbers $\alpha \geq 0$ with $\alpha^* = \sup A$. Then we have the equivalence*

$$g \in \mathbb{L}^\alpha \ \textit{for all}\ \alpha \in A \qquad \Longleftrightarrow \qquad g \in \mathbb{L}^{\alpha^*}.$$

Proof. The assertion follows in the same way as for Lemma 1.5 ∎

Definition 2.6. Let be $g \in \mathbb{L}_s$. Then the number

$$\alpha^* = \alpha_L^*(g) := \sup \{\alpha : g \in \mathbb{L}^\alpha\}$$

is called *L-index* of g. (Operators $g \in \mathbb{L}^0 \setminus \mathbb{L}_s$ obtain the L-index 0.)

Remark 2.7. The L-index $\alpha_L^*(g)$ of $g \in \mathbb{L}^0$ is given by

$$\alpha_L^*(g) = \inf \left\{ \frac{\|y-x\|^2 - \|g(y)-g(x)\|^2}{\|g'(y)-g'(x)\|^2} \,\middle|\, y,x \in Q \text{ with } g'(y) \neq g'(x) \right\}.$$

By Lemma 2.5 we get $\alpha_L^*(g) = \max\{\alpha : g \in \mathbb{L}^\alpha\}$. Moreover, the set

$$\mathbb{L}_*^\alpha := \mathbb{L}^\alpha \setminus \cup_{\beta>\alpha} \mathbb{L}^\beta$$

contains all operators with L-index $\alpha_L^*(g) = \alpha$.

Theorem 2.8. *For arbitrary $\alpha \geq 0$ it holds*

$$\mathbb{L}^\alpha(M) \subseteq \mathbb{F}^\alpha(M),$$

that is, α-strongly non-expansive operators with fixed points are α-strongly Fejér monotone.

Proof. The class $\mathbb{L}^\alpha(M)$ contains operators g with $M = F(g) \neq \emptyset$ (see Remark 2.4). Let g be such an operator. Then we obtain for $x \in M$, $y \in Q$ and $z = g(y)$ in view of $g(x) = x$ the identity

$$\|y-x\|^2 - \|g(y)-g(x)\|^2 - \alpha\|g'(y)-g'(x)\|^2 = \|y-x\|^2 - \|z-x\|^2 - \alpha\|z-y\|^2.$$

This shows the assertion if the Definitions 1.2 and 2.3 concerning $g \in \mathbb{F}^\alpha$ and $g \in \mathbb{L}^\alpha$ are taken into account. The choice of M implies that (3) can be omitted here (see Remark 1.3) ∎

Lemma 2.9. *For $g \in \mathbb{L}_r$ we get the index relation $\alpha_F^*(g) \geq \alpha_L^*(g)$.*

Proof. We have $g \in \mathbb{L}^\alpha(M)$ for $\alpha = \alpha_L^*(g)$ and $M = F(g)$. By Theorem 2.8, we get $g \in \mathbb{F}^\alpha(M)$ with this α. Using Definition 1.6 of the F-index the assertion follows immediately ∎

There are examples with $\alpha_F^*(g) > \alpha_L^*(g)$ (see [9]).

3. Auxiliary results

First we list some auxiliary statements which are useful to show the results of the next two sections. An important part will play the functional $d^\alpha : H \times H \to \mathbb{R}$ defined for $\alpha \in \mathbb{R}$ by

$$d^\alpha(u,v) := \|u\|^2 - \|v\|^2 - \alpha\|u-v\|^2 \qquad (u,v \in H) \tag{6}$$

which is related to (2) in Definition 1.2 and to (5) in Definition 2.3.

Lemma 3.1. *The functional d^α in (6) has the equivalent form*

$$d^\alpha(u,v) = (1-\alpha)\,\|u\|^2 - (1+\alpha)\,\|v\|^2 + 2\alpha\langle u,v\rangle.$$

Proof. The assertion is an immediate consequence of the identity $\|u-v\|^2 = \langle u-v, u-v\rangle = \|u\|^2 - 2\langle u,v\rangle + \|v\|^2$ ∎

Next let V be a linear space. We define for two elements $u, v \in V$ and $\lambda \in \mathbb{R}$ the affine combination

$$w_\lambda = w_\lambda(u,v) := (1-\lambda)\,u + \lambda\,v \tag{7}$$

which is related to the relaxation (1) in Definition 1.1, namely $g_\lambda = w_\lambda(I,g)$ with the identity I. In the special case $V = \mathbb{R}$ we use the notation w_λ^*. For fixed u and v with $u \neq v$ the affine combinations generate the straight line through u and v. The operation w_λ has simple, but interesting properties listed in the next lemma.

Lemma 3.2. *Let V be a linear space. The following relations hold for elements $u, v, u', v' \in V$ and numbers $\lambda, \mu, \nu, a, b \in \mathbb{R}$:*

a) $w_0(u,v) = u,\ w_1(u,v) = v,\ w_\lambda(u,v) = w_{1-\lambda}(v,u)$

b) $w_\lambda(u,u) = u,\ w_\lambda(\nu\,u, \nu\,v) = \nu\, w_\lambda(u,v)$

c) $w_\lambda(u+u', v+v') = w_\lambda(u,v) + w_\lambda(u',v')$

d) $w_\lambda(u,v) - w_\mu(u,v) = (\lambda-\mu)(v-u)$

e) $w_{a\,\lambda+b\,\mu}(u,v) = a\,w_\lambda(u,v) + b\,w_\mu(u,v)$ *for* $a+b=1$

f) $w_{\lambda\mu}(u,v) = w_\mu(u, w_\lambda(u,v))$

g) $\|w_\lambda(u,v)\|^2 = w_\lambda^*(\|u\|^2, \|v\|^2) - \lambda(1-\lambda)\,\|u-v\|^2$.

Proof. It is easy to check the above identities by use of the definition (7) ∎

Now, returning to a Hilbert space H, we study for fixed but arbitrary elements $u, v \in H$ the non-negative function

$$r(\lambda) = r(\lambda; u, v) := \|w_\lambda(u,v)\| \tag{8}$$

which describes the distance of straight line points from 0. Because of Lemma 3.2/e) and norm properties $r(\lambda)$ is convex. Observing that H is strictly normed, $r(\lambda)$ is even strictly convex if and only if u and v are linearly independent. In this case there exists a unique positive minimum at

$$\lambda^* = \lambda^*(u,v) := \frac{\langle u, u-v\rangle}{\|u-v\|^2}. \tag{9}$$

This can easily be checked if the quadratic function

$$s(\lambda) = s(\lambda; u, v) := r^2(\lambda; u, v)$$

instead of $r(\lambda)$ is considered which has the same minimizing argument. So $\lambda^*(u,v)$ is the zero of the derivative

$$s'(\lambda) = \langle w_\lambda, v - u\rangle = \langle u, v - u\rangle + \lambda \|u - v\|^2.$$

If $v = c\,u$ is satisfied for a number c, then

$$r(\lambda) = |1 + (c-1)\lambda|\,\|u\|.$$

For $u \neq v$, that is $c \neq 1$, $r(\lambda)$ has a unique minimum, too, and it is attained at

$$\lambda^*(u,v) := \frac{1}{1-c}.$$

A check shows that this value is the specification of (9). Finally, $r(\lambda) = \|u\|$ arizes for $u = v$. The properties of $r(\lambda)$ can also be revealed, if Lemma 3.2/g) is applied and quadratic completion is joined. By use of (9) we get

$$\begin{aligned} s(\lambda) &= (1-\lambda)\|u\|^2 + \lambda\|v\|^2 - \lambda(1-\lambda)\|u-v\|^2 \\ &= \|u-v\|^2\lambda^2 - 2\langle u, u-v\rangle\lambda + \|u\|^2 \\ &= \|u-v\|^2\big((\lambda-\lambda^*)^2 - (\lambda^*)^2\big) + \|u\|^2 \end{aligned}$$

and for $u \neq v$ the minimal value

$$s(\lambda^*) = \|u\|^2 - (\lambda^*)^2\|u-v\|^2 = \|u\|^2 - \frac{\langle u, u-v\rangle^2}{\|u-v\|^2}.$$

(If $u = v$, then λ^* is not unique, but arbitrary.) For all u and v we recognize the symmetry property

$$r(\lambda^* - \lambda) = r(\lambda^* + \lambda).$$

Further, $r(\lambda^*)$ is (strictly) positive if u and v are linearly independent. Namely, in this case the Schwarz inequality for scalar products holds strictly (see, e.g., [5: p. 155]). For $u \neq v$ the minimizing argument λ^* separates the left strictly monotone decreasing part of $r(\lambda)$ from the right strictly monotone increasing part. If symmetry of $r(\lambda)$ with respect to λ^* is observed, then we get the (extended) monotony relations

$$r(\lambda) \le r(\mu) \qquad \text{if } \mu \le \lambda \le 2\lambda^- - \mu, \tag{10}$$

for the left part, where $\lambda^- \le \lambda^*$. For $u \neq v$ the inequality sign in (10) can be replaced by the strict one.

Further, we can show important relations combining the functions $r(\lambda)$ in (8) and d^α in (6), the latter considered in dependence on the superscript α.

Lemma 3.3. *The following equation is fulfilled for arbitrary* λ *and* μ*:*

$$r^2(\lambda; u, v) - r^2(\mu; u, v) = -(\lambda - \mu)\, d^{\lambda+\mu-1}(u, v).$$

Besides we have for $\lambda > \mu$ *the equivalence*

$$r(\lambda; u, v) \le r(\mu; u, v) \quad \Longleftrightarrow \quad d^{\lambda+\mu-1}(u, v) \ge 0,$$

where the equality in one of the relations implies also the equality in the other.

Proof. Using the properties g) and d) in Lemma 3.2 for w_λ and w_λ^*, respectively, we obtain for $r^2(\lambda) = r^2(\lambda; u, v) = \|w_\lambda(u, v)\|^2$ the transformations

$$\begin{aligned} &r^2(\lambda) - r^2(\mu) \\ &\quad = w_\lambda^*(\|u\|^2, \|v\|^2) - w_\mu^*(\|u\|^2, \|v\|^2) + \big(\mu(1-\mu) - \lambda(1-\lambda)\big)\|u - v\|^2 \\ &\quad = (\lambda - \mu)(\|v\|^2 - \|u\|^2) + (\mu - \lambda - \mu^2 + \lambda^2)\|u - v\|^2 \\ &\quad = (\lambda - \mu)\big(\|v\|^2 - \|u\|^2 + (\lambda + \mu - 1)\|u - v\|^2\big) \\ &\quad = -(\lambda - \mu)\, d^{\lambda+\mu-1}(u, v). \end{aligned}$$

This is the first assertion. The second assertion is a simple consequence ∎

Remark 3.4. Obviously, $d^\alpha(u, v)$ is for fixed and different elements u, v a linear function of the superscript α with the zero

$$\alpha^* = \alpha^*(u, v) := \frac{\|u\|^2 - \|v\|^2}{\|u - v\|^2} = \frac{\langle u + v, u - v\rangle}{\|u - v\|^2}.$$

Observe that α^* is the greatest α fulfilling $d^\alpha(u, v) \ge 0$. Now Lemma 3.3 supplies for $\lambda \ne \mu$ the equivalence

$$r(\lambda) = r(\mu) \quad \Longleftrightarrow \quad \lambda + \mu = \alpha^* + 1.$$

But, using (9) and considering

$$\alpha^* + 1 = \frac{\langle u + v, u - v\rangle + \langle u - v, u - v\rangle}{\|u - v\|^2} = 2\frac{\langle u, u - v\rangle}{\|u - v\|^2} = 2\lambda^*, \tag{11}$$

the symmetry of $r(\lambda)$ relative to λ^* is again verified. Besides, Lemma 3.3 allows to compute the derivative $s'(\lambda)$ of $s(\lambda) = r^2(\lambda)$. A limit transition shows $s'(\lambda; u, v) = -d^{2\lambda-1}(u, v)$. This confirms again the relation (11).

The next result shows how the superscript of d is influenced by changing the second argument from v to w_λ. Besides, δ will play the part of a perturbation.

Lemma 3.5. *Let be* $\lambda \ne 0$ *and* $\delta \in \mathbb{R}$*. Then the formula*

$$d^{\beta+\delta}(u, w_\lambda) = \lambda\, d^{\alpha+\lambda\delta}(u, v)$$

holds with $w_\lambda = w_\lambda(u,v)$ *if the parameters* α, β *and* λ *fulfil the condition* $(1+\beta)\,\lambda = 1+\alpha$. *Moreover, we get under this condition for* $\lambda > 0$ *the equivalence*

$$d^{\beta+\delta}(u,w_\lambda) \geq 0 \qquad \Longleftrightarrow \qquad d^{\alpha+\lambda\delta}(u,v) \geq 0.$$

Proof. At first suppose $\delta = 0$. Then we start with the right-hand side of the formula and substitute $v = \frac{\lambda-1}{\lambda}\,u + \frac{1}{\lambda}\,w_\lambda$ from (7). Using Lemma 3.1 this leads to

$$\begin{aligned}
\lambda\, d^\alpha(u,v) &= \lambda(1-\alpha)\,\|u\|^2 - \lambda(1+\alpha)\left\|\frac{\lambda-1}{\lambda}\,u + \frac{1}{\lambda}\,w_\lambda\right\|^2 \\
&\quad + 2\,\lambda\,\alpha\left\langle u, \frac{\lambda-1}{\lambda}\,u + \frac{1}{\lambda}\,w_\lambda\right\rangle \\
&= \left((1-\alpha)\lambda - (1+\alpha)\frac{(\lambda-1)^2}{\lambda} + 2\alpha(\lambda-1)\right)\|u\|^2 \\
&\quad - (1+\alpha)\,\frac{1}{\lambda}\,\|w_\lambda\|^2 + 2\left(\alpha - (1+\alpha)\,\frac{\lambda-1}{\lambda}\right)\langle u, w_\lambda\rangle \\
&= \frac{2\lambda - 1 - \alpha}{\lambda}\,\|u\|^2 - \frac{1+\alpha}{\lambda}\,\|w_\lambda\|^2 + 2\,\frac{1+\alpha-\lambda}{\lambda}\,\langle u, w_\lambda\rangle \\
&= (1-\beta)\,\|u\|^2 - (1+\beta)\,\|w_\lambda\|^2 + 2\,\beta\langle u, w_\lambda\rangle \\
&= d^\beta(u,w_\lambda).
\end{aligned}$$

This is the first assertion for $\delta = 0$. Now let δ be arbitrary. Then the general formula follows immediately if $(1+\beta+\delta)\lambda = (1+\beta)\lambda + \delta\lambda = 1+\alpha+\lambda\delta$ is considered. The second assertion is a simple consequence ∎

For $\alpha \geq 0$ and $\beta \geq 0$ the relation $(1+\beta)\lambda = 1+\alpha$ shows that λ can vary in the interval $(0, 1+\alpha]$.

4. Norm relations for relaxations

We want to characterize the considered mappings $g : Q \to \mathbb{P}(Q)$ by certain norm relations of corresponding relaxations. Therefore we choose a mapping $g \in \mathbb{F}_r(M)$ and study for arbitrary elements of the non-empty set

$$J_g = \Big\{(x,y,z) : x \in M,\, y \in Q \setminus M,\, z \in g(y)\Big\}, \qquad M = F_-(g)$$

the properties of the specialized functions

$$r(\lambda) = r(\lambda; y-x, z-x) = \|w_\lambda(y-x, z-x)\| = \|z_\lambda - x\|, \tag{12}$$

where $z_\lambda = w_\lambda(y,z) \in g_\lambda(y)$, arizing from (8) with $u = y-x$ and $v = z-x$. Namely, by Lemma 3.2/b) and c) we obtain

$$w_\lambda(y-x, z-x) = w_\lambda(y,z) - w_\lambda(x,x) = z_\lambda - x.$$

Now observe $r(0) = \|y-x\|$ and $r(1) = \|z-x\|$ such that $r(1) \le r(0)$ for all $(x,y,z) \in J_g$ holds if $g \in \mathbb{F}_r(M)$. The excluded case $y \in M$ corresponds to $z = y$ (compare Remark 1.3) and produces only the uninteresting constant function $r(\lambda) = \|y-x\|$. For $z \neq y$ and all the more for $(x,y,z) \in J_g$ the unique minimum of $r(\lambda)$ is attained at

$$\lambda^* = \lambda^*(y-x, z-x) = \frac{(y-x, y-z)}{\|y-z\|^2}$$

(see (9) and the following remarks) which is non-negative for $g \in \mathbb{F}_r(M)$. Thus we can define the characterizing number

$$\lambda_F^*(g) := \inf\Big\{\lambda^*(y-x, z-x) : (x,y,z) \in J_g\Big\} \tag{13}$$

for g. Then the (extended) monotony property (10) of $r(\lambda)$ supplies with the specification (12)

$$\|z_\lambda - x\| \le \|z_\mu - x\| \qquad \text{for all } \begin{cases} (\lambda,\mu) \text{ with } \mu \le \lambda \le 2\,\lambda_F^*(g) - \mu \\ (x,y,z) \in J_g \end{cases}.$$

This norm relation between relaxations can also be used to include a mapping $g : Q \to \mathbb{P}(Q)$ with weak fixed points (i.e. $M = F_-(g) \neq \emptyset$) into a certain class $\mathbb{F}^\alpha$. At first we get for $\alpha \ge 0$ the characterization

$$g \in \mathbb{F}^\alpha \iff g \in \mathbb{F}^\alpha(M) \iff d^\alpha(y-x, z-x) \ge 0 \text{ for all } (x,y,z) \in J_g \tag{14}$$

by the functional d^α in (6) (see Definition 1.2 and Remark 1.3). Observe that $y \in M$ is also admissible. Hence J_g can be replaced by the extension

$$J_g^0 = \Big\{(x,y,z) : x \in M,\ y \in Q,\ z \in g(y)\Big\}.$$

Now the central statement of this section follows which can be exploited in various ways.

Theorem 4.1. *Let be $\lambda - \mu > 0$ and $\lambda + \mu \ge 1$. If $M = F_-(g) \neq \emptyset$ is satisfied for a mapping $g : Q \to \mathbb{P}(Q)$, then the equivalence*

$$g \in \mathbb{F}^{\lambda+\mu-1} \iff \|z_\lambda - x\| \le \|z_\mu - x\| \text{ for all } (x,y,z) \in J_g$$

holds.

Proof. With the above mentioned choices $u = y - x$ and $v = z - x$, where $(x,y,z) \in J_g$, Lemma 3.3 reads for $\lambda > \mu$ as

$$r(\lambda) \le r(\mu) \iff d^{\lambda+\mu-1}(y-x, z-x) \ge 0.$$

But, if the left part of the equivalence holds for all $(x,y,z) \in J_g$, then also the right part. By the definition (12) and the characterization (14) the latter corresponds for $\lambda + \mu \ge 1$ to the assertion. Observe that the assumption $M = F_-(g) \neq \emptyset$ is necessary for $J_g \neq \emptyset$ ∎

Remark 4.2. The set J_g can be extended to J_g^0 in Theorem 4.1 (see the remarks before Theorem 4.1). The equality $r(\lambda) = r(\mu)$ for (12) holds on J_g only in special cases (see Remark 3.4). But for $(x, y, z) \in J_g^0 \setminus J_g$ and $z = y$, respectively, the equation $r(\lambda) = r(\mu)$ is fulfilled automatically.

Lemma 4.3. *Suppose $M = F_-(g) \neq \emptyset$ for a mapping $g : Q \to \mathbb{P}(Q)$. Then we obtain:*

a) *For $\mu \in (0,1)$: $g \in \mathbb{F}^\mu \iff \|z - x\| \leq \|z_\mu - x\|$ for all $(x, y, z) \in J_g$.*

b) *For $\lambda \in (1, \infty)$: $g \in \mathbb{F}^{\lambda-1} \iff \|z_\lambda - x\| \leq \|y - x\|$ for all $(x, y, z) \in J_g$.*

c) *For $\mu \in (0,1)$: $g \in \mathbb{F}^1 \iff \|z_{2-\mu} - x\| \leq \|z_\mu - x\|$ for all $(x, y, z) \in J_g$.*

Proof. The assertions follow immediately if we choose $\lambda = 1, \mu = 0$ and $\lambda + \mu = 2$ in Theorem 4.1 ∎

Theorem 4.4. *If we introduce the sets*

$$\Lambda_\alpha^* := \Big\{(\lambda, \mu) : \lambda - \mu > 0 \ \text{ and } \ \lambda + \mu = \alpha + 1\Big\}$$
$$\Lambda_\alpha := \Big\{(\lambda, \mu) : \lambda - \mu > 0 \ \text{ and } \ \lambda + \mu \leq \alpha + 1\Big\},$$

then we get for a mapping $g : Q \to \mathbb{P}(Q)$ with $M = F_-(g) \neq \emptyset$, any $\alpha \geq 0$ and any subset Λ_α^- of Λ_α with the property $\sup\{\lambda + \mu : (\lambda, \mu) \in \Lambda_\alpha^-\} = \alpha + 1$:

$$\begin{aligned} g \in \mathbb{F}^\alpha \quad &\iff \quad \exists (\lambda, \mu) \in \Lambda_\alpha^* \ \forall (x, y, z) \in J_g : \ \|z_\lambda - x\| \leq \|z_\mu - x\| \\ &\iff \quad \forall (\lambda, \mu) \in \Lambda_\alpha^- \ \forall (x, y, z) \in J_g^0 : \ \|z_\lambda - x\| \leq \|z_\mu - x\|. \end{aligned}$$

Proof. The assertions follow from Theorem 4.1 if the substitution $\alpha = \lambda + \mu - 1$ is used. For the second equivalence we need additionally Lemma 1.5 and Remark 4.2 ∎

Corollary 4.5. *Suppose a mapping $g : Q \to \mathbb{P}(Q)$ with $M = F_-(g) \neq \emptyset$ and $\alpha \geq 0$. Then each of the following conditions is equivalent to the statement $g \in \mathbb{F}^\alpha$:*

a) *$\|z_\lambda - x\| \leq \|z_\mu - x\|$ for all λ, μ with $\mu < \lambda < \frac{\alpha+1}{2}$ and all $(x, y, z) \in J_g^0$.*

b) *$\|z_{\frac{\alpha+1}{2}} - x\| \leq \|z_\mu - x\|$ for all $\mu < \frac{\alpha+1}{2}$ and all $(x, y, z) \in J_g^0$.*

c) *$\|z_\lambda - x\| \leq \|z_{\frac{\alpha-1}{2}} - x\|$ for all $\lambda \in (\frac{\alpha-1}{2}, \frac{\alpha+3}{2})$ and all $(x, y, z) \in J_g^0$.*

Proof. In assertion a) the pairs (λ, μ) form a set fulfilling the properties of Λ_α^- in Theorem 4.4. The same is true for the pairs $(\frac{\alpha+1}{2}, \mu)$ in assertion b) and $(\lambda, \frac{\alpha-1}{2})$ in assertion c). So Corollary 4.5 is a direct consequence of Theorem 4.4 ∎

The mentioned properties of $r(\lambda) = \|z_\lambda - x\|$ show that the inequalities for r in Corollary 4.5 hold even strictly if the triples (x, y, z) vary in J_g instead of J_g^0. Namely, $r(\lambda) < r(\mu)$ for $\lambda + \mu < \alpha + 1 \leq \alpha^* + 1$ in view of Remark 3.4. On the other hand, if the equality for r is admitted, then by Theorem 4.4 the endpoints of the open intervals regarding the arguments λ and μ can be included. Besides, the range can be modified as long as the pairs (λ, μ) form a set of the kind Λ_α^- from Theorem 4.4.

Lemma 4.6. *The mappings $g \in \mathbb{F}^1$ can be characterized by the following properties:*

a) $\|z_\lambda - x\| \le \|z_\mu - x\|$ *for all* λ, μ *with* $0 \le \mu \le \lambda \le 1$ *and all* $(x,y,z) \in J_g^0$.

b) $\|z_\lambda - x\| \le \|z_{2-\lambda} - x\| \le \|y - x\|$ *for all* $\lambda \in [1,2]$ *and all* $(x,y,z) \in J_g^0$.

c) $\|z_{2-\lambda} - x\| \le \|z_\lambda - x\|$ *for all* $\lambda \in [0,1]$ *and all* $(x,y,z) \in J_g^0$.

Proof. All assertions arize if Theorem 4.4 is used with $\alpha = 1$. Compare the above assertion a) also with Corollary 4.5/a) and the above assertion c) with Lemma 4.3/c) ∎

The properties listed in Lemma 4.6 are described in [1: p. 85 - 86] for special relaxations, namely for so-called *transfer operators* g_λ of *simultaneous projectors* g which belong to $\mathbb{F}^1$ (see [9] and Example 6.4).

Corollary 4.5 means that for $g \in \mathbb{F}^\alpha$ and $(x,y,z) \in J_g$ the function $r(\lambda)$ is monotone decreasing up to $\frac{\alpha+1}{2}$ and attains in $[\frac{\alpha-1}{2}, \frac{\alpha+3}{2}]$ its maximum at the left endpoint. So the relation $\frac{\alpha+1}{2} \le \lambda_F^*(g)$ is true (see (9) and (13)). This suggests that perhaps the equality is satisfied for the index $\alpha = \alpha_F^*(g)$. Indeed, the next theorem will show this.

Theorem 4.7. *For $g \in \mathbb{F}_r$ the formula $\alpha_F^*(g) + 1 = 2\,\lambda_F^*(g)$ holds.*

Proof. By (11) we obtain $\alpha^*(y-x, z-x)+1 = 2\,\lambda^*(y-x, z-x)$ for all $(x,y,z) \in J_g$. Now the assertion follows if we take the infimum over these elements. Namely, consider $\alpha_F^*(g) = \inf\{\alpha^*(y-x, z-x) : (x,y,z) \in J_g\}$ (Remarks 1.7 and 3.4) and the definition (13) of $\lambda_F^*(g)$ ∎

Analogous results are obtained for the classes of non-expansivity. We start with a non-expansive operator $g : Q \to Q$ (i.e. $g \in \mathbb{L}^0$) and introduce the set

$$J_g' = \Big\{(x,y) : x,y \in Q \ \text{ with } \ g'(x) \neq g'(y)\Big\}$$

where $g' = I - g$ again denotes the complement of g. Now we consider for arbitrary elements of J_g' the specialized functions

$$\begin{aligned} r(\lambda) &= r\big(\lambda; y-x, g(y)-g(x)\big) \\ &= \big\|w_\lambda\big(y-x, g(y)-g(x)\big)\big\| = \|g_\lambda(y) - g_\lambda(x)\| \end{aligned} \tag{15}$$

from (8) with $u = y - x$ and $v = g(y) - g(x)$. The latter representation of r is verified by Lemma 3.2/b) and c) which yields

$$w_\lambda\big(y-x, g(y)-g(x)\big) = w_\lambda(y, g(y)) - w_\lambda(x, g(x)) = g_\lambda(y) - g_\lambda(x).$$

The excluded case $g'(y) = g'(x)$ supplies again the uninteresting constant function $r(\lambda) = \|y-x\|$. Observe that for $x \in F(g)$ we arrive at $r(\lambda) = \|z_\lambda - x\|$ with $z_\lambda = g_\lambda(y)$ such that for $g \in \mathbb{L}_r$ the family of functions r in (15) is an extension of the previously studied family of functions r in (12). Taking the special values $r(0) = \|y-x\|$ and $r(1) = \|g(y) - g(x)\|$ the inequality $r(1) \le r(0)$ holds for all $(x,y) \in J_g'$. Further, we can define the characterizing number

$$\lambda_L^*(g) := \inf\Big\{\lambda^*\big(y-x, g(y)-g(x)\big) : (x,y) \in J_g'\Big\}$$

which again turns out to be finite. The functional characterization of $\mathbb{L}^\alpha$ is given by

$$g \in \mathbb{L}^\alpha \iff d^\alpha(y - x, g(y) - g(x)) \geq 0 \text{ for all } (x,y) \in J'_g. \tag{16}$$

In this characterization $Q \times Q$ can be used instead of J'_g. Now we could list analogous results replacing formally $\mathbb{F}_r = \mathbb{F}^0$ by $\mathbb{L}^0$, $\mathbb{F}^\alpha$ by $\mathbb{L}^\alpha$, J_g by J'_g, J^0_g by $Q \times Q$, r in (12) by r in (15), $\alpha^*_F(g)$ by $\alpha^*_L(g)$, $\lambda^*_F(g)$ by $\lambda^*_L(g)$ and so on. But we need not suppose a non-empty fixed point set. Here we restrict us to a short selection. For instance, Theorem 4.1 can be reformulated as follows.

Theorem 4.1'. *Let be $\lambda - \mu > 0$ and $\lambda + \mu \geq 1$. Further, suppose $g : Q \to Q$. Then*

$$g \in \mathbb{L}^{\lambda+\mu-1} \iff \|g_\lambda(y) - g_\lambda(x)\| \leq \|g_\mu(y) - g_\mu(x)\| \text{ for all } (x,y) \in J'_g.$$

The analogue of Corollary 4.5/a) reads for $\alpha = 1$:

$$g \in \mathbb{L}^1 \iff \begin{cases} \|g_\lambda(y) - g_\lambda(x)\| \leq \|g_\mu(y) - g_\mu(x)\| \\ \text{for all } \lambda, \mu \text{ with } 0 \leq \mu \leq \lambda \leq 1 \text{ and all } (x,y) \in Q \times Q \end{cases}$$

if the endpoints of the λ-interval are included and μ is restricted to the non-negative domain (see remarks after Corollary 4.5). This yields with the specification $\lambda = 1$ a remarkable characterization of $\mathbb{L}^1$:

Theorem 4.8. *For an operator $g : Q \to Q$ the equivalence*

$$g \in \mathbb{L}^1 \iff \begin{cases} \|g(y) - g(x)\| \leq \|g_\mu(y) - g_\mu(x)\| \\ \text{for all } \mu \in [0,1] \text{ and all } (x,y) \in Q \times Q \end{cases}$$

is fulfilled.

Operators g with this property on the right-hand side of the equivalence play an important part in the fixed point theory and are called there *firmly non-expansive* (see [4: p. 41 - 44]). So these operators turn out to be in our context nothing else than strongly non-expansive (with L-indices at least 1).

Finally, corresponding to Theorem 4.7, the equation

$$\alpha^*_L(g) + 1 = 2\,\lambda^*_L(g)$$

holds for $g \in \mathbb{L}^0$. Because of Theorem 4.7 and the index relation $\alpha^*_L(g) \leq \alpha^*_F(g)$ from Lemma 2.9 we get as a byproduct $\lambda^*_L(g) \leq \lambda^*_F(g)$.

5. Determination of parameters for relaxations

As seen above, relaxations supply characterizations of mapping classes $\mathbb{F}^\alpha$ and $\mathbb{L}^\alpha$, respectively. But they also open the possibility to change between these classes. This is interesting if mappings with a certain α are needed. We investigate this possibility below. Again we formulate the results for $\mathbb{F}^\alpha$. The transformation to $\mathbb{L}^\alpha$ can be realized without difficulties.

Theorem 5.1. *For a mapping $g : Q \to \mathbb{P}(Q)$ and parameters $\alpha \geq 0, \beta \geq 0$ and $\lambda > 0$ connected by the equation $(1+\beta)\lambda = 1 + \alpha$ the statement*

$$g \in \mathbb{F}^\alpha(M) \iff g_\lambda \in \mathbb{F}^\beta(M)$$

holds. Moreover, this correspondence is also fulfilled for F-indices, i.e.

$$\alpha = \alpha_F^*(g) \iff \beta = \alpha_F^*(g_\lambda).$$

Proof. We consider a mapping $g : Q \to \mathbb{P}(Q)$. In view of (7), (1) and Lemma 3.2/b) and c) we have for $x \in M, y \in Q$ and $z \in g(y)$ the equations

$$w_\lambda(y-x, z-x) = z_\lambda - x,$$

where $z_\lambda = w_\lambda(y,z) \in g_\lambda(y)$ (see also (12) and the passage after it). Putting $\delta = 0$ in Lemma 3.5 the equivalence

$$d^\alpha(y-x, z-x) \geq 0 \iff d^\beta(y-x, z_\lambda - x) \geq 0$$

follows if the parameters satisfy the conditions given in this theorem. Now, if $(x,y,z) \in J_g$ is related to $(x,y,z_\lambda) \in J_{g_\lambda}$, then a bijective mapping between these two triple sets is established. Considering the characterization (14) of $g \in \mathbb{F}^\alpha$ and $M = F_-(g) = F_-(g_\lambda)$ this corresponds to the first assertion. Choosing $\delta > 0$ in Lemma 3.5 the statement

$$d^{\alpha+\delta\lambda}(y-x, z-x) \geq 0 \iff d^{\beta+\delta}(y-x, z_\lambda - x) \geq 0$$

for the above listed elements shows also the index result. Namely, assume that $\beta + \delta = \alpha_F^*(g_\lambda)$ holds for $\alpha = \alpha_F^*(g)$. But this leads by (14) to the consequence $\alpha_F^*(g) \geq \alpha + \delta\lambda > \alpha$ and yields a contradiction. The reversed direction of the index assertion can be handled in the same way ∎

Corollary 5.2. *Let be $\lambda \in (0, i+1]$, where $i \in \mathbb{N}$. Then the relation*

$$g \in \mathbb{F}^i(M) \iff g_\lambda \in \mathbb{F}^\beta(M) \ \text{ for } \beta = \frac{1+i-\lambda}{\lambda}$$

is fulfilled.

Proof. A rearrangement of the relation $(1+\beta)\lambda = 1+\alpha$ in Theorem 5.1 supplies $\beta = \frac{1}{\lambda}(1+\alpha-\lambda)$. The assumption $\lambda \in (0, i+1]$ ensures $\beta \geq 0$. Now the assertion follows immediately by putting $\alpha = i$ in Theorem 5.1 ∎

Corollary 5.2 is of special interest for $i = 0$ and $i = 1$. Now we present a symmetric version of Theorem 5.1.

Corollary 5.3. *For a mapping $g : Q \to \mathbb{P}(Q)$ and parameters $\alpha \geq 0, \beta \geq 0, \lambda > 0$ and $\mu > 0$ connected by the equation $(1+\alpha)\lambda = (1+\beta)\mu$ the statement*

$$g_\lambda \in \mathbb{F}^\alpha(M) \quad \Longleftrightarrow \quad g_\mu \in \mathbb{F}^\beta(M)$$

holds.

Proof. By Lemma 3.2/f) and $g_\lambda = w_\lambda(I, g)$ we have $g_\mu = (g_\lambda)_{\frac{\mu}{\lambda}}$. If we use Theorem 5.1 with g_λ instead of g and with $\mu' = \frac{\mu}{\lambda}$ instead of λ, then the asserted equivalence is fulfilled for the parameter relation $\mu'(1+\beta) = 1+\alpha$. But this corresponds to $(1+\alpha)\lambda = (1+\beta)\mu$ ∎

The next theorem shows that $\lambda = \lambda^*_F(g)$ is just that parameter for g which supplies the relaxation g_λ with F-index 1.

Theorem 5.4. *Let be $g \in \mathbb{F}_r$. For $\lambda^* = \lambda^*_F(g)$ the relation*

$$g_{\lambda^*} \in \mathbb{F}^1_*, \qquad \text{that is} \qquad \alpha^*_F(g_{\lambda^*}) = 1$$

holds.

Proof. There is a number $\alpha = \alpha^* = \alpha^*_F(g) \geq 0$ such that $g \in \mathbb{F}^\alpha_*$. Then the relation $g_{\lambda^*} \in \mathbb{F}^\beta$ is fulfilled by Theorems 5.1 and 4.7 with

$$\alpha^*_F(g_{\lambda^*}) = \beta = \frac{1}{\lambda^*}(1+\alpha^* - \lambda^*) = \frac{1+\alpha^*}{\lambda^*} - 1 = 1.$$

But this is the assertion ∎

Now we turn to the sets $\mathbb{F}^\alpha_*(M)$ which contain mappings $g \in \mathbb{F}_r(M)$ with F-index α (see Remark 1.7).

Theorem 5.5. *The sets $\mathbb{F}^\alpha_*(M)$ are non-empty for all $\alpha \geq 0$.*

Proof. At first, the set $\mathbb{F}^1_*(M)$ is non-empty by Example 6.2. Namely, the metric projector P_M onto M has the F-index 1. If we choose $g \in \mathbb{F}^1_*(M)$, then Theorem 5.1 implies $g_\lambda \in \mathbb{F}^\alpha_*(M)$ for arbitrary $\alpha > 0$ and $\lambda = \frac{2}{1+\alpha}$. So the sets $\mathbb{F}^\alpha_*(M)$ are all non-empty ∎

In view of Theorem 5.5 the proper subset relation is satisfied in Remark 1.4 for the sets $\mathbb{F}^\alpha(M)$. We want to show now that appropriate relaxations g_λ of g create a complete system of representatives for the family $\{\mathbb{F}^\alpha\}$.

Theorem 5.6. *Let be $g \in \mathbb{F}^\gamma(M)$ with $\gamma \geq 0$ and $J = (0, 1+\gamma]$. Then $\{g_\lambda : \lambda \in J\}$ is a choice set of $\{\mathbb{F}^\alpha(M) : \alpha \geq 0\}$, that means, there is a bijective mapping $\lambda : [0, \infty) \to J$ such that $g_\lambda \in \mathbb{F}^\alpha(M)$ for $\lambda = \lambda(\alpha)$.*

Proof. We suppose $g \in \mathbb{F}^\gamma(M)$. If α and β are replaced by γ and α, respectively, then the relation $g_\lambda \in \mathbb{F}^\alpha(M)$ holds by Theorem 5.1 for the bijective mapping $\lambda = \lambda(\alpha) := \frac{1+\gamma}{1+\alpha}$ with the range J ∎

The second part of Theorem 5.1 shows that $\alpha_F^*(g_\lambda) = \alpha$ for $\alpha_F^*(g) = \gamma$. This means $g_\lambda \in \mathbb{F}_*^\alpha(M)$ for $g \in \mathbb{F}_*^\gamma(M)$ and the above λ. So $\{g_\lambda : \lambda \in J\}$ is a choice set of $\{\mathbb{F}_*^\alpha(M) : \alpha \geq 0\}$, too.

Corollary 5.7. *Let be $g \in \mathbb{F}_*^\gamma$. Then the relaxations g_λ of g with $\lambda \in (0, 1+\gamma]$ are pairwise different.*

Proof. By the foregoing remarks g_λ is for pairwise different $\lambda \in (0, 1+\gamma]$ in pairwise disjoint sets $\mathbb{F}_*^\alpha$. So the assertion follows immediately ∎

As mentioned at the beginning of this section, analogous results can be formulated for the classes $\mathbb{L}^\alpha$ of strongly non-expansive operators. For instance, we have corresponding to Theorem 5.1 the equivalence

$$g \in \mathbb{L}^\alpha \quad \Longleftrightarrow \quad g_\lambda \in \mathbb{L}^\beta$$

for $\alpha \geq 0, \beta \geq 0$ and $\lambda > 0$ if $(1+\beta)\,\lambda = 1 + \alpha$ is fulfilled.

6. Applications

The following examples illustrate the theory.

Example 6.1. Let $b : Q \to \mathbb{R}$ be convex and continuous. Then the set $N(b) = \{x \in Q : b(x) \leq 0\}$ is convex and closed. We assume $N(b)$ to be non-empty. Further, the *subgradient* ∂b is defined on Q. If b^+ denotes the positive part of b, we define for elements $y \in Q$ and $v \in H$

$$\mu(b, y, v) := \begin{cases} \frac{b^+(y)v}{\|v\|^2} & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases} \qquad \text{and} \qquad t_b(y) := \{\mu(b, y, v) : v \in \partial b(y)\}. \tag{17}$$

Then the mapping g_b given by $g_b(y) = y - t_b(y)$ is 1-strongly $N(b)$-Fejér monotone, i.e. $\alpha_F^*(g_b) \geq 1$ (see [9]). So the results of Sections 4 and 5 hold for $g := g_b \in \mathbb{F}^1(N(b))$.

Example 6.2. For a convex and closed set $M \subset Q \subseteq H$ the metric projector $P_M : Q \to Q$ onto M is well-defined. Moreover, P_M is 1-strongly non-expansive. More precisely, even

$$\alpha_L^*(P_M) = \alpha_F^*(P_M) = 1$$

(see [9]). Hence, results of Sections 4 and 5 can be applied to $g := P_M \in \mathbb{L}_*^1(M)$.

Example 6.3 (Relaxations). We consider the mapping $g(y) = g_b(y) = y - t_b(y)$ with t_b given in (17). If we study the relaxed form (1)

$$g_\lambda(y) = (1-\lambda)\,y + \lambda\, g(y) = y - \lambda\, t_b(y) \qquad (\lambda \in (0,2)),$$

then the functions

$$r(\lambda) = \|z_\lambda - x\| \qquad \big(z_\lambda \in g_\lambda(y),\ x \in N(b)\big)$$

from (12) fulfil the properties of Theorem 4.4 and Corollary 4.5 with $\alpha = 1$. Theorem 4.7 says that $\lambda_F^*(g_b) \geq 1$ holds. Moreover, Corollary 5.2 yields for $i = 1$ that g_λ is α-strongly Fejér monotone with $\alpha = \frac{2-\lambda}{\lambda}$. This is a generalization of a result in [2: p. 308], where only the so-called strict Fejér monotony is proven which stands between the classes $\mathbb{F}_r$ and $\mathbb{F}_s$. Similarly, the relaxed projector

$$P_\lambda(y) := (1-\lambda)\,y + \lambda\, P_M(y) = y - \lambda\,(y - P_M(y)) \qquad (\lambda \in (0,2))$$

generates the functions

$$r(\lambda) = \|P_\lambda(y) - P_\lambda(x)\|$$

from (15) which fulfil the corresponding properties outlined for strongly non-expansive operators g. Besides, P_λ is α-strongly non-expansive with

$$\alpha = \frac{2-\lambda}{\lambda} = \alpha_L^*(P_\lambda) = \alpha_F^*(P_\lambda)$$

by analogues of Theorem 5.1 and Corollary 5.2 for $\mathbb{L}$-classes. This again generalizes results in [2: p. 307] and [10: p. 47], where P_λ is only proven to be strictly Fejér monotone and non-expansive in this case, respectively.

Example 6.4 (Convex intersection problem). Let M_i $(i = 1, \ldots, m)$ be convex and closed sets with the non-empty intersection $M := \cap_{i=1}^m M_i$. Further, consider for mappings g_i $(i = 1, \ldots, m)$ the *sequential* or *successive* combination

$$g := g_m g_{m-1} \cdots g_1$$

and the *parallel* or *simultaneous* combination

$$g := \gamma_1\, g_1 + \gamma_2\, g_2 + \ldots + \gamma_m\, g_m \in \mathbb{F}^\alpha(M)$$

where

$$\gamma_i \geq 0 \quad (i = 1, \ldots, m) \qquad \text{and} \qquad \gamma_1 + \gamma_2 + \ldots + \gamma_m = 1.$$

If $g_i \in \mathbb{F}_i^\alpha(M_i)$ $(i = 1, \ldots, m)$, then

$$g \in \mathbb{F}^\alpha(M) \qquad \text{for } \alpha := \frac{1}{2^{m-1}} \min\{\alpha_i : i = 1, \ldots, m\}$$

in the sequential case and

$$g \in \mathbb{F}^\alpha(M) \qquad \text{for } \alpha := \min\{\alpha_i : i = 1, \ldots, m\}$$

in the parallel case (see [9]). Finally, we start from the projectors P_i onto M_i and the corresponding relaxations

$$g_i = (1-\lambda_i)\, I + \lambda_i\, P_i \qquad (0 < \lambda_i < 2).$$

Then we have the relation $g \in \mathbb{F}^\alpha(M)$ with the above α and $\alpha_i = \frac{2-\lambda_i}{\lambda_i}$ by Example 6.3. A further relaxation of g leads to $g_\lambda \in \mathbb{F}^\beta(M)$ with β according to Theorem 5.1. Observe that g_λ then represents for parallelly generated g a so-called *transfer operator* of *simultaneous projectors* (see [1]).

References

[1] Butnariu, D. and Y. Censor: *On the behavior of a block-iterative projection method for solving convex feasibility problems.* Int. J. Comp. Math. 34 (1990), 79 - 94.

[2] Elsner, L., Koltracht, I. and M. Neumann: *Convergence of sequential and asynchronous nonlinear paracontractions.* Numer. Math. 62 (1992), 305 - 319.

[3] Eremin, I. I. and V. D. Mazurov: *Nonstationary Processes of Optimization* (in Russian). Moscow: Nauka 1979.

[4] Goebel, K. and S. Reich: *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings.* New York: Marcel Dekker 1984.

[5] Werner, D.: *Funktionalanalysis.* Berlin: Springer-Verlag 1995.

[6] Schott, D.: *A general iterative scheme with applications to convex optimization and related fields.* Optimization 22 (1991), 885 - 902.

[7] Schott, D.: *Iterative solution of convex problems by Fejér monotone methods.* Num. Funct. Anal. Optim. 16 (1995), 1323 - 1357.

[8] Schott, D.: *Basic properties of Fejér monotone mappings.* Rostock. Math. Kolloq. 50 (to appear).

[9] Schott, D.: *Case studies and geometry of strongly Fejér monotone mappings.* Convex Analysis (submitted).

[10] Youla, D. C.: *Mathematical theory of image restoration by the method of convex projections.* In : Image Recovery: Theory and Applications (ed.: H. Stark). New York: Academic Press 1987.

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