Oscillation and Non-Oscillation Theorems for a Class of Second Order Quasilinear Difference Equations

E. Thandapani and R. Arul

Abstract. In this paper there are established necesssary and sufficient conditions for the second order quasilinear difference equation There are established necessary and sufficience equation
difference equation
 $\Delta(p_n\varphi(\Delta y_n)) + f(n,y_{n+1}) = 0$ $(n \in \mathbb{N}_0)$

$$
\Delta(p_n\varphi(\Delta y_n))+f(n,y_{n+1})=0 \qquad (n\in\mathbb{N}_0)
$$

to have various types of non-oscillatory solutions. In addition, in the case that the equation is either strongly superlinear or strongly sublinear, there are established necessary and sufficient conditions for all solutions to oscillate. **LET UP:** The set of socillate.

Let us to oscillate.

Let difference equations, oscillation, non-oscillatory solutions

cation: 39 A 10

Let the second order quasilinear difference equation
 $\Delta(p_n\varphi(\Delta y_n)) + f(n,y_{n+1}) = 0$

Keywords: *Quasilincar difference equations, oscillation, non-oscillatory solutions* AMS subject classification: 39 A 10

1. Introduction

In this paper we consider the second order quasilinear difference equation

$$
\Delta(p_n\varphi(\Delta y_n)) + f(n, y_{n+1}) = 0 \qquad (n \in \mathbb{N}_0)
$$
 (1)

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and Δ is the forward difference operator defined by $\Delta y_n =$ $y_{n+1} - y_n$. Further we assume the following:

(a) ${p_n}_{n>0}$ is a real sequence with $p_n > 0$.

(b) $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, strictly increasing function with sgn $\varphi(u) = \text{sgn } u$ and $\varphi(\mathbb{R}) = \mathbb{R}$. *p* increasing function with $sgn \varphi(u) = sgn u$
 p and p increasing function with $sgn \varphi(u) = sgn u$
 p and $f(n, \cdot)$ is

the conditions (a) - (c) is
 $\beta_{n+1} = 0$ ($n \in \mathbb{N}_0$) (2)
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(c) $f: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $uf(n, u) > 0$ for $u \neq 0$, and $f(n, \cdot)$ is non-decreasing for each fixed $n \in \mathbb{N}_0$.

A prototype of equation (1) satisfying the conditions $(a) - (c)$ is

$$
\Delta((\Delta y_n)^{\alpha_*}) + q_n y_{n+1}^{\beta_*} = 0 \qquad (n \in \mathbb{N}_0)
$$
 (2)

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where α and β are positive constants, $\{q_n\}_{n>0}$ is a positive real sequence and u^{λ_*} = $|u|^{\lambda}$ sgn u for any $\lambda > 0$.

By a *solution* of equation (1) we mean a non-trivial sequence $\{y_n\}_{n>1}$ satisfying equation (1) for all $n \in \mathbb{N}$. A solution $\{y_n\}_{n\geq 1}$ of equation (1) is said to be *non*oscillatory if it is either eventually positive or eventually negative, and *oscillatory* otherwise.

The literature on oscillation criteria of difference equations is vast (see, e.g., [1, 12], which cover a large number of recent papers on this topic). In particular, we refer to [16 - 23, 251, where oscillations of equations similar to equation (1) have been discussed. We note that an equation related to the continuous version astants, $\{q_n\}_{n\geq0}$ is a positive real sequence and $u^{\lambda_*} =$

(1) we mean a non-trivial sequence $\{y_n\}_{n\geq1}$ satisfying

(a solution $\{y_n\}_{n\geq1}$ of equation (1) is said to be *non*-

illy positive or eventually

$$
(p(t)\varphi(y'))' + f(t,y) = 0 \tag{3}
$$

of (1) where $p(t) > 0$ has been the subject matter of many recent investigations (see, e.g., [2, 3, 6, 8 - 11, 13 - 15, 24]). Further, the oscillation results obtained for equation (3) can be applied to derive similar properties for solutions of certain partial differential equations. Hence, the study of oscillatory and non-oscillatory behaviour of solutions of equation (1) extends beyond the obvious self interest.

Our objective here is to investigate in detail the oscillatory and non-oscillatory Our objective here is to investigate in detail the oscillatory and non-oscillatory
behaviour of solutions of equation (1). Under additional hypotheses on p_n , φ and f ,
first we study the structure of the set of non first we study the structure of the set of non-oscillatory solutions of equation (1), and then establish criteria for all solutions of equation (1) to be oscillatory. Thus, we are which the oscillation of all solutions can be completely characterized. Let us a finite of solutions of solutions of
vious self interest.
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(1). Under additional hypotheses on p_n , φ and f ,
set of non-oscillatory solutions of equation

2. Existence of non-oscillatory solutions

Throughout the paper we make the following assumptions without further mention:

2. Existence of non-oscillatory solutions
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$$
\sum_{n=0}^{\infty} \left| \varphi^{-1} \left(\frac{k}{p_n} \right) \right| = \infty \quad \text{for every constant } k \neq 0,
$$
 (4)
where $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$ denotes the inverse function of φ , and

Then the stationary criteria for an solutions of equation (1) to be essentially. Thus, we are able to indicate a wide class of equations of the form (1), including (2) with
$$
\alpha \neq \beta
$$
, for which the oscillation of all solutions can be completely characterized.

\n**2. Existence of non-oscillatory solutions**

\nThroughout the paper we make the following assumptions without further mention:

\n
$$
\sum_{n=0}^{\infty} \left| \varphi^{-1} \left(\frac{k}{p_n} \right) \right| = \infty \qquad \text{for every constant } k \neq 0,
$$
\nwhere $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$ denotes the inverse function of φ , and\n
$$
\Phi_{k,N}(p;n) = \sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{k}{p_s} \right) \qquad (n \geq N)
$$
\nwhere $N \in \mathbb{N}_0$ and $\sum_{s=N}^{N-1} \cdots = 0$. From (4) and (5) it is clear that\n
$$
\Phi_{k,N}(p;N) = 0
$$
\nlim $|\Phi_{k,N}(p;n)| = \infty$, for every $k \neq 0$.

where $N \in \mathbb{N}_0$ and $\sum_{s=N}^{N-1}$:= 0. From (4) and (5) it is clear that

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$$
\sum_{n=0}^{\infty} \left| \varphi^{-1} \left(\frac{k}{p_n} \right) \right| = \infty \quad \text{for every constant } k \neq 0,
$$
\n
$$
{}^{1}: \mathbb{R} \to \mathbb{R} \text{ denotes the inverse function of } \varphi, \text{ and}
$$
\n
$$
\Phi_{k,N}(p;n) = \sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{k}{p_s} \right) \quad (n \geq N)
$$
\n
$$
\Phi_k(p;n) = \Phi_{k,0}(p;n) \quad (n \geq 0)
$$
\n
$$
\in \mathbb{N}_0 \text{ and } \sum_{s=N}^{N-1} := 0. \text{ From (4) and (5) it is clear that}
$$
\n
$$
\Phi_{k,N}(p;N) = 0
$$
\n
$$
\lim_{n \to \infty} |\Phi_{k,N}(p;n)| = \infty \quad \text{for every } k \neq 0
$$
\n
$$
|\Phi_{k,N}(p;n)| > |\Phi_{m,N}(p;n)| \quad (n > N) \quad \text{for } |k| > |m| \text{ with } km > 0
$$
\n
$$
\lim_{k \to 0} \Phi_{k,N}(p;n) = 0 \quad \text{for each } n \geq N.
$$

We begin by classifying all possible non-oscillatory solutions of equation (1) according to their asymptotic behaviour as $n \to \infty$.

Lemma 1. *Each non-oscillatory solution* $\{y_n\}_{n\geq 0}$ of equation (1) *must belong to one of the following three types:*

- **(I)** $\lim_{n \to \infty} p_n \varphi(\Delta y_n) = \text{const} \neq 0$
- (II) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ and $\lim_{n\to\infty} |y_n| = \infty$.
- (III) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ *and* $\lim_{n\to\infty} y_n = \text{const} \neq 0$.

Proof. Let $\{y_n\}_{n>0}$ be a non-oscillatory solution of equation (1). Without loss of generality, we may assume that $y_n > 0$ for $n \geq n_0 \in \mathbb{N}$. From equation (1), it follows that $\Delta(p_n \varphi(\Delta y_n)) < 0$ for $n \ge n_0$, and therefore the sequence $\{p_n \varphi(\Delta y_n)\}_{n \ge n_0}$ is decreasing. We claim that $p_n \varphi(\Delta y_n) > 0$ for $n \ge n_0$, so that $\lim_{n \to \infty} p_n \varphi(\Delta y_n) \ge 0$. Oscillation and Non-Oscillation Theorems 751

one of the following three types:

(1) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = \text{const} \neq 0$

(11) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ and $\lim_{n\to\infty} |y_n| = \infty$.

(111) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ and \lim_{n **Example 1.** Each non-oscillatory solution {y_n}_n}_n}o of equation (1) must belong to

one of the following three types:

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(II) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ and $\lim_{n\to\infty} |y_n| = \infty$ **Lemma 1.** Each non-oscillatory solution $\{y_n\}_{n\geq 0}$ of equation (1) must belong to

one of the following three types:

(I) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = \text{const} \neq 0$

(II) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ and $\lim_{n\to\infty} |y_n| = \infty$.
 for $n \geq n_1$, so $\Delta y_n \leq \varphi^{-1}(-\frac{k}{p_n})$ for $n \geq n_1$. Summing up the last inequality from *n₁* to *n*, we see in view of (4) that $y_n \to -\infty$ as $n \to \infty$. But this contradicts the assumed positivity of y_n . Hence, $p_n\varphi(\Delta y_n) > 0$ for $n \geq n_0$, as claimed. A concequence of this observation is that $\Delta y_n > 0$ for $n \ge n_0$, that is, the sequence $\{y_n\}_{n \ge 0}$ is strictly increasing. (II) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ and $\lim_{n\to\infty} y_n = \text{const} \neq 0$.

(III) $\lim_{n\to\infty} p_n \varphi(\Delta y_n) = 0$ and $\lim_{n\to\infty} y_n = \text{const} \neq 0$.
 Proof. Let $\{y_n\}_{n\geq 0}$ be a non-oscillatory solution of equation (1). Without loss or

r

The limit $\lim_{n\to\infty}p_n\varphi(\Delta y_n)$ is either positive or zero. In the first case, the sequence $\{y_n\}_{n>0}$ is unbounded, since there are positive constants k_1 and k_2 with $k_1 < k_2$ and an integer n_0 such that $\Phi_{k_1,n_0}(p;n) \leq y_n - y_{n_0} \leq \Phi_{k_2,n_0}(p;n)$ for all $n \geq n_0$. In the second case, since $\{y_n\}_{n>0}$ is increasing, y_n tends to a positive limit, finite or infinite, as $n \to \infty$ $-\infty$ as $n \to \infty$. But this contradicts the
 > 0 for $n \ge n_0$, as claimed. A concequence
 n_0 , that is, the sequence $\{y_n\}_{n\geq 0}$ is strictly

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tive constants k_1 a $p_n\varphi(\Delta y_n) > 0$ for $n \ge n_0$, as claimed. A

0 for $n \ge n_0$, that is, the sequence $\{y_n\}_n$

is either positive or zero. In the first case, i.e.

i.e. are positive constants k_1 and k_2 with
 $n \ge y_n - y_{n_0} \le \Phi_{k_2,n_0$

Theorem 2. *Assume that, for each fixed* $k \neq 0$ *and* $N \in \mathbb{N}_0$,

$$
\lim_{m \to 0 \ (mk > 0)} \frac{\Phi_{m,N}(p;n)}{\Phi_{k,N}(p;n)} = 0
$$
 (6)

uniformly for all $n \geq N_1 > N$ *. Then a necessary and sufficient condition for the equation* (1) *to have a non-oscillatory type* (I) *solution* $\{y_n\}_{n>0}$ *is that*

$$
\sum_{n=0}^{\infty} |f(n, c \Phi(p; n+1))| < \infty \tag{7}
$$

for some constants $k \neq 0$ *and* $c > 0$.

Proof. Necessity: Let $\{y_n\}_{n>0}$ be a non-oscillatory type (I) solution of the equation (1). We may assume that $y_n > 0$ for $n \geq n_0 \in \mathbb{N}_0$ since a similar argument holds if $\{y_n\}_{n\geq 0}$ is eventually negative. There exist positive constants c_1 and k_1 such that for some constants $k \neq 0$ and $c > 0$.
 Proof. Necessity: Let $\{y_n\}_{n\geq 0}$ be a non-oscillatory type (I) solution of the equation (1). We may assume that $y_n > 0$ for $n \geq n_0 \in \mathbb{N}_0$ since a similar argument holds $(n_1)_{n\geq 0}$ be a non-oscillatory type (I) solution of the equation $n > 0$ for $n \geq n_0 \in \mathbb{N}_0$ since a similar argument holds if
ive. There exist positive constants c_1 and k_1 such that
Summation of equation (1) cillatory type (I) solution of the equation
 \in N₀ since a similar argument holds if

positive constants c_1 and k_1 such that

uation (1) yields $\sum_{s=n}^{\infty} f(n, y_{n+1}) < \infty$

s to $\sum_{n=n_0}^{\infty} f(n, c_1 \Phi_{k_1}(p; n+1))$

Sufficiency: Assume that (7) holds for some constants $c > 0$ and $k > 0$. Because of (6) we can choose some $m > 0$ and an integer $N > 0$ such that $m < \frac{k}{2}$ and

$$
\sum_{n=N}^{\infty} f(n, \Phi_{2m}(p; n+1)) \leq m. \tag{8}
$$

Consider the Banach space B_N of all real sequences $y = \{y_n\}_{n \ge N}$ with the supremum norm $||y|| = \sup_{n > N} |y_n|$. We define a set S as

$$
S = \Big\{ y \in B_N : \Phi_{m,N}(p;n) \leq y_n \leq \Phi_{2m,N}(p;n) \quad (n \geq N) \Big\}.
$$

Clearly, S is a bounded, closed and convex subset of B_N . Now, define an operator $T: S \rightarrow B_N$ as

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\nBanach space
$$
B_N
$$
 of all real sequences $y = \{y_n\}_{n \geq N}$ with the supremum $p_{n \geq N} |y_n|$. We define a set S as

\n
$$
\bar{S} = \left\{ y \in B_N : \Phi_{m,N}(p;n) \leq y_n \leq \Phi_{2m,N}(p;n) \quad (n \geq N) \right\}.
$$

\nto bounded, closed and convex subset of B_N . Now, define an operator

\n
$$
Ty_n = \sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{1}{p_s} \left(m + \sum_{t=s}^{\infty} f(t, y_{t+1}) \right) \right) \qquad (n \geq N).
$$

\n(9)

\nothers this operator T is continuous. If $y \in S$, then since

\n
$$
\sum_{n=N}^{\infty} f(n, y_{n+1}) \leq \sum_{n=N}^{\infty} f(n, \Phi_{2m}(p;n+1)) \leq m \qquad (N_1 \geq N)
$$

\n(9)

\n
$$
\sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{m}{p_s} \right) \leq Ty_n \leq \sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{2m}{p_s} \right) \qquad (n \geq N)
$$

\n
$$
TS \subset S
$$
. Therefore, by the Schauder fixed point theorem, T has a fixed

From the hypotheses this operator T is continuous. If
$$
y \in S
$$
, then since
\n
$$
0 \le \sum_{n=N_1}^{\infty} f(n, y_{n+1}) \le \sum_{n=N}^{\infty} f(n, \Phi_{2m}(p; n+1)) \le m \qquad (N_1 \ge N)
$$
\nwe obtain from (9)
\n
$$
\sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{m}{p_s}\right) \le Ty_n \le \sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{2m}{p_s}\right) \qquad (n \ge N)
$$
\nimplying that $TS \subset S$. Therefore, by the Schauder fixed point theorem, T is given by $U \in S$. It is clear that $y = \{y_n\}_{n \ge 0}$ is a positive solution of the equation

we obtain from (9)

$$
\sum_{s=N}^{n-1} \varphi^{-1}\left(\frac{m}{p_s}\right) \leq Ty_n \leq \sum_{s=N}^{n-1} \varphi^{-1}\left(\frac{2m}{p_s}\right) \qquad (n \geq N)
$$

implying that $TS \subset S$. Therefore, by the Schauder fixed point theorem, *T* has a fixed point $y \in S$. It is clear that $y = \{y_n\}_{n>0}$ is a positive solution of the equation (1), and it is obviously of type $(I) \blacksquare$

If (7) holds for some constants $k < 0$ and $c > 0$, then a similar argument can be used to construct a negative type (I) solution of the equation (1).

Theorem **3.** *A necessary and sufficient condition for the equation* (1) *to have a non-oscillatory type* (III) *solution* $\{y_n\}_{n\geq 0}$ *is that*

Therefore, by the Schauder fixed point theorem, T has a fixed
$$
y = \{y_n\}_{n\geq 0}
$$
 is a positive solution of the equation (1), and instants $k < 0$ and $c > 0$, then a similar argument can be e type (I) solution of the equation (1). *ary and sufficient condition for the equation* (1) *to have a lution* $\{y_n\}_{n\geq 0}$ *is that*\n
$$
\sum_{n=0}^{\infty} \left| \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, c) \right) \right| < \infty
$$
\n(10)

for some constant $c \neq o$ *.*

Proof. Necessity: Let $\{y_n\}_{n\geq 0}$ be a positive type (III) solution of the equation (1).
 n there are a positive constant c_1 and an $N_0 \in \mathbb{N}_0$ such that $y_n \geq c_1$ for $n \geq N_0$.
 Pn(Y_n) $= \sum_{s=n}^{\infty} f(s$ Then there are a positive constant c_1 and an $N_0 \in \mathbb{N}_0$ such that $y_n \geq c_1$ for $n \geq N_0$. Summing up the equation (1) from *n* to ∞ , we get

$$
p_n\varphi(\Delta y_n)=\sum_{s=n}^{\infty}f(s,y_{s+1})\qquad(n\geq N_0)
$$

which implies

$$
p_n \varphi(\Delta y_n) = \sum_{s=n}^{\infty} f(s, y_{s+1}) \qquad (n \ge N_0)
$$

$$
\Delta y_n = \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, y_{s+1}) \right) \qquad (n \ge N_0).
$$

Summing up this equation again and using $y_n \geq c_1$, we find that

$$
\Delta y_n = \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, c_1) \right) < \infty.
$$

is eventually negative is similar.
that (10) holds for some constant c

$$
e \le N \in \mathbb{N} \text{ sufficiently large so that}
$$

$$
\sum_{s=1}^{\infty} \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=1}^{\infty} f(s, c) \right) \le \frac{c}{2}.
$$

The proof when $\{y_n\}_{n\geq 0}$ is eventually negative is similar.

Sufficiency: Assume that (10) holds for some constant $c > 0$ (a similar argument will hold if $c < 0$). Choose $N \in \mathbb{N}$ sufficiently large so that

eventually negative is similar.
at (10) holds for some constant

$$
N \in \mathbb{N}
$$
 sufficiently large so that

$$
\sum_{n=N}^{\infty} \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=N}^{\infty} f(s, c) \right) \leq \frac{c}{2}.
$$

Let B_N be the same Banach space as in the proof of Theorem 2, and let

$$
S=\Big\{y\in B_N:\,\frac{c}{2}\leq y_n\leq c\ \ (n\geq N)\Big\}.
$$

Clearly, S is a bounded, closed and convex subset of B_N . Now, we define an operatorT: $S \rightarrow B_N$ as

$$
\sum_{n=N}^{\infty} \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=N}^{\infty} f(s, c) \right) \le \frac{c}{2}.
$$

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$$
S = \left\{ y \in B_N : \frac{c}{2} \le y_n \le c \quad (n \ge N) \right\}.
$$

undefined, closed and convex subset of B_N . Now, we defi:

$$
Ty_n = c - \sum_{s=N}^{\infty} \varphi^{-1} \left(\frac{1}{p_s} \sum_{t=s}^{\infty} f(t, y_{t+1}) \right) \qquad (n \ge N).
$$

so continuous, and it is easy to see that $TS \subset S$. Hence

This operator *T* is continuous, and it is easy to see that $TS \subset S$. Hence, by the Schauder fixed point theorem, *T* has a fixed point $y \in S$. This is the desired type (III) solution of the equation (1)

Next we give sufficient conditions for the existence of type (II) solutions of the equation (1).

Theorem 4. Suppose the condition (6) holds. Then the equation (1) *has a non***i oscillatory type (II) solution if condition (7) holds for some constants** $k \neq 0$ **and** $c > 0$ **
and** $\sum_{n=0}^{\infty} \left| \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, d) \right) \right| = \infty$ **(11)** *and*

conditions for the existence of type (II) solutions of the
\nthe condition (6) holds. Then the equation (1) has a non-
\nn if condition (7) holds for some constants
$$
k \neq 0
$$
 and $c > 0$
\n
$$
\sum_{n=0}^{\infty} \left| \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, d) \right) \right| = \infty
$$
\n(11)
\n(d such that $kd > 0$.

for every non-zero constant d *such that* $kd > 0$ *.*

Proof. It is enough to consider the case where $k > 0$ and $d > 0$. Let $a > 0$ be an arbitrary fixed constant, and choose $m > 0$ small enough and $N \in \mathbb{N}_0$ large enough so *for every non-zero constant a such that* $kd > 0$.
Proof. It is enough to consider the case where $m > 0$ smale that $a + \Phi_m(p; n) \leq c \Phi_k(p; n)$ for all $n \geq N$ and $\sum_{n=0}^{\infty}$ | φ^{-1} | $\frac{1}{p}$
 t d such that

consider the

consider the

consider the number

(6) and of the number

$$
\sum_{n=N}^{\infty} f(n, a+\Phi_m(p; n+1)) \leq m.
$$

This is possible because of (6) and of the fact that $\lim_{n\to\infty} \Phi_k(p;n) = \infty$. Let B_N be the same Banach space as in the proofs of the Theorems 2 and 3, and let the set S be defined as

$$
S = \big\{ y \in B_N : a \leq y_n \leq a + \Phi_m(p;n) \mid (n \geq N) \big\}.
$$

Clearly, S is a bounded, closed and convex subset of B_N . Define an operator $T: S$ B_N as

apani and R. Arul
\nbounded, closed and convex subset of
$$
B_N
$$
. Define an operator $T : S \to$
\n
$$
Ty_n = a + \sum_{s=N}^{n-1} \varphi^{-1} \left(\frac{1}{p_s} \sum_{t=s}^{\infty} f(t, y_{t+1}) \right) \qquad (n \ge N). \tag{12}
$$
\nis continuous, and as earlier it is easy to see that $TS \subset S$. Therefore, by
\nd point theorem, T has a fixed point $y \in S$. It is clear that $y = \{y_n\}_n > 0$

This operator *T* is continuous, and as earlier it is easy to see that $TS \subset S$. Therefore, by the Schauder fixed point theorem, *T* has a fixed point $y \in S$. It is clear that $y = \{y_n\}_{n>0}$ is a positive solution of the equation (1). From (12) we see also that

$$
p_n\varphi(\Delta y_n)=\sum_{s=n}^{\infty}f(s,y_{s+1})\to 0\qquad\text{as}\ \ n\to\infty,
$$

and by (11) that

$$
s = N \qquad \cdots \qquad t = s
$$
\n
$$
T \text{ is continuous, and as earlier it is easy to see that } TS \subset S.
$$
\n2. Let $TS \subset S$ be the function of the equation (1). From (12) we see also that

\n
$$
p_n \varphi(\Delta y_n) = \sum_{s=n}^{\infty} f(s, y_{s+1}) \to 0 \qquad \text{as } n \to \infty,
$$
\n3. Let $y_n \geq a + \sum_{n=N}^{\infty} \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, a) \right) \to \infty \qquad \text{as } n \to \infty.$

\n4. Let $y_n \geq a + \sum_{n=N}^{\infty} \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, a) \right) \to \infty \qquad \text{as } n \to \infty.$

\n5. Let $y_n \geq a + \sum_{n=N}^{\infty} \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, a) \right) \to \infty$

It follows therefore that $\{y_n\}_{n>0}$ is a type (II) solution of the equation (1) **I**

Example 1. Consider the equation (2) where $\alpha > 0$ and $\beta > 0$, and where $\{q_n\}_{n>0}$ is a positive real sequence. This is a special case of the equation (1) in which $p_n = 1$, $\varphi(u) = u^{\alpha}$ and $f(n, v) = q_n v^{\beta}$, and we have $\varphi^{-1}(u) = u^{\frac{1}{\alpha}}$ and $\Phi_{k,N}(p; n) = k^{\frac{1}{\alpha}}(n -$

N), so that the conditions (4) and (6) are satisfied for the equation (2).

The possible types of asymptotic behaviour at infinity of non-oscil

of the equation (2) are as follows:

(I) $\lim_{n\to\infty} \frac{y_n}{n} = \text{const} \neq 0$. The possible types of asymptotic behaviour at infinity of non-oscillatory solutions of the equation (2) are as follows:

(1)
$$
\lim_{n\to\infty} \frac{y_n}{n} = \text{const} \neq 0.
$$

(II) $\lim_{n\to\infty} \frac{y_n}{n} = 0$ and $\lim_{n\to\infty} |y_n| = \infty$.

(III) $\lim_{n\to\infty} y_n = \text{const} \neq 0$.

From Theorems 2 and 3 it follows that the equation (2) has a type (I) solution if and only if

c behaviour at infinity of non-oscillatory solutions
\n
$$
\log |y_n| = \infty.
$$
\n
$$
\log |y_n| \leq \infty,
$$
\n
$$
\sum_{n=0}^{\infty} n^{\beta} q_n < \infty,
$$
\n(13)

and that the equation (2) has a type (III) solution if and only if

0000 **(1: qs)** <00. (14) n=O **E (E** *qs)* (15)

Theorem 4 implies that the conditions (13) and

$$
\sum_{n=0}^{\infty} \left(\sum_{s=n}^{\infty} q_s \right)^{\frac{1}{\alpha}} = \infty \tag{15}
$$

are sufficient for the existence of type (II) solutons of the equation (2).

The conditions (13) and (15) are not always consistent. In fact, let $q_n = (n+1)^{\lambda}$ for some constant λ . Then, condition (13) holds if and only if $\lambda < -1 - \beta$, and condition (15) holds if and only if $\lambda \ge -1 - \alpha$. Hence these two conditions are inconsistent if $\alpha \leq \beta$. Thus, if $\alpha > \beta$ so that $-1 - \alpha < -1 - \beta$, then there exists a type (II) solution for the equation $\Delta((\Delta y_n)^{\alpha_*}) + (n+1)^{\lambda} y_{n+1}^{\beta_*} = 0.$

3. Oscillation of all solutions

In this section we study the oscillatory behaviour of solutions of the equation (1). In view of the results of Hooker and Patula [5] and those of Kulenovic and Budincevic [7], it is reasonable to expect that a characterization of oscillation for the equation (1) can be obtained under suitable additional conditions on the nonlinear functions f and φ . of Hooker and Patula [5] and those
spect that a characterization of countable additional conditions on the
equation (1) is said to be
ritinear if there is a constant $\gamma > 0$
each fixed n and
 $\int_{0}^{\infty} \frac{ds}{(s-1)(s\gamma)} < \infty$ **f all solutior**
 dy the oscillator
 f Hooker and Patu

pect that a chara
 f the equation (1) is
 flinear if there is a

each fixed *n* and
 $\int_{0}^{\infty} \frac{ds}{\varphi^{-1}(s^{\gamma})} < \infty$ of the equation (1). In
ovic and Budincevic [7],
for the equation (1) can
ear functions f and φ .
the function $|\cdot|^{-\gamma} f(n, \cdot)$
 $\lt \infty$ (16) of Hooker and Patula [5] and tho
xpect that a characterization of
suitable additional conditions on
The equation (1) is said to be
verlinear if there is a constant $\gamma >$
or each fixed *n* and
 $\int_{M}^{\infty} \frac{ds}{\varphi^{-1}(s^{\gamma})$

Definition 5. The equation (1) is said to be

 $\bf (i)$ *strongly superlinear* if there is a constant $\gamma > 0$ such that the function is non-decreasing for each fixed *n* and

The equation (1) is said to be
\n*perlinear* if there is a constant
$$
\gamma > 0
$$
 such that the function $|\cdot|^{-\gamma} f(n, \cdot)$
\nor each fixed *n* and
\n
$$
\int_{M}^{\infty} \frac{ds}{\varphi^{-1}(s^{\gamma})} < \infty \quad \text{and} \quad \int_{-M}^{-\infty} \frac{ds}{\varphi^{-1}(s^{\gamma_{\bullet}})} < \infty \quad (16)
$$
\n*blinear* if there is a constant $\delta > 0$ such that the function $|\cdot|^{-\delta} f(n, \cdot)$
\nor each fixed *n* and
\n
$$
\int_{N}^{N} \frac{ds}{(\varphi^{-1}(s))^{\gamma}} < \infty \quad \text{and} \quad \int_{N}^{0} \frac{ds}{(\varphi^{-1}(s))^{\delta_{\bullet}}} < \infty \quad (17)
$$

for any $M > 0$;

(ii) *strongly sublinear* if there is a constant $\delta > 0$ such that the function is non-increasing for each fixed *n* and

1 5. The equation (1) is said to be
\ny superlinear if there is a constant
$$
\gamma > 0
$$
 such that the function $|\cdot|^{-\gamma} f(n, \cdot)$
\nng for each fixed *n* and
\n
$$
\int_{M}^{\infty} \frac{ds}{\varphi^{-1}(s^{\gamma})} < \infty \quad \text{and} \quad \int_{-M}^{-\infty} \frac{ds}{\varphi^{-1}(s^{\gamma})} < \infty \quad (16)
$$
\n
$$
\vdots
$$
\ny sublinear if there is a constant $\delta > 0$ such that the function $|\cdot|^{-\delta} f(n, \cdot)$
\nng for each fixed *n* and
\n
$$
\int_{0}^{N} \frac{ds}{(\varphi^{-1}(s))^{\gamma}} < \infty \quad \text{and} \quad \int_{-N}^{0} \frac{ds}{(\varphi^{-1}(s))^{\delta_{\bullet}}} < \infty \quad (17)
$$
\nto the above definition, the equation (2) is strongly superlinear if $\alpha < \beta$
\nublinear if $\alpha > \beta$.
\n6. Assume the equation (1) is strongly superlinear. Assume further that
\n
$$
\varphi^{-1}(uv) \geq \varphi^{-1}(u)\varphi^{-1}(v) \quad \text{for all } u \text{ and } v \text{ with } uv > 0. \quad (18)
$$
\nonso of the equation (1) are oscillatory if and only if

for any $N > 0$.

According to the above definition, the equation (2) is strongly superlinear if $\alpha < \beta$ and strongly sublinear if $\alpha > \beta$.

Theorem 6. *Assume the equation* (1) *is strongly superlinear. Assume further that*

$$
\varphi^{-1}(uv) \geq \varphi^{-1}(u)\varphi^{-1}(v) \qquad \text{for all } u \text{ and } v \text{ with } uv > 0. \tag{18}
$$

Theorem 6. Assume the equation (1) is strongly superlinear. Assume further that
\n
$$
\varphi^{-1}(uv) \ge \varphi^{-1}(u)\varphi^{-1}(v) \qquad \text{for all } u \text{ and } v \text{ with } uv > 0. \tag{18}
$$
\nThen all solutions of the equation (1) are oscillatory if and only if\n
$$
\sum_{n=0}^{\infty} \left| \varphi^{-1}\left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s,c)\right) \right| = \infty \tag{19}
$$

for every non-zero constant c.

Proof. The necessity part follows from Theorem 3. To prove the sufficiency, suppose that the equation (1) has a non-oscillatory solution $\{y_n\}_{n\geq 0}$, say $y_n > 0$
for $n \geq n_0 \in \mathbb{N}_0$. Summing up the equation (1) from *n* to ∞ and noting that
 $\lim_{n\to\infty} p_n \varphi(\Delta y_n) \geq 0$, we have
 $p_n \varphi(\Delta$ for $n \ge n_0 \in \mathbb{N}_0$. Summing up the equation (1) from *n* to ∞ and noting that $\lim_{n\to\infty}p_n\varphi(\Delta y_n)\geq 0$, we have

$$
p_n\varphi(\Delta y_n)\geq \sum_{s=n}^{\infty}f(s,y_{s+1})\qquad (n\geq n_0)
$$

which implies

equation (1) has a non-oscillatory solution
$$
\{y\}
$$

\nSumming up the equation (1) from n to 0, we have
\n
$$
p_n\varphi(\Delta y_n) \geq \sum_{s=n}^{\infty} f(s, y_{s+1}) \qquad (n \geq n_0)
$$

\n
$$
\Delta y_n \geq \varphi^{-1}\left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, y_{s+1})\right) \qquad (n \geq n_0).
$$

\ninequality by $\varphi^{-1}(y_{n+1}^{\gamma}),$ where $\gamma > 0$ is the

which implies
 $\Delta y_n \ge \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty}f(s, y_{s+1})\right)$ $(n \ge n_0)$.

Now divide the last inequality by $\varphi^{-1}(y_{n+1}^{\gamma})$, where $\gamma > 0$ is the constant of strong

superlinearity of the equation (1), and use (18) to o superlinearity of the equation (1), and use (18) to obtain

$$
\overline{s=n}
$$
\nwhich implies

\n
$$
\Delta y_n \ge \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} f(s, y_{s+1}) \right) \qquad (n \ge n_0).
$$
\nwe divide the last inequality by $\varphi^{-1}(y_{n+1}^{\gamma})$, where $\gamma > 0$ is the constant of strong
perlinearity of the equation (1), and use (18) to obtain

\n
$$
\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \ge \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} \frac{f(s, y_{s+1})}{y_{n+1}^{\gamma}} \right) \ge \varphi^{-1} \left(\frac{1}{p_n} \sum_{s=n}^{\infty} \frac{f(s, y_{s+1})}{y_{s+1}^{\gamma}} \right) \qquad (n \ge n_0).
$$
\nsince $y_n \ge c_0$ $(n \ge n_0)$ for some constant $c_0 > 0$ we have, in view of the strong
transitive of the equation (1) $\sqrt{7}$ $f(n, y) \ge \pi^{-\gamma} f(n, y)$ $(n \ge n)$ so that

Since $y_n \geq c_0$ ($n \geq n_0$) for some constant $c_0 > 0$ we have, in view of the strong superlinearity of the equation (1), $y_{n+1}^{\gamma} f(n, y_{n+1}) \ge c_0^{-\gamma} f(n, c_0)$ ($n \ge c_0$) so that

$$
\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \geq \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty}\frac{f(s,y_{s+1})}{y_{n+1}^{\gamma}}\right) \geq \varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty}\frac{f(s,y_{s+1})}{y_{s+1}^{\gamma}}\right) \qquad (n \geq n_0).
$$

Since $y_n \geq c_0$ $(n \geq n_0)$ for some constant $c_0 > 0$ we have, in view of the strong superlinearity of the equation (1), $y_{n+1}^{\gamma}f(n,y_{n+1}) \geq c_0^{-\gamma}f(n,c_0)$ $(n \geq c_0)$ so that

$$
\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \geq \varphi^{-1}\left(c_0^{-\gamma}\frac{1}{p_n}\sum_{s=n}^{\infty}f(s,c_0)\right) \geq \varphi^{-1}(c_0^{-\gamma})\varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty}f(s,c_0)\right) \quad (n \geq n_0).
$$

Summing up the last inequality from n_0 to $n-1$, we obtain

Summing up the last inequality from n_0 to $n-1$, we obtain

للمرا

$$
-1\left(c_0^{-\gamma}\frac{1}{p_n}\sum_{s=n}^{\infty}f(s,c_0)\right) \geq \varphi^{-1}(c_0^{-\gamma})\varphi^{-1}\left(\frac{1}{p_n}\sum_{s=n}^{\infty}f(s,c_0)\right) \quad (n \geq n_0).
$$

\ne last inequality from n_0 to $n-1$, we obtain
\n
$$
\varphi^{-1}(c_0^{-\gamma})\sum_{s=n_0}^{n-1}\varphi^{-1}\left(\frac{1}{p_n}\sum_{t=s}^{\infty}f(t,c_0)\right) \leq \sum_{s=n_0}^{n-1}\frac{\Delta y_s}{\varphi^{-1}(y_{s+1}^{\gamma})}. \qquad (20)
$$

\n
$$
\frac{1}{\varphi^{-1}(x^{\gamma})} \geq \frac{1}{\varphi^{-1}(y_{n+1}^{\gamma})} \qquad \text{for} \quad y_n \leq x \leq y_{n+1},
$$

Since

$$
\frac{1}{\varphi^{-1}(x^{\gamma})} \ge \frac{1}{\varphi^{-1}(y_{n+1}^{\gamma})} \quad \text{for } y_n \le x \le y_{n+1},
$$

$$
\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \le \int^{y_{n+1}} \frac{dx}{\varphi^{-1}(x^{\gamma})}.
$$

we have

$$
\geq \frac{1}{\varphi^{-1}(y_{n+1}^{\gamma})} \quad \text{for } y_n \leq x \leq y_{n+1},
$$
\n
$$
\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \leq \int_{y_n}^{y_{n+1}} \frac{dx}{\varphi^{-1}(x^{\gamma})}.
$$
\nwe obtain\n
$$
(21)
$$

Substituting (21) into (20), we obtain

$$
\frac{1}{\varphi^{-1}(x^{\gamma})} \ge \frac{1}{\varphi^{-1}(y_{n+1}^{\gamma})} \quad \text{for } y_n \le x \le y_{n+1},
$$
\n
$$
\frac{\Delta y_n}{\varphi^{-1}(y_{n+1}^{\gamma})} \le \int_{y_n}^{y_{n+1}} \frac{dx}{\varphi^{-1}(x^{\gamma})}.
$$
\n21) into (20), we obtain\n
$$
\varphi^{-1}(c_0^{-\gamma}) \sum_{s=n_0}^{n-1} \varphi^{-1}\left(\frac{1}{p_n} \sum_{t=s}^{\infty} f(t, c_0)\right) \le \int_{y_{n_0}}^{y_n} \frac{dx}{\varphi^{-1}(x^{\gamma})} < \infty,
$$

which is a contradiction **I**

Theorem 7. *Let the equation (1) be strongly sublinear and suppose the conditions (6) and (7) hold. Then all solutions of the equation (1) are oscillatory if and only if*

\n Oscillation and Non-Oscillation Theorems
\n *z* to 757
\n *uation (1) be strongly sublinear and suppose the conditions*
\n *olutions of the equation (1) are oscillatory if and only if*
\n
$$
\sum_{n=0}^{\infty} |f(n, c\Phi_k(p; n+1))| = \infty
$$

\n *can be a specific result.*
\n (22)\n

for every non-zero constant c.

Proof. First, note that the necessity of the condition (22) follows from Theorem 2. Next, let $\{y_n\}_{n\geq 0}$ be a non-oscillatory solution of the equation (1), say $y_n > 0$ for $n \geq n_0 \in \mathbb{N}_0$. First note that $\Delta y_n > 0$ for $n \geq n_0$, and mstant c.

te that the necessity of the condition

be a non-oscillatory solution of the obte that $\Delta y_n > 0$ for $n \ge n_0$, and
 $y_n \ge \varphi^{-1}\left(\frac{1}{p_n}\right) \varphi^{-1}(p_n \varphi(\Delta y_n))$ *y*_n, hote that the necessity of the condition (22) follows from Theorem y_n , y_n be a non-oscillatory solution of the equation (1), say $y_n > 0$ for y_n is that $\Delta y_n > 0$ for $n \ge n_0$, and $\Delta y_n \ge \varphi^{-1}\left(\frac{1}{p_n}\right)\varphi^{-$

$$
\Delta y_n \geq \varphi^{-1}\left(\frac{1}{p_n}\right)\varphi^{-1}\big(p_n\,\varphi(\Delta y_n)\big) \qquad (n\geq n_0).
$$

Summing up the last inequality from n_0 to $n-1$ and using the decreasing nature of the sequence $\{p_n \varphi(\Delta y_n)\}_{n\geq 0}$, we get *y_n* $\geq \varphi^{-1}\left(\frac{1}{p_n}\right)\varphi^{-1}(p_n\varphi(\Delta y_n))$ *(n* $\geq n_0$).
 *i*e last inequality from n_0 to $n-1$ and using the decreasing nature of the $(\Delta y_n)\}_{{n \geq 0}}$, we get
 y_n $\geq y_n - y_{n_0} \geq \varphi^{-1}(p_n\varphi(\Delta y_n))\Phi_{1,n_0}(p;n)$

$$
y_n \ge y_n - y_{n_0} \ge \varphi^{-1}(p_n \varphi(\Delta y_n)) \Phi_{1,n_0}(p;n) \qquad (n \ge n_0).
$$
 (23)

sequence $\{p_n \varphi(\Delta y_n)\}_{n\geq 0}$, we get
 $y_n \geq y_n - y_{n_0} \geq \varphi^{-1}(p_n \varphi(\Delta y_n))\Phi_{1,n_0}(p)$

From the strong sublinearity and the inequality $y_n \leq c_0 > 0$ is constant, it follows that $c_0\Phi_{k,n_0}(p;n)$ $(n \geq n_0)$, where $c_0 > 0$ is constant, it follows that

$$
y_{n+1}^{-\delta} f(n, y_{n+1}) \ge c_0^{-\delta} (\Phi_{k,n_0}(p; n))^{-\delta} f(n, \Phi_{k,n_0}(p; n+1))
$$
 (24)

for $n \geq n_0$. From (23) and (24), and using the inequality

$$
r_{n+1} \geq c_0^{-\delta} (\Phi_{k,n_0}(p;n))^{-\delta} f(n, \Phi_{k,n_0}(p;n))
$$

and (24), and using the inequality

$$
\frac{\Phi_{1,n_0}(p;n)}{\Phi_{k,n_0}(p;n)} \geq \varphi^{-1} \left(\frac{1}{k}\right) \qquad (n > n_0)
$$

which follows from (18), we have

$$
y_{n+1}^{-\delta} f(n, y_{n+1}) \ge c_0^{-\delta} (\Phi_{k,n_0}(p; n))^{-\delta} f(n, \Phi_{k,n_0}(p; n+1))
$$
(24)
com (23) and (24), and using the inequality

$$
\frac{\Phi_{1,n_0}(p; n)}{\Phi_{k,n_0}(p; n)} \ge \varphi^{-1} \left(\frac{1}{k}\right) \qquad (n > n_0)
$$

from (18), we have

$$
\frac{f(n, y_{n+1})}{(\varphi^{-1}(p_n \varphi(\Delta y_n)))^{\delta}} = \frac{y_{n+1}^{-\delta} f(n, y_{n+1}) y_{n+1}^{\delta}}{(\varphi^{-1}(p_n \varphi(\Delta y_n)))^{\delta}}
$$
(25)

$$
\ge c_0^{-\delta} \varphi^{-1} \left(\frac{1}{k}\right) f(n, c_0 \Phi_{k,n_0}(p; n+1))
$$

com the equation (1) and (25) we obtain

$$
z^{-1} \left(\frac{1}{k}\right) \sum_{s=n_0}^{n-1} f(n, c_0 \Phi_{k,n_0}(p; s+1)) \le \sum_{s=n_0}^{n-1} \frac{-\Delta(p_s \varphi(\Delta y_s))}{(\varphi^{-1}(p_s \varphi(\Delta y_s)))^{\delta}}
$$
(26)
nce

$$
\frac{1}{(\varphi^{-1}(x))^{\delta}} \ge \frac{1}{(\varphi^{-1}(p_n \varphi(\Delta y_n)))^{\delta}}
$$

for $n \geq n_0$. From the equation (1) and (25) we obtain

$$
\geq c_0^{-\theta} \varphi^{-1}\left(\frac{1}{k}\right) f\left(n, c_0 \Phi_{k,n_0}(p; n+1)\right)
$$

0. From the equation (1) and (25) we obtain

$$
c_0^{-\delta} \varphi^{-1}\left(\frac{1}{k}\right) \sum_{s=n_0}^{n-1} f\left(n, c_0 \Phi_{k,n_0}(p; s+1)\right) \leq \sum_{s=n_0}^{n-1} \frac{-\Delta(p_s \varphi(\Delta y_s))}{\left(\varphi^{-1}(p_s \varphi(\Delta y_s))\right)^{\delta}}
$$
(26)
0. Since

$$
\frac{1}{\left(\varphi^{-1}(x)\right)^{\delta}} \geq \frac{1}{\left(\varphi^{-1}(p_n \varphi(\Delta y_n))\right)^{\delta}}
$$

for $n \geq n_0$. Since

$$
f(n, c_0\Phi_{k,n_0}(p; s+1)) \leq \sum_{s=n_0} \frac{1}{(\varphi)}
$$

$$
\frac{1}{(\varphi^{-1}(x))^\delta} \geq \frac{1}{(\varphi^{-1}(p_n\,\varphi(\Delta y_n)))^\delta}
$$

for $p_n \varphi(\Delta y_n) \ge x \ge p_{n+1} \varphi(\Delta y_{n+1}),$ we have

$$
i \text{ and } R. \text{ Arul}
$$
\n
$$
p_{n+1}\varphi(\Delta y_{n+1}), \text{ we have}
$$
\n
$$
-\frac{p_n\varphi(y_n)}{(\varphi^{-1}(p_n\varphi(y_n)))^\delta} < \int_{p_{n+1}\varphi(y_{n+1})}^{p_n\varphi(y_n)} \frac{dx}{(\varphi^{-1}(x))^\delta}.
$$
\n
$$
= o(26), \text{ we obtain}
$$
\n
$$
(27)
$$

Substituting (27) into (26), we obtain

E. Thandapani and R. Arul
\n
$$
\Delta y_n) \geq x \geq p_{n+1} \varphi(\Delta y_{n+1}), \text{ we have}
$$
\n
$$
-\frac{p_n \varphi(y_n)}{(\varphi^{-1}(p_n \varphi(y_n)))^{\delta}} < \int_{p_{n+1} \varphi(y_{n+1})}^{p_n \varphi(y_n)} \frac{dx}{(\varphi^{-1}(x))^{\delta}}.
$$
\n
$$
\text{ting (27) into (26), we obtain}
$$
\n
$$
c_0^{-\delta} \varphi^{-1} \left(\frac{1}{k}\right) \sum_{s=n_0}^{n-1} f(n, c_0 \Phi_{k,n_0}(p; s+1)) < \int_{p_n \varphi(y_n)}^{p_{n_0} \varphi(y_{n_0})} \frac{dx}{(\varphi^{-1}(x))^{\delta}} < \infty,
$$
\na contradiction **Im**
\n**mple 2.** Consider the equation (2) again. Since\n
$$
\varphi^{-1}(u) = u^{\frac{1}{\alpha*}} \quad \text{and} \quad f(n, v) = q_n v^{\beta*},
$$
\n
$$
\text{if } \varphi(\beta) \text{ is the complex variables, } \beta \text{ is the complex variable, } \beta \text{ is the complex variable.}
$$

which is a contradiction \blacksquare

Example 2. Consider the equation (2) again. Since

$$
\varphi^{-1}(u) = u^{\frac{1}{\alpha_{\bullet}}} \quad \text{and} \quad f(n,v) = q_n v^{\beta_{\bullet}},
$$

the equation (2) is strongly superlinear or strongly sublinear according as $\alpha < \beta$ or $\alpha > \beta$. Therefore, from Theorems 6 and 7, it follows that a necessary and sufficient condition for the oscillation of all solutions of the equation (2) is

is a contradiction
\nis a contradiction
\n**Example 2.** Consider the equation (2) again. Since
\n
$$
\varphi^{-1}(u) = u^{\frac{1}{\alpha_{\bullet}}} \quad \text{and} \quad f(n, v) = q_n v^{\beta_{\bullet}},
$$
\n
$$
\varphi^{-1}(u) = u^{\frac{1}{\alpha_{\bullet}}} \quad \text{and} \quad f(n, v) = q_n v^{\beta_{\bullet}},
$$
\n
$$
\varphi^{-1}(u) = u^{\frac{1}{\alpha_{\bullet}}} \quad \text{and} \quad f(n, v) = q_n v^{\beta_{\bullet}},
$$
\n
$$
\varphi^{-1}(u) = u^{\frac{1}{\alpha_{\bullet}}} \quad \text{and} \quad f(n, v) = u^{\beta_{\bullet}} \quad \text{and} \quad f
$$

Remark. Our results reduce to some of the results of He [4] when $\varphi(u) = u$. Also, our results generalize some of the results obtained in [19] in the sense that we do not require the condition $p_n \to \infty$ as $n \to \infty$.

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