The Skew Derivative Problem
in the Exterior of Open Curves in a Plane

P. A. Krutitskii

Abstract. The skew derivative problem for the Laplace equation in the exterior of open curves in the plane is reduced to the Fredholm integral equation of the second kind, which is uniquely solvable.

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1. Introduction

Boundary value problems for 2-dimensional partial differential equations are mostly treated in domains, bounded by closed curves. A small number of studies are devoted to problems in the exterior of open curves, for instance, Dirichlet and Neumann problems for Helmholtz and Laplace equations were treated in [1, 4, 6 – 13, 15 – 16]. Similar problems have great significance, because open curves model cracks, screens or wings. The present note is an attempt to consider a skew derivative problem for 2-dimensinal harmonic functions in the exterior of open curves. This problem arises in the physics of semiconductors [5].

2. Formulation of the problem

In the plane $x = (x_1, x_2) \in \mathbb{R}^2$ we consider simple open curves $\Gamma_1, ..., \Gamma_N \in C^{2,\lambda}$ $(\lambda \in (0, 1])$, so that they do not have common points. We put

$$\Gamma = \bigcup_{n=1}^{N} \Gamma_n.$$

We assume that each curve Γ_n is parametrized by the arc length s:

$$\Gamma_n = \left\{ x : x = x(s) = (x_1(s), x_2(s)) \text{ for } s \in [a_n, b_n] \right\} \quad (n = 1, ..., N)$$

P. A. Krutitskii: Moscow State University, Faculty of Physics, Department of Mathematics, Moscow 119899, Russia

so that $a_1 < b_1 < ... < a_N < b_N$. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence. Below the sets of the intervals on the Os-axis

$$\bigcup_{n=1}^{N} [a_n, b_n]$$

will be denoted by Γ also.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$ and $\sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\frac{\pi}{2}$.

We say, that the function $u = u(x)$ belongs to the smoothness class **K** if the following conditions are satisfied:

1) $u \in C^0(\overline{\mathbb{R}^2\backslash\Gamma}) \cap C^2(\mathbb{R}^2\backslash\Gamma)$, and u is continuous at the ends of Γ.

2) $\nabla u \in C^0(\overline{\mathbb{R}^2\backslash\Gamma}\backslash X)$, where X is a point set, consisting of the endpoints of Γ:

$$X = \bigcup_{n=1}^{N} \big(x(a_n) \cup x(b_n) \big).$$

3) In the neighbourhood of any point $x(d) \in X$, for some constants $C > 0$ and $\varepsilon > -1$, the inequality

$$|\nabla u| \leq C|x - x(d)|^\varepsilon \tag{1}$$

holds where $x \to x(d)$ and $d = a_n$ or $d = b_n$ for $n = 1, ..., N$.

Remark. In the definition of the class **K** we consider Γ as a set of cuts in a plane. In particular, the notation $C^0(\overline{\mathbb{R}^2\backslash\Gamma})$ denotes a class of functions, which are continuously extended on Γ from the left and right, but their values on Γ from the left and right can be different, so that the functions may have a jump on Γ.

Let us formulate the *skew derivative problem* for the Laplace equation in $\mathbb{R}^2\backslash\Gamma$:

Problem U. To find a function $u = u(x)$ of the class **K** which satisfies the Laplace equation

$$\Delta u(x) = 0 \qquad (x \in \mathbb{R}^2\backslash\Gamma; \ \Delta = \partial_{x_1}^2 + \partial_{x_2}^2), \tag{2_a}$$

the boundary condition

$$\left(\frac{\partial}{\partial\mathbf{n}_x} u(x(s)) + \beta\frac{\partial}{\partial\tau_x} u(x(s)) \right)\bigg|_{\Gamma} = f(s) \tag{2_b}$$

and the conditions at infinity

$$|u(x)| \leq \text{const} \quad \text{and} \quad |\nabla u(x)| = o(|x|^{-1}) \qquad \text{as} \quad |x| = \sqrt{x_1^2 + x_2^2} \to \infty. \tag{2_c}$$

We suppose that β is a real constant. All conditions of the problem **U** must be satisfied in the classical sense.

The problem **U** arises, for instance, in the mathematical models of magnetized semiconductors [5]. The Neumann problem for the Laplace equation in the exterior of open curves is a particular case of our problem when $\beta = 0$.

On the basis of energy equalities [14: Sections 21.2, 28.2 and 31.1] we can easily prove the following assertion.

Theorem 1. *Let* $\Gamma \in C^{2,\lambda}$ $(\lambda \in (0,1])$. *If a solution of the problem* **U** *exists, then it is defined up to an arbitrary additive constant.*

Let us show that if $u_0 = u_0(x)$ is a solution of the homogeneous problem **U**, then $u_0(x) \equiv$ const. To prove this with the help of energy equalities, we envelope open curves by closed contours, tend contours to the curves and use the smoothness of the solution of the problem **U**. In this way we obtain

$$\|\nabla u_0\|^2_{L_2(\mathbb{R}^2\backslash\Gamma)} = \lim_{r\to\infty} \|\nabla u_0\|^2_{L_2(C_r\backslash\Gamma)} = \int_\Gamma u_0^+ \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^+ ds - \int_\Gamma u_0^- \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^- ds,$$

where the conditions (1) and (2)$_c$ are taken into account and C_r is the circle of the radius r with the center in the origin. Besides, in the latter formula we consider Γ as a set of cuts. The side of Γ which is on the left, when the parameter s increases we denote by Γ^+ and the opposite side we denote by Γ^-. In a similar manner, by the superscripts "+" and "−" we denote the limit values of functions on Γ^+ and Γ^-, respectively.

By $\int_\Gamma \ldots d\sigma$ we mean $\sum_{n=1}^N \int_{a_n}^{b_n} \ldots d\sigma$. Using the homogeneous boundary condition (2)$_b$ we obtain from the latter formula

$$\|\nabla u_0\|^2_{L_2(\mathbb{R}^2\backslash\Gamma)} = -\beta \left\{ \int_\Gamma u_0^+ \left(\frac{\partial u_0}{\partial \tau_x}\right)^+ ds - \int_\Gamma u_0^- \left(\frac{\partial u_0}{\partial \tau_x}\right)^- ds \right\}$$

$$= -\beta\frac{1}{2}\left\{ \sum_{m=1}^N \left([u_0^+(x(b_m))]^2 - [u_0^+(x(a_m))]^2\right) \right.$$

$$\left. - \sum_{m=1}^N \left([u_0^-(x(b_m))]^2 - [u_0^-(x(a_m))]^2\right) \right\} = 0$$

since $u_0^+(x(b_m)) = u_0^-(x(b_m))$ and $u_0^+(x(a_m)) = u_0^-(x(a_m))$ for $m = 1,...,N$ in accordance with the smoothness properties of the function u_0, which belongs to the class **K**. Thus, $\nabla u_0 \equiv 0$ and $u_0 \equiv$ const, and the theorem is proved due to the linearity of the problem **U** ∎

3. Integral equations at the boundary

Below we assume that $f = f(s)$ from (2)$_b$ is an arbitrary function from the Banach space $C^{0,\lambda}(\Gamma)$, with Hölder exponent $\lambda \in (0,1]$. We consider the angular potential from [2] for the equation (2)$_a$ on Γ

$$v[\mu](x) = -\frac{1}{2\pi} \int_\Gamma \mu(\sigma)V(x,\sigma)\,d\sigma. \tag{3}$$

The kernel $V(x,\sigma)$ is defined (up to indeterminacy $2\pi m$, with $m = \pm1, \pm2, ...$) by the formulae

$$\cos V(x,\sigma) = \frac{x_1 - y_1(\sigma)}{|x - y(\sigma)|} \qquad \text{and} \qquad \sin V(x,\sigma) = \frac{x_2 - y_2(\sigma)}{|x - y(\sigma)|},$$

where

$$y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma$$

$$|x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}.$$

One can see that $V(x,\sigma)$ is the angle between the vector $\overrightarrow{y(\sigma)x}$ and the direction of the Ox_1-axis. More precisely, $V(x,\sigma)$ is a many-valued harmonic function of x connected with $\ln|x - y(\sigma)|$ by the Cauchy-Riemann relations.

Below by $V(x,\sigma)$ we denote an arbitrary fixed branch of this function, which varies continuously with σ along each curve Γ_n $(n = 1, ..., N)$ for given fixed $x \notin \Gamma$. Under this definition of $V(x,\sigma)$, the potential $v[\mu](x)$ is a many-valued function. In order that the potential $v[\mu](x)$ be single-valued it is necessary to impose the additional conditions

$$\int_{a_n}^{b_n} \mu(\sigma)\, d\sigma = 0 \qquad (n = 1, ..., N). \tag{4}$$

Below we suppose that the density $\mu(s)$ belongs to the Banach space $C_q^\omega(\Gamma)$ $(\omega \in (0, 1], q \in [0, 1))$ and satisfies the conditions (4). We say that $\mu \in C_q^\omega(\Gamma)$ if the function h defined by

$$h(s) = \mu(s) \prod_{n=1}^{N} |s - a_n|^q |s - b_n|^q$$

belongs to the Hölder space $C^{0,\omega}(\Gamma)$ with exponent ω and $\|\mu\|_{C_q^\omega(\Gamma)} = \|h\|_{C^{0,\omega}(\Gamma)}$. As shown in [2, 6], for such $\mu(s)$ the angular potential $v[\mu](x)$ belongs to the class **K**. In particular, the inequality (1) holds with $\varepsilon = -q$, if $q \in (0, 1)$. Moreover, integrating $v[\mu](x)$ by parts and using (4) we express the angular potential in terms of a double layer potential

$$v[\mu](x) = \frac{1}{2\pi} \int_\Gamma \rho(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x - y(\sigma)|\, d\sigma \tag{5}$$

with density

$$\rho(\sigma) = \int_{a_n}^{\sigma} \mu(\xi)\, d\xi \qquad (\sigma \in [a_n, b_n]; \; n = 1, ..., N). \tag{6}$$

Consequently, $v[\mu](x)$ satisfies both equation $(2)_a$ outside Γ and the conditions at infinity $(2)_c$.

Let us construct a solution of the problem **U**. Such solution can be obtained with the help of potential theory for the equation $(2)_a$. We seek a solution of the problem in the form

$$u[\mu](x) = v[\mu](x) - \beta w[\mu](x) + C , \tag{7}$$

where C is an arbitrary constant, $v[\mu](x)$ is given by (3) and (5), and

$$w[\mu](x) = -\frac{1}{2\pi} \int_\Gamma \mu(\sigma) \ln|x - y(\sigma)|\, d\sigma.$$

As mentioned above, we will seek $\mu(s)$ from the Banach space $C_q^\omega(\Gamma)$ ($\omega \in (0,1]$, $q \in [0,1)$). We note that $\mu(s)$ must satisfy the conditions (4). For such μ the function (7) belongs to the class **K** and satisfies all conditions of the problem **U** except the boundary condition $(2)_b$. In particular, the conditions at infinity hold due to (4). To satisfy the boundary condition we put (7) into $(2)_b$, use the limit formulas for the angular potential from [2, 6] and arrive at the singular integral equation [11: Section 96] for the density μ

$$-\frac{1+\beta^2}{2\pi} \int_\Gamma \mu(\sigma) \frac{\sin\varphi_0\big(x(s), y(\sigma)\big)}{|x(s) - y(\sigma)|} \, d\sigma = f(s) \qquad (s \in \Gamma) \tag{8}$$

where $\varphi_0(x, y)$ is the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x, y)$ is taken to be positive if it is measured anticlockwise from $\mathbf{n}_x$, and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x, y)$ is continuous in $x, y \in \Gamma$ if $x \neq y$.

Thus, if $\mu(s)$ is a solution of the equations (4), (8) from the space $C_q^\omega(\Gamma)$ ($\omega \in (0,1]$, $q \in [0,1)$), then the potential (7) satisfies all conditions of the problem **U**. The following theorem holds.

Theorem 2. *If $\Gamma \in C^{2,\lambda}$ and $f \in C^{0,\lambda}(\Gamma)$ ($\lambda \in (0,1]$), equation (8) has a solution $\mu(s)$ from the Banach space $C_q^\omega(\Gamma)$ ($\omega \in (0,1]$, $q \in [0,1)$), and the conditions (4) hold, then the function (7) is a solution of the problem **U**.*

Our further treatment will be aimed to the proof of the solvability of the system (4), (8) in the Banach space $C_q^\omega(\Gamma)$. Moreover, we reduce the system (4), (8) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

It can be easily proved that

$$Y(s, \sigma) = \frac{1}{\pi} \left(\frac{\sin\varphi_0\big(x(s), y(\sigma)\big)}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - s} \right) \in C^{0,\lambda}(\Gamma \times \Gamma)$$

(see [6, 7] for details). Therefore we can rewrite (8) in the form

$$\frac{1}{\pi} \int_\Gamma \mu(\sigma) \frac{d\sigma}{\sigma - s} + \int_\Gamma \mu(\sigma) Y(s, \sigma) \, d\sigma = -\frac{2}{1+\beta^2} f(s) \qquad (s \in \Gamma). \tag{9}$$

4. The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator into (9) we arrive at the following integral equation of the second kind [11: Section 99]:

$$\mu(s) + \frac{1}{Q(s)} \int_\Gamma \mu(\sigma) A_0(s, \sigma) \, d\sigma + \frac{1}{Q(s)} \sum_{n=0}^{N-1} G_n s^n = \frac{1}{Q(s)} \Phi_0(s) \qquad (s \in \Gamma), \tag{10}$$

where

$$A_0(s,\sigma) = -\frac{1}{\pi} \int_\Gamma \frac{Y(\xi,\sigma)}{\xi - s} Q(\xi)\, d\xi$$

$$Q(s) = \prod_{n=1}^{N} |\sqrt{s - a_n}\sqrt{b_n - s}|\,\mathrm{sign}(s - a_n)$$

$$\Phi_0(s) = \frac{1}{1 + \beta^2}\frac{1}{\pi} \int_\Gamma \frac{2Q(\sigma)f(\sigma)}{\sigma - s}\, d\sigma$$

and $G_0, ..., G_{N-1}$ are arbitrary constants. To derive equations for these constants we substitute $\mu(s)$ from (10) into the conditions (4). Then we obtain

$$\int_\Gamma \mu(\sigma)l_n(\sigma)\, d\sigma + \sum_{m=0}^{N-1} B_{nm}G_m = H_n \qquad (n = 1, ..., N), \tag{11}$$

where

$$\left.\begin{array}{l} l_n(\sigma) = -\displaystyle\int_{a_n}^{b_n} Q^{-1}(s)A_0(s,\sigma)\, ds \\[2em] H_n = -\displaystyle\int_{a_n}^{b_n} Q^{-1}(s)\Phi_0(s)\, ds \\[2em] B_{nm} = -\displaystyle\int_{a_n}^{b_n} Q^{-1}(s)s^m\, ds. \end{array}\right\} \tag{12}$$

By B we denote the $(N \times N)$-matrix with the elements B_{nm} from (12). As shown in [7], this matrix is invertible. The elements of the inverse matrix will be called $(B^{-1})_{nm}$. Inverting the matrix B in (11) we express the constants $G_0, ..., G_{N-1}$ in terms of μ as

$$G_n = \sum_{m=1}^{N} (B^{-1})_{nm}\left[H_m - \int_\Gamma \mu(\sigma)l_m(\sigma)\, d\sigma\right].$$

We substitute G_n into (10) and obtain the following integral equation for $\mu(s)$ on Γ:

$$\mu(s) + \frac{1}{Q(s)} \int_\Gamma \mu(\sigma)A(s,\sigma)\, d\sigma = \frac{1}{Q(s)}\Phi(s) \qquad (s \in \Gamma), \tag{13}$$

where

$$A(s,\sigma) = A_0(s,\sigma) - \sum_{n=0}^{N-1} s^n \sum_{m=1}^{N} (B^{-1})_{nm}l_m(\sigma)$$

$$\Phi(s) = \Phi_0(s) - \sum_{n=0}^{N-1} s^n \sum_{m=1}^{N} (B^{-1})_{nm}H_m.$$

It can be shown using the properties of singular integrals (see [3: Section 5] and [11: Section 18]) that $\Phi_0(s)$ and $A_0(s,\sigma)$ are Hölder functions if $s \in \Gamma$ and $\sigma \in \Gamma$. Therefore,

$\Phi(s)$ and $A(s,\sigma)$ are also Hölder functions if $s \in \Gamma$ and $\sigma \in \Gamma$. Consequently, any solution of the equation (13) belongs to $C_{\frac{1}{2}}^{\omega}(\Gamma)$, and below we look for $\mu(s)$ on Γ in this space. Moreover, it follows ¿from our treatment that any solution of the equation (13) meets the conditions (4).

Instead of $\mu \in C_{\frac{1}{2}}^{\omega}(\Gamma)$ we introduce the new unknown function $\mu_* \in C^{0,\omega}(\Gamma)$ defined by $\mu_*(s) = \mu(s)Q(s)$ and rewrite equation (13) in the form

$$\mu_*(s) + \int_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)\,d\sigma = \Phi(s) \qquad (s \in \Gamma). \tag{14}$$

Thus, the system of equations (4), (8) for $\mu(s)$ has been reduced to the equation (14) for the function $\mu_*(s)$. It is clear from our consideration that any solution of equation (14) gives a solution of the system (4), (8).

As noted above, $\Phi(s)$ and $A(s,\sigma)$ are Hölder functions if $s \in \Gamma$ and $\sigma \in \Gamma$. More precisely (see [7] and [11: Section 18]), $\Phi \in C^{0,p}(\Gamma)$ $(p = \min\{\lambda, \frac{1}{2}\})$ and $A(\cdot, \sigma) \in C^{0,p}(\Gamma)$ uniformly with respect to $\sigma \in \Gamma$. We arrive at the following assertion.

Lemma 1. *If $\Gamma \in C^{2,\lambda}$ $(\lambda \in (0,1])$, $\Phi \in C^{0,p}(\Gamma)$ $(p = \min\{\lambda, \frac{1}{2}\})$ and $\mu_* \in C^0(\Gamma)$ satisfies the equation (14), then $\mu_* \in C^{0,p}(\Gamma)$.*

The condition $\Phi \in C^{0,p}(\Gamma)$ holds if $f \in C^{0,\lambda}(\Gamma)$. Hence below we will seek $\mu_*(s)$ from $C^0(\Gamma)$.

Since the function $A(s,\sigma)$ belongs to $C^0(\Gamma \times \Gamma)$, the integral operator from (14)

$$\mathbf{A}\mu_*(s) = \int_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)\,d\sigma$$

is compact and maps $C^0(\Gamma)$ into itself. Therefore, (14) is a Fredholm equation of the second kind in the Banach space $C^0(\Gamma)$.

Let us show that the homogeneous equation (14) has only a trivial solution. Then, according to Fredholm's theorems, the inhomogeneous equation (14) has a unique solution for any right-hand side. We will prove this by a contradiction. Let $\mu_*^0(s)$ from $C^0(\Gamma)$ be a non-trivial solution of the homogeneous equation (14). According to Lemma 1, $\mu_*^0 \in C^{0,p}(\Gamma)$ $(p = \min\{\lambda, \frac{1}{2}\})$. Therefore the function $\mu^0(s) = \mu_*^0(s)Q^{-1}(s)$ belongs to $C_{\frac{1}{2}}^p(\Gamma)$ and converts the homogeneous equation (13) into an identity. Using the homogeneous identity (13) we check that $\mu^0(s)$ satisfies the conditions (4). Besides, acting on the homogeneous identity (13) with a singular operator with the kernel $\frac{1}{s-t}$ we find that $\mu^0(s)$ satisfies the homogeneous equation (9). Consequently, $\mu^0(s)$ satisfies the homogeneous equation (8). On the basis of Theorem 2, $u[\mu^0](x)$ is a solution of the homogeneous problem **U**. According to Theorem 1, $u[\mu^0](x) \equiv \text{const}$ $(x \in \mathbb{R}^2 \backslash \Gamma)$. Using the limit formulas for the tangent and normal derivatives of potentials [2, 6], we obtain

$$\lim_{x \to x(s) \in (\Gamma)^+} \left\{ \beta \frac{\partial}{\partial n_x} u[\mu^0](x) - \frac{\partial}{\partial \tau_x} u[\mu^0](x) \right\}$$
$$- \lim_{x \to x(s) \in (\Gamma)^-} \left\{ \beta \frac{\partial}{\partial n_x} u[\mu^0](x) - \frac{\partial}{\partial \tau_x} u[\mu^0](x) \right\} = -(1 + \beta^2)\mu^0(s) \equiv 0 \qquad (s \in \Gamma).$$

By Γ^+ we denote the side of Γ which is on the left as the parameter s increases and by Γ^- we denote the other side. Consequently, if $s \in \Gamma$, then $\mu^0(s) \equiv 0$ and $\mu_*^0(s) = \mu^0(s)Q^{-1}(s) \equiv 0$, and we arrive at a contradiction to the assumption that $\mu_*^0(s)$ is a nontrivial solution of the homogeneous equation (14). Thus, the homogeneous Fredholm equation (14) has only a trivial solution in $C^0(\Gamma)$.

We have proved the following assertion.

Theorem 3. *If $\Gamma \in C^{2,\lambda}$ ($\lambda \in (0,1]$), then (14) is a Fredholm equation of the second kind in the space $C^0(\Gamma)$. Moreover, equation (14) has a unique solution $\mu_* \in C^0(\Gamma)$ for any $\Phi \in C^0(\Gamma)$.*

As a consequence of Theorem 3 and Lemma 1 we obtain the following corollary.

Corollary. *If $\Gamma \in C^{2,\lambda}$ ($\lambda \in (0,1]$) and $\Phi \in C^{0,p}(\Gamma)$, where $p = \min\{\lambda, \frac{1}{2}\}$, then the unique solution of equation (14) in $C^0(\Gamma)$, ensured by Theorem 3, belongs to $C^{0,p}(\Gamma)$.*

We recall that $\Phi(s)$ belongs to the class of smoothness required in the corollary if $f \in C^{0,\lambda}(\Gamma)$. As mentioned above, if $\mu_* \in C^{0,p}(\Gamma)$ is a solution of equation (14), then $\mu(s) = \mu_*(s)Q^{-1}(s)$ is a solution of the system (4), (8) in the space $C_{\frac{1}{2}}^p(\Gamma)$. We obtain the following statement.

Proposition. *If $\Gamma \in C^{2,\lambda}$ and $f \in C^{0,\lambda}(\Gamma)$ ($\lambda \in (0,1]$), then the system of equations (4),(8) has a solution $\mu \in C_{\frac{1}{2}}^p(\Gamma)$, $p = \min\{\lambda, \frac{1}{2}\}$, which is expressed by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$, where $\mu_* \in C^{0,p}(\Gamma)$ is the unique solution of the Fredholm equation (14) in $C^0(\Gamma)$.*

Thus, the system (4), (8) is solvable for any $f \in C^{0,\lambda}(\Gamma)$. On the basis of Theorem 2 we arrive at the following final result.

Theorem 4. *If $\Gamma \in C^{2,\lambda}$ and $f \in C^{0,\lambda}(\Gamma)$ ($\lambda \in (0,1]$), then the solution of problem U exists and is given by (7), where $\mu(s)$ is a solution of the equations (4),(8) from $C_{\frac{1}{2}}^p(\Gamma)$, $p = \min\{\lambda, \frac{1}{2}\}$ ensured by Proposition.*

It can be checked directly that the solution of the problem U satisfies condition (1) with $\varepsilon = -\frac{1}{2}$. Explicit expressions for singularities of the solution gradient at the end-points of the open curves can be easily obtained with the help of formulas presented in [6, 7].

Theorem 4 ensures existence of a classical solution of the problem U when $\Gamma \in C^{2,\lambda}$ and $f \in C^{0,\lambda}(\Gamma)$. On the basis of our consideration we suggest the following scheme for solving the problem U. First, we find the unique solution $\mu_*(s)$ of the Fredholm equation (14) from $C^0(\Gamma)$. This solution automatically belongs to $C^{0,p}(\Gamma)$, with $p = \min\{\lambda, \frac{1}{2}\}$. Second, we construct the solution of equations (4), (8) from $C_{\frac{1}{2}}^p(\Gamma)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. This solution automatically belongs to $C_{\frac{1}{2}}^p(\Gamma)$. Finnaly, substituting $\mu(s)$ into (7) we obtain the solution of the problem U.

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