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Oscillation and Non-Oscillation Theorems for a Class of Second Order Quasilinear Difference Equations

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Abstract. In this paper there are established necesssary and sufficient conditions for the second order quasilinear difference equation

$$\Delta\big(p_n\varphi(\Delta y_n)\big) + f(n, y_{n+1}) = 0 \qquad (n \in \mathbb{N}_0)$$

to have various types of non-oscillatory solutions. In addition, in the case that the equation is either strongly superlinear or strongly sublinear, there are established necessary and sufficient conditions for all solutions to oscillate.

Keywords: *Quasilinear difference equations, oscillation, non-oscillatory solutions*

AMS subject classification: 39 A 10

1. Introduction

In this paper we consider the second order quasilinear difference equation

$$\Delta\big(p_n\varphi(\Delta y_n)\big) + f(n, y_{n+1}) = 0 \qquad (n \in \mathbb{N}_0) \tag{1}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$. Further we assume the following:

(a) $\{p_n\}_{n\geq 0}$ is a real sequence with $p_n > 0$.

(b) $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, strictly increasing function with $\operatorname{sgn}\varphi(u) = \operatorname{sgn} u$ and $\varphi(\mathbb{R}) = \mathbb{R}$.

(c) $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $uf(n,u) > 0$ for $u \neq 0$, and $f(n,\cdot)$ is non-decreasing for each fixed $n \in \mathbb{N}_0$.

A prototype of equation (1) satisfying the conditions (a) - (c) is

$$\Delta\big((\Delta y_n)^{\alpha *}\big) + q_n y_{n+1}^{\beta *} = 0 \qquad (n \in \mathbb{N}_0) \tag{2}$$

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where α and β are positive constants, $\{q_n\}_{n\geq 0}$ is a positive real sequence and $u^{\lambda *} = |u|^\lambda \operatorname{sgn} u$ for any $\lambda > 0$.

By a *solution* of equation (1) we mean a non-trivial sequence $\{y_n\}_{n\geq 1}$ satisfying equation (1) for all $n \in \mathbb{N}$. A solution $\{y_n\}_{n\geq 1}$ of equation (1) is said to be *non-oscillatory* if it is either eventually positive or eventually negative, and *oscillatory* otherwise.

The literature on oscillation criteria of difference equations is vast (see, e.g., [1, 12], which cover a large number of recent papers on this topic). In particular, we refer to [16 - 23, 25], where oscillations of equations similar to equation (1) have been discussed. We note that an equation related to the continuous version

$$\big(p(t)\varphi(y')\big)' + f(t,y) = 0 \tag{3}$$

of (1) where $p(t) > 0$ has been the subject matter of many recent investigations (see, e.g., [2, 3, 6, 8 - 11, 13 - 15, 24]). Further, the oscillation results obtained for equation (3) can be applied to derive similar properties for solutions of certain partial differential equations. Hence, the study of oscillatory and non-oscillatory behaviour of solutions of equation (1) extends beyond the obvious self interest.

Our objective here is to investigate in detail the oscillatory and non-oscillatory behaviour of solutions of equation (1). Under additional hypotheses on p_n, φ and f, first we study the structure of the set of non-oscillatory solutions of equation (1), and then establish criteria for all solutions of equation (1) to be oscillatory. Thus, we are able to indicate a wide class of equations of the form (1), including (2) with $\alpha \neq \beta$, for which the oscillation of all solutions can be completely characterized.

2. Existence of non-oscillatory solutions

Throughout the paper we make the following assumptions without further mention:

$$\sum_{n=0}^{\infty} \left|\varphi^{-1}\left(\frac{k}{p_n}\right)\right| = \infty \qquad \text{for every constant } k \neq 0, \tag{4}$$

where $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$ denotes the inverse function of φ, and

$$\left.\begin{aligned} \Phi_{k,N}(p;n) &= \sum_{s=N}^{n-1} \varphi^{-1}\left(\frac{k}{p_s}\right) \qquad (n \geq N) \\ \Phi_k(p;n) &= \Phi_{k,0}(p;n) \qquad (n \geq 0) \end{aligned}\right\} \tag{5}$$

where $N \in \mathbb{N}_0$ and $\sum_{s=N}^{N-1} \cdot := 0$. From (4) and (5) it is clear that

$$\Phi_{k,N}(p;N) = 0$$

$$\lim_{n\to\infty} |\Phi_{k,N}(p;n)| = \infty \quad \text{for every} \quad k \neq 0$$

$$|\Phi_{k,N}(p;n)| > |\Phi_{m,N}(p;n)| \quad (n > N) \quad \text{for} \quad |k| > |m| \quad \text{with} \quad km > 0$$

$$\lim_{k\to 0} \Phi_{k,N}(p;n) = 0 \quad \text{for each} \quad n \geq N.$$

We begin by classifying all possible non-oscillatory solutions of equation (1) according to their asymptotic behaviour as $n \to \infty$.

Lemma 1. *Each non-oscillatory solution* $\{y_n\}_{n\geq 0}$ *of equation* (1) *must belong to one of the following three types:*

(I) $\lim_{n\to\infty} p_n\,\varphi(\Delta y_n) = \text{const} \neq 0$

(II) $\lim_{n\to\infty} p_n\,\varphi(\Delta y_n) = 0$ *and* $\lim_{n\to\infty} |y_n| = \infty$.

(III) $\lim_{n\to\infty} p_n\,\varphi(\Delta y_n) = 0$ *and* $\lim_{n\to\infty} y_n = \text{const} \neq 0$.

Proof. Let $\{y_n\}_{n\geq 0}$ be a non-oscillatory solution of equation (1). Without loss of generality, we may assume that $y_n > 0$ for $n \geq n_0 \in \mathbb{N}$. From equation (1), it follows that $\Delta(p_n\,\varphi(\Delta y_n)) < 0$ for $n \geq n_0$, and therefore the sequence $\{p_n\,\varphi(\Delta y_n)\}_{n\geq n_0}$ is decreasing. We claim that $p_n\,\varphi(\Delta y_n) > 0$ for $n \geq n_0$, so that $\lim_{n\to\infty} p_n\,\varphi(\Delta y_n) \geq 0$. If $p_{n_1}\varphi(\Delta y_{n_1}) = -k < 0$ for some integer $n_1 \geq n_0$ and $k > 0$, then $p_n\varphi(\Delta y_n) \leq -k$ for $n \geq n_1$, so $\Delta y_n \leq \varphi^{-1}(-\frac{k}{p_n})$ for $n \geq n_1$. Summing up the last inequality from n_1 to n, we see in view of (4) that $y_n \to -\infty$ as $n \to \infty$. But this contradicts the assumed positivity of y_n. Hence, $p_n\varphi(\Delta y_n) > 0$ for $n \geq n_0$, as claimed. A concequence of this observation is that $\Delta y_n > 0$ for $n \geq n_0$, that is, the sequence $\{y_n\}_{n\geq 0}$ is strictly increasing.

The limit $\lim_{n\to\infty} p_n\varphi(\Delta y_n)$ is either positive or zero. In the first case, the sequence $\{y_n\}_{n\geq 0}$ is unbounded, since there are positive constants k_1 and k_2 with $k_1 < k_2$ and an integer n_0 such that $\Phi_{k_1,n_0}(p;n) \leq y_n - y_{n_0} \leq \Phi_{k_2,n_0}(p;n)$ for all $n \geq n_0$. In the second case, since $\{y_n\}_{n\geq 0}$ is increasing, y_n tends to a positive limit, finite or infinite, as $n \to \infty$ ∎

Theorem 2. *Assume that, for each fixed* $k \neq 0$ *and* $N \in \mathbb{N}_0$,

$$\lim_{m\to 0\ (mk>0)} \frac{\Phi_{m,N}(p;n)}{\Phi_{k,N}(p;n)} = 0 \tag{6}$$

uniformly for all $n \geq N_1 > N$. *Then a necessary and sufficient condition for the equation* (1) *to have a non-oscillatory type* (I) *solution* $\{y_n\}_{n\geq 0}$ *is that*

$$\sum_{n=0}^{\infty} \left| f(n, c\,\Phi(p;n+1)) \right| < \infty \tag{7}$$

for some constants $k \neq 0$ *and* $c > 0$.

Proof. Necessity: Let $\{y_n\}_{n\geq 0}$ be a non-oscillatory type (I) solution of the equation (1). We may assume that $y_n > 0$ for $n \geq n_0 \in \mathbb{N}_0$ since a similar argument holds if $\{y_n\}_{n\geq 0}$ is eventually negative. There exist positive constants c_1 and k_1 such that $c_1\Phi_{k_1}(p;n) \leq y_n$ for $n \geq n_0$. Summation of equation (1) yields $\sum_{s=n}^{\infty} f(n, y_{n+1}) < \infty$ which combined with the above inequality leads to $\sum_{n=n_0}^{\infty} f\big(n, c_1\Phi_{k_1}(p;n+1)\big) < \infty$.

Sufficiency: Assume that (7) holds for some constants $c > 0$ and $k > 0$. Because of (6) we can choose some $m > 0$ and an integer $N > 0$ such that $m < \frac{k}{2}$ and

$$\sum_{n=N}^{\infty} f\big(n, \Phi_{2m}(p;n+1)\big) \leq m. \tag{8}$$

Consider the Banach space B_N of all real sequences $y = \{y_n\}_{n\geq N}$ with the supremum norm $\|y\| = \sup_{n\geq N} |y_n|$. We define a set S as

$$S = \Big\{ y \in B_N : \Phi_{m,N}(p;n) \leq y_n \leq \Phi_{2m,N}(p;n) \quad (n \geq N) \Big\}.$$

Clearly, S is a bounded, closed and convex subset of B_N. Now, define an operator $T : S \to B_N$ as

$$Ty_n = \sum_{s=N}^{n-1} \varphi^{-1}\left(\frac{1}{p_s}\left(m + \sum_{t=s}^{\infty} f(t, y_{t+1})\right)\right) \qquad (n \geq N). \tag{9}$$

From the hypotheses this operator T is continuous. If $y \in S$, then since

$$0 \leq \sum_{n=N_1}^{\infty} f(n, y_{n+1}) \leq \sum_{n=N}^{\infty} f\big(n, \Phi_{2m}(p; n+1)\big) \leq m \qquad (N_1 \geq N)$$

we obtain from (9)

$$\sum_{s=N}^{n-1} \varphi^{-1}\left(\frac{m}{p_s}\right) \leq Ty_n \leq \sum_{s=N}^{n-1} \varphi^{-1}\left(\frac{2m}{p_s}\right) \qquad (n \geq N)$$

implying that $TS \subset S$. Therefore, by the Schauder fixed point theorem, T has a fixed point $y \in S$. It is clear that $y = \{y_n\}_{n\geq 0}$ is a positive solution of the equation (1), and it is obviously of type (I) ∎

If (7) holds for some constants $k < 0$ and $c > 0$, then a similar argument can be used to construct a negative type (I) solution of the equation (1).

Theorem 3. *A necessary and sufficient condition for the equation* (1) *to have a non-oscillatory type* (III) *solution* $\{y_n\}_{n\geq 0}$ *is that*

$$\sum_{n=0}^{\infty} \left| \varphi^{-1}\left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, c)\right) \right| < \infty \tag{10}$$

for some constant $c \neq o$.

Proof. Necessity: Let $\{y_n\}_{n\geq 0}$ be a positive type (III) solution of the equation (1). Then there are a positive constant c_1 and an $N_0 \in \mathbb{N}_0$ such that $y_n \geq c_1$ for $n \geq N_0$. Summing up the equation (1) from n to ∞, we get

$$p_n \varphi(\Delta y_n) = \sum_{s=n}^{\infty} f(s, y_{s+1}) \qquad (n \geq N_0)$$

which implies

$$\Delta y_n = \varphi^{-1}\left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, y_{s+1})\right) \qquad (n \geq N_0).$$

Summing up this equation again and using $y_n \geq c_1$, we find that

$$\Delta y_n = \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty} f(s,c_1)\right) < \infty.$$

The proof when $\{y_n\}_{n\geq 0}$ is eventually negative is similar.

Sufficiency: Assume that (10) holds for some constant $c > 0$ (a similar argument will hold if $c < 0$). Choose $N \in \mathbb{N}$ sufficiently large so that

$$\sum_{n=N}^{\infty} \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=N}^{\infty} f(s,c)\right) \leq \frac{c}{2}.$$

Let B_N be the same Banach space as in the proof of Theorem 2, and let

$$S = \left\{y \in B_N : \frac{c}{2} \leq y_n \leq c \quad (n \geq N)\right\}.$$

Clearly, S is a bounded, closed and convex subset of B_N. Now, we define an operator$T : S \to B_N$ as

$$Ty_n = c - \sum_{s=N}^{\infty} \varphi^{-1}\left(\frac{1}{p_s}\sum_{t=s}^{\infty} f(t,y_{t+1})\right) \qquad (n \geq N).$$

This operator T is continuous, and it is easy to see that $TS \subset S$. Hence, by the Schauder fixed point theorem, T has a fixed point $y \in S$. This is the desired type (III) solution of the equation (1) ∎

Next we give sufficient conditions for the existence of type (II) solutions of the equation (1).

Theorem 4. *Suppose the condition* (6) *holds. Then the equation* (1) *has a non-oscillatory type* (II) *solution if condition* (7) *holds for some constants* $k \neq 0$ *and* $c > 0$ *and*

$$\sum_{n=0}^{\infty}\left|\varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty} f(s,d)\right)\right| = \infty \tag{11}$$

for every non-zero constant d *such that* $kd > 0$.

Proof. It is enough to consider the case where $k > 0$ and $d > 0$. Let $a > 0$ be an arbitrary fixed constant, and choose $m > 0$ small enough and $N \in \mathbb{N}_0$ large enough so that $a + \Phi_m(p;n) \leq c\,\Phi_k(p;n)$ for all $n \geq N$ and

$$\sum_{n=N}^{\infty} f\big(n, a + \Phi_m(p;n+1)\big) \leq m.$$

This is possible because of (6) and of the fact that $\lim_{n\to\infty} \Phi_k(p;n) = \infty$. Let B_N be the same Banach space as in the proofs of the Theorems 2 and 3, and let the set S be defined as

$$S = \left\{y \in B_N : a \leq y_n \leq a + \Phi_m(p;n) \quad (n \geq N)\right\}.$$

Clearly, S is a bounded, closed and convex subset of B_N. Define an operator $T : S \to B_N$ as

$$Ty_n = a + \sum_{s=N}^{n-1} \varphi^{-1}\left(\frac{1}{p_s}\sum_{t=s}^{\infty} f(t, y_{t+1})\right) \qquad (n \geq N). \tag{12}$$

This operator T is continuous, and as earlier it is easy to see that $TS \subset S$. Therefore, by the Schauder fixed point theorem, T has a fixed point $y \in S$. It is clear that $y = \{y_n\}_{n\geq 0}$ is a positive solution of the equation (1). From (12) we see also that

$$p_n\varphi(\Delta y_n) = \sum_{s=n}^{\infty} f(s, y_{s+1}) \to 0 \qquad \text{as } n \to \infty,$$

and by (11) that

$$y_n \geq a + \sum_{n=N}^{\infty} \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty} f(s, a)\right) \to \infty \qquad \text{as } n \to \infty.$$

It follows therefore that $\{y_n\}_{n\geq 0}$ is a type (II) solution of the equation (1) ∎

Example 1. Consider the equation (2) where $\alpha > 0$ and $\beta > 0$, and where $\{q_n\}_{n\geq 0}$ is a positive real sequence. This is a special case of the equation (1) in which $p_n = 1$, $\varphi(u) = u^{\alpha *}$ and $f(n, v) = q_n v^{\beta *}$, and we have $\varphi^{-1}(u) = u^{\frac{1}{\alpha *}}$ and $\Phi_{k,N}(p; n) = k^{\frac{1}{\alpha *}}(n - N)$, so that the conditions (4) and (6) are satisfied for the equation (2).

The possible types of asymptotic behaviour at infinity of non-oscillatory solutions of the equation (2) are as follows:

(I) $\lim_{n\to\infty} \frac{y_n}{n} = \text{const} \neq 0$.

(II) $\lim_{n\to\infty} \frac{y_n}{n} = 0$ and $\lim_{n\to\infty} |y_n| = \infty$.

(III) $\lim_{n\to\infty} y_n = \text{const} \neq 0$.

From Theorems 2 and 3 it follows that the equation (2) has a type (I) solution if and only if

$$\sum_{n=0}^{\infty} n^{\beta} q_n < \infty, \tag{13}$$

and that the equation (2) has a type (III) solution if and only if

$$\sum_{n=0}^{\infty}\left(\sum_{s=n}^{\infty} q_s\right)^{\frac{1}{\alpha}} < \infty. \tag{14}$$

Theorem 4 implies that the conditions (13) and

$$\sum_{n=0}^{\infty}\left(\sum_{s=n}^{\infty} q_s\right)^{\frac{1}{\alpha}} = \infty \tag{15}$$

are sufficient for the existence of type (II) solutons of the equation (2).

The conditions (13) and (15) are not always consistent. In fact, let $q_n = (n+1)^\lambda$ for some constant λ. Then, condition (13) holds if and only if $\lambda < -1-\beta$, and condition (15) holds if and only if $\lambda \geq -1-\alpha$. Hence these two conditions are inconsistent if $\alpha \leq \beta$. Thus, if $\alpha > \beta$ so that $-1-\alpha < -1-\beta$, then there exists a type (II) solution for the equation $\Delta((\Delta y_n)^{\alpha_*}) + (n+1)^\lambda y_{n+1}^{\beta_*} = 0$.

3. Oscillation of all solutions

In this section we study the oscillatory behaviour of solutions of the equation (1). In view of the results of Hooker and Patula [5] and those of Kulenovic and Budincevic [7], it is reasonable to expect that a characterization of oscillation for the equation (1) can be obtained under suitable additional conditions on the nonlinear functions f and φ.

Definition 5. The equation (1) is said to be

(i) *strongly superlinear* if there is a constant $\gamma > 0$ such that the function $|\cdot|^{-\gamma} f(n,\cdot)$ is non-decreasing for each fixed n and

$$\int_M^\infty \frac{ds}{\varphi^{-1}(s^\gamma)} < \infty \qquad \text{and} \qquad \int_{-M}^{-\infty} \frac{ds}{\varphi^{-1}(s^{\gamma_*})} < \infty \tag{16}$$

for any $M > 0$;

(ii) *strongly sublinear* if there is a constant $\delta > 0$ such that the function $|\cdot|^{-\delta} f(n,\cdot)$ is non-increasing for each fixed n and

$$\int_0^N \frac{ds}{(\varphi^{-1}(s))^\gamma} < \infty \qquad \text{and} \qquad \int_{-N}^0 \frac{ds}{(\varphi^{-1}(s))^{\delta_*}} < \infty \tag{17}$$

for any $N > 0$.

According to the above definition, the equation (2) is strongly superlinear if $\alpha < \beta$ and strongly sublinear if $\alpha > \beta$.

Theorem 6. *Assume the equation* (1) *is strongly superlinear. Assume further that*

$$\varphi^{-1}(uv) \geq \varphi^{-1}(u)\varphi^{-1}(v) \qquad \textit{for all } u \textit{ and } v \textit{ with } uv > 0. \tag{18}$$

Then all solutions of the equation (1) *are oscillatory if and only if*

$$\sum_{n=0}^\infty \left| \varphi^{-1}\left(\frac{1}{p_n} \sum_{s=n}^\infty f(s,c) \right) \right| = \infty \tag{19}$$

for every non-zero constant c.

Proof. The necessity part follows from Theorem 3. To prove the sufficiency, suppose that the equation (1) has a non-oscillatory solution $\{y_n\}_{n\geq 0}$, say $y_n > 0$ for $n \geq n_0 \in \mathbb{N}_0$. Summing up the equation (1) from n to ∞ and noting that $\lim_{n\to\infty} p_n\varphi(\Delta y_n) \geq 0$, we have

$$p_n\varphi(\Delta y_n) \geq \sum_{s=n}^{\infty} f(s, y_{s+1}) \qquad (n \geq n_0)$$

which implies

$$\Delta y_n \geq \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty} f(s, y_{s+1})\right) \qquad (n \geq n_0).$$

Now divide the last inequality by $\varphi^{-1}(y_{n+1}^{\gamma})$, where $\gamma > 0$ is the constant of strong superlinearity of the equation (1), and use (18) to obtain

$$\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \geq \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty}\frac{f(s, y_{s+1})}{y_{n+1}^{\gamma}}\right) \geq \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty}\frac{f(s, y_{s+1})}{y_{s+1}^{\gamma}}\right) \qquad (n \geq n_0).$$

Since $y_n \geq c_0$ $(n \geq n_0)$ for some constant $c_0 > 0$ we have, in view of the strong superlinearity of the equation (1), $y_{n+1}^{\gamma} f(n, y_{n+1}) \geq c_0^{-\gamma} f(n, c_0)$ $(n \geq c_0)$ so that

$$\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \geq \varphi^{-1}\left(c_0^{-\gamma}\frac{1}{p_n}\sum_{s=n}^{\infty} f(s, c_0)\right) \geq \varphi^{-1}(c_0^{-\gamma})\varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty} f(s, c_0)\right) \quad (n \geq n_0).$$

Summing up the last inequality from n_0 to $n-1$, we obtain

$$\varphi^{-1}(c_0^{-\gamma})\sum_{s=n_0}^{n-1}\varphi^{-1}\left(\frac{1}{p_n}\sum_{t=s}^{\infty} f(t, c_0)\right) \leq \sum_{s=n_0}^{n-1}\frac{\Delta y_s}{\varphi^{-1}(y_{s+1}^{\gamma})}. \tag{20}$$

Since

$$\frac{1}{\varphi^{-1}(x^{\gamma})} \geq \frac{1}{\varphi^{-1}(y_{n+1}^{\gamma})} \qquad \text{for } y_n \leq x \leq y_{n+1},$$

we have

$$\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \leq \int_{y_n}^{y_{n+1}} \frac{dx}{\varphi^{-1}(x^{\gamma})}. \tag{21}$$

Substituting (21) into (20), we obtain

$$\varphi^{-1}(c_0^{-\gamma})\sum_{s=n_0}^{n-1}\varphi^{-1}\left(\frac{1}{p_n}\sum_{t=s}^{\infty} f(t, c_0)\right) \leq \int_{y_{n_0}}^{y_n} \frac{dx}{\varphi^{-1}(x^{\gamma})} < \infty,$$

which is a contradiction ∎

Theorem 7. *Let the equation* (1) *be strongly sublinear and suppose the conditions* (6) *and* (7) *hold. Then all solutions of the equation* (1) *are oscillatory if and only if*

$$\sum_{n=0}^{\infty} \left| f(n, c\,\Phi_k(p; n+1)) \right| = \infty \tag{22}$$

for every non-zero constant c.

Proof. First, note that the necessity of the condition (22) follows from Theorem 2. Next, let $\{y_n\}_{n\geq 0}$ be a non-oscillatory solution of the equation (1), say $y_n > 0$ for $n \geq n_0 \in \mathbb{N}_0$. First note that $\Delta y_n > 0$ for $n \geq n_0$, and

$$\Delta y_n \geq \varphi^{-1}\left(\frac{1}{p_n}\right) \varphi^{-1}(p_n\,\varphi(\Delta y_n)) \qquad (n \geq n_0).$$

Summing up the last inequality from n_0 to $n-1$ and using the decreasing nature of the sequence $\{p_n\,\varphi(\Delta y_n)\}_{n\geq 0}$, we get

$$y_n \geq y_n - y_{n_0} \geq \varphi^{-1}(p_n\,\varphi(\Delta y_n))\,\Phi_{1,n_0}(p; n) \qquad (n \geq n_0). \tag{23}$$

From the strong sublinearity and the inequality $y_n \leq c_0 \Phi_{k,n_0}(p; n)$ $(n \geq n_0)$, where $c_0 > 0$ is constant, it follows that

$$y_{n+1}^{-\delta} f(n, y_{n+1}) \geq c_0^{-\delta} \left(\Phi_{k,n_0}(p; n)\right)^{-\delta} f\left(n, \Phi_{k,n_0}(p; n+1)\right) \tag{24}$$

for $n \geq n_0$. From (23) and (24), and using the inequality

$$\frac{\Phi_{1,n_0}(p; n)}{\Phi_{k,n_0}(p; n)} \geq \varphi^{-1}\left(\frac{1}{k}\right) \qquad (n > n_0)$$

which follows from (18), we have

$$\begin{aligned} \frac{f(n, y_{n+1})}{\left(\varphi^{-1}(p_n\,\varphi(\Delta y_n))\right)^{\delta}} &= \frac{y_{n+1}^{-\delta} f(n, y_{n+1})\, y_{n+1}^{\delta}}{\left(\varphi^{-1}(p_n\,\varphi(\Delta y_n))\right)^{\delta}} \\ &\geq c_0^{-\delta} \varphi^{-1}\left(\frac{1}{k}\right) f\left(n, c_0 \Phi_{k,n_0}(p; n+1)\right) \end{aligned} \tag{25}$$

for $n \geq n_0$. From the equation (1) and (25) we obtain

$$c_0^{-\delta} \varphi^{-1}\left(\frac{1}{k}\right) \sum_{s=n_0}^{n-1} f\left(n, c_0 \Phi_{k,n_0}(p; s+1)\right) \leq \sum_{s=n_0}^{n-1} \frac{-\Delta\left(p_s\,\varphi(\Delta y_s)\right)}{\left(\varphi^{-1}(p_s\,\varphi(\Delta y_s))\right)^{\delta}} \tag{26}$$

for $n \geq n_0$. Since

$$\frac{1}{\left(\varphi^{-1}(x)\right)^{\delta}} \geq \frac{1}{\left(\varphi^{-1}(p_n\,\varphi(\Delta y_n))\right)^{\delta}}$$

for $p_n\,\varphi(\Delta y_n) \geq x \geq p_{n+1}\varphi(\Delta y_{n+1})$, we have

$$-\frac{p_n\,\varphi(y_n)}{\left(\varphi^{-1}\left(p_n\,\varphi(y_n)\right)\right)^{\delta}} < \int\limits_{p_{n+1}\varphi(y_{n+1})}^{p_n\varphi(y_n)} \frac{dx}{\left(\varphi^{-1}(x)\right)^{\delta}}. \tag{27}$$

Substituting (27) into (26), we obtain

$$c_0^{-\delta}\varphi^{-1}\left(\frac{1}{k}\right)\sum_{s=n_0}^{n-1} f\big(n, c_0\Phi_{k,n_0}(p;s+1)\big) < \int\limits_{p_n\varphi(y_n)}^{p_{n_0}\varphi(y_{n_0})} \frac{dx}{\left(\varphi^{-1}(x)\right)^{\delta}} < \infty,$$

which is a contradiction ∎

Example 2. Consider the equation (2) again. Since

$$\varphi^{-1}(u) = u^{\frac{1}{\alpha_*}} \qquad \text{and} \qquad f(n,v) = q_n v^{\beta_*},$$

the equation (2) is strongly superlinear or strongly sublinear according as $\alpha < \beta$ or $\alpha > \beta$. Therefore, from Theorems 6 and 7, it follows that a necessary and sufficient condition for the oscillation of all solutions of the equation (2) is

$$\sum_{n=0}^{\infty}\left(\sum_{s=n}^{\infty} q_s\right)^{\frac{1}{\alpha}} = \infty \quad \text{if } \alpha < \beta \qquad \text{and} \qquad \sum_{n=0}^{\infty} n^{\beta} q_n = \infty \quad \text{if } \alpha > \beta.$$

Remark. Our results reduce to some of the results of He [4] when $\varphi(u) = u$. Also, our results generalize some of the results obtained in [19] in the sense that we do not require the condition $p_n \to \infty$ as $n \to \infty$.

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