On the Local Property of Factored Fourier Series

H. Bor

Abstract. In this paper two theorems on $|\bar{N}, p_n; \delta|_k$ -summability methods have been proved. These theorems include some known results.

Keywords: *Sumrnability factors, Fourier series, local property* **AMS subject classification: 40 D** 15, 42 A 24, 42 A 28

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums s_n . Let (p_n) be a sequence of positive numbers such that

This paper two theorems on
$$
|\{N, p_n, v\}_*
$$
-summability factors, Fourier series, local property
\nclassification: 40 D 15, 42 A 24, 42 A 28
\n\naction

\na given infinite series with partial sums s_n . Let (p_n) be a sequence of
\ners such that

\n
$$
P_n = \sum_{v=0}^n p_v \to \infty \text{ as } n \to \infty \quad (P_{-i} = p_{-i} = 0 \ (i \ge 1)). \qquad (1)
$$
\nto-sequence transformation

\n
$$
t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \qquad (2)
$$
\nquence (t_n) of (\bar{N}, p_n) -means of the sequence (s_n) generated by the se-

The sequence-to-sequence transformation

$$
t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v
$$
 (2)

defines the sequence (t_n) of (\bar{N}, p_n) -means of the sequence (s_n) generated by the sequence of coefficients (p_n) . The series $\sum a_n$ is said to be $|\bar{N}, p_n|_k$ -summable $(k \ge 1)$ if $($ see $[2]$ $)$

$$
\Rightarrow \infty \text{ as } n \to \infty \qquad (P_{-i} = p_{-i} = 0 \ (i \ge 1)). \tag{1}
$$
\n
$$
\text{transformation}
$$
\n
$$
t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \qquad (2)
$$
\n
$$
\text{of } (\bar{N}, p_n) \text{-means of the sequence } (s_n) \text{ generated by the se-}
$$
\n
$$
\text{The series } \sum a_n \text{ is said to be } |\bar{N}, p_n|_k \text{-summable } (k \ge 1) \text{ if}
$$
\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty, \tag{3}
$$
\n
$$
\delta|_k \text{-summable } (k \ge 1 \text{ and } \delta \ge 0) \text{ if (see [3])}
$$
\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}
$$

and it is said to be $|\bar{N}, p_n; \delta|_k$ -summable $(k \ge 1$ and $\delta \ge 0)$ if (see [3])

$$
n=1 \quad (1 \quad n)
$$
\n
$$
;\delta|_{k-summable} \ (k \ge 1 \text{ and } \delta \ge 0) \text{ if (see [3])}
$$
\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}
$$

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In the special case $\delta = 0$, $|\bar{N}, p_n; \delta|_k$ -summability is the same as $|\bar{N}, p_n|_k$ -summability, and in the case $\delta = 0$ and $\hat{k} = 1$, $|\bar{N}, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n|$. summability. $|\bar{N}, p_n; \delta|_k$ -summability is the same

and $k = 1, |\bar{N}, p_n; \delta|_k$ -summability
 $\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}$

rated by a function f with period 2π $\tilde{a}_0 = 0$, $|\bar{N}, p_n; \delta|_k$ -summability is the same as $|\bar{N}, p_n|_k$ -summability,
 $= 0$ and $k = 1$, $|\bar{N}, p_n; \delta|_k$ -summability is the same as $|\bar{N}, p_n|$ -
 $a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t)$ (5)

Let

$$
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(t)
$$
 (5)

be the Fourier series generated by a function f with period 2π which is Lebesgue integrable over $(-\pi, \pi)$. It is familiar that the convergence of the Fourier series at $t = x$ is a local property of *f* (i.e. it depends only on the behaviour of *f* in an arbitrarily small neighbourhood of *x*), and hence the summability of the Fourier series at $t = x$ by any neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Borwein [4] has proved the following theorems for $|N, p_n|$ -summability methods. $\begin{aligned} &f_{n}\sin nt)=\frac{1}{2}a_{0}+\sum_{n=1}^{n}\sin ft\text{ with period }2\pi\ \text{e convergence of the}\ &\text{on the behaviour of}\ &\text{and }x\sin nt\ \text{a local property of }f\text{ by the Fourier transform }\ &\text{the boundary of }\sin nt\ \text{bounded}\ &\text{and }\left(\lambda_{n}\right)\text{ is}\ &\text{the following }\left|\lambda_{n}\right|<\infty\ &\text{the following }\left|\lambda_{n}\right|<\infty\ \end{aligned}$

Theorem A. If the sequence (s_n) is bounded and (λ_n) is a sequence such that

$$
\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\lambda_n| < \infty \tag{6}
$$

$$
\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty \tag{7}
$$

where $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$, then the series $\sum_{n=1}^{\infty} a_n \lambda_n$ is $|\bar{N}, p_n|$ -summable.

 $\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\lambda_n| < \infty$ (6)
 $\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty$ (7)
 Theorem B. The IN, p_n l-summability of the series $\sum_{n=1}^{\infty} a_n \lambda_n$ is $|\bar{N}, p_n|$ -summable.
 Theorem B. The $|\bar{N}, p_n|$ -summability of the series *local property of the generating function if the conditions (6) and (7) are satisfied.*

It may be remarked that the above theorems have been proved by Lal [5) previously.

2. The results

The aim of this paper is to generalize the above theorems for $|\bar{N}, p_n; \delta|_k$ -summability methods. We shall prove the following theorems.

Theorem 1. Let $k \geq 1$ and $0 \leq \delta k < 1$. If the sequence (s_n) is bounded and (λ_n) *is a sequence such that*

-summability of the series
$$
\sum_{n=1}^{\infty} A_n(t)\lambda_n
$$
 at a point is a
\nfunction if the conditions (6) and (7) are satisfied.
\nwe above theorems have been proved by Lal [5] previously.
\n**generalize the above theorems for** $|\bar{N}, p_n; \delta|_k$ -summability
\nallowing theorems.
\nand $0 \le \delta k < 1$. If the sequence (s_n) is bounded and (λ_n)
\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |\lambda_n|^k < \infty
$$
\n(8)
\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta \lambda_n| < \infty
$$

$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta \lambda_n| < \infty \tag{9}
$$

and

Intends. We shall prove the following theorems.

\nTheorem 1. Let
$$
k \geq 1
$$
 and $0 \leq \delta k < 1$. If the sequence (s_n) is bounded and (λ_n) is a sequence such that

\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |\lambda_n|^k < \infty \tag{8}
$$
\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta \lambda_n| < \infty \tag{9}
$$
\nand

\n
$$
\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right),
$$
\nthen the series

\n
$$
\sum_{n=1}^{\infty} a_n \lambda_n
$$
 is $|\bar{N}, p_n; \delta|_k$ -summable.\nThen the series

\n
$$
\sum_{n=1}^{\infty} a_n \lambda_n
$$
 is $|\bar{N}, p_n; \delta|_k$ -summable.

 i s $|\bar{N}, p_n; \delta|_k$ -summable.

On the Local Property of Factored Fourier Series 771
Theorem 2. Let $k \ge 1$ and $0 \le \delta k < 1$. The $|\bar{N}, p_n; \delta|_k$ -summability of the series On the Local Property of Factored Fourier Series 771
 Theorem 2. Let $k \ge 1$ and $0 \le \delta k < 1$. The $|\bar{N}, p_n; \delta|_k$ -summability of the series
 $A_n(t)\lambda_n$ at a point is a local property of the generating function if the cond $\overline{(8)} - (10)$ are satisfied.

If we take $k = 1$ and $\delta = 0$ in our Theorems 1 and 2, then we get Theorems A and B, respectively. Furthermore, Theorem 2 includes as particular case well-known results due to Bhatt [1], Matsumota [6] and Mohanty [7].

3. Proof of the theorems

Proof of Theorem 1. Let (T_n) be the sequence of the (\bar{N}, p_n) -means of the series $\sum a_n \lambda_n$. Then, by definition,

Furthermore, Theorem 2 includes as particular case

\nMatsumota [6] and Mohanty [7].

\nthe theorems

\nsorem 1. Let
$$
(T_n)
$$
 be the sequence of the (\bar{N}, p_n) - T definition,

\n
$$
T_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v \lambda_v
$$
\nen, for $n \geq 1$, we get that

\n
$$
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v \lambda_v \qquad (P_{-1} = 0).
$$
\nTherefore, the result is a function of T and T is a function of T .

where
$$
\lambda_0 = 0
$$
. Then, for $n \ge 1$, we get that

$$
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v \lambda_v \qquad (P_{-1} = 0).
$$

By Abel's transformation, we have

Then, by definition,
\n
$$
T_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v \lambda_v
$$
\n
$$
= 0.
$$
 Then, for $n \ge 1$, we get that
\n
$$
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v \lambda_v \qquad (P_{-1} = 0).
$$
\n's transformation, we have
\n
$$
T_n - T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v + \frac{1}{s_n \lambda_n p_n}
$$
\n
$$
= T_{n,1} + T_{n,2} + T_{n,3}.
$$
\n(rowski's inequality for $k > 1$, to complete the proof of Theorem 1 it is sufficient that
\nthat
\n
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,r}|^k < \infty \quad \text{for } r = 1, 2, 3.
$$
\n(11)
\nplying Hölder's inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, by (8), (10)
\n $\in O(1)$ we have

By Minkowski's inequality for $k > 1$, to complete the proof of Theorem 1 it is sufficient to show that

$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,r}|^k < \infty \qquad \text{for } r = 1, 2, 3. \tag{11}
$$

Now, applying Hölder's inequality with indices *k* and *k'* where $\frac{1}{k} + \frac{1}{k'} = 1$, by (8), (10) and $s_n = O(1)$ we have plying Hölder's in
 \overline{C} , $O(1)$ we have
 $\sum_{k=1}^{m+1} \left(\frac{P_n}{P_k}\right)^{\delta k+k-1}$

6.1.
$$
P(n, p) = \frac{1}{n-1} \left(\frac{p_n}{p_n} \right)^{\delta k + k - 1}
$$

\n6.1.
$$
\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,r}|^k < \infty \quad \text{for } r = 1, 2, 3.
$$

\n7.
$$
P(n, p) = \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,r}|^k
$$

\n8.1.
$$
P(n, p) = \sum_{n=2}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,1}|^k
$$

\n9.1.
$$
\sum_{n=2}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \left\{ \sum_{n=1}^{\infty} |s_n|^k p_n |\lambda_n|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{n=1}^{n-1} p_n \right\}^{k - 1}
$$

\n10.1.
$$
P(n, p) = O(1) \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{n=1}^{\infty} p_n |\lambda_n|^k
$$

\n21.
$$
P(n, p) = O(1) \sum_{n=1}^{\infty} p_n |\lambda_n|^k \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}}
$$

\n32.
$$
P(n, p) = O(1) \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} |\lambda_n|^k
$$

\n43.
$$
P(n, p) = O(1)
$$

\n5.
$$
P(n, p) = O(1)
$$

\n6.
$$
P(n, p) = O(1)
$$

\n7.
$$
P(n, p) = O(1)
$$

\n8. <math display="</p>

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as $m \to \infty$. Again, by (9), (10) and $s_n = O(1)$ we have

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\nas
$$
m \to \infty
$$
. Again, by (9), (10) and $s_n = O(1)$ we have
\n
$$
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,2}|^k
$$
\n
$$
\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} |s_v|^k P_v |\Delta \lambda_v| \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \right\}^{k - 1}
$$
\n
$$
= O(1) \sum_{n=2}^{m} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \right\}
$$
\n
$$
= O(1) \sum_{v=1}^{m} P_v |\Delta \lambda_v| \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}}
$$
\n
$$
= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{\delta k} |\Delta \lambda_v|
$$
\n
$$
= O(1)
$$
\nas $m \to \infty$. Finally, by (8) and $s_n = O(1)$ we have
\n
$$
\sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,3}|^k = \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} |\lambda_n|^k = O(1)
$$
\nas $m \to \infty$. Therefore, we have

ally, by (8) and
$$
s_n = O(1)
$$
 we have
\n
$$
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,3}|^k = \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |\lambda_n|^k = O(1)
$$

as $m \to \infty$. Therefore, we have

$$
\left(\begin{array}{cc} p_n & \cdots & \cdots & \cdots \\ p_{n-1} & p_n & \cdots & \cdots \end{array}\right)
$$

ore, we have

$$
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,r}|^k = O(1) \qquad (r = 1, 2, 3)
$$

as $m \to \infty$ this completes the proof of Theorem 1

Proof of Theorem 2. Since the convergence of the Fourier series at a point is a local property of its generating function *f,* Theorem 2 follows immediately from Theorem 1 **I** $\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,r}|^k = O(1)$ ($r = 1, 2, 3$
 $\rightarrow \infty$ this completes the proof of Theorem 1
 Proof of Theorem 2. Since the convergence of the Fourier

al property of its generating function f, Theorem 2

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