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On the Local Property of Factored Fourier Series

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Abstract. In this paper two theorems on $|\bar{N}, p_n; \delta|_k$-summability methods have been proved. These theorems include some known results.

Keywords: *Summability factors, Fourier series, local property*

AMS subject classification: 40 D 15, 42 A 24, 42 A 28

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums s_n. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \qquad \big(P_{-i} = p_{-i} = 0 \ (i \geq 1)\big). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of $(\bar{N}, p_n)$-*means* of the sequence (s_n) generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be $|\bar{N}, p_n|_k$-*summable* $(k \geq 1)$ if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty, \tag{3}$$

and it is said to be $|\bar{N}, p_n; \delta|_k$-*summable* $(k \geq 1$ and $\delta \geq 0)$ if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}$$

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In the special case $\delta = 0$, $|\bar{N}, p_n; \delta|_k$-summability is the same as $|\bar{N}, p_n|_k$-summability, and in the case $\delta = 0$ and $k = 1$, $|\bar{N}, p_n; \delta|_k$-summability is the same as $|\bar{N}, p_n|$-summability.

Let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(t) \tag{5}$$

be the Fourier series generated by a function f with period 2π which is Lebesgue integrable over $(-\pi, \pi)$. It is familiar that the convergence of the Fourier series at $t = x$ is a local property of f (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Borwein [4] has proved the following theorems for $|\bar{N}, p_n|$-summability methods.

Theorem A. *If the sequence (s_n) is bounded and (λ_n) is a sequence such that*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\lambda_n| < \infty \tag{6}$$

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty \tag{7}$$

where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, then the series $\sum_{n=1}^{\infty} a_n\lambda_n$ is $|\bar{N}, p_n|$-summable.

Theorem B. *The $|\bar{N}, p_n|$-summability of the series $\sum_{n=1}^{\infty} A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (6) and (7) are satisfied.*

It may be remarked that the above theorems have been proved by Lal [5] previously.

2. The results

The aim of this paper is to generalize the above theorems for $|\bar{N}, p_n; \delta|_k$-summability methods. We shall prove the following theorems.

Theorem 1. *Let $k \geq 1$ and $0 \leq \delta k < 1$. If the sequence (s_n) is bounded and (λ_n) is a sequence such that*

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |\lambda_n|^k < \infty \tag{8}$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta\lambda_n| < \infty \tag{9}$$

and

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right), \tag{10}$$

then the series $\sum_{n=1}^{\infty} a_n\lambda_n$ is $|\bar{N}, p_n; \delta|_k$-summable.

Theorem 2. *Let $k \geq 1$ and $0 \leq \delta k < 1$. The $|\bar{N}, p_n; \delta|_k$-summability of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (8) – (10) are satisfied.*

If we take $k = 1$ and $\delta = 0$ in our Theorems 1 and 2, then we get Theorems A and B, respectively. Furthermore, Theorem 2 includes as particular case well-known results due to Bhatt [1], Matsumota [6] and Mohanty [7].

3. Proof of the theorems

Proof of Theorem 1. Let (T_n) be the sequence of the $(\bar{N}, p_n)$-means of the series $\sum a_n\lambda_n$. Then, by definition,

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v$$

where $\lambda_0 = 0$. Then, for $n \geq 1$, we get that

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v \qquad (P_{-1} = 0).$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v + \frac{1}{s_n\lambda_n p_n} \\ &= T_{n,1} + T_{n,2} + T_{n,3}. \end{aligned}$$

By Minkowski's inequality for $k > 1$, to complete the proof of Theorem 1 it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,r}|^k < \infty \qquad \text{for } r = 1, 2, 3. \tag{11}$$

Now, applying Hölder's inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, by (8), (10) and $s_n = O(1)$ we have

$$\begin{aligned} &\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,1}|^k \\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}|s_v|^k p_v |\lambda_v|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1} \\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k - 1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v|\lambda_v|^k \\ &= O(1)\sum_{v=1}^{m} p_v|\lambda_v|^k \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k - 1}\frac{1}{P_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k - 1}|\lambda_v|^k \\ &= O(1) \end{aligned}$$

as $m \to \infty$. Again, by (9), (10) and $s_n = O(1)$ we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,2}|^k
&\le \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} |s_v|^k P_v |\Delta\lambda_v|\right\} \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v|\right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} P_v |\Delta\lambda_v|\right\} \\
&= O(1) \sum_{v=1}^{m} P_v |\Delta\lambda_v| \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} |\Delta\lambda_v| \\
&= O(1)
\end{aligned}$$

as $m \to \infty$. Finally, by (8) and $s_n = O(1)$ we have

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,3}|^k = \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |\lambda_n|^k = O(1)$$

as $m \to \infty$. Therefore, we have

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,r}|^k = O(1) \qquad (r = 1, 2, 3)$$

as $m \to \infty$ this completes the proof of Theorem 1 ∎

Proof of Theorem 2. Since the convergence of the Fourier series at a point is a local property of its generating function f, Theorem 2 follows immediately from Theorem 1 ∎

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