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Convergence Results for Discrete Trigonometric Collocation Methods with Product Integration in Hölder-Zygmund Spaces

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Abstract. In this paper convergence results with respect to Hölder-Zygmund norms – including also maximum norm error estimates – are derived for the fully discrete trigonometric collocation method presented earlier by Saranen and Vainikko for solution of boundary integral equations on smooth closed curves. Approximation of the integral operator is based on product integration for which the explicit Fourier representation of the main part is not needed, and still the convergence of arbitrarily high rate for smooth solutions can be achieved. Saranen and Vainikko have given their analysis with respect to Sobolev norms yielding results that do not imply pointwise error estimates of optimal order. In this work the approach is based on the use of Hölder-Zygmund norms, and the optimal order maximum norm estimates are accomplished.

Keywords: *Boundary integrals, trigonometric collocation, product integration*

AMS subject classification: Primary 65 R 20, secondary 45 L 10

1. Introduction

Saranen and Vainikko introduced in [12] for solution of boundary integral equations a fully discrete trigonometric collocation method based on product integration. Discretization of the integral operator using product integration has been discussed before by several authors (see, for example, the references given in [11: Section 3] and [12]). But the approach of [12] gives us a new efficient scheme of applying this technique. Previously, the operator was supposed to have a specific structure as a decomposition of the main part with an explicit Fourier representation and a smoothing perturbation. This form, however, is not necessary immediately available for operators appearing in applications, and in order to use the discrete method the proper decomposition has to be derived first. In [12] an expansion of more general form is now allowed, and the product integration is applied directly without the exact Fourier representation of the main part, making the method easier to employ in practical computations. Moreover, it gives a high convergence rate, being even an exponential one in the case of infinitely smooth solutions. In addition to solution of a single equation the method can naturally be applied also to systems of boundary integral equations; an application to solving of systems connected with the biharmonic clamped plate problem is presented in [2].

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The basic trigonometric collocation method was discussed in [3], and fully discretized versions in [8] for operators of order 0 and in [1, 4, 11] for operators of arbitrary order. For other full discretizations that have been presented for operators of some particular types see, e.g., the references of [11, 12]. The method of [8] is actually a quadrature method based on the use of integral representation of the operator and any Fourier representation is not needed for applying the scheme. For application of the methods of [1, 4] the previously mentioned specific decomposition of the operator is essential since the main part is discretized based on its Fourier representation; the smooth perturbation is replaced by the trapezoidal rule approximation. The method of [11] is applicable to an operator of a form more general than it is the case in [1, 3, 4], but there also the explicit Fourier representation of the main part is needed. The analysis of the methods of [1, 8, 11, 12] is given with respect to Sobolev norms, and the maximum norm error estimates of optimal order, which means the convergence of the same order as for trigonometric interpolation, are not achieved. In [3, 4] Hölder-Zygmund norms are used, and pointwise error estimates of optimal order are derived for the methods involved in the case of operators of integer order.

In this work we analyze the fully discrete trigonometric method of [12] analyzing it by applying the Hölder-Zygmund norms. Moreover, we present maximum norm error estimates in the case of boundary integral operators of integer order. Concerning the error analysis, we utilize the approach relying on the concepts of stability, consistency and convergence known from [6, 8] and also from [9, 10, 11]. Our analysis is different from that of [3, 4] especially because of the consistency estimates, describing the accuracy of approximation when discretizing operators by using product integration. The methods of [3, 4] are included here with an extension that covers also ε-collocation, $\varepsilon \in [0,1)$. For basic results of Hölder-Zygmund spaces, mapping properties of operators with respect to these norms, and for some results of approximation theory, as well, we refer to [3 - 5]. The statements of this work can be found in a less detailed form in [13] where, however, approximation and consistency results are given without proofs.

2. Preliminaries

We consider the approximate solution of the equation

$$Lu = f \tag{2.1}$$

where u and f are 1-periodic functions. For application of the discretization to be presented in the following, we need f to be continuous. A 1-periodic function (distribution) u has the Fourier representation

$$u(t) = \sum_{k \in \mathbf{Z}} \hat{u}(k) e^{ik2\pi t} \qquad \text{with} \;\; \hat{u}(k) = \int_0^1 u(t) e^{-ik2\pi t} dt.$$

The form of the operator L to be described next covers elliptic boundary integral equations appearing in applications; for the interpretion of the representation, see [12]. The properties $(2.2)_a - (2.2)_h$ of L following below imply the unique solvability of the equation (2.1).

Now, L is supposed to be a pseudodifferential operator of the form

$$L = \sum_{j=0}^{r} A_j \qquad \text{with } (A_j u)(t) = \int_0^1 k_j(t,s)u(s)ds. \tag{$(2.2)_a$}$$

The main part A_0 of L is assumed to have a 1-biperiodic kernel k_0 given by

$$k_0(t,s) = \kappa_0^+(t-s)a_0^+(t,s) + \kappa_0^-(t-s)a_0^-(t) \tag{$(2.2)_b$}$$

where $a_0^+ \in C_1^\infty(\mathbb{R}\times\mathbb{R})$ is 1-biperiodic and $a_0^- \in C_1^\infty(\mathbb{R})$ is 1-periodic, both functions being infinitely smooth. For the Fourier coefficients of 1-periodic functions $\kappa_0^\pm$ we assume the existence of $\beta \in \mathbb{R}$ and $\gamma > \frac{1}{2}$ such that

$$\begin{aligned} \left|\hat{\kappa}_0^+(l) - |l|^\beta\right| &\le C|l|^{\beta-\gamma} \\ \left|\hat{\kappa}_0^-(l) - \operatorname{sign}(l)|l|^\beta\right| &\le C|l|^{\beta-\gamma} \end{aligned} \qquad (l \neq 0). \tag{$(2.2)_c$}$$

As usual, $C > 0$ denotes here and in the following a generic constant. The coefficients $a_0^+(t) := a_0^+(t,t)$ and a_0^- are supposed to satisfy two conditions. The first one is the *ellipticity condition*

$$(a_0^+(t))^2 \neq (a_0^-(t))^2 \qquad (t \in \mathbb{R}). \tag{$(2.2)_d$}$$

The second one concerns the *winding number* which for a function a is denoted by $\mathrm{w}(a)$ and defined by $\mathrm{w}(a) = \frac{1}{2\pi}[\Delta \arg a(t)]_{[0,1]}$ where $[\Delta\arg a(t)]_{[0,1]}$ means the change of the argument of the complex values $a(t)$ when t increases from 0 to 1. We assume that

$$\mathrm{w}(a_0^+ + a_0^-) = \mathrm{w}(a_0^+ - a_0^-). \tag{$(2.2)_e$}$$

The operators $A_j \quad (1 \le j \le r)$ have biperiodic kernels k_j of the form

$$k_j(t,s) = \kappa_j(t-s)a_j(t,s) \qquad \text{with } |\hat{\kappa}_j(l)| \le C|l|^{\beta_j} \quad (l \neq 0,\ 1 \le j \le r) \tag{$(2.2)_f$}$$

where the real values β_j are such that

$$\beta_r \le \beta_{r-1} \le \ \dots\ \le \beta_1 < \beta - \tfrac{1}{2}. \tag{$(2.2)_g$}$$

Finally, we set for L the requirement

$$Lu = 0 \quad \text{for } u \in C_1^\infty(\mathbb{R}) \qquad \Longrightarrow \qquad u = 0. \tag{$(2.2)_h$}$$

The analysis of the methods applied to the equation (2.1) is carried out with respect to the Hölder-Zygmund norms. To define these norms we present first some notations. Let $C^m \quad (m \in \mathbb{N}_0)$ be the space of continuously m-differentiable 1-periodic functions with the norm

$$\|u\|_{C^m} = \sum_{j=0}^{m} \|D^j u\|_C \qquad \text{where } \|u\|_C = \max_{t\in\mathbb{R}} |u(t)| \text{ and } D = \frac{d}{dt}.$$

Moreover, denote

$$[u]_\alpha = \begin{cases} \sup_{h>0} \frac{\|\Delta_h u\|_C}{h^\alpha} & \text{if } 0 < \alpha < 1 \\ \sup_{h>0} \frac{\|\Delta_h^2 u\|_C}{h^\alpha} & \text{if } \alpha = 1 \end{cases}$$

with

$$(\Delta_h u)(t) = u(t+h) - u(t) \qquad \text{and} \qquad \Delta_h^2 = \Delta_h \circ \Delta_h.$$

Now, the Hölder-Zygmund space H^σ for real values $\sigma > 0$ is defined by

$$H^\sigma = \left\{ u \in C^m \middle| [D^m u]_\alpha < \infty \right\} \qquad (\sigma = m + \alpha \in \mathbb{N}_0 + (0,1])$$

with the corresponding norm $\|u\|_{H^\sigma} = \|u\|_{C^m} + [D^m u]_\alpha$. Introducing the notations Λ_η^+ and Λ_η^- for $\eta \in \mathbb{R}$ as

$$\begin{aligned} (\Lambda_\eta^+ u)(t) &= \hat{u}(0) + \sum_{k \in \mathbb{Z}^*} |k|^\eta \hat{u}(k) e^{ik2\pi t} \\ (\Lambda_\eta^- u)(t) &= \sum_{k \in \mathbb{Z}^*} \operatorname{sign}(k) |k|^\eta \hat{u}(k) e^{ik2\pi t} \end{aligned} \tag{2.3}$$

where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, we extend the definition for non-positive values of σ by applying the Bessel potential Λ_η^+. The operator $\Lambda_\eta^+ : H^\sigma \to H^{\sigma-\eta}$ is an isomorphism when $\sigma > \max(0,\eta)$ (see Lemma 2.1). Choosing η such that $\eta < \sigma \leq 0$, we set

$$\|u\|_{H^\sigma} = \|\Lambda_\eta^+ u\|_{H^{\sigma-\eta}} \tag{2.4}$$

yielding with different values of η a family of norms $\|\cdot\|_{H^\sigma}$ equivalent to each other. Accordingly, we define the Hölder-Zygmund space H^σ for indices $\sigma \leq 0$ to be the set of 1-periodic distributions u such that the norm (2.4) is finite.

In the following lemma we present some mapping properties of the Bessel potentials $\Lambda_\eta^\pm$ $(\eta \in \mathbb{R})$. The property (2.5) and the invertibility result of Λ_η^+ are given in [3 - 5] (without proofs – the results are classical and they are based on the works of Noether and Stein (for accurate references see [5])). The estimates (2.6) (a related result is given in [5: Lemma 4.7]) and (2.7) are verified in Appendix.

Lemma 2.1. *For $\Lambda_\eta^\pm$ $(\eta \in \mathbb{R})$ given by* (2.3) *there holds*

$$\Lambda_\eta^\pm : H^\sigma \to H^{\sigma-\eta} \qquad (\sigma \in \mathbb{R}). \tag{2.5}$$

The operator $\Lambda_\eta^+ : H^\sigma \to H^{\sigma-\eta}$ is an isomorphism for $\sigma > \max(0,\eta)$ and consequently, with the extension (2.4), *for all $\sigma \in \mathbb{R}$, such that the inverse is $(\Lambda_\eta^+)^{-1} = \Lambda_{-\eta}^+$. With the notation*

$$\|\phi\|_\mu^* = |\hat{\phi}(0)| + \sum_{k \in \mathbb{Z}^*} |k|^\mu |\hat{\phi}(k)| \qquad (\mu \in \mathbb{R})$$

we have

$$\|\phi \Lambda_\eta^\pm u\|_{H^{\sigma-\eta}} \leq \begin{cases} C\|\phi\|_{H^{\sigma-\eta}} \|u\|_{H^\sigma} & \text{if } \sigma > \eta \\ C\|\phi\|_\nu^* \|u\|_{H^\sigma} & \text{if } \sigma \leq \eta,\ \nu > |\sigma - \eta| + 1. \end{cases} \tag{2.6}$$

Furthermore, there holds

$$\|(\phi\Lambda_\eta^\pm - \Lambda_\eta^\pm\phi)u\|_{H^{\sigma-\eta}} \le C\|\phi\|_\nu^* \|u\|_{H^{\sigma-1}} \qquad (2.7)$$

for $\sigma \in \mathbb{R}$ *and* $\nu > \max(|\sigma-\eta-1|, |\sigma-1|)+2$.

The next lemma gives the mapping property for the operator

$$(Au)(t) = \int_0^1 \kappa(t-s)a(t,s)u(s)\,ds \qquad (2.8)_a$$

where κ is a 1-periodic function such that

$$|\hat{\kappa}(l)| \le C|l|^\eta \qquad (l \ne 0;\ \eta \in \mathbb{R}) \qquad (2.8)_b$$

and a is a 1-biperiodic function (sufficiently smooth). For a with the representation

$$a(t,s) = \sum_{k\in\mathbb{Z}} a_k(t)e^{ik2\pi s}$$

we define the norm $\|\cdot\|_{\nu,\mu}^{*;*}$ by

$$\|a\|_{\nu,\mu}^{*;*} = \|a_0\|_\nu^* + \sum_{k\in\mathbb{Z}^*} |k|^\mu \|a_k\|_\nu^* \qquad (\nu,\mu \in \mathbb{R}).$$

Lemma 2.2. *Let A be an operator of the form* (2.8). *Then the estimate*

$$\|Au\|_{H^{\sigma-\eta}} \le C\|a\|_{\nu,\mu}^{*;*} \|u\|_{H^{\sigma+\delta}} \qquad (2.9)$$

with $\sigma \in \mathbb{R}$, $\delta > \frac{1}{2}$, $\nu > |\sigma-\eta|+1$ *and* $\mu > |\sigma|$ *holds.*

As shown in Appendix, analogously to [12], the following property of L can be derived from Lemmas 2.1 and 2.2.

Theorem 2.3. *For the operator L given by* $(2.2)_a - (2.2)_h$ *there holds that*

$$L : H^\sigma \to H^{\sigma-\beta} \qquad (\sigma \in \mathbb{R}) \qquad (2.10)$$

is an isomorphism.

To describe the approximate solution of the equation (2.1), we first define the N-dimensional space T_N $(N \in \mathbb{N})$ of trigonometric polynomials

$$v(t) = \sum_{l\in\Lambda_N} v_l e^{il2\pi t} \qquad (v_l \in \mathbb{C})$$

with

$$\Lambda_N = \left\{ l \in \mathbb{Z} \,\middle|\, -\frac{N}{2} < l \le \frac{N}{2} \right\}.$$

Let Q_N^ε $(0 \le \varepsilon < 1)$ be the trigonometric interpolation operator such that

$$Q_N^\varepsilon u = v \in T_N : \quad v(\tfrac{l+\varepsilon}{N}) = u(\tfrac{l+\varepsilon}{N}) \quad (l \in \Lambda_N) \tag{2.11}$$

where u is a continuous 1-periodic function. For L given by (2.2) we apply a discretization L_N of the form

$$\left.\begin{aligned} L_N u &= \sum_{j=0}^{r} A_{j,N} \\ (A_{0,N}u)(t) &= \int_0^1 \kappa_0^+(t-s)\big[Q_{N,\xi}\big(a_0(t,\xi)u(\xi)\big)\big](s)\,ds \\ &\quad + a_0^-(t)\int_0^1 \kappa_0^-(t-s)\big[Q_{N,\xi}u(\xi)\big](s)\,ds \\ (A_{j,N}u)(t) &= \int_0^1 \kappa_j(t-s)\big[Q_{N,\xi}\big(a_j(t,\xi)u(\xi)\big)\big](s)\,ds \end{aligned}\right\} \quad (1 \le j \le r). \tag{2.12}$$

Here $Q_{N,\xi}$ denotes the interpolation operator such that the trigonometric interpolation is applied with respect to the variable ξ at the points $\frac{j}{N}$ $(j \in \Lambda_N)$. The equation (2.1) is now solved by replacing L with L_N and collocating at the points $\frac{j+\varepsilon}{N}$ $(j \in \Lambda_N)$, which may equivalently be written as

$$u_N \in T_N : \quad Q_N^\varepsilon L_N u_N = Q_N^\varepsilon f. \tag{2.13}$$

Finally, for analysis we define the trigonometric operator $P_N : H^\sigma \to T_N$ $(\sigma \in \mathbb{R})$ by

$$(P_N u)(t) = \sum_{k \in \Lambda_N} \hat{u}(k) e^{ik2\pi t}. \tag{2.14}$$

3. Lemmas

In this section we present lemmas needed in the analysis of (2.13). The first lemma gives properties of the trigonometric operators P_N and Q_N^ε. Concerning the estimate (3.1) for positive indices we refer, for instance, to [5, 7] and the references given there; for the remaining indices the result is immediate by (2.4). The estimate (3.2) is obtained by extending in an obvious way the corresponding result of [3: Theorem 2.1] such that in addition to $\varepsilon = 0$ also the values $\varepsilon \in (0,1)$ are covered. The new estimates (3.3) and (3.4), that give better results than the ones implied by the general property (3.2), are verified in Appendix. Versions of the estimates (3.3) and (3.4) with respect to Sobolev norms can be found in [10: Lemma 3.3], [11: Lemma 4.1] and [12: Propositions 4 and 5]. For Hölder-Zygmund norms an approximation result related to (3.4) has been derived in [5: Lemma 4.8] for P_N in case of positive indices. From now on, we assume $N \ge 2$.

Lemma 3.1. *If $\tau, \sigma \in \mathbb{R}$ with $\tau \le \sigma$, then there holds*

$$\|(I - P_N)u\|_{H^\tau} \le CN^{\tau-\sigma} \ln N \|u\|_{H^\sigma}. \tag{3.1}$$

For $0 < \tau \le \sigma$ we have

$$\|(I - Q_N^\varepsilon)u\|_{H^\tau} \le CN^{\tau-\sigma} \ln N \|u\|_{H^\sigma}. \tag{3.2}$$

Assume $v \in T_N$. Then we obtain the estimates

$$\|Q_N^\varepsilon(\phi v)\|_{H^\tau} \le C\|\phi\|_\nu^* \|v\|_{H^\tau} \qquad (\tau \in \mathbb{R},\, \nu > |\tau| + 1) \tag{3.3}$$

and

$$\|(I - Q_N^\varepsilon)(\phi v)\|_{H^\tau} \le CN^{\tau-\sigma} \|\phi\|_\nu^* \|v\|_{H^\sigma} \quad \left({\tau,\sigma\in\mathbb{R} \atop \nu > |\tau|+1+\max(\sigma-\tau,0)} \right). \tag{3.4}$$

We give next variants of the estimates (3.1) - (3.3) needed for showing the error estimates with respect to the maximum norms. For this introduce the operators P^+ and P^- by

$$(P^+u)(t) = \sum_{k\ge 0} \hat{u}(k) e^{ik2\pi t} \qquad \text{and} \qquad (P^-u)(t) = \sum_{k\le -1} \hat{u}(k) e^{ik2\pi t}.$$

They can be written also in the form

$$P^+ = \frac{1}{2}(\Lambda_0^+ + \Lambda_0^- + J) \qquad \text{and} \qquad P^- = \frac{1}{2}(\Lambda_0^+ - \Lambda_0^- - J)$$

with $Ju = \hat{u}(0)$. The inequality (3.1)′ below follows from the facts that $P^\pm P_N = P_N P^\pm$ and

$$\|(I - P_N)u\|_{C^\tau} \le CN^{\tau-\sigma} \ln N \|u\|_{H^\sigma} \qquad (\tau \in \mathbb{N}_0,\, \sigma \in \mathbb{R},\, \tau < \sigma)$$

(see, e.g., [5, 7]). The essential result for proving (3.2)′, namely

$$\|P^\pm(Q_N^\varepsilon - P_N)u\|_{C^\tau} \le CN^{\tau-\sigma} \ln N \|u\|_{H^\sigma} \qquad (\tau \in N_0,\, \sigma \in \mathbb{R},\, \tau < \sigma)$$

is shown in the proof of [5: Lemma 5.2] (there with $\varepsilon = 0$; the extension for $\varepsilon \in (0,1)$ is obvious). The third result (3.3)′ is shown in Appendix.

Lemma 3.1′. *Assume $\tau \in \mathbb{N}_0$ and $\sigma \in \mathbb{R}$ with $\tau < \sigma$. Then*

$$\left\|P^\pm(I - P_N)u\right\|_{C^\tau} \le CN^{\tau-\sigma} \ln N \|u\|_{H^\sigma} \tag{3.1$'$}$$

$$\left\|P^\pm(I - Q_N^\varepsilon)u\right\|_{C^\tau} \le CN^{\tau-\sigma} \ln N \|u\|_{H^\sigma}. \tag{3.2$'$}$$

Moreover, for $\phi \in C_1^\infty(\mathbb{R})$ and $v \in T_N$ there holds

$$\|P^\pm Q_N^\varepsilon(\phi v)\|_{C^\tau} \le C\big(\|P^+v\|_{C^\tau} + \|P^-v\|_{C^\tau}\big). \tag{3.3$'$}$$

The next lemma contains consistency properties for the approximation of an operator $A: H^\sigma \to H^{\sigma-\beta-\delta}$ ($\sigma, \beta \in \mathbb{R}$ and $\delta > \frac{1}{2}$) (see Lemma 2.2) given by

$$(Au)(t) = \int_0^1 \kappa(t-s)a(t,s)u(s)\,ds \qquad (|\hat{\kappa}(l)| \le C|l|^\beta,\ l \neq 0) \tag{3.5}$$

where a is a sufficiently smooth 1-periodic function. Furthermore, let A_N be the approximation

$$(A_N u)(t) = \int_0^1 \kappa(t-s)\big[Q_{N,\xi}\big(a(t,\xi)u(\xi)\big)\big](s)\,ds. \tag{3.6}$$

Lemma 3.2. *For A and A_N given by* (3.5) *and* (3.6), *respectively, there holds*

$$\big\|Q_N^\varepsilon(A - A_N)v\big\|_{H^{\tau-\beta}} \le CN^{\tau-\sigma}\|a\|^{*,*}_{\nu,\mu}\|v\|_{H^\sigma} \qquad (v \in T_N;\ \tau,\sigma \in \mathbb{R}) \tag{3.7}$$

where

$$\nu > |\tau - \beta| + 1 \qquad \text{and} \qquad \mu > |\beta| + |\tau - \beta| + \max(\sigma - \tau, 0) + 2.$$

Especially, if $\varepsilon = 0$ and

$$\big|\hat{\kappa}(l) - |l|^\beta\big| \le C|l|^{\beta-\gamma} \qquad (l \neq 0 \text{ for some } \gamma > 0), \tag{3.8}$$

then

$$\big\|Q_N^\varepsilon(A - A_N)v\big\|_{H^{\tau-\beta}} \le CN^{\tau-\sigma-\min\{1,\gamma\}}\|\phi\|^{*,*}_{\nu,\tilde{\mu}}\|v\|_{H^\sigma} \qquad \left(\begin{smallmatrix} v\in T_N \\ \tau,\sigma\in\mathbb{R}\end{smallmatrix}\right) \tag{3.9}$$

with

$$\tilde{\mu} > |\beta| + |\tau - \beta| + \max(\sigma - \tau, 0) + 2 + \min(1, \gamma).$$

On the other hand, if $\varepsilon \in [0,1)$ and in (3.5) *$a(t,s) =: a(t)$ depends only on t, then*

$$Q_N^\varepsilon A_N v = Q_N^\varepsilon A v \qquad (v \in T_N).$$

Finally, if in (3.5) *and* (3.6) *$\kappa(t) = 1$ $(t \in \mathbb{R})$ and $a \in C_1^\infty(\mathbb{R} \times \mathbb{R})$, then we have for any $\lambda \in \mathbb{R}$*

$$\big\|Q_N^\varepsilon(A - A_N)v\big\|_{H^{\tau-\beta}} \le CN^\lambda\|v\|_{H^\sigma} \qquad (v \in T_N;\ \tau,\sigma \in \mathbb{R}). \tag{3.10}$$

Also these estimates are proved in Appendix. The corresponding results of [11: Lemmas 3.4 and 3.5] and [12: Lemmas 2 and 3] in Sobolev spaces, and the related result [4: Lemma 3.2] of (3.10) are referred to. From Lemma 3.2 we deduce the consistency property for the approximation L_N of L, given by (2.12) and (2.2), respectively.

Theorem 3.3. *Let L have the form* (2.2), *and assume that $\varepsilon = 0$, or that $\varepsilon \in (0,1)$ and in* $(2.2)_b$ *$a_0^+(t,s) = a_0^+(t)$ depends only on the variable t. Then, we get for the approximation L_N given by* (2.12) *the estimate*

$$\big\|Q_N^\varepsilon(L - L_N)v\big\|_{H^{\tau-\beta}} \le CN^{\tau-\sigma-\min(1,\gamma,\beta-\beta_1)}\|v\|_{H^\sigma} \qquad \left(\begin{smallmatrix} v\in T_N \\ \tau,\sigma\in\mathbb{R}\end{smallmatrix}\right) \tag{3.11}$$

where $\gamma > 0$ is the value given in $(2.2)_c$.

4. Stability and convergence

For the analysis of the fully discrete method (2.13) we use the stability result of the corresponding trigonometric ε-collocation method, $\varepsilon \in [0,1)$,

$$u_N^c \in T_N : \quad Q_N^\varepsilon L u_N^c = Q_N^\varepsilon f \tag{4.1}$$

where L has the form $(2.2)_a - (2.2)_h$.

Theorem 4.1. *For sufficiently large values of $N \in \mathbb{N}$, the method (4.1) is stable, i.e. there holds the estimate*

$$\|v\|_{H^\tau} \leq C\|Q_N^\varepsilon L v\|_{H^{\tau-\beta}} \qquad (\tau > \beta,\, v \in T_N) \tag{4.2}$$

and consequently, the equation (4.1) is uniquely solvable if $f \in H^{\sigma-\beta}$ $(\sigma > \beta)$. Moreover, for $\beta < \tau \leq \sigma$ we obtain the convergence estimate

$$\|u - u_N^c\|_{H^\tau} \leq C N^{\tau-\sigma} \ln N \|u\|_{H^\sigma}. \tag{4.3}$$

Proof. By [12: Theorem 2] (with the extension to all the values of $\varepsilon \in [0,1)$), the equation (4.1) is uniquely solvable for large enough values of N. We derive next the stablility estimate (4.2). As shown in Appendix (see the formula (A.14)), by the procedure of [12] applying the results of [6], we get for L the form

$$L = c(\tilde{A} + \tilde{B}) \qquad (c \in C_1^\infty(\mathbb{R}),\, c(t) \neq 0 \text{ for } t \in \mathbb{R}) \tag{4.4}$$

where $\tilde{A} : H^\sigma \to H^{\sigma-\beta}$ $(\sigma \in \mathbb{R})$ is such that $\tilde{A}v \in T_N$ for all $v \in T_N$ and $\tilde{B} : H^\sigma \to H^{\sigma-\beta+\delta}$ $(\sigma \in \mathbb{R})$ is bounded for some value $\delta > 0$. Hence we obtain

$$Lv = c\,Q_N^\varepsilon(c^{-1} Q_N^\varepsilon L v) + c(I - Q_N^\varepsilon)\tilde{B}v \qquad (v \in T_N) \tag{4.4$'$}$$

which, when applying the invertibility of L, as well as (2.6), (3.2) and (3.3), yields

$$\begin{aligned} \|v\|_{H^\tau} &\leq C\|Lv\|_{H^{\tau-\beta}} \\ &\leq C\Big(\|Q_N^\varepsilon L v\|_{H^{\tau-\beta}} + \|(I - Q_N^\varepsilon)Bv\|_{H^{\tau-\beta}}\Big) \qquad (\tau > \beta). \\ &\leq C\Big(\|Q_N^\varepsilon L v\|_{H^{\tau-\beta}} + N^{-\delta}\ln N \|v\|_{H^\tau}\Big) \end{aligned}$$

If $N \in \mathbb{N}$ is sufficiently large (let us say, for instance, that $N^{-\delta}\ln N < \frac{1}{2C}$), then (4.2) is implied. By (2.1) the equation $Q_N^\varepsilon L u_N^c = Q_N^\varepsilon L u$ is equivalent to (4.1). By this, as well as by (4.2) we get the error estimate

$$\|u - u_N^c\|_{H^\tau} \leq \|(I - P_N)u\|_{H^\tau} + C\|Q_N^\varepsilon L(P_N - I)u\|_{H^{\tau-\beta}} \tag{4.5}$$

if N is large enough. Then, we make use of the decomposition

$$Q_N^\varepsilon L(P_N - I)u = L(P_N - I)u + (I - Q_N^\varepsilon)Lu + (Q_N^\varepsilon - I)LP_N u$$

and the associated inequalities

$$\begin{aligned} \left\| L(I-P_N)u \right\|_{H^{\tau-\beta}} &\le CN^{\tau-\sigma}\ln N\|u\|_{H^\sigma} \\ \left\| (I-Q_N^\varepsilon)Lu \right\|_{H^{\tau-\beta}} &\le CN^{\tau-\sigma}\ln N\|u\|_{H^\sigma} \qquad (\beta<\tau\le\sigma). \\ \left\| (Q_N^\varepsilon - I)LP_N u \right\|_{H^{\tau-\beta}} &\le CN^{\tau-\sigma}\ln N\|u\|_{H^\sigma} \end{aligned}$$

Here the first two inequalities are direct consequences of (3.1) and (3.2) and the last one follows from the representation (4.4) by (3.1), (3.2), (2.6) and (3.4) (for N large enough). Thus, we get

$$\left\| Q_N^\varepsilon L(P_N - I)u \right\|_{H^{\tau-\beta}} \le CN^{\tau-\sigma}\ln N\|u\|_{H^\sigma} \qquad (\beta<\tau\le\sigma). \tag{4.6}$$

The convergence estimate (4.3) is achieved by (4.5), (3.1) and (4.6) ∎

Theorem 4.1 supplements the stability and convergence results of [3] and [4], where (4.1) is considered for $\varepsilon = 0$ in case of an operator L being of a form simpler than (2.2). Recalling in this connection the analogy to [11: Theorem 3.6] and [12: Theorem 3], we deduce next from the result of Theorem 4.1 and from the consistency property (3.11) stability and convergence results.

Theorem 4.2. *Assume that L has the form (2.2), and either $\varepsilon = 0$, or $\varepsilon \in (0,1)$ and additionally in $(2.2)_b$ $a_0^+(t,s) = a_0^+(t)$ depends only on the variable t. Assume also $f \in \dot{H}^{\sigma-\beta}$ $(\sigma > \beta)$. Then, the equation (2.13) has a unique solution $u_N \in T_N$ if only N is large enough. Moreover, if $\beta < \tau \le \sigma$, there holds the asymptotic error estimate*

$$\|u - u_N\|_{H^\tau} \le CN^{\tau-\sigma}\ln N\|u\|_{H^\sigma}. \tag{4.7}$$

Proof. By [12: Theorem 3] the equation (2.13) is uniquely solvable. Writing

$$Q_N^\varepsilon Lv = Q_N^\varepsilon L_N v + Q_N^\varepsilon(L - L_N)v \qquad (v \in T_N) \tag{4.4$''$}$$

we obtain, for sufficiently large value of N, by (4.2) and (3.11)

$$\begin{aligned} \|v\|_{H^\tau} &\le C\{\|Q_N^\varepsilon L_N v\|_{H^{\tau-\beta}} + \|Q_N^\varepsilon(L-L_N)v\|_{H^{\tau-\beta}}\} \\ &\le C\{\|Q_N^\varepsilon L_N v\|_{H^{\tau-\beta}} + N^{-\tilde{\delta}}\|v\|_{H^\tau}\} \end{aligned}$$

where $\tilde{\delta} = \min(1,\gamma,\beta-\beta_1) > \frac{1}{2}$. Choosing again N large enough, we get from the previous inequality the stability estimate

$$\|v\|_{H^\tau} \le C\|Q_N^\varepsilon L_N v\|_{H^{\tau-\beta}} \qquad (v \in T_N,\ \tau > \beta). \tag{4.8}$$

Application of (2.13) yields the equality

$$Q_N^\varepsilon L_N(P_N u - u_N) = Q_N^\varepsilon(L_N - L)P_N u + Q_N^\varepsilon L(P_N - I)u \tag{4.9}$$

which together with (4.8), (3.1), (3.11) and (4.6) implies

$$\begin{aligned} \|u - u_N\|_{H^\tau} &\le \|(I-P_N)u\|_{H^\tau} \\ &\quad + C\Big\{\|Q_N^\varepsilon(L_N - L)P_N u\|_{H^{\tau-\beta}} + \|Q_N^\varepsilon L(P_N - I)u\|_{H^{\tau-\beta}}\Big\} \\ &\le CN^{\tau-\sigma}\ln N\|u\|_{H^\sigma} \end{aligned}$$

(for $\beta < \tau \le \sigma$) which completes the proof ∎

Now, by modifying the proof of Theorem 4.2, a result analogous to [4: Theorem 2.2/Formula (2.18)] with respect to the maximum norm can be shown.

Theorem 4.3. *Let the assumptions of Theorem 4.2 be valid, and in addition, assume that $\beta \in \mathbb{Z}$. Then, for sufficiently large value of $N \in \mathbb{N}$, we have for the method* (2.13) *the error estimate*

$$\|u - u_N\|_{C^\tau} \le CN^{\tau-\sigma} \ln N \|u\|_{H^\sigma} \qquad \left(\substack{\tau\in\mathbb{N}_0,\,\sigma\in\mathbb{R}\\ \max(0,\beta)\le\tau<\sigma}\right). \tag{4.10}$$

Proof. The stability estimate

$$\|v\|_{C^\tau} \le C\Big(\|P^+(Q_N^\epsilon Lv)\|_{C^{\tau-\beta}} + \|P^-(Q_N^\epsilon Lv)\|_{C^{\tau-\beta}}\Big) \qquad \left(\substack{v\in T_N\\ \tau\in\mathbb{N}_0}\right) \tag{4.11}$$

of the trigonometric collocation method (4.1) follows, for sufficiently large N, from

$$\|v\|_{C^\tau} \le C\Big(\|P^+(Lv)\|_{C^{\tau-\beta}} + \|P^-(Lv)\|_{C^{\tau-\beta}}\Big)$$

implied by [4: Lemma 5.1]. Moreover, we need the representation (4.4)$'$, the estimates

$$\|\phi u\|_{C^m} \le C\|u\|_{C^m} \qquad (\phi \in C_1^\infty(\mathbb{R}),\, m \in \mathbb{N}_0),$$

(3.3)$'$, (2.6), (2.7), (3.2) and (3.3). Essential for the stability result of (2.13) is the consistency estimate

$$\big\|P^\pm Q_N^\epsilon(L - L_N)v\big\|_{C^{\tau-\beta}} \le CN^{\tau-\sigma-\lambda}\|v\|_{H^\sigma} \qquad (v \in T_N,\, \sigma \in \mathbb{R}) \tag{4.12}$$

with $0 < \lambda < \min(1,\gamma,\beta-\beta_1)$, obtained by applying Theorem 3.3. Now, this inequality with the choice $\sigma = \tau$ and (4.11) yield

$$\|v\|_{C^\tau} \le C\Big(\|P^+(Q_N^\epsilon L_N v)\|_{C^{\tau-\beta}} + \|P^-(Q_N^\epsilon L_N v)\|_{C^{\tau-\beta}}\Big) \qquad \left(\substack{v\in T_N\\ \tau\in\mathbb{N}_0}\right) \tag{4.13}$$

if N is large enough. The convergence estimate (4.10) can be shown by modifying the proof of Theorem 4.2. For estimation of

$$\|P^\pm Q_N^\epsilon L(P_N - I)u\|_{C^{\tau-\beta}}$$

we need the representation (A.13) (given in the proof of Theorem 2.2 in Appendix), and particularly the equality

$$\begin{aligned} &Q_N^\epsilon(a_0^+\Lambda_\beta^+ + a_0^-\Lambda_\beta^-)(P_N - I)u \\ &\qquad = Q_N^\epsilon a_0^+(P_N - Q_N^\epsilon)\Lambda_\beta^+ u + Q_N^\epsilon a_0^-(P_N - Q_N^\epsilon)\Lambda_\beta^- u. \end{aligned}$$

Hence, by the mapping property of B_1 in (A.13), we get (4.10) when applying (3.1), (3.1)$'$ - (3.3)$'$ and (4.12) ∎

5. Appendix

Next we give the proofs for lemmas and theorems of Chapters 2 and 3.

Proof of Lemma 2.1. Before verifying (2.6), (2.7) and (2.10) we give for $\sigma > 0$ and $\sigma = m + \alpha$ with $m \in \mathbb{Z}^+ \cup \{0\}$ and $\alpha \in (0,1]$ the results

$$\|\phi u\|_{H^\sigma} \le C\|\phi\|_{H^\sigma}\|u\|_{H^\sigma} \qquad (A.1)$$

$$\|\phi u\|_{C^m} \le C\|\phi\|_{C^m}\|u\|_{C^m} \qquad (A.1)'$$

which follow directly from

$$D^m(\phi u) = \sum_{j=0}^{m} \binom{m}{j} \phi^{(j)} u^{(m-j)} \qquad \text{and} \qquad [\phi u]_\alpha \le C\|\phi\|_{H^\alpha}\|u\|_{H^\alpha}.$$

For estimation of $\|\phi u\|_{H^\sigma}$, in the case of $\sigma \le 0$, we apply the invertibility of Λ_η^+ with a choice of $\eta = \sigma - \rho$, and consider only the norms $\|\cdot\|_{H^\rho}$ $(0 < \rho < 1)$. We write first

$$\Lambda_\eta^+(e_k u) = e_k \sum_{l\in\mathbb{Z}} \big(\max(1, |k+l|)\big)^\eta \hat{u}(l) e_l = e_k(\Lambda_\eta^+ u + C_{\eta k} u) \qquad (k \in \mathbb{Z}) \qquad (A.2)$$

where $e_k(t) = e^{ik2\pi t}$ and

$$C_{\eta k} u = \sum_{l\in\mathbb{Z}} \Big[\big(\max(1,|k+l|)\big)^\eta - \big(\max(1,|l|)\big)^\eta\Big] \hat{u}(l) e_l.$$

By the formula

$$|k+l|^\eta - |l|^\eta = \eta \xi_{kl}^{\eta-1} \Delta(k,l) \qquad (l \ne 0, -k)$$

with ξ_{kl} being a real value between $|k+l|$ and $|l|$ and $\Delta(k,l) = |k+l| - |l|$, we obtain

$$\begin{aligned} \big||k+l|^\eta - |l|^\eta\big| &\le C|k| \max\big(|k+l|^{\eta-1}|l|^{-\eta}, |l|^{-1}\big)|l|^\eta \\ &\le C|k|^{|\eta|+1} \max\big(|k+l|^{-1}, |l|^{-1}\big)|l|^\eta \end{aligned} \qquad (l \ne 0, -k)$$

where the last upper bound is achieved by applying Peetre's inequality

$$|j|^r |k|^{-r} \le 2^{|r|} |j-k|^{|r|} \qquad (j,k \in \mathbb{Z}^* \text{ with } j \ne k). \qquad (A.3)$$

Making use of the Hölder inequality yields

$$\begin{aligned} \|C_{\eta k} u\|_C &\le C|k|^{|\eta|+1} \Big\{ \|\Lambda_\eta^+ u\|_C + \Big(\sum_{l \ne 0,-k} \max\big(|k+l|^{-2}, |l|^{-2}\big) \Big)^{\frac{1}{2}} \|\Lambda_\eta^+ u\|_{L_2} \Big\} \\ &\le C|k|^{|\eta|+1} \|\Lambda_\eta^+ u\|_C. \end{aligned} \qquad (A.4)$$

In the same way we can show $\|\Delta_h(C_{\eta k} u)\|_C \le C|k|^{|\eta|+1} \|\Delta_h \Lambda_\eta^+ u\|_C$ which implies

$$[C_{\eta k} u]_\rho \le C \max\big(1, |k|^{|\eta|+1}\big) [\Lambda_\eta^+ u]_\rho \qquad (0 < \rho < 1)$$

giving with (A.4)

$$\|C_{\eta k}u\|_{H^\rho} \le C|k|^{|\eta|+1}\|\Lambda_\eta^+ u\|_{H^\rho}. \tag{A.5}$$

Applying (A.1) with $\sigma = \rho$ and

$$\|e_k\|_{H^\rho} \le 1 + 2\pi^\rho |k|^\rho \tag{A.6}$$

as well as (A.5), we get, based on the representation (A.2),

$$\begin{aligned}\|\Lambda_\eta^+(e_k u)\|_{H^\rho} &\le C\|e_k\|_{H^\rho}\big(\|\Lambda_\eta^+ u\|_{H^\rho} + \|C_{\eta k}u\|_{H^\rho}\big)\\ &\le C\max\big(1, |k|^{|\eta|+1+\rho}\big)\|\Lambda_\eta^+ u\|_{H^\rho}.\end{aligned}$$

So, we obtain the estimate

$$\|\Lambda_\eta^+(\phi u)\|_{H^\rho} \le C\|\phi\|^*_{|\eta|+1+\rho}\|\Lambda_\eta^+ u\|_{H^\rho} \qquad (0<\rho<1) \tag{A.7}$$

which with $\eta = \sigma - \rho$ implies for $\sigma \le 0$

$$\|\phi u\|_{H^\sigma} = \|\Lambda_{\sigma-\rho}^+(\phi u)\|_{H^\rho} \le C\|\phi\|^*_{|\sigma|+1+2\rho}\|\Lambda_{\sigma-\rho}^+ u\|_{H^\rho}.$$

Consequently, for any $\nu > |\sigma| + 1$, when choosing $\rho = \frac{\nu-|\sigma|-1}{2}$, we have

$$\|\phi u\|_{H^\sigma} \le C\|\phi\|_\nu^*\|u\|_{H^\sigma} \qquad (\sigma \le 0). \tag{A.8}$$

Now, by (A.1) and (A.8), the estimate (2.6) follows.

Applying the expansion

$$|k+l|^\eta - |l|^\eta = \eta|l|^{\eta-1}\Delta(k,l) + \frac{\eta(\eta-1)}{2}\zeta_{lk}^{\eta-2}(\Delta(k,l))^2 \qquad (l \ne 0, -k)$$

where ζ_{lk} is between $|k+l|$ and $|l|$, we have $(\Lambda_\eta^+ e_0 - e_0\Lambda_\eta^+)u = 0$ and, for $k \ne 0$,

$$\Lambda_\eta^+ e_k - e_k\Lambda_\eta^+ = e_k(\eta k\Lambda_{\eta-1}^- + D_{\eta k})$$

with

$$\begin{aligned}D_{\eta k}u = (|k|^\eta - 1)\hat{u}(0) &- \operatorname{sign}(k)\big((\eta-1) + |k|^{-\eta}\big)(\widehat{\Lambda_{\eta-1}^- u})(-k)e_{-k}\\ &- 2\eta\sum_{l(k+l)<0}\operatorname{sign}(l)|l|^{\eta-1}(k+l)\hat{u}(l)e_l\\ &+ \frac{\eta(\eta-1)}{2}\sum_{l\ne 0,-k}\zeta_{lk}^{\eta-2}(\Delta(l,k))^2\hat{u}(l)e_l.\end{aligned}$$

Estimating in the already described way, we obtain $\|D_{\eta k}u\|_{H^\rho} \le C|k|^{|\eta-1|+2}\|\Lambda_{\eta-1}^- u\|_{H^\rho}$ $(0<\rho<1)$ which further gives

$$\big\|(\Lambda_\eta^+\phi - \phi\Lambda_\eta^+)u\big\|_{H^\rho} \le C(\eta,\rho)\|\phi\|^*_{|\eta-1|+2+\rho}\|u\|_{H^{\rho+\eta-1}} \qquad \left({0<\rho<1 \atop \eta\in\mathbb{R}}\right). \tag{A.9}$$

Using $\Lambda_\lambda^+\Lambda_\mu^+ = \Lambda_{\lambda+\mu}^+$ $(\lambda,\mu\in\mathbb{R})$ we write

$$\Lambda_{\sigma-\eta-\rho}^+(\phi\Lambda_\eta^+ - \Lambda_\eta^+\phi)u = \big(\Lambda_{\sigma-\eta-\rho}^+\phi - \phi\Lambda_{\sigma-\eta-\rho}^+\big)\Lambda_\eta^+ u + \big(\phi\Lambda_{\sigma-\rho}^+ - \Lambda_{\sigma-\rho}^+\phi\big)u$$

and then by (A.9) obtain

$$\begin{aligned}&\big\|(\phi\Lambda_\eta^+ - \Lambda_\eta^+\phi)u\big\|_{H^{\sigma-\eta}}\\ &\quad\le C\Big\{\|\phi\|^*_{|\sigma-\eta-1-\rho|+2+\rho}\|u\|_{H^{\sigma-1}} + \|\phi\|^*_{|\sigma-1-\rho|+2+\rho}\|u\|_{H^{\sigma-1}}\Big\}\\ &\quad\le C\|\phi\|^*_{\tilde\nu}\|u\|_{H^{\sigma-1}}\end{aligned}\tag{A.10}$$

with any $\sigma\in\mathbb{R}$ and $\tilde\nu > \max(|\sigma-\eta-1|,|\sigma-1|)+2$, verifying (2.7) in the case of Λ_η^+.

To show (2.7) for Λ_η^- $(\eta\in\mathbb{R})$ we consider first

$$\Lambda_\eta^+(\Lambda_0^- e_k - e_k\Lambda_0^-)u = e_k E_{\eta k} \quad \text{with} \quad E_{\eta k}u = -2\sum_{l(k+l)<0}\operatorname{sign}(l)|k+l|^\eta\hat u(l)e_l.$$

Again by (A.1) and (A.6), and by

$$\|E_{\eta k}u\|_{H^\rho} \le C|k|^{|\eta-1|+2}\|\Lambda_{\eta-1}^+u\|_{H^\rho} \qquad (0<\rho<1)$$

it follows

$$\big\|\Lambda_\eta^+(\Lambda_0^-\phi - \phi\Lambda_0^-)u\big\|_{H^\rho} \le C\|\phi\|^*_{|\eta-1|+2+\rho}\|\Lambda_{\eta-1}^+u\|_{H^\rho}$$

which implies

$$\big\|(\Lambda_0^-\phi - \phi\Lambda_0^-)u\big\|_{H^\sigma} \le C\|\phi\|^*_\lambda\|u\|_{H^{\sigma-1}} \qquad (\sigma\in\mathbb{R},\ \lambda>|\sigma-1|+2).\tag{A.11}$$

Now, decomposing

$$\big(\Lambda_\eta^-\phi - \phi\Lambda_\eta^-\big)u = \Lambda_0^-\big(\Lambda_\eta^+\phi - \phi\Lambda_\eta^+\big)u + \big(\Lambda_0^-\phi - \phi\Lambda_0^-\big)\Lambda_\eta^+u$$

and applying (A.10) and (A.11) gives

$$\big\|(\Lambda_\eta^-\phi - \phi\Lambda_\eta^-)u\big\|_{H^{\sigma-\eta}} \le C\|\phi\|^*_\nu\|u\|_{H^{\sigma-1}} \qquad \big(\nu>\max(|\sigma-\eta-1|,|\sigma-1|)+2\big)$$

and thus the proof of (2.7) is complete ∎

Proof of Lemma 2.2. For proving (2.9) we utilize the representations

$$a(t,s) = \sum_{k\in\mathbf{Z}} a_k(t)e_k(s), \quad Au = \sum_{k\in\mathbf{Z}} a_k K(e_k u), \quad (Ku)(t) = \int_0^1 \kappa(t-s)u(s)\,ds$$

and show

$$\|K(e_k u)\|_{H^{\sigma-\eta}} \le C\big(\max(1,|k|)\big)^\mu\|u\|_{H^{\sigma+\delta}} \qquad \big(\delta>\tfrac12,\ \mu>|\sigma|\big).\tag{A.12}$$

This estimate is achieved by using

$$\Lambda_{\sigma-\eta-\rho}^+(Ke_k u) = e_k\sum_{l\in\mathbf{Z}}\hat\kappa(l+k)\big(\max(1,|l+k|)\big)^{\sigma-\eta-\rho}\hat u(l)e_l,$$

(A.3), the Hölder inequality and (A.6). From (2.6) and (A.12), we get

$$\|Au\|_{H^{\sigma-\eta}} \le C\|a\|^{*;*}_{\nu,\mu}\|u\|_{H^{\sigma+\delta}} \qquad \big(\delta>\tfrac12,\ \nu>|\sigma-\eta|+1,\ \mu>|\sigma|\big)$$

and consequently, the statements of this lemma are verified ∎

Proof of Theorem 2.3. By $(2.2)_f$ we have, for $1 \le j \le r$, $|\hat{\kappa}_j(l)| \le C|l|^{\beta-\gamma_j}$ $(l \ne 0)$ with $\gamma_j = \beta - \beta_j > \frac{1}{2}$. Thus, (2.9) implies that $\|A_j u\|_{H^{\sigma-\beta}} \le C\|u\|_{H^{\sigma-\delta_j}}$ $(\sigma \in \mathbb{R})$ for some value $\delta_j > 0$. Therefore, $\delta' = \min_{1\le j\le r}(\delta_j) > 0$ exists such that

$$\sum_{j=1}^{r} A_j : H^\sigma \to H^{\sigma-\beta+\delta'} \qquad (\sigma \in \mathbb{R})$$

is bounded. As shown in [12], A_0 decomposes to the form

$$A_0 = A_{01} + A_{02}$$

with

$$A_{01} = a_0^+\Lambda_\beta^+ + a_0^-\Lambda_\beta^- \qquad \text{and} \qquad (A_{02}u)(t) = \int_0^1 k_{02}(t,s)u(s)\,ds,$$

where

$$k_{02}(t,s) = \kappa_{02}^1(t-s)c(t,s) + \kappa_{02}^2(t-s)a_0^+(t) + \kappa_{02}^3(t-s)a_0^-(t)$$

with

$$c(t,s) = \frac{a_0^+(t,s) - a_0^+(t)}{e^{i2\pi(s-t)} - 1} \in C_1^\infty(\mathbb{R}\times\mathbb{R})$$

and, for $l \ne 0$,

$$|\widehat{\kappa_{02}^1}(l)| \le C|l|^{\beta-\min(\gamma,1)}, \qquad |\widehat{\kappa_{02}^2}(l)| \le C|l|^{\beta-\gamma}, \qquad |\widehat{\kappa_{02}^3}(l)| \le C|l|^{\beta-\gamma}$$

where $\gamma > \frac{1}{2}$ is given by $(2.2)_c$. So, by (2.9), for some value $\tilde{\delta} \in \left(0, \min(\gamma,1) - \frac{1}{2}\right)$, the operator $A_{02} : H^\sigma \to H^{\sigma-\beta+\tilde{\delta}}$ $(\sigma \in \mathbb{R})$ is bounded. Gathering these results together, we obtain the representation

$$\begin{aligned} L &= A_{01} + B_1 \\ A_{01} &= a_0^+\Lambda_\beta^+ + a_0^-\Lambda_\beta^- : H^\sigma \to H^{\sigma-\beta} \\ B_1 &= A_{02} + \sum_{j=1}^{r} A_j : H^\sigma \to H^{\sigma-\beta+\delta} \end{aligned} \qquad (\sigma \in \mathbb{R}) \qquad (A.13)$$

with $\delta = \min(\delta', \tilde{\delta}) > 0$, verifying the mapping property of L. Next we present a further decomposition of L that implies $L : H^\sigma \to H^{\sigma-\beta}$ $(\sigma \in \mathbb{R})$ to be isomorphic. Moreover, the form is crucial for the proof of Theorem 4.1. A detailed discussion can be found in [6] and in [3, 7, 12]. Because of the assumptions $(2.2)_d$ and $(2.2)_e$ we have for the quotient $\frac{\sigma_+}{\sigma_-}$ with $\sigma_+ = a_0^+ + a_0^-$ and $\sigma_- = a_0^+ - a_0^-$ the factorization $\frac{\sigma_+}{\sigma_-} = c_+c_-$ such that

$$c_+ \in \left\{c \in C_1^\infty(\mathbb{R}) | \hat{c}(l) = 0 \;\; (l \le -1)\right\} \quad \text{and} \quad c_- \in \left\{c \in C_1^\infty(\mathbb{R}) | \hat{c}(l) = 0 \;\; (l \ge 1)\right\}$$

allowing for A_{01} the representation

$$A_{01} = \sigma_- c_+ \tilde{A} + B_2$$

with

$$\tilde{A} = (P^+ c_- + P^- c_+^{-1})\Lambda_\beta^+$$

and

$$B_2 = \left[\sigma_- c_+ (c_- P^+ - P^+ c_-)\Lambda_\beta^+\right] + \left[\sigma_- c_+ (c_+^{-1} P^- - P^- c_+^{-1})\Lambda_\beta^+\right] - a_0^- J\Lambda_\beta^+$$

where $Ju = \hat{u}(0)$. Here $\tilde{A} : H^\sigma \to H^{\sigma-\beta}$ $(\sigma \in \mathbb{R})$ is an isomorphism with the property $\tilde{A}v \in T_N$ for all $v \in T_N$. Furthermore, by Lemma 2.1 it holds $B_2 : H^\sigma \to H^{\sigma-\beta+1}$ $(\sigma \in \mathbb{R})$. Hence we finally get when denoting $c = \sigma_- c_+$ and $\tilde{B} = c^{-1}(B_1 + B_2)$ the form

$$L = c(\tilde{A} + \tilde{B}) \tag{A.14}$$

where $c \in C_1^\infty(\mathbb{R})$ with $c(t) \neq 0$ for all $t \in \mathbb{R}$ and $\tilde{B} : H^\sigma \to H^{\sigma-\beta+\min(\delta,1)}$ $(\sigma \in \mathbb{R})$.

Assume now that $Lu = 0$. We may write this equation by (A.14) and by the properties of $\tilde{A}$ and $\tilde{B}$ equivalently in the form $u = \tilde{A}^{-1}\tilde{B}u$. If $u \in H^\sigma$ for some value of $\sigma \in \mathbb{R}$, then $u \in H^{\sigma+\min(\delta,1)}$, and consequently $u \in C_1^\infty(\mathbb{R})$. Therefore, by $(2.2)_h$ we deduce $u = 0$, and the proof is complete ∎

Proof of Lemma 3.1. We start by showing (3.4), since (3.3) is a direct consequence of (3.4) and (2.6). Assuming $\phi = \sum_{k\in\mathbb{Z}} \hat{\phi}(k)e_k$ and $v = \sum_{p\in\Lambda_N} \hat{v}(p)e_p$, we write

$$\Lambda_\eta^+(Q_N^\epsilon - I)\phi v = \sum_{|k|\leq \frac{N}{4}} \hat{\phi}(k) w_{k,\epsilon}^1 + \sum_{|k|>\frac{N}{4}} \hat{\phi}(k) w_{k,\epsilon}^2 \tag{A.15}$$

with

$$w_{k,\epsilon}^1 = \begin{cases} \sum\limits_{\substack{p\in\Lambda_N \\ p+k>\frac{N}{2}}} \hat{v}(p)\left[|p+k-N|^\eta e^{i2\pi\epsilon} e_{p+k-N} - |p+k|^\eta e_{p+k}\right] & \text{if } 0 < k \leq \frac{N}{4} \\ \sum\limits_{\substack{p\in\Lambda_N \\ p+k\leq -\frac{N}{2}}} \hat{v}(p)\left[|p+k+N|^\eta e^{-i2\pi\epsilon} e_{p+k+N} - |p+k|^\eta e_{p+k}\right] & \text{if } -\frac{N}{4} \leq k < 0 \\ 0 & \text{if } k = 0 \end{cases}$$

and

$$w_{k,\epsilon}^2 = \sum_{p\in\Lambda_N} \hat{v}(p)\left[\left(\max(1,|n_{pk}|)\right)^\eta e^{i l_{pk} 2\pi\epsilon} e_{n_{pk}} - \left(\max(1,|p+k|)\right)^\eta e_{p+k}\right]$$

where $n_{pk} \in \Lambda_N$ and $l_{pk} \in \mathbb{Z}$ are such that $p + k = n_{pk} + l_{pk}N$. For estimation of $\|w_{k,\epsilon}^1\|_{H^\rho}$ $(0 < \rho < 1)$ in the case of $0 < k \leq \frac{N}{4}$ we decompose

$$\begin{aligned} \Delta_h e_{p+k} &= (\Delta_h e_p) e_k e^{ik2\pi h} + e_p(\Delta_h e_k) \\ \Delta_h e_{p+k-N} &= -(\Delta_h e_{-p}) e_{2p+k-N} \\ &= (\Delta_h e_{-p}) e_{2p+k-N} + e_{-p}(\Delta_h e_{2p+k-N}) \\ &= -(\Delta_h e_p) e_{k-N} e^{i(p+k-N)2\pi h} + e_{-p}(\Delta_h e_{2p+k-N}). \end{aligned} \tag{A.16}$$

Moreover, we need the inequalities

$$|p+k-N|^{\eta} \le CN^{\eta-\mu}|p|^{\mu} \quad \text{and} \quad |p+k|^{\eta} \le CN^{\eta-\mu}|p|^{\mu} \qquad (\eta,\mu\in\mathbb{R}) \tag{A.17}$$

implied by the facts that when $p+k > \frac{N}{2}$ and $0 < k \le \frac{N}{4}$, there holds

$$\frac{N}{4} \le |p+k-N| < \frac{N}{2} < |p+k| \le \frac{3N}{4} \quad \text{and} \quad \frac{N}{4} \le \frac{N}{2} - k \le p \le \frac{N}{2}. \tag{A.18}$$

Since $|2p+k-N| \le k$, we obtain by (A.16), (A.17) and (A.6) that

$$\|w^1_{k,\epsilon}\|_{H^\rho} \le CN^{\eta-\mu}|k|^{1+\rho}\|\Lambda^+_\mu v\|_{H^\rho} \tag{A.19}$$

for $0 < k \le \frac{N}{4}$. Estimate (A.19) for values $-\frac{N}{4} \le k < 0$ can be shown very analogously. In a straightforward way, when applying (A.3), (A.16) (the first equality) and (A.6), we achieve the estimate

$$\|w^2_{k,\epsilon}\|_{H^\rho} \le C\big(\max(1,|k|)\big)^{|\eta|+\max(\mu-\eta,0)+1+\rho}\|\Lambda^+_\mu v\|_{H^\rho} \qquad (|k| > \tfrac{N}{4}). \tag{A.20}$$

Consequently, from (A.15), (A.19) and (A.20) it follows

$$\big\|\Lambda^+_\eta(Q^\epsilon_N - I)\phi v\big\|_{H^\rho} \le CN^{\eta-\mu}\|\phi\|^*_\lambda\|\Lambda^+_\mu v\|_{H^\rho} \tag{A.21}$$

with $\lambda \ge |\eta| + \max(\mu-\eta,0)+1+\rho$. Choosing $\eta = \tau-\rho$ and $\mu = \sigma-\rho$ with $\rho\in(0,1)$ such that

$$\rho < \begin{cases} \tau & \text{if } \tau > 0 \\ \frac{1}{2}(\nu+\tau-\max(\sigma-\tau,0)-1) & \text{if } \tau \le 0 \end{cases}$$

we get

$$\begin{aligned} \|(Q^\epsilon_N - I)\phi v\|_{H^\tau} &\le C\|\Lambda^+_{\tau-\rho}(Q^\epsilon_N - I)\phi v\|_{H^\rho} \\ &\le CN^{\tau-\sigma}\|\phi\|^*_\nu\,\|\Lambda^+_{\sigma-\rho}v\|_{H^\rho} \qquad (\tau,\sigma\in\mathbb{R}) \\ &\le CN^{\tau-\sigma}\|\phi\|^*_\nu\,\|v\|_{H^\sigma} \end{aligned}$$

with $\nu > |\tau| + 1 + \max(\sigma-\tau,0)$ verifying (3.4). The estimate (3.3) is now obtained from

$$\|Q^\epsilon_N(\phi v)\|_{H^\tau} \le \|(Q^\epsilon_N - I)\phi v\|_{H^\tau} + \|\phi v\|_{H^\tau} \le C\|\phi\|^*_\nu\,\|v\|_{H^\tau} \qquad (\nu > |\tau|+1)$$

where (3.4) as well as (2.6) have been used ∎

Proof of Lemma 3.1′. Very analogously to the proof of (3.4), we can show

$$\big\|D^jP^\pm(Q^\epsilon_N - I)\phi v\big\|_C \le C\|\phi\|^*_{j+1}\|D^jv\|_C \qquad (j\in\mathbb{N}_0,\ v\in T_N).$$

Additionally, we need the inequality

$$\big\|D^jP^\pm(\phi v)\big\|_C \le C\|\phi\|^*_{j+1}\big(\|P^+v\|_{C^j} + \|P^-v\|_{C^j}\big) \qquad (j\in\mathbb{N}_0,\ v\in T_N)$$

implied by (2.7) and $(A.1)'$. The estimate $(3.3)'$ now follows from these two estimates ∎

Proof of Lemma 3.2. Denoting $a_k(t) = \int_0^1 a(t,s)e^{-ik2\pi s}\,ds$ $(t \in \mathbb{R})$ we have

$$Q_N^\varepsilon(A_N - A)v = Q_N^\varepsilon\Big[\sum_{|k|\le N/4} a_k z_{k,\varepsilon}^1 + \sum_{|k|>N/4} a_k z_{k,\varepsilon}^2\Big]$$

with

$$z_{k,\varepsilon}^1 = \begin{cases} \sum_{\substack{p\in\Lambda_N \\ p+k>\frac{N}{2}}} \Big[\hat\kappa(p+k-N) - \hat\kappa(p+k)e^{i2\pi\varepsilon}\Big]\hat v(p)e_{p+k-N} & \text{if } 0<k\le\frac{N}{4} \\ \sum_{\substack{p\in\Lambda_N \\ p+k\le-\frac{N}{2}}} \Big[\hat\kappa(p+k+N) - \hat\kappa(p+k)e^{-i2\pi\varepsilon}\Big]\hat v(p)e_{p+k+N} & \text{if } -\frac{N}{4}\le k<0 \\ 0 & \text{if } k=0 \end{cases}$$

and with

$$z_{k,\varepsilon}^2 = \sum_{p\in\Lambda_N}\Big(\hat\kappa(n_{pk}) - \hat\kappa(p+k)e^{il_{pk}2\pi\varepsilon}\Big)\hat v(p)e_{n_{pk}}$$

where $n_{pk} \in \Lambda_N$ and $l_{pk} \in \mathbb{Z}^*$ are as described before. Using (3.3), we obtain

$$\|Q_N^\varepsilon(A_N - A)v\|_{H^{\tau-\beta}} \le C\Big\{\sum_{|k|\le N/4}\|a_k\|_\nu^*\,\|z_{k,\varepsilon}^1\|_{H^{\tau-\beta}} + \sum_{|k|>N/4}\|a_k\|_\nu^*\,\|z_{k,\varepsilon}^2\|_{H^{\tau-\beta}}\Big\} \tag{A.22}$$

with $\nu > |\tau-\beta|+1$. When determining the upper bound for $\|z_{k,\varepsilon}^1\|_{H^{\tau-\beta}}$, it is most essential to consider

$$\tilde\Delta(p+k-N, p+k, \varepsilon) \quad \text{for} \quad 0\le k\le\frac{N}{4},\ p+k>\frac{N}{2}$$

$$\tilde\Delta(p+k+n, p+k, -\varepsilon) \quad \text{for} \quad -\frac{N}{4}\le k<0,\ p+k\le-\frac{N}{2}$$

with the notation $\tilde\Delta(l,m,x) = \hat\kappa(l) - \hat\kappa(m)e^{i2\pi x}$. Assume again $0<\rho<1$. In general, we have, by (3.5) and (A.17), for the first expression

$$\big|\tilde\Delta(p+k-N, p+k, \varepsilon)\big| \le CN^{\beta-\sigma+\rho}|p|^{\sigma-\rho} \qquad \left(\substack{0<k\le\frac{N}{4} \\ p+k>\frac{N}{2}}\right) \tag{A.23}$$

and, analogously,

$$\big|\tilde\Delta(p+k+N, p+k, -\varepsilon)\big| \le CN^{\beta-\sigma+\rho}|p|^{\sigma-\rho} \qquad \left(\substack{-\frac{N}{4}\le k<0 \\ p+k\le-\frac{N}{2}}\right). \tag{A.24}$$

Therefore, applying again (A.17), (A.16) (as well as the analogous results for the values $-\frac{N}{4}\le k<0$) and (A.6) we obtain

$$\|z_{k,\varepsilon}^1\|_{H^{\tau-\beta}} \le C\|\Lambda_{\tau-\beta-\rho}^+ z_{k,\varepsilon}^1\|_{H^\rho} \le CN^{\tau-\sigma}|k|^{1+\rho}\|v\|_{H^\sigma} \qquad \left(\substack{|k|\le\frac{N}{4} \\ \tau,\sigma\in\mathbb{R}}\right). \tag{A.25}$$

In the special case of $\varepsilon = 0$ and (3.8) being valid, we can improve the estimates of (A.23) and (A.24). Namely, for $0<k\le\frac{N}{4}$ and $p+k>\frac{N}{2}$ there exists a value ξ_{pk}

between $N-p-k$ and $p+k$ such that $|p+k-N|^{\beta}-|p+k|^{\beta}=\beta\xi_{pk}^{\beta-1}(2p+2k-N)$. Consequently, by (3.8) we get, when estimating as before,

$$\left|\tilde{\Delta}(p+k-N,p+k,0)\right| \leq CN^{\beta-\min(\gamma,1)-\sigma+\rho}|k|\,|p|^{\sigma-\rho} \qquad \binom{0<k\leq\frac{N}{4}}{p+k>\frac{N}{2}}.$$

Similarly, we have

$$\left|\tilde{\Delta}(p+k+N,p+k,0)\right| \leq CN^{\beta-\min(\gamma,1)-\sigma+\rho}|k|\,|p|^{\sigma-\rho} \qquad \binom{-\frac{N}{4}\leq k<0}{p+k\leq-\frac{N}{2}}.$$

As a consequence of the previous two estimates, we can deduce

$$\|z_{k,0}^{1}\|_{H^{\tau-\beta}} \leq CN^{\tau-\sigma-\delta}|k|^{2+\rho}\|v\|_{H^{\sigma}} \qquad (\tau,\sigma\in\mathbb{R}) \tag{A.26}$$

with $\delta=\min(\gamma,1)$. Because of

$$\left|\tilde{\Delta}(n_{pk},p+k,l_{pk}\varepsilon)\right|\big(\max(1,|n_{pk}|)\big)^{\tau-\beta-\rho} \leq CN^{\tau-\sigma-1-\delta}|k|^{\mu}\big(\max(1,|p|)\big)^{\sigma-\rho}$$

for $|k|>\frac{N}{4}$, with $\mu>|\beta|+|\tau-\beta|+\max(\sigma-\tau,0)+2+\delta$ (also the equality is valid if $\tau\geq\beta$), there holds

$$\|z_{k,\varepsilon}^{2}\|_{H^{\tau-\beta}} \leq CN^{\tau-\sigma-\delta}|k|^{\mu}\|v\|_{H^{\sigma}}. \tag{A.27}$$

Thus the estimate (3.7) follows from (A.22) by (A.25) and (A.27), and the estimate (3.9) by (A.26) and (A.27). Finally, in the case of $\kappa(t)=1$ $(t\in\mathbb{R})$ the condition $|\hat{\kappa}(l)|\leq C|l|^{r}$ $(l\neq0)$ is valid for any $r\in\mathbb{R}$. Assuming $a\in C_1^{\infty}(\mathbb{R}\times\mathbb{R})$ the application of (3.7) gives

$$\left\|Q_N^{\varepsilon}(A_N-A)v\right\|_{H^{\tau-\beta}} \leq CN^{\lambda}\|v\|_{\tau-\beta+r-\lambda} \qquad (\lambda\in\mathbb{R},\ \tau\in\mathbb{R})$$

and hence choosing $r=\sigma-\tau+\lambda+\beta$, we obtain (3.10) ∎

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