Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 16 (1997), No. 3, 761–768

Approximate Solutions for Multiple Stochastic Equations with Respect to Semimartingales

C. Tudor and M. Tudor

Abstract. In this paper we investigate different types of approximations for a class of multiple stochastic equations driven by semimartingales. This class includes in particular integro-differential equations and Volterra stochastic equations.

Keywords: *Multiple stochastic integrals, approximation schemes of stochastic equations*

AMS subject classification: Primary 60 H 20, secondary 60 H 30

1. Preliminaries

Multiple stochastic integrals with respect to the Brownian motion were introduced by Itô [3] for deterministic integrals, Meyer [7] for martingale differentials and random integrands, and Ruiz de Chaves [8] for a class of semimartingale differentials. As far as we know there are no results about associated non-anticipating stochastic equations. Of course, such equations will include the classical Itô equations, as well as integro-differential and Volterra equations. In the present paper we consider some approximation schemes associated to this class of stochastic equations.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \le t \le 1})$ be a filtered probability space and let $S^i = M^i + A^i$ $(1 \le i \le n)$ be a family of real semimartingales such that

$$d\langle M^i \rangle_t = m_t^i \, dt, \qquad dA_t^i = a_t^i \, dt, \qquad |m_t^i| \le c, \qquad \int_0^1 (a_t^i)^2 dt \le c. \tag{1.1}$$

Define the sets

$$C_n = \Big\{ \underline{u} = (u_1, ..., u_n) \in [0,1]^n : 0 < u_1 < ... < u_n \le 1 \Big\}$$

and

$$C_n(t) = C_n \cap \{u_n \le t\} \qquad (t \in [0,1]).$$

C. Tudor: Univ. Bucharest, Fac. Math., Str. Academiei 14, 70109 Bucharest, Romania
M. Tudor: Acad. Econ. Stud., Dept. Math., Str. Piata Romana 6, 70167 Bucharest, Romania

ISSN 0232-2064 / $ 2.50

A process $\{H(\underline{u})\}_{\underline{u}\in C_n}$ is said to be *predictable simple* if it has the representation

$$H(\underline{u}) = \sum_{j=1}^{m} h^j \mathbf{1}_{(a_1^j, b_1^j]}(u_1) \cdots \mathbf{1}_{(a_n^j, b_n^j]}(u_n) \qquad (1.2)$$

where $0 < a_1^j < b_1^j < a_2^j < b_2^j < \ldots < b_n^j$, h^j are bounded, $\mathcal{F}_{a_1^j}$ are measurable, and $\mathbf{1}_M$ denotes the characteristic function of a set M. We denote by $\mathcal{P}^n$ the σ-algebra generated by the set $\mathcal{E}^n$ of predictable simple processes.

The next result is due to Meyer [7] in the case $S^1 = \ldots = S^n$ where S^1 is a martingale with the increasing process $\langle S^1 \rangle_t = t$, and to Ruiz de Chaves [8] when S^i are possible different semimartingales which satisfy (1.1).

Theorem 1.1. *The mapping* $J_n : \mathcal{E}^n \to L^2(\Omega, \mathcal{F}, P)$ *defined by*

$$J_n(H) = \sum_{j=1}^{m} h^j (S^1_{b_1^j} - S^1_{a_1^j}) \cdots (S^n_{b_n^j} - S^n_{a_n^j})$$

for H *of the form* (1.2) *has a unique extension* (*denoted also by* J_n)

$$J_n : L^2(\mathcal{P}^n, \lambda_n \otimes P) \to L^2(\Omega, \mathcal{F}, P)$$

(*where* λ_n *is the Lebesgue measure on* C_n) *with the property*

$$E(|J_n(H)|^2) \le 4^n c^n E\left(\int_{C_n} |H(\underline{u})|^2 d\underline{u}\right). \qquad (1.3)$$

We call $J_n(H)$ the *n-multiple stochastic integral* of H with respect to $S^1, \ldots, S^n$ and we denote it by

$$\int_{C_n} H(\underline{u})\, dS^1_{u_1} \cdots dS^n_{u_n}.$$

We also put

$$\int_{C_n(t)} H(\underline{u})\, dS^1_{u_1} \cdots dS^n_{u_n} = \int_{C_n} H(\underline{u}) \mathbf{1}_{\{u_n \le t\}}\, dS^1_{u_1} \cdots dS^n_{u_n}.$$

Remark 1.2. The process $\left\{ \int_{C_n(t)} H(\underline{u})\, dS^1_{u_1} \cdots dS^n_{u_n} \right\}_{t\in[0,1]}$ is a càdlàg semimartingale (resp. martingale if every S^i is a martingale) and we have the inequality

$$E\left(\sup_{0\le t\le 1} \left| \int_{C_n(t)} H(\underline{u})\, dS^1_{u_1} \cdots dS^n_{u_n} \right|^2 \right) \le 5.2^{2n-1} c^n E\left(\int_{C_n} |H(\underline{u})|^2 d\underline{u}\right). \qquad (1.4)$$

Let now

$$Z^j = (S_1^j, \ldots, S_n^j) = (M_1^j + A_1^j, \ldots, M_n^j + A_n^j) \qquad (1 \le j \le m) \qquad (1.5)$$

where $S_1^j, \ldots, S_n^j$ satisfy (1.1). Let

$$H(\underline{u}) = \{H_{ij}(\underline{u})\}_{\substack{1 \le j \le m \\ 1 \le i \le d}} \qquad \text{with} \qquad H_{ij}(\underline{u}) \in L^2(\mathcal{P}^n, \lambda_n \otimes P). \tag{1.6}$$

Then we define

$$\int_{C_n} H(\underline{u})\, dZ_{\underline{u}}^1 \otimes \cdots \otimes dZ_{\underline{u}}^m = \left(\sum_{j=1}^m \int_{C_n} H_{ij}\, dS_1^j \cdots dS_n^j \right)_{1 \le i \le d}$$

and

$$\int_{C_n(t)} H(\underline{u})\, dZ_{\underline{u}}^1 \otimes \cdots \otimes dZ_{\underline{u}}^m$$

in a similar way.

Remark 1.3. From [7: Theorem 49] it follows that multiple stochastic integrals can be viewed as iterated stochastic integrals (altghout in [7] only the case of martingales is treated, the semimartingale case follows similarly).

2. Main results

We consider a multiple stochastic equation of the form

$$X_t = H(t) + \int_{C_n(t)} F(\underline{u}, X_{u_1})\, dZ_{\underline{u}}^1 \otimes \cdots \otimes Z_{\underline{u}}^m \qquad (t \in [0,1]) \tag{2.1}$$

where H is a càdlàg R^d-valued process, Z^j are of the form (1.5) and F is as in (1.6).

Remark 2.1. Obviously, the Itô equations are particular cases of (2.1). Moreover, the characterization of multiple stochastic integrals as iterated stochastic integrals implies that also the stochastic integro-differential equations (and in particular stochastic Volterra equations) studied by Berger and Mizel [2] are covered by (2.1).

The proof of the following existence and uniqueness theorem follows as in the case of classical Itô equations and thus is based on the method of successive approximations (the details are left to the reader).

Theorem 2.2. *Assume the following hypotheses are satisfied:*

(1) *H is continuous and*

$$E\left(\sup_{0 \le t \le 1} |H(t)|^2 \right) < \infty. \tag{2.2}$$

(2) *Z^j are continuous semimartingales which satisfy* (1.5).

(3) *$F(\underline{u}, x)$ is measurable and satisfies the Lipschitz and the growth conditions in x.*

Then there exists a pathwise unique continuous solution $\{X_t\}_{t\in[0,1]}$ *of equation* (2.1) *such that*

$$E\left(\sup_{0\le t\le 1}|X_t|^2\right)<\infty. \tag{2.3}$$

Remark 2.3. The above result covers [2: Theorem 2.E].

Next we shall investigate the convergence of several types of approximations to the solution of equation (2.1). For simplicity we shall assume that $d=m=1$, $n=2$ and $H(t)\equiv c$, that is we consider the double stochastic equation

$$X_t=c+\int_{0<u<v\le t}F(u,v,X_u)\,dS_u^1dS_v^2 \qquad (0\le t\le 1). \tag{2.4}$$

For it we introduce the following approximations (the assumptions needed in order that these approximations to be well defined are given within the statement of Theorems 2.5 and 2.6):

(a) *Carathéodory approximations* $\{X_t^n\}_{-1\le t\le 1}$ defined by

$$X_t^n=\begin{cases} c & \text{if } -1\le t\le 0\\ c+\int_{0<u<v\le t}F(u,v,X_{u-\frac{1}{n}}^n)\,dS_u^1dS_v^2 & \text{if } 0\le t\le 1.\end{cases} \tag{2.5}$$

(b) *Chaplygin approximations* $\{Y_t^n\}_{0\le t\le 1}$ defined by

$$\begin{aligned} Y_t^0&=c\\ Y_t^{n+1}&=c+\int_{0<u<v\le t}F(u,v,Y_u^n)\,dS_u^1dS_v^2\\ &\quad+\int_{0<u<v\le t}\frac{\partial F}{\partial x}(u,v,Y_u^n)(Y_u^{n+1}-Y_u^n)\,dS_u^1dS_v^2. \end{aligned} \tag{2.6}$$

(c) Let $\mathcal{B}$ be the Banach space of all continuous adapted processes $\{\varphi(t)\}_{0\le t\le 1}$ with respect to the norm

$$\|\varphi\|_{\mathcal{B}}^2=E\left(\sup_{0\le t\le 1}|\varphi(t)|^2\right).$$

Consider the operator $\Phi:\mathcal{B}\to\mathcal{B}$ defined by

$$\Phi(h)_t=h(t)-h(0)-\int_{0<u<v\le t}F(u,v,h(u))\,dS_u^1dS_v^2. \tag{2.7}$$

The *Newton approximations* $\{Z_t^n\}_{0\le t\le 1}$ associated to (2.4) (or to the operator Φ) are defined by

$$\left.\begin{aligned} Z_t^0&=c\\ Z_t^{n+1}&=Z_t^n-d\Phi^{-1}(Z^n)(\Phi(Z^n))(t) \qquad (0\le t\le 1)\end{aligned}\right\} \tag{2.8}$$

where, for $h,z\in\mathcal{B}$ with $z(0)=0$, by $d\Phi^{-1}(h)(z)$ we mean the pathwise unique solution of the linear double stochastic equation

$$\left.\begin{aligned} y(0)&=0\\ z(t)&=d\Phi(h)(y)_t \qquad (0\le t\le 1)\end{aligned}\right\} \tag{2.9}$$

(here $d\Phi(h)(y)$ is the Gâteaux derivative of Φ at $h\in\mathcal{B}$ computed in the point $y\in\mathcal{B}$).

Remark 2.4. In the case of Itô equations, the convergence of the Carathéodory approximations is examined by Bell and Mohamed [1] and Mao [5,6], and the Newton method is examined by Kawabata and Yamada [4].

The following two theorems represent extensions to multiple stochastic equations of the above mentioned results.

Theorem 2.5. *Assume the hypotheses of Theorem 2.2 hold. Let $\{X_t\}_{0\le t\le 1}$ and $\{X_t^n\}_{0\le t\le 1}$ be the solution of equation (2.4) and the Carathéodory approximations associated to equation (2.4), respectively. Then there exists a constant $K>0$ such that*

$$E\left(\sup_{0\le t\le 1}|X_t^n - X_t|^2\right) \le \frac{K}{n} \qquad \text{for all } n\ge 1. \tag{2.10}$$

Theorem 2.6. *Suppose*

(i) $E(|c|^2)<\infty$

(ii) $F(\underline{u},x)$ *and* $\frac{\partial F(\underline{u},x)}{\partial x}$ *are continuous, and* $\left|\frac{\partial F(\underline{u},x)}{\partial x}\right|\le K_1$.

Then the following conclusions hold:

(a) *The Chaplygin approximations $\{Y_t^n\}_{0\le t\le 1}$ associated to equation (2.4) coincide with the Newton approximations $\{Z_t^n\}_{0\le t\le 1}$.*

(b) (Local convergence). *There exists $t_0\in(0,1)$ such that*

$$\lim_{n\to\infty} E\left(\sup_{0\le t\le t_0}|Y_t^n - X_t|^2\right) = 0. \tag{2.11}$$

(c) (Global convergence). *We have*

$$\lim_{n\to\infty} E\left(\sup_{0\le t\le 1}|Y_t^n - X_t|^2\right) = 0 \tag{2.12}$$

if and only if

$$\sup_n E\left(\sup_{0\le t\le 1}|Y_t^n|^2\right) = L<\infty. \tag{2.13}$$

We need the following lemma.

Lemma 2.7. *Let $\{Y_t\}_{0\le t\le 1}$ be a continuous process of the form*

$$Y_t = H_t + \int_{0<u<v\le t} G(u,v)\,dS_u^1 dS_v^2 \qquad (0\le t\le 1). \tag{2.14}$$

We assume the following:

(a) *H is a continuous process and*

$$E\left(\sup_{0\le t\le 1}|Y_t|^2\right)<\infty \qquad \text{and} \qquad E\left(\sup_{0\le t\le 1}|H_t|^2\right)<\infty. \tag{2.15}$$

(b) *G is a predictable process such that, for some constants $K_1 \geq 0$ and $K_2 \geq 0$,*

$$E(|G(u,v)|^2) \leq K_1 + K_2 E\left(\sup_{0\leq s\leq u} |Y_s|^2\right) \qquad \text{for all } 0 < u < v \leq 1. \tag{2.16}$$

Then if $0 \leq t_0 < t_1 \leq 1$, we have

$$E\left(\sup_{t_0\leq t\leq t_1} |Y_t|^2\right) \leq C_1 \left\{E(|Y_{t_0}|^2) + E\left(\sup_{t_0\leq t\leq t_1} |H_t - H_{t_0}|^2\right) + \left[K_1 + K_2 E\left(\sup_{0\leq s\leq t_0} |Y_s|^2\right)\right](t_1 - t_0)\right\} e^{K_2(t_1-t_0)} \tag{2.17}$$

where C_1 is a constant independent of K_1, K_2 and t_0, t_1. Moreover, if $K_2 = 0$, then

$$E\left(\sup_{t_0\leq t\leq t_1} |Y_t - Y_{t_0}|^2\right) \leq C_2 \left\{E\left(\sup_{t_0\leq t\leq t_1} |H_t - H_{t_0}|^2\right) + K_1(t_1 - t_0)\right\} \tag{2.18}$$

where C_2 is a constant independent of K_1, K_2 and t_0, t_1.

Proof. If $t \geq t_0$, we have

$$Y_t = Y_{t_0} + (H_t - H_{t_0}) + \int_{t_0<u<v\leq t} G(u,v)\, dS_u^1 dS_v^2 + \int_{C_n} 1_{0<u\leq t_0<v\leq t} G(u,v)\, dS_u^1 dS_v^2. \tag{2.19}$$

Then, by using (1.4) and (2.16), we obtain

$$E\left(\sup_{t_0\leq t\leq t_1} |Y_t|^2\right) \leq C_1 \left\{E(|Y_{t_0}|^2) + E\left(\sup_{t_0\leq t\leq t_1} |H_t - H_{t_0}|^2\right) + \left[K_1 + K_2 E\left(\sup_{0\leq s\leq t_0} |Y_s|^2\right)\right](t_1 - t_0) + K_2 \int_{t_0}^{t_1} E\left(\sup_{t_0\leq u\leq v} |Y_u|^2\right) dv\right\}$$

where (2.17) is a consequence of the Gronwall lemma. The inequality (2.18) follows in a similar manner from (2.19) ∎

Proof of Theorem 2.5. From (2.3) it follows that F satisfies (2.16) with $K_2 = 0$ and hence by (2.18) we deduce

$$E(|X_t - X_s|^2) \leq C_2(t - s) \qquad \text{for all } s < t. \tag{2.20}$$

Next we have

$$X_t^n - X_t = H_n(t) + \int_{0<u<v\leq t} \left[F(u,v,X^n_{u-\frac{1}{n}}) - F(u,v,X_{u-\frac{1}{n}})\right] dS_u^1 dS_v^2$$

with

$$H_n(t) = \int_{0<u<v\leq t} \left[F(u,v,X_{u-\frac{1}{n}}) - F(u,v,X_u)\right] dS_u^1 dS_v,$$

and thus by (1.4), the Lipschitz condition and (2.20) we deduce

$$E\left(\sup_{0\leq s\leq t} |H_n(t)|^2\right) \leq C_3 \sup_{|u-v|\leq \frac{1}{n}} E(|X_u - X_v|^2) \leq \frac{C_4}{n}. \tag{2.21}$$

Now the conclusion follows from (2.21) and (2.17) ∎

Proof of Theorem 2.6. **(a)** First we note that from (1.4) Φ maps $\mathcal{B}$ into itself. Then we have

$$\frac{\Phi(h+\varepsilon hy(t)-\Phi(h)(t)}{\varepsilon}=G(t)+R_\varepsilon(t) \qquad \text{for all } \ h,y\in\mathcal{B} \tag{2.22}$$

where

$$G(t)=y(t)-y(0)-\int_{0<u<v\le t}\frac{\partial F(u,v,h(u))}{\partial x}\,y(u)\,dS^1_u dS^2_v \tag{2.23}$$

and

$$R_\varepsilon(t)=\int_{0<u<v\le t}\left[\frac{\partial F(u,v,h(u))}{\partial x}-\frac{\partial F(u,v,h(u)+\varepsilon\theta(u))}{\partial x}\right]y(u)\,dS^1_u dS^2_v \tag{2.24}$$

and θ is a measurable process with $0\le\theta\le 1$. It is easily seen that $\lim_{\varepsilon\to 0}\|R_\varepsilon\|_{\mathcal{B}}=0$. Therefore the Gâteaux derivative $d\Phi(h)(y)$ is given by

$$d\Phi(h)y=\lim_{\varepsilon\to 0}\frac{\Phi(h+\varepsilon y)-\Phi(h)}{\varepsilon}=G \qquad \text{in } \ \mathcal{B}. \tag{2.25}$$

By Theorem 2.2 it follows that the linear double stochastic equation (2.9) has a pathwise unique continuous solution $d\Phi^{-1}(h)(y)\in\mathcal{B}$ (the fact that the coefficients are now random do not change the result and the proof of Theorem 2.2). Now (2.8) and (2.9) imply the equality $Y^n=Z^n$.

(b) Denote

$$\|h\|_t^2 E\left(\sup_{0\le s\le t}|h(s)|^2\right) \qquad (0<t\le 1,\ h\in\mathcal{B}).$$

We have the equality

$$Y_t^{n+1}-X_t=H_n(t)+\int_{0<u<v\le t}\frac{\partial F(u,v,Y_u^n)}{\partial x}(Y_u^{n+1}-X_u)\,dS^1_u dS^2_v$$

where

$$H_n(t)=\int_{0<u<v\le t}\left[F(u,v,Y_u^n)-F(u,v,X_u)+\frac{\partial F(u,v,Y_u^n)}{\partial x}(X_u-Y_u^n)\right]dS^1_u dS^2_v.$$

It is easily seen that $\|H_n\|_t^2\le C_4t\|Y^n-X\|_t^2$, and hence by (2.17) we deduce

$$\|Y^{n+1}-X\|_t^2\le C_5te^{K_5t}\|Y^n-X\|_t^2. \tag{2.26}$$

Choose now $t_0\in(0,1)$ such that $\alpha=C_5t_0e^{C_5t_0}<1$. Then from (2.26) we have

$$\|Y^{n+1}-X\|_{t_0}^2\le\alpha\|Y^n-X\|_{t_0}^2$$

and hence (2.11) follows.

(c) Assume now (2.13) is satisfied and define

$$t_1 = \sup\left\{0 \le t \le 1 : \|Y^n - X\|_t^2 \to 0 \text{ as } n \to \infty\right\}.$$

Of course, from (2.11) we have $0 < t_0 \le t_1 \le 1$. For $0 < \varepsilon < t_1$ we have by (2.18)

$$E\left(\sup_{t_1-\varepsilon\le t\le t_1} |X_t - X_{t_1-\varepsilon}|^2\right) \le C_6\varepsilon,$$

$$\sup_n E\left(\sup_{t_1-\varepsilon\le t\le t_1} |Y_t^n - Y_{t_1-\varepsilon}^n|^2\right) \le C_7\varepsilon$$

and thus

$$E\left(\sup_{t_1-\varepsilon\le t\le t_1} |Y_t^n - X_t|^2\right) \le C_8\varepsilon$$

for n large enough. If we assume that $t_1 < 1$, then proceeding as in Part (b) we get a $\delta \in (0, 1 - t_1)$ such that

$$E\left(\sup_{0\le t\le t_1+\delta} |Y_t^n - X_t|^2\right) \to 0 \qquad \text{as } n \to \infty$$

and this contradicts the definition of t_1 ∎

References

[1] Bell, D. R. and S. E. A. Mohamed: *On the solution of stochastic ordinary differential equations via small delays.* Stochastics and Stochastics Reports 29 (1989), 293 - 299.

[2] Berger, M. and V. Mizel: *Volterra equations with Itô integrals.* Part I. J. Int. Equ. 2 (1980), 187 - 245.

[3] Itô, K.: *Multiple Wiener integrals.* J. Math. Soc. Japan 3 (1951), 157 - 169.

[4] Kawabata, S. and T. Yamada: *On Newton's method for stochastic differential equation.* Lect. Notes Math. 1485 (1991), 121 - 137.

[5] Mao, X.: *Approximate solutions for a class of stochastic evolution equations with variable delays.* Numer. Funct. Anal. Optimiz. 12 (1991), 525 - 533.

[6] Mao, X.: *Approximate solutions for stochastic differential equations with pathwise uniqueness.* Stoch. Anal. Appl. 12 (1994), 355 - 367.

[7] Meyer, P. A.: *Notions sur les intégrales multiples.* Lect. Notes Math. 511 (1976), 321 - 331.

[8] Ruiz de Chaves, J.: *Sur les intégrales stochastiques multiples.* Lect. Notes Math. 1123 (1985), 248 - 257.

Received 03.07.1996; in revised form 24.02.1997