Weighted Norm Inequalities for Riemann-Liouville Fractional Integrals of Order Less than One

Y. Rakotondratsimba

Abstract. Necessary and sufficient condition on weight functions $u(\cdot)$ and $v(\cdot)$ are derived in order that the Riemann-Liouville fractional integral operator R_{α} ($0 < \alpha < 1$) is bounded from the weighted Lebesgue spaces $L^p((0,\infty),v(x)dx)$ into $L^q((0,\infty),u(x)dx)$ whenever $1 <$ $p \le q < \infty$ or $1 < q < p < \infty$. As a consequence for monotone weights then a simple characterization for this boundedness is given whenever $p \le q$. Similar problems for convolution operators, acting on the whole real axis $(-\infty, \infty)$, are also solved.

Keywords: *Weighted inequalities, Riernann-Liouville operators, convolution operators* AMS subject classification: *42B25, 26A33*

1. Introduction

The Riemann-Liouville and Weyl fractional integral operators are defined, up to normalizing constants, respectively by

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\nrespectively by
\n
$$
(R_{\alpha}f)(x) = \int_0^x (x - y)^{\alpha - 1} f(y) dy \qquad (\alpha > 0)
$$
\n
$$
(W_{\alpha}f)(x) = \int_x^{\infty} (y - x)^{\alpha - 1} f(y) dy \qquad (\alpha > 0)
$$

and

$$
(W_{\alpha}f)(x) = \int_{x}^{\infty} (y-x)^{\alpha-1} f(y) dy \qquad (\alpha > 0)
$$

for all locally integrable functions $f(\cdot)$ on $(0, \infty)$. One of our purposes is to study weighted inequalities of the form

$$
(W_{\alpha}f)(x) = \int_{x}^{\infty} (y - x)^{\alpha - 1} f(y) dy \qquad (\alpha > 0)
$$

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hted inequalities of the form

$$
\left(\int_{0}^{\infty} (Tf)^{q}(x)u(x) dx\right)^{\frac{1}{q}} \leq C \left(\int_{0}^{\infty} f^{p}(x)v(x) dx\right)^{\frac{1}{p}} \qquad \text{for all } f(\cdot) \geq 0 \qquad (1.1)
$$

where *T* is either R_{α} or W_{α} $(0 < \alpha < 1, 1 < p, q < \infty)$, $u(\cdot)$ and $v(\cdot)$ are non-negative weight functions, and $C > 0$ is a constant depending only on p, q, $u(\cdot)$ and $v(\cdot)$. For convenience (1.1) is also denoted by

$$
T: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx).
$$

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The boundedness (1.1) for R_{α} or W_{α} is very useful in Real Analysis. For instance, it can be used in order to derive analogous weighted inequalities for the Laplace transform and the Edélyi-Kober operators [1, 21. Inequalities like (1.1) find also applications in studying boundedness of fractional maximal and integral operators on amalgam spaces with weights [3].

For the range $\alpha \geq 1$ and $1 < p \leq q < \infty$, a characterization of weights $u(\cdot)$ and $v(\cdot)$ for which (1.1) holds, was due to F. Martin-Reyes and E. Sawyer [8], and independently by Stepanov [12] who also solved the problem for $1 < q < p < \infty$.

So in this paper our study will be focused for the case $0 < \alpha < 1$ and with $1 < p, q <$ ∞ which is from now assumed. In such a setting, problem (1.1) remains open in full generality. For a large class of weight functions, and particularly for monotone weights, we will completely solve this problem by using very simple characterizing conditions. which is from now assumed. In such a setting, problem (1.1) remains open in full
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we will $u(x) dx$, with $p < \frac{1}{\alpha}$ and $\frac{1}{p^2} = \frac{1}{p} - \alpha$, was found by K. Andersen and E. Sawyer [2]. For the range $\alpha \ge 1$ and $1 < p \le q < \infty$, a characterization of weights $u(\cdot)$ and $v(\cdot)$
for which (1.1) holds, was due to F. Martin-Reyes and E. Sawyer [8], and independently
by Stepanov [12] who also solved the problem For $u(\cdot) \neq v(\cdot)$ the boundedness W_{α} : $L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$ can be characterized whenever $u(\cdot) \in A_{\infty}^+$, i.e. For a large class of weight functions, and particularly for monoton
pletely solve this problem by using very simple characterizing cor
sary and sufficient condition for $W_{\alpha} : L^p((0, \infty), u(x) dx) \to L$
ith $p < \frac{1}{\alpha}$ and $\frac{$

$$
\left(\frac{1}{\varepsilon}\int_{b-\varepsilon}^{b} u(z) dz\right)^{\frac{1}{t}} \left(\frac{1}{\varepsilon}\int_{b}^{b+\varepsilon} u^{1-t'}(z) dz\right)^{\frac{1}{t'}} \leq A \qquad \text{for all } 0 < \varepsilon < b
$$

for some fixed constants $t > 1$, $A > 0$ and with $t' = \frac{t}{t-1}$. Indeed, for such a weight $u(\cdot)$, it is known in [7] that $\int_0^\infty (W_\alpha f)^q(x)u(x) dx \approx \int_0^\infty (M_\alpha^+ f)^q(x)u(x) dx$, where M_α^+ is the right-sided fractional maximal operator studied by F. Martin-Reyes and A. de la Torre [9]. Thus (1.1) (with $T = W_{\alpha}$) becomes equivalent to M_{α}^{+} : $L^p((0,\infty),v(x) dx) \rightarrow$ $L^q((0,\infty), u(x) dx)$ whose a characterization was also given by these authors. However, note that the characterizing condition is often difficult to use for explicit computations, since it is expressed in terms of the maximal operator itself and integrations over (special) arbitrary intervalls. Later M. Lorente and A. de la Torre [6] found a simpler characterizing condition for the range $p < q$. More details on their condition will be discussed in the next Section 2. for some fixed constants $t > 1$, $A > 0$ and with $t' = \frac{t}{t-1}$. Indeed, for such a weight $u(\cdot)$,
it is known in [7] that $\int_0^\infty (W_\alpha f)^q(x)u(x) dx \approx \int_0^\infty (M_\alpha^+ f)^q(x)u(x) dx$, where M_α^+ is the
right-sided fractional maxima

Without any further assumptions on $u(\cdot)$ and $v(\cdot)$ a result due to K. Andersen and whenever, for some $\varepsilon \in [0,1]$ and $A > 0$, H. Heinig [1] asserts that, for $p \leq q$ then $R_{\alpha}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$

$$
\left(\int_{R}^{\infty} (y-R)^{(\alpha-1)\epsilon q} u(y) dy\right)^{\frac{1}{q}} \left(\int_{0}^{R} (R-y)^{(\alpha-1)(1-\epsilon)p'} v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \leq A \qquad (1.2)
$$

for all $R > 0$. Here and in the sequel $p' = \frac{p}{p-1}$. Condition (1.2) is only a sufficient one (generally not necessary) for (1.1) to hold. It will be seen in the next section that (1.2) cannot be used to treat the limiting case $\frac{1}{q} = \frac{1}{p} - \alpha$, and many weight functions $u(\cdot)$ are excluded. Also the case $q < p$ is not treated in [1]. These facts lead us to consider and study again inequality (1.1).

As we will see below, a necessary condition for $R_{\alpha}: L^p\big((0,\infty), v(x)\,dx\big)\to L^q\big((0,\infty)\big)$ $u(x) dx$) when $p \leq q$ is

we will see below, a necessary condition for
$$
R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty))
$$

\nwhen $p \le q$ is
\n
$$
\left(\int_{2R}^{\infty} y^{(\alpha-1)q} u(y) dy\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0. \quad (1.3)
$$

So a main question, answered in this paper, is to find another companion condition [see Theorem 2.1] such that both of them are necessary and sufficient for R_c $L^p((0,\infty),v(x)dx) \to L^q((0,\infty),u(x)dx)$. This companion condition is expressed in $\left(\int_{2R} y^{(\alpha-1)q} u(y) dy\right)^{r} \left(\int_{0} v^{1-p'}(y) dy\right)^{r} \leq A$ for all $R > 0$. (1.3)
So a main question, answered in this paper, is to find another companion condi-
tion [see Theorem 2.1] such that both of them are necessary and acterizing condition for this last boundedness remains open, surprisingly [see Proposition 2.3] a simple (pointwise) sufficient condition can be derived. This last is not far from a suitable necessary condition for $R_{\alpha}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)$. As • consequence [in Proposition *2.51* we will see that (1.3) *is* a necessary and sufficient condition for (1.1) [with $T = R_{\alpha}$] for monotone weights. And really [see Corollary 2.6] this characterization remains true for a large class of weights. Similar results for the operator W_{α} as well as in the range $q < p$ are also obtained in Theorem 2.2, Proposition *2.4* and Corollary *2.7.* Examples showing the computabilities of our conditions and also the gain over past results will be presented in Corollaries *2.8 - 2.10.*

The second purpose of this paper is the generalization of results for R_{α} and W_{α} to the case of convolution operators (with decreasing kernel) which act on the whole real axis ($-\infty$, ∞). These general results will be stated in Section 3. The last Section 4 is devoted to the proof of our results.

While this paper is typesetted, I receive a preprint from V. Kokilashvili *[5]* announcing a full characterization for R_{α} : $L^p((0,\infty),v(x)dx) \rightarrow L^q((0,\infty),u(x)dx)$ with $p < q$. So from his collaboration with I. Genebashvili and A. Gogatishvili [4] then it is known that this boundedness holds if and only if, for all $0 < \varepsilon < b$, Form and results we general results when \int of our results.
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ashvili and A. Gogatishvili [4] then

if, for all $0 < \varepsilon < b$,
 $v^{1-p'}(y) dy$ $\overrightarrow{r'} \leq A$ (1.4)
 $v^{1-p'}(y) dy$ $\overrightarrow{r'} \leq A$ (1.5)

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from his collaboration with I. Genebashvili and A. Gogatishvili [4] then
at this boundedness holds if and only if, for all
$$
0 < \varepsilon < b
$$
,

$$
\left(\int_{b-\varepsilon}^{b+\varepsilon} u(y) dy \right)^{\frac{1}{q}} \left(\int_{0}^{b-\varepsilon} (b-y)^{(\alpha-1)p'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A \qquad (1.4)
$$

and

$$
\left(\int_{b+\epsilon}^{\infty} (y-b)^{(\alpha-1)q} u(y) \, dy\right)^{\frac{1}{q}} \left(\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) \, dy\right)^{\frac{1}{p'}} \leq A \tag{1.5}
$$

where $A > 0$ is a fixed constant. However, in [5] and [4] the reader would be aware of confusions in the range of integrations.

In comparison with their result, one of the contributions of the present paper is the treatement of the forbidden case $q \leq p$. On the other hand, for the case $p < q$ the interest in our results can be found on the computabilities of the conditions introduced. The difficulties which appear in checking, for instance, condition (1.5) are alluded in the next Section *2.* \mathcal{A} :

2. Results for the Riemann-Liouville and Weyl **operators**

This section is devoted to the statement of our results (see Section 3) for the usal Riemann-Liouville and Weyl operators when they act on $(0,\infty)$. In this paper, it wil be assumed that *0<a<1, l<p,q<cx, p'=--,* **Paramelet Allen Weyl open**
 P results (see Section
 P act on $(0, \infty)$. In the $\frac{p}{p-1}$, $q' = \frac{q}{q-1}$
 P and non-negative function

$$
0 < \alpha < 1, \qquad 1 < p, q < \infty, \qquad p' = \frac{p}{p-1}, \qquad q' = \frac{q}{q-1}, \qquad q' = \frac{q}{q-1}, \qquad q' = \frac{q}{q-1}.
$$

and

are locally integrable and non-negative functions.

 $0 < \alpha < 1$, $1 < p, q < \infty$, $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$,
and
 $u(\cdot), v^{1-p'}(\cdot)$ are locally integrable and non-negative functions.
First we give a necessary and sufficient condition for the boundednesses $R_{\alpha}: L^p((0, \infty), u(x) dx) \$ whenever $p \leq q$ or $q < p$.

Theorem 2.1. *Suppose* $p \leq q$ *. The boundedness*

$$
R_\alpha:\,L^p\big((0,\infty),v(x)\,dx\big)\rightarrow L^q\big((0,\infty),u(x)\,dx\big)
$$

and
\n
$$
u(\cdot), v^{1-p'}(\cdot) \qquad \text{are locally integrable and non-negative functions.}
$$
\nFirst we give a necessary and sufficient condition for the boundednesses $R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$ and $W_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$
\nwhenever $p \le q$ or $q < p$.
\nTheorem 2.1. Suppose $p \le q$. The boundedness
\n
$$
R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)
$$

\nholds if and only if for some constant $A > 0$
\n
$$
\left(\int_{2R}^{\infty} x^{(\alpha-1)q} u(x) dx\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(x) dx\right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0 \quad (2.1)
$$

\nand

and

$$
\widetilde{R}_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

where \widetilde{R}_{α} is the restricted operator given by

$$
(\widetilde{R}_{\alpha}f)(x)=\int_{\frac{1}{2}x}^{x}(x-y)^{\alpha-1}f(y)\,dy.
$$

Analogously,

$$
W_\alpha: L^p((0,\infty),v(x)\,dx\big)\to L^q((0,\infty),u(x)\,dx\big)
$$

if and only if

$$
I_n = \int_{\alpha}^{R} W_{\alpha} : L^p((0, \infty), v(x) \, dx) \to L^q((0, \infty), u(x) \, dx)
$$
\n
$$
I_n = \int_{2R}^{\infty} \int_{\alpha}^{2\alpha} x^{(\alpha - 1)p'} v^{1-p'}(x) \, dx \Big)^{\frac{1}{p'}} \left(\int_0^R u(x) \, dx \right)^{\frac{1}{q}} \leq A \quad \text{for all } R > 0 \tag{2.2}
$$

and

$$
\overline{W}_{\alpha}:\,L^p\big((0,\infty),v(x)\,dx\big)\rightarrow L^q\big((0,\infty),u(x)\,dx\big)
$$

where

$$
(\widetilde{W}_{\alpha}f)(x)=\int_{x}^{2x}(y-x)^{\alpha-1}f(y)\,dy.
$$

Theorem 2.2. *Let* $q < p$ *and* $r = \frac{qp}{p-q}$. *The boundedness* $R + IP((0, \infty), v(x) dx) \to I^q((0, \infty), v(x) dx)$

$$
R_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

holds if and only if for some constant A > 0:

$$
\begin{aligned}\n\text{reorem 2.2.} \quad & Let \ q < p \ \text{and} \ r = \frac{qp}{p-q}. \ \text{The boundedness} \\
& R_{\alpha} : L^p((0, \infty), v(x) \, dx) \to L^q((0, \infty), u(x) \, dx) \\
\text{and only if for some constant } A > 0: \\
\int_0^\infty \left[\left(\int_{2x}^\infty y^{(\alpha-1)q} u(y) \, dy \right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y) \, dy \right)^{\frac{1}{q'}} \right] \Bigg[v^{1-p'}(x) \, dx < A^r \end{aligned} \tag{2.3}
$$

 \sim *and*

$$
\widetilde{R}_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx).
$$

Similarly,

$$
W_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

if and only if

$$
\widetilde{R}_{\alpha}: L^{p}((0, \infty), v(x) dx) \to L^{q}((0, \infty), u(x) dx).
$$
\n
$$
W_{\alpha}: L^{p}((0, \infty), v(x) dx) \to L^{q}((0, \infty), u(x) dx)
$$
\nif\n
$$
\int_{0}^{\infty} \left[\left(\int_{2x}^{\infty} y^{(\alpha-1)p'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \left(\int_{0}^{x} u(y) dy \right)^{\frac{1}{p}} \right]^{r} u(x) dx < A^{r}
$$
\n(2.4)

and

$$
\widetilde{W}_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx).
$$

Therefore, the real difficulty to derive $R_{\alpha}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x))$ *dx*) is on getting the weighted inequality $\widetilde{R}_{\alpha}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$ for the restricted operator \widetilde{R}_{α} . A characterization of weights $u(\cdot)$ and $v(\cdot)$ for which this last boundedness holds is an open problem. However, it is possible to give a sufficient condition as stated in the following two propositions.

Proposition 2.3. Let $p \le q$. For $p < q$ and $p < \frac{1}{\alpha}$ it is also assumed that $q \le p^*$ *with* $\frac{1}{n^*} = \frac{1}{n} - \alpha$. The boundedness

$$
\widetilde{R}_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

holds whenever, for a constant $A > 0$,

$$
\text{troposition 2.3. Let } p \le q. \text{ For } p < q \text{ and } p < \frac{1}{\alpha} \text{ it is also assumed that } q \le p^*
$$
\n
$$
\frac{1}{\alpha^*} = \frac{1}{p} - \alpha. \text{ The boundedness:}
$$
\n
$$
\widetilde{R}_{\alpha}: L^p((0, \infty), v(x) \, dx) \to L^q((0, \infty), u(x) \, dx)
$$
\n
$$
\text{whenever, for a constant } A > 0,
$$
\n
$$
R^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\sup_{R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1 - p'}(y) \right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0. \tag{2.5}
$$

Similarly,

$$
\widetilde{W}_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

whenever

$$
R^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\sup_{R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \tag{2.5}
$$
\n
$$
i\left\{ \text{and} \, 0 \right\}.
$$
\n
$$
\widetilde{W}_{\alpha} : L^p((0, \infty), v(x) \, dx) \to L^q((0, \infty), u(x) \, dx)
$$
\n
$$
n\left\{ \text{and} \, 0 \right\}.
$$
\n
$$
R^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\sup_{\frac{1}{2}R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{R < y < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \tag{2.6}
$$
\n
$$
i\left\{ \text{and} \, 0 \right\}.
$$
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A *necessari, condition for Ra L"((0,c*),v(x)dx) —i L((0,),u(x)dx)* is - <a. **I**

Indeed, this boundedness implies

Y. Rakotondratsimba
\n*necessary condition for*
$$
R_{\alpha}
$$
: $L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$
\n $\frac{1}{p} - \frac{1}{q} \leq \alpha$.
\n1, this boundedness implies
\n $t^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\frac{1}{t} \int_{a+t}^{a+2t} u(z) dz \right)^{\frac{1}{q}} \left(\frac{1}{t} \int_a^{a+t} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$ for all $a, t > 0$
\nwith a fixed constant $A > 0$. Therefore, by the Lebesgue differentiation the
\n $\frac{1}{q} - \frac{1}{p} < 0$, then necessarily $u(\cdot) = 0$ or $v^{1-p'}(\cdot) = 0$ a.e.

and with a fixed constant $A > 0$. Therefore, by the Lebesgue differentiation theorem, if $\alpha + \frac{1}{q} - \frac{1}{p} < 0$, then necessarily $u(\cdot) = 0$ or $v^{1-p'}(\cdot) = 0$ a.e.

The condition (2.5) is not too far from a necessary one for R_{α} : $L^p((0,\infty),v(x) dx)$ \rightarrow $L^{q}((0,\infty),u(x)\,dx).$ Indeed, this last boundedness implies that for a fixed constant $A > 0$, α i $L^p((0, \infty))$
 A for a fixe
 A for all $R > 0$ $\frac{1}{\pi} - \frac{1}{p} < 0$, then necessarily $u(\cdot) = 0$ or $v^{1-p}(\cdot) = 0$ a.e.
 e condition (2.5) is not too far from a necessary one for $R_{\alpha} : L^p(\cdot) = 0$
 $R^{\alpha-1} \left(\int_R^{2R} u(z)dz\right)^{\frac{1}{q}} \left(\int_{\frac{1}{2}R}^R v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \leq$

$$
R^{\alpha-1} \left(\int_{R}^{2R} u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{2}R}^{R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0
$$

and

$$
\int_{R}^{2R} u(z)dz \leq R \sup_{R < z < 2R} u(z), \qquad \int_{\frac{1}{2}R}^{R} v^{1-p'}(y) \, dy \leq R \sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y).
$$

Of course, analogous observations can be made for condition (2.6).

Proposition 2.4. Let $q < p$ and $r = \frac{qp}{p-q}$. The boundedness

$$
\widetilde{R}_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

$$
\int_{R} u(z)dz \leq R \sup_{R < z < 2R} u(z), \qquad \int_{\frac{1}{2}R} v^{1-p'}(y) dy \leq R \sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y).
$$

Of course, analogous observations can be made for condition (2.6).
Proposition 2.4. Let $q < p$ and $r = \frac{qp}{p-q}$. The boundedness
 $\widetilde{R}_{\alpha}: L^{p}((0, \infty), v(x) dx) \to L^{q}((0, \infty), u(x) dx)$
holds whenever, for a constant $A > 0$ and a sequence $(\mathcal{B}(n))_{n \in \mathbb{Z}}$,
 $2^{n[\alpha + \frac{1}{q} - \frac{1}{p}]} \left(\sup_{2^{n} < y < 2^{n+1}} u(y) \right)^{\frac{1}{q}} \left(\sup_{2^{n-1} < y < 2^{n+1}} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq \mathcal{B}(n) \quad \forall n \in \mathbb{Z}$ (2.7)
and

$$
\sum_{n \in \mathbb{Z}} [\mathcal{B}(n)]^{r} < A^{r}.
$$
 (2.8)
Conditions (2.7) and (2.8) are not too far from the necessary condition

and

$$
\sum_{n\in\mathbb{Z}}[\mathcal{B}(n)]^r < A^r. \tag{2.8}
$$

Conditions (2.7) and (2.8) are not too far from the necessary condition

$$
\sum_{n\in\mathbb{Z}}[\mathcal{A}(n)]^r < c\tag{2.9}
$$

for the boundedness $R_{\alpha}: L^p\big((0,\infty),v(x)\,dx\big)\to L^q\big((0,\infty),u(x)\,dx\big),$ where

$$
\overbrace{\mathcal{A}(n)}^{\text{adness}} R_{\alpha}: L^{p}((0,\infty), v(x) dx) \to L^{q}((0,\infty), u(x) dx), \text{ wh}
$$

$$
\mathcal{A}(n) = 2^{n(\alpha-1)} \bigg(\int_{2^{n}}^{2^{n+1}} u(z) dz \bigg)^{\frac{1}{q}} \bigg(\int_{2^{n-1}}^{2^{n+1}} v^{1-p'}(y) dy \bigg)^{\frac{1}{p'}}.
$$

The fact that (2.9) is a required condition can be seen by taking

Weighted Norm In
\n2.9) is a required condition can be seen by taking
\n
$$
f(x) = \sum_{k=-M}^{N} 2^{k(\alpha-1)\frac{r}{p}} \left(\int_{2^n}^{2^{n+1}} u(z) dz \right)^{\frac{r}{pq}}
$$
\n
$$
\times \left(\int_{2^{n-1}}^{x} v^{1-p'}(y) dy \right)^{\frac{r}{pq'}} v^{1-p'}(x) 1_{(2^{n-1},2^n)}(x)
$$

in the corresponding inequality to R_{α} : $L^p((0,\infty),v(x) dx) \rightarrow L^q((0,\infty),u(x) dx)$ where *N* and *M* are arbitrary non-negative integers.

For monotone weight functions, Theorem 2.1 and Proposition 2.3 can be used to get easy characterizations for the above boundednesses.

Proposition 2.5. *Let* $p \leq q$, with $q \leq p^*$ for $p < q$ and $p < \frac{1}{q}$. Suppose that $u(\cdot)$ and $v^{1-p'}(.)$ are monotone functions.

Condition (2.1) is a necessary and sufficient one for

$$
R_{\alpha}: L^p((0,\infty),v(x)\,dx) \to L^q((0,\infty),u(x)\,dx)
$$

whenever $u(\cdot)$ *is an increasing function or whenever* $v^{1-p'}(\cdot)$ *is a decreasing function.*

The above equivalence remains true if $u(\cdot)$ *is decreasing and* $v^{1-p'}(\cdot)$ *increasing with* $u(x) \leq c u(2x)$ or $v^{1-p'}(2x) \leq c v(x)$, for a fixed constant $c > 0$.

Condition (2.2) is a necessary and sufficient one for

$$
W_\alpha: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

whenever $v^{1-p'}(\cdot)$ *is an increasing function or whenever* $u(\cdot)$ *is a decreasing function.* This last equivalence remains true if $v^{1-p'}(\cdot)$ is decreasing and $u(\cdot)$ increasing with $v^{1-p'}(x) \leq c v(2x)$ or $u(2x) \leq c u(x)$. increasing function or whenever $v^{1-p'}(\cdot)$ is a decreasing function.

alence remains true if $u(\cdot)$ is decreasing and $v^{1-p'}(\cdot)$ increasing with
 $1-p'(2x) \leq cv(x)$, for a fixed constant $c > 0$.

is a necessary and sufficient *is an increasince remains to* $r u(2x) \leq c v$
 $\text{weight} \text{ function}$
 $\text{sup} \quad w(z) \leq$
 $R < z < 2R$
 $\text{and } N \text{ (an in } \mathbb{R})$

Each monotone weight function $w(\cdot)$ satisfies the growth condition

$$
\sup_{\frac{1}{2}R < z < 2R} w(z) \le C \frac{1}{R} \int_{2^{-N}R}^{2^{N}R} w(y) \, dy \qquad \text{for all} \ \ R > 0 \tag{C}
$$

where both $C > 0$ and N (an integer greater than 2) depend only on $w(\cdot)$. It will be denoted that $w(\cdot) \in \mathcal{C}$. For a monotone weight, then $w(\cdot) \in \mathcal{C}$ with the constant $N = 2$. There are also non-necessarily monotone weights for which this property is fulfilled. Indeed, it can be shown that $w(\cdot) \in \mathcal{C}$ whenever $w(\cdot) = w_0(\cdot) \mathbf{1}_{[0,1]}(\cdot) + w_1(\cdot) \mathbf{1}_{(1,\infty)}(\cdot)$ where $w_0(\cdot)$ or $w_1(\cdot)$ is an increasing or decreasing weight function, respectively. R both $C > 0$ and
ted that $w(\cdot) \in C$.
e are also non-nee
cd, it can be show
 $e w_0(\cdot)$ or $w_1(\cdot)$ is
'or $u(\cdot), v^{1-p'}(\cdot) \in$
 $L^q((0, \infty), u(x))$
 $R^{\alpha-1} \bigg(\int_{2^{-2(N+1)}R}^{R}$
(2.10) is a sufficien

 $\text{For } u(\cdot), v^{1-p'}(\cdot) \in \mathcal{C}, \text{ by Proposition 2.3, then the boundedness } \widetilde{R}_{\boldsymbol{\alpha}}: L^{\boldsymbol{p}}\big((0,\infty), v(x)\big)$ $dx) \rightarrow L^q((0,\infty),u(x) dx)$ holds whenever

R R u(x)dx) **(***1***²** *-* **² (N+I)R** ^v 1_P' (x)dx) *^A*V *^R***> 0. (2.10) ²2(N4-1)R** Also (2.10) is a sufficient condition which ensures *L ((0,* oo), *v(x) dx) - L ((0,* oo),

 $u(x) dx$.

Corollary 2.6. Let $p \leq q$, with $q \leq p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Suppose that $u(\cdot)$, $v^{1-p'}(.) \in \mathcal{C}$. Then (2.1) is a necessary and sufficient condition for

$$
R_\alpha: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

whenever (2.10) is satisfied. Also under (2.10), then (2.2) is a necessary and sufficient condition for

$$
W_{\alpha}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx).
$$

In order to state an analogous result for the case $q < p,$ it is convenient to introduce the condition

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\nCorollary 2.6. Let
$$
p \le q
$$
, with $q \le p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Suppose that $u(\cdot)$,
\n $p'(\cdot) \in C$. Then (2.1) is a necessary and sufficient condition for
\n $R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$
\nenever (2.10) is satisfied. Also under (2.10), then (2.2) is a necessary and sufficient
\ndition for
\n $W_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$:
\nIn order to state an analogous result for the case $q < p$, it is convenient to introduce
\ncondition
\n $R^{\alpha-1} \Biggl(\int_{2-(N+1)R}^{2^{(N+1)R}} u(y) dy \Biggr)^{\frac{1}{q}} \Biggl(\int_{2-(N+1)R}^{2^{(N+1)R}} v^{1-p'}(y) dy \Biggr)^{\frac{1}{r'}}$
\n $\leq c \Biggl(\int_{4R}^{\infty} y^{(\alpha-1)q} u(y) dy \Biggr)^{\frac{1}{q}} \Biggl(\int_{0}^{R} v^{1-p'}(y) dy \Biggr)^{\frac{1}{q'}} \Biggl(\int_{R}^{2R} v^{1-p'}(y) dy \Biggr)^{\frac{1}{r}}$
\nCorollary 2.7. Let $q < p$ and $r = \frac{qp}{p-q}$. Suppose that $u(\cdot), v^{1-p'}(\cdot) \in C$. Then
\n3) is a necessary and sufficient condition for
\n $R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$
\nenever (2.11) is satisfied.
\nTo illustrate these results, some examples are now given

Corollary 2.7. Let $q < p$ and $r = \frac{qp}{p-q}$. Suppose that $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{C}$. Then *(2.3) is a necessary and sufficient condition for*

$$
R_{\alpha}: L^p((0,\infty),v(x)\,dx) \to L^q((0,\infty),u(x)\,dx)
$$

whenever (2.11) is satisfied.

To illustrate these results, some examples are now given.

Corollary 2.8. Let $p \leq q$, with $q \leq p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Define $u(x) = x^{\beta-1}$ *and* $v(x) = x^{\delta-1}$ *. The boundedness*

$$
R_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

holds if and only if both $\delta < p$ *and* $\beta < (1 - \alpha)q$ *and*

$$
\alpha + \frac{\beta}{q} = \frac{\delta}{p}.
$$
 (2.12)

Similarly,

$$
W_{\alpha}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

if and only if both $0 < \beta$ *and* $\alpha p < \delta$ *and (2.12) is satisfied.*

As mentioned in the introduction, according to K. Andersen and H. Heinig [1), then $R_{\alpha}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$ whenever, for some $\varepsilon \in [0,1]$ and

$$
\text{trly,}
$$
\n
$$
W_{\alpha}: L^{p}((0, \infty), v(x) dx) \to L^{q}((0, \infty), u(x) dx)
$$
\n
$$
\text{only if both } 0 < \beta \text{ and } \alpha p < \delta \text{ and } (2.12) \text{ is satisfied.}
$$
\nis mentioned in the introduction, according to K. Andersen and H. Heinig

\n
$$
R_{\alpha}: L^{p}((0, \infty), v(x) dx) \to L^{q}((0, \infty), u(x) dx) \text{ whenever, for some } \varepsilon \in [0, 1]
$$
\n,

\n
$$
\left(\int_{R}^{\infty} (y - R)^{(\alpha - 1)\varepsilon q} u(y) dy\right)^{\frac{1}{q}} \left(\int_{0}^{R} (R - y)^{(\alpha - 1)(1 - \varepsilon)p'} v^{1 - p'}(y) dy\right)^{\frac{1}{p'}} \leq A
$$

Wei
for all $R > 0$. Taking $u(y) = y^{\beta - 1}$ and $v(y) = y^{\delta - 1}$ then for all $R > 0$.

Weighted Norm Inequalities
\nII
$$
R > 0
$$
. Taking $u(y) = y^{\beta - 1}$ and $v(y) = y^{\delta - 1}$ then
\n
$$
\int_{R}^{\infty} (y - R)^{(\alpha - 1)\epsilon q} u(y) dy = \int_{R}^{2R} + \int_{2R}^{\infty} (y - R)^{(\alpha - 1)\epsilon q} y^{\beta - 1} dy
$$
\n
$$
\approx R^{\beta - 1} \int_{0}^{R} t^{(\alpha - 1)\epsilon q} dt + \int_{2R}^{\infty} y^{(\alpha - 1)\epsilon q + \beta - 1} dt.
$$

Consequently, to continue the computations it is required that $\beta < (1 - \alpha)\varepsilon q < 1$. And analogously

$$
\approx R^{\beta-1} \int_0^t t^{(\alpha-1)\epsilon q} dt + \int_{2R} y^{(\alpha-1)\epsilon q + \beta - 1} dt.
$$

quently, to continue the computations it is required that $\beta < (1 - \alpha)\epsilon q < 1$

$$
\int_0^R (R - y)^{(\alpha-1)(1-\epsilon)p'} v^{1-p'}(y) dy
$$

$$
= \int_0^{\frac{1}{2}R} + \int_{\frac{1}{2}R} (R - y)^{(\alpha-1)(1-\epsilon)p'} v^{1-p'}(y) dy
$$

$$
\approx R^{(\alpha-1)(1-\epsilon)p'} \int_0^{\frac{1}{2}R} y^{(1-p')(\delta-1)} dy + R^{(1-p')(\delta-1)} \int_0^{\frac{1}{2}R} t^{(\alpha-1)(1-\epsilon)p'} dt
$$

and it requires that $\delta < p$ and $(1 - \alpha)(1 - \varepsilon)p' < 1$. So the real ε must satisfy $-\alpha < \varepsilon(1 - \alpha) < \frac{1}{q}$. Therefore, this Andersen-Heinig's result can be only applied

nenever $\beta < 1$ and $\frac{1}{p} - \alpha < \frac{1}{q}$. In view of Corollary 2.8, these restrictions are not

cded since it is necessary that $\beta < (1$ whenever β < 1 and $\frac{1}{p} - \alpha < \frac{1}{q}$. In view of Corollary 2.8, these restrictions are not needed since it is necessary that $\beta < (1 - \alpha)q$ and $\frac{1}{n} - \alpha \leq \frac{1}{n}$. \int_0^{π}

at $\delta < p$ and $(1 - \alpha)(1)$
 $\leq \frac{1}{q}$. Therefore, this An
 $\ln \frac{1}{p} - \alpha < \frac{1}{q}$. In view α

ecessary that $\beta < (1 - \alpha)$

introduction, for $u(\cdot) \in A$
 \int_0^{π}
 $h + \epsilon$
 $v^{1-p'}(y) dy$
 $\int_0^{\frac{1}{p'}} \left(\int_0^{b$

As seen in the introduction, for $u(\cdot) \in A_{\infty}^{+}$ and $p < q$, by a result due to M. Lorente and A. de la Torre [6], the boundedness $W_{\alpha}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$ is equivalent to $\in A_{\infty}^{+}$ i
 V_{α} : L^{p}
 $\int_{0}^{b-\epsilon}$ (*b*)

$$
\left(\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \left(\int_0^{b-\epsilon} (b-y)^{(\alpha-1)q} u(y) dy\right)^{\frac{1}{q}} \leq A
$$

for all *b* and ε with $0 < \varepsilon < b$.

Compared to (2.2) and (2.6) [see the proof of Corollary 2.8] this last condition is more delicate to check. To justify this claim consider again the case of power weights $u(y)$ = As seen in the introduction, for $u(\cdot) \in A_{\infty}^{+}$ and $p < q$, by a result due to

nd A. de la Torre [6], the boundedness $W_{\alpha}: L^{p}((0, \infty), v(x) dx) \to L^{q}((0, \infty))$

equivalent to
 $\left(\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \left(\int_{0}^{b-\epsilon} ($ $\left(\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \left(\int_0^{b-\epsilon} (b-y)^{(\alpha-1)q} u(y) dy\right)^{\frac{1}{q}} \leq A$
for all b and ϵ with $0 < \epsilon < b$.
Compared to (2.2) and (2.6) [see the proof of Corollary 2.8] this last condition is more
delicate to check. To case it is used that $v(y) \approx b^{b-1}$ for $b-\varepsilon < y < b+\varepsilon$. The term $\int_0^{b-\varepsilon} (b-y)^{(\alpha-1)q} u(y) dy$ is more subtle to estimate than the first one. For $\frac{1}{2}b \leq \varepsilon < b$ the main point is $(b-y) \approx b$ the proof of Corollary 2.8] this last con

im consider again the case of power w
 $\frac{1-e}{e}v^{1-p'}(y) dy$ is evaluated following
 $\frac{b+e}{e}v^{1-p'}(y) dy \leq \int_0^{2b} v^{1-p'}(y) dy$ and
 $-e < y < b + e$. The term $\int_0^{b-\epsilon} (b-y)^{c}$

irst one. *e* Corollary 2.8] this last condition is more again the case of power weights $u(y) = dy$ is evaluated following $\frac{1}{2}b \le \varepsilon < b$ or $dy \le \int_0^{2b} v^{1-p'}(y) dy$ and in the second $+ \varepsilon$. The term $\int_0^{b-\varepsilon} (b-y)^{(\alpha-1)q} u(y) dy$ $y^{\beta-1}$ and $v(y) = y^{\delta-1}$. The term $\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) dy$ is evaluated following $\frac{1}{2}b \leq \epsilon < b$ or $0 < \epsilon < \frac{1}{2}b$. In the first case then $\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) dy \leq \int_0^{2b} v^{1-p'}(y) dy$ and in the second case it is us for $0 < \varepsilon < \frac{1}{2}b$ then ared to (2.2) and (2.6) [see the proof of $v(y) = y^{\delta-1}$. The term $\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) dy$.
 b. In the first case then $\int_{b-\epsilon}^{b+\epsilon} v^{1-p'}(y) dy$

ased that $v(y) \approx b^{\delta-1}$ for $b-\epsilon < y < b$

btle to estimate than the first one. for $0 < y < b - \varepsilon \leq \frac{1}{2}b$ and then $\int_0^{b-\varepsilon} (b-y)^{(\alpha-1)q} u(y) dy \leq c b^{(\alpha-1)q} \int_0^b u(y) dy$. And m consider again the case of pow
 $\int_{\epsilon}^{\epsilon} v^{1-p'}(y) dy$ is evaluated follow
 $\int_{\epsilon}^{\epsilon} v^{1-p'}(y) dy \leq \int_{0}^{2b} v^{1-p'}(y) dy$
 $\epsilon \in \epsilon \leq y < b + \epsilon$. The term $\int_{0}^{b-\epsilon} (b - y)^{(\alpha-1)q} u(y) dy \leq c b^{(\alpha-1)}$
 $\int_{\frac{1}{2}b}^{b} + \int_{\frac{1}{2}b}^{$ *y*) *dy* is evaluated followi
 y) *dy* $\leq \int_0^{2b} v^{1-p'}(y) dy$ a
 b + ε . The term $\int_0^{b-\epsilon} (b-\cot \frac{1}{2}b) \leq \varepsilon < b$ the main p
 $(\alpha-1)q(u(y)) dy \leq c b^{(\alpha-1)q}$
 $\qquad \qquad -\epsilon$
 $(b-y)^{(\alpha-1)q}u(y) dy$
 $\int_0^{\frac{1}{2}b} u(y) dy + b^{(\beta-1$

$$
\frac{1}{2} \frac{1
$$

So the sequel of computations depends on the sign of $(\alpha - 1)q + 1$. For instance, if $(\alpha - 1)q + 1 < 0$, then $\int_{\epsilon}^{\frac{1}{2}b} t^{(\alpha - 1)q} dt \approx \epsilon^{(\alpha - 1)q+1}$ for $\epsilon \to 0$.

All of these considerations lead to think that the conditions used in our results are quite easy to apply for explicit computations compared to known results.

Weights which are not necessarily of power type can be treated by the above results.

Corollary 2.9. Let $p \leq q$, with $q \leq p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Define the weight *functions* reply for explicit computations compared to known results.

which are not necessarily of power type can be treated by the a

ry 2.9. Let $p \le q$, with $q \le p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Defin
 $u(x) = x^{\beta-1}1_{(0,\frac{1}{2})}(x) +$ *v*(*x*) = $x^{\beta-1}1_{(0,\frac{1}{2})}(x) + x^{\gamma-1}1_{(\frac{1}{2},\infty)}(x)$ *with* $(1-\alpha)q < \beta$
 v(*x*) = $x^{p-1}\ln^p(x^{-1})1_{(0,\frac{1}{2})}(x) + x^{\theta-1}1_{(\frac{1}{2},\infty)}(x)$ *with* $\theta < p$.
 v(*x*) = $x^{p-1}\ln^p(x^{-1})1_{(0,\frac{1}{2})}(x) + x^{\theta-1}1_{(\frac{1}{2},\infty)}($

$$
u(x) = x^{\beta - 1} \mathbf{1}_{(0, \frac{1}{2})}(x) + x^{\gamma - 1} \mathbf{1}_{(\frac{1}{2}, \infty)}(x) \quad \text{with} \quad (1 - \alpha)q < \beta
$$

and

$$
v(x) = x^{p-1} \ln^p (x^{-1}) \mathbf{1}_{(0,\frac{1}{2})}(x) + x^{\theta-1} \mathbf{1}_{(\frac{1}{2},\infty)}(x) \quad \text{with} \ \theta < p.
$$

The boundedness

$$
R_\alpha:\ L^p\big((0,\infty),v(x)\,dx\big)\rightarrow L^q\big((0,\infty),u(x)\,dx\big)
$$

holds if and only if $\gamma < (1 - \alpha)q < \beta$ and

$$
\alpha + \frac{\gamma}{q} \le \frac{\theta}{p}.\tag{2.13}
$$

What is remarkable in this example is the fact that $\int_0^R v^{(1-p')\epsilon}(y) dy = \infty$ for $R <$ and $\epsilon > 1$. So this boundedness cannot be treated by using a bumping condition like

$$
u(x) = x^{\beta - 1} 1_{(0, \frac{1}{2})}(x) + x^{\gamma - 1} 1_{(\frac{1}{2}, \infty)}(x) \quad \text{with } (1 - \alpha)q < \beta
$$
\n
$$
v(x) = x^{p-1} \ln^p(x^{-1}) 1_{(0, \frac{1}{2})}(x) + x^{\theta - 1} 1_{(\frac{1}{2}, \infty)}(x) \quad \text{with } \theta < p.
$$
\nboundedness

\n
$$
R_{\alpha} : L^p((0, \infty), v(x) \, dx) \to L^q((0, \infty), u(x) \, dx)
$$
\nif and only if $\gamma < (1 - \alpha)q < \beta$ and

\n
$$
\alpha + \frac{\gamma}{q} \leq \frac{\theta}{p}.
$$
\nWhat is remarkable in this example is the fact that $\int_0^R v^{(1-p')\epsilon}(y) dy = \infty$ for l :

\n
$$
z > 1. So this boundedness cannot be treated by using a bumping condition
$$
\n
$$
t^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\frac{1}{t} \int_{x_0 - t}^{x_0 + t} u^{\epsilon}(z) dz \right)^{\frac{1}{\epsilon q}} \left(\frac{1}{t} \int_{x_0 - t}^{x_0 + t} v^{(1-p')\epsilon}(z) dz \right)^{\frac{1}{\epsilon p'}} \leq A \quad \forall \ t, x_0 > 0
$$
\nis introduced and used in [11] to treat weighted inequalities for the two

\nators $I_{\alpha} = R_{\alpha} + W_{\alpha}$.

as it is introduced and used in [11] to treat weighted inequalities for the two-sided operators $I_{\alpha} = R_{\alpha} + W_{\alpha}$.

Finally, we give an example for the case $q < p$ which is new since it seems there is no available papers which treats the problem for this case.

Corollary 2.10. Let $q < p$ and $r = \frac{qp}{p-q}$. Let $u(x) = x^{\beta-1}1_{(0,1)}(x) + x^{\gamma-1}1_{(1,\infty)}(x)$ *and* $v(x) = x^{\delta-1}$ *. The boundedness Let u*(x)* = $x^{(1-q)(\delta-1)}$ *Let u*(x)* = $x^{(1-q)(\delta-1)}$ *Let u*(x)* = $x^{(1-q)(\delta-1)}$ *Let* $p(x) = x^{(1-q)(\delta-1)}$ *<i>Let u*(x)* = $x^{(1-q)(\delta-1)}$ *Let u*(x)* = $x^{(1-q)(\delta-1)}$ *Let u*(x)* = $x^{(1-q)(\delta-1)}$ *Let u*(x)* = $x^{(1-q)(\delta-1)}$ *L* Finally, we give an example for the case $q < p$ which is new sine
 D available papers which treats the problem for this case.
 Corollary 2.10. Let $q < p$ and $r = \frac{qp}{p-q}$. Let $u(x) = x^{\beta-1}1_{(0,1)}(dx)$
 nd $v(x) = x^{\delta-1}$

$$
R_{\alpha}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)
$$

holds whenever both $\delta < p$ and $\gamma < (1 - \alpha)q$ and

$$
\alpha + \frac{\gamma}{q} < \frac{\delta}{p} < \alpha + \frac{\beta}{q}.\tag{2.14}
$$

boundedness

$$
W_{\alpha}: L^p((0,\infty),v^*(x) dx) \to L^q((0,\infty),u^*(x) dx)
$$

holds whenever both $\delta < q'$ and $\gamma < (1 - \alpha)p'$ and

$$
\alpha + \frac{\gamma}{p'} < \frac{\delta}{q'} < \alpha + \frac{\beta}{p'}.\tag{2.15}
$$

For an explicit example suppose

$$
\alpha + \frac{\gamma}{p'} < \frac{\delta}{q'} < \alpha + \frac{\beta}{p'}.
$$
\nFor an explicit example, suppose

\n
$$
0 < \alpha < \frac{2}{8}, \quad u(x) = x^{\beta - 1} \mathbf{1}_{[0,1]}(x) + x^{\gamma - 1} \mathbf{1}_{[1,\infty)}(x), \quad \gamma = \frac{1}{4}, \quad \beta > 2, \quad v(x) = x^{\frac{1}{2}}.
$$
\nThen

\n
$$
R_{\alpha}: L^{4}((0, \infty), v(x) \, dx) \to L^{2}((0, \infty), u(x) \, dx).
$$
\n(2.20)

3. Results for convolution operators

In this section the results in Section *2* are generalized for convolution operators like

$$
(Tf)(x) = \int_{-\infty}^{x} K(x-y)f(y) dy
$$

where *K is* a non-negative kernel quasi-decreasing, i.e.

It is in Section 2 are generalized for convolution operators like

\n
$$
(Tf)(x) = \int_{-\infty}^{x} K(x - y)f(y) \, dy
$$
\nitive kernel quasi-decreasing, i.e.

\n
$$
K(R_2) \le c \, K(R_1), \qquad \text{for all } 0 < R_1 \le R_2 \tag{3.1}
$$
\nwith condition

and satisfying the growth condition

$$
K(R) \le c K(2R) \qquad \text{for all} \quad R > 0. \tag{3.2}
$$

for all $0 < R_1 \le R_2$ (3.1)

for all $R > 0$. (3.2)

only on the kernel $K(\cdot)$. Without (3.2)

(*R*) or $K(x)$ is replaced by $K(CR)$ or In (3.1) and (3.2) the constant $c > 0$ depends only on the kernel $K(\cdot)$. Without (3.2) our results remain true if in each occurence $K(R)$ or $K(x)$ is replaced by $K(CR)$ or $K(Cx)$, respectively, where $C > 0$ is a constant depending only on $K(\cdot)$, *p* and *q*.

Our purpose, in this section, is to study the boundedness

$$
T: L^p((-\infty,\infty),v(x)\,dx) \to L^q((-\infty,\infty),u(x)\,dx)
$$

which means

$$
\left(\int_{-\infty}^{\infty} (Tf)^{q}(x)u(x) dx\right)^{\frac{1}{q}} \le C \left(\int_{-\infty}^{\infty} f^{p}(x)v(x) dx\right)^{\frac{1}{p}} \quad \text{for all } f(\cdot) \ge 0. \tag{3.3}
$$

Of course, here $C > 0$ is a fixed constant. For shortness, we will restrict to the range

 $p \leq q$.

The case *q < p* can be also treated as it is done in Section *2* for the Riemann-Liouville and Weyl operators.

First a necessary and sufficient conditions for $T : L^p((-\infty,\infty), v(x) dx) \to L^q((-\infty,$ ∞ , $u(x) dx$) is stated.

Theorem 3.1. *The boundedness*

$$
T: L^p((-\infty,\infty),v(x) dx) \to L^q((-\infty,\infty),u(x) dx)
$$

holds if and only if, for a constant A > 0, the three conditions

by equations:

\nLet
$$
A = \text{c} \cos(\ln x)
$$
 and $A = \int_{0}^{1} f(x) \, dx$ for $T: L^{p}((-\infty, \infty), v(x) \, dx) \to L^{q}((-\infty, \infty), v(x) \, dx) \to L^{q}((-\infty, \infty), v(x) \, dx) \to L^{q}((-\infty, \infty), u(x) \, dx)$

\nand only if, for $a \text{ constant } A > 0$, the three conditions

\n
$$
\left(\int_{2R}^{\infty} K^{q}(x) u(x) \, dx \right)^{\frac{1}{q}} \left(\int_{0}^{R} [v^{1-p'}(x) + v^{1-p'}(-x)] dx \right)^{\frac{1}{p'}} \leq A \quad \forall \, R > 0 \quad (3.4)
$$
\n
$$
\left(\int_{0}^{R} [u(x) + u(-x)] dx \right)^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x) v^{1-p'}(-x) dx \right)^{\frac{1}{p'}} \leq A \quad \forall \, R > 0 \quad (3.5)
$$
\n
$$
K(R) \left(\int_{\frac{1}{2}R}^{2R} u(x) dx \right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(-x) dx \right)^{\frac{1}{p'}} \leq A \quad \forall \, R > 0 \quad (3.6)
$$

$$
\left(\int_0^R [u(x) + u(-x)]dx\right)^{\frac{1}{q}} \left(\int_{2R}^\infty K^{p'}(x)v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \leq A \quad \forall \ R > 0 \quad (3.5)
$$

$$
K(R)\bigg(\int_{\frac{1}{2}R}^{2R} u(x)\,dx\bigg)^{\frac{1}{q}}\bigg(\int_0^R v^{1-p'}(-x)\,dx\bigg)^{\frac{1}{p'}}\leq A \quad \forall R>0 \quad (3.6)
$$

and

$$
\widetilde{T}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)
$$

$$
\widetilde{T}^*: L^p((0,\infty),v(-x) dx) \to L^q((0,\infty),u(-x) dx)
$$

are satisfied, where

$$
\widetilde{T}^* : L^p((0,\infty), v(-x) dx) \to L^q((0,\infty), u(-x) dx)
$$

tisfied, where

$$
(\widetilde{T}f)(x) = \int_{\frac{1}{2}x}^{x} K(x-y)f(y) dy \quad and \quad (\widetilde{T}^*f)(x) = \int_{x}^{2x} K(z-x)f(z) dz.
$$

general, the three conditions (3.4) - (3.6) do not overlap. Indeed, take for in

 $\mathcal{O}(\mathcal{O}(\mathcal{O}(\log n)^{1/2}))$

In general, the three conditions (3.4) \cdot (3.6) do not overlap. Indeed, take for instance $K(x) = |x|^{\alpha-1}$ and $u(x) = |x|^{\beta-1}$. Then (3.4) can only be held whenever at least $\beta < q(1-\alpha)$. For (3.5) it is needed that $\beta > 0$ which is not a priori the case for (3.6).

Although a characterization of weights $u(\cdot)$ and $v(\cdot)$ for which $\widetilde{T}: L^p((0,\infty),v(x))$ $dx) \rightarrow L^q((0,\infty), u(x) dx)$ is an open problem, it is not too difficult to derive a sufficient *are satisfied, where*
 $(\widetilde{T}f)(x) = \int_{\frac{1}{2}x}^{x} K(x - y)f(y) dy$ and $(\widetilde{T}^*f)(x) =$

In general, the three conditions (3.4) - (3.6) do not over
 $K(x) = |x|^{\alpha - 1}$ and $u(x) = |x|^{\beta - 1}$. Then (3.4) can only
 $\beta < q(1 - \alpha)$. For (3. $|z| \propto |x|^{\alpha-1}$ and $u(x) = |x|^{\beta-1}$. Then $(1-\alpha)$. For (3.5) it is needed that β
 d D D (0, \comparity) a characterization of weight $\rightarrow L^q((0, \infty), u(x) dx)$ is an open probtion. This last one depends highly cs, goi

Proposition 3.2. Assume that, for some $\varepsilon \in [0,1]$ and $c > 0$,

$$
dx_j \to L^1((0, \infty), u(x_j) dx)
$$
 is an open problem, it is not too a
condition. This last one depends highly on further properties of the kernel K. So two
results, going in this direction, are given.
Proposition 3.2. Assume that, for some $\varepsilon \in [0, 1]$ and $\varepsilon > 0$,

$$
\int_0^R K^{\varepsilon q}(z)dz \le cR \times K^{\varepsilon q}(R), \qquad \int_0^R K^{(1-\varepsilon)p'}(z)dz \le cR \times K^{(1-\varepsilon)p'}(R).
$$
 (3.7)
The boundedness
 $\widetilde{T}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$

The boundedness

$$
\widetilde{T}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)
$$

holds whenever for a constant A > 0

$$
K^{\epsilon q}(z)dz \le cR \times K^{\epsilon q}(R), \qquad \int_{0}^{R} K^{(1-\epsilon)p'}(z)dz \le cR \times K^{(1-\epsilon)p'}(R). \tag{3.7}
$$

\n
$$
\tilde{T}: L^{p}((0,\infty), v(x) dx) \to L^{q}((0,\infty), u(x) dx)
$$

\n
$$
e^{i\pi + \frac{1}{p'}} K(R) \left(\sup_{R \le z \le 2R} u(z)\right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y)\right)^{\frac{1}{p'}} \le A \quad \forall R > 0. \tag{3.8}
$$

\n
$$
\tilde{T}^* : L^{p}((0,\infty), v(-x) dx) \to L^{q}((0,\infty), u(-x) dx)
$$

Similarly,

$$
\widetilde{T}^*: L^p((0,\infty),v(-x)\,dx)\to L^q((0,\infty),u(-x)\,dx)
$$

holds whenever

$$
R^q \xrightarrow{p'} K(R) \begin{pmatrix} \sup_{R < z < 2R} u(z) & \sup_{\frac{1}{2}R < y < 2R} v^{1-p} \left(y \right) \end{pmatrix} \leq A \quad \forall R > 0. \tag{3.8}
$$
\n
$$
arly,
$$
\n
$$
\widetilde{T}^* : L^p \left((0, \infty), v(-x) \, dx \right) \to L^q \left((0, \infty), u(-x) \, dx \right)
$$
\n
$$
wherever
$$
\n
$$
R^{\frac{1}{q} + \frac{1}{p'}} K(R) \begin{pmatrix} \sup_{\frac{1}{2}R < z < 2R} u(-z) \end{pmatrix}^{\frac{1}{q'}} \begin{pmatrix} \sup_{R < y < 2R} v^{1-p'}(-y) \end{pmatrix}^{\frac{1}{p'}} \leq A \quad \forall R > 0. \tag{3.9}
$$
\n
$$
\text{or } K(x) = x^{\alpha - 1} \quad (0 < \alpha < 1) \text{ then (3.7) can hold whenever}
$$
\n
$$
\left(\frac{1}{p} - \alpha \right) < (1 - \alpha) \varepsilon < \frac{1}{q}. \tag{3.10}
$$
\n
$$
\text{for } \frac{1}{\alpha} \leq p, \text{ condition (3.7) is always satisfied whenever } 0 \leq \varepsilon < \min(1, \frac{1}{1 - \alpha} \frac{1}{q}). \text{ And}
$$

For $K(x) = x^{\alpha-1}$ $(0 < \alpha < 1)$ then (3.7) can hold whenever

$$
\left(\frac{1}{p}-\alpha\right) < (1-\alpha)\varepsilon < \frac{1}{q}.\tag{3.10}
$$

For $K(x) = x^{\alpha - 1}$ $(0 < \alpha < 1)$ then (3.7) can hold whenever
 $\left(\frac{1}{p} - \alpha\right) < (1 - \alpha)\varepsilon < \frac{1}{q}$ (3.10)

Thus for $\frac{1}{\alpha} \leq p$, condition (3.7) is always satisfied whenever $0 \leq \varepsilon < \min(1, \frac{1}{1 - \alpha} \frac{1}{q})$. And

for p for $p < \frac{1}{\alpha}$, a necessary condition for (3.10) is $\frac{1}{p^*} = \frac{1}{p} - \alpha < \frac{1}{q}$ or $q < p^*$. Consequently, the boundedness \widetilde{T} : $L^p((0,\infty),v(x) dx) \to L^{p^*}((0,\infty),u(x) dx)$ cannot be decided *from Proposition* 3.2, *and* another kind *of criterion is needed.*

Proposition 3.3. Assume that for some $\bar{p} \geq q$ and $c > 0$

Weighted Norm Inequalities
\n
$$
\text{Using } \mathcal{L} \text{ is a function of } \mathcal{L} \text{ with } \mathcal{L} \text{ is a function of } \mathcal{L} \text{ with } \mathcal
$$

and

Weighted Norm Inequalities 813
\n*ssume that for some*
$$
\overline{p} \ge q
$$
 and $c > 0$
\n $T: L^p((0, \infty), dx) \to L^{\overline{p}}((0, \infty), dx)$ (3.11)
\n $1 \le c R^{\frac{1}{p'} + \frac{1}{p}} K(R)$ for all $R > 0$. (3.12)
\n*lines the boundedness*

Then condition (3.8) implies the boundedness

$$
\widetilde{T}: L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx).
$$

Similarly, condition (3.9) yields

$$
\cdot \widetilde{T}^{\bullet} : L^p\big((0,\infty),v(-x)\,dx\big) \to L^q\big((0,\infty),u(-x)\,dx\big)
$$

whenever

 $1 \le c R^{\frac{1}{p'} + \frac{1}{p}} K(R)$ for all $R > 0$. (3.12)
 Ites the boundedness
 $L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)$.
 yields
 $f'(0,\infty),v(-x) dx) \to L^q((0,\infty),u(-x) dx)$
 $T^* : L^p((0,\infty),dx) \to L^{\overline{p}}((0,\infty),dx)$ (3.13)

only introduced in orde

and (3.12) is satisfied.

Hypothesis *(3.12) is* only introduced in order to have the same sufficient conditions in Propositions *3.2* and *3.3.* Without *(3.12)* it will be seen in the proof that [with (3.11)] $\text{the boundedness $\widetilde T:\, L^p\big((0,\infty),v(x)\,dx\big)\to L^q\big((0,\infty),u(x)\,dx\big)$ holds whenever}$ 2) is only introduced in order to have the same sufficient c
and 3.3. Without (3.12) it will be seen in the proof that [wi
 $\therefore L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$ holds whenev

sup $u(z)$) $\int_0^1 \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y)\right$ *R<z<2R H<y<2H*

$$
R^{\frac{1}{q}-\frac{1}{p}}\Big(\sup_{R0.
$$

Now these results are applied to the case of even and quasi-monotone weights. Here $w(\cdot)$ is said to be an even and quasi-monotone weight if $w(x) = w_0(x)$ for $x > 0$, $w(-x) = w_0(x)$ and where $w_0(\cdot)$ is quasi-monotone on $(0, \infty)$. Remind that the quasidecrease is taken in the sense of (3.1) [and $\varphi(\cdot)$ is quasi-increasing if $\frac{1}{\varphi}(\cdot)$ is quasidecreasing].

Proposition 3.4. *Assume that property (3.7) is fulfilled or all three conditions* $(3.11) - (3.13)$ are satisfied (so in this last case $p \leq q \leq \overline{p}$). Suppose that $u(\cdot)$ and $v^{1-p'}$.) are even and quasi-monotone weight functions. The boundedness

$$
T: L^p((-\infty,\infty),v(x) dx) \to L^q((-\infty,\infty),u(x) dx)
$$

holds if and only if the following three conditions are satisfied:

position 3.4. Assume that property (3.7) is fulfilled or all three conditions

\n(3.13) are satisfied (so in this last case
$$
p \leq q \leq \overline{p}
$$
). Suppose that $u(\cdot)$ and $u(\cdot)$ and $u(\cdot)$ and $u(\cdot) = \int_0^\infty f(x) \, dx$.

\n
$$
T: L^p((-\infty, \infty), v(x) \, dx) \to L^q((-\infty, \infty), u(x) \, dx)
$$

\nand only if the following three conditions are satisfied:

\n
$$
\left(\int_{2R}^{\infty} K^q(x) u(x) \, dx\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(x) \, dx\right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0 \quad (3.14)
$$

\n
$$
\left(\int_0^R u(x) \, dx\right)^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x) v^{1-p'}(x) \, dx\right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0 \quad (3.15)
$$

$$
\left(\int_0^R u(x)\,dx\right)^{\frac{1}{q}}\left(\int_{2R}^\infty K^{p'}(x)v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}}\leq A\qquad\text{for all }R>0\qquad(3.15)
$$

$$
\left(\int_{2R}^{R} h'(x)u(x) dx\right) \left(\int_{0}^{R} v'(x) dx\right) \leq A \quad \text{for all } R > 0 \quad (3.14)
$$
\n
$$
\left(\int_{0}^{R} u(x) dx\right)^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x) v^{1-p'}(x) dx\right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0 \quad (3.15)
$$
\n
$$
K(R) \left(\int_{\frac{1}{2}R}^{2R} u(x) dx\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(x) dx\right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \quad (3.16)
$$

Better, if $u(\cdot)$ *is quasi-increasing or* $v^{1-p'}(\cdot)$ *is quasi-decreasing, then*

$$
T: L^p((-\infty,\infty),v(x)\,dx\big)\to L^q((-\infty,\infty),u(x)\,dx\big)
$$

if and only both (3.14) and (3.15) are satisfied. This last equivalence remains true whenever both $u(\cdot)$ and $v^{1-p'}(\cdot)$ are quasi-decreasing with $u(x) \leq c u(2x)$ or $v^{1-p'}(x) \leq$ $cv^{1-p'}(2x)$, respectively, for a fixed constant $c > 0$. *both* (3.14) and (3.15) are satisfied. This last
h $u(\cdot)$ and $v^{1-p'}(\cdot)$ are quasi-decreasing with $u(\cdot)$
respectively, for a fixed constant $c > 0$.
t the conditions (3.14) and (3.16) can be combin
 $\left(\int_{\frac{1}{2}R}^{\infty}$

Note that the conditions *(3.14)* and *(3.16)* can be combined as

$$
\left(\int_{\frac{1}{2}R}^{\infty} K^{q}(x)u(x)\,dx\right)^{\frac{1}{q}}\left(\int_{0}^{R} v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \quad \text{for all } R>0.
$$

Since the class C (see Section 2) is larger than that of quasi-monotone weights, it would be interesting to state results for weights belonging to this class.

Proposition 3.5. *Assume that property (3.7) is fulfilled or the conditions (3.11)-* (3.13) are satisfied (so in this last case $p \le q \le \overline{p}$). Suppose that $u(\cdot)$, $v^{1-p'}(\cdot)$, $u(-)$, $v^{1-p'}(-) \in \mathcal{C}$ with the (integer) constant $N \geq 2$. Then $(3.4) - (3.6)$ are necessary and *sufficient conditions for the boundedness n* this last case $p \le q \le \overline{p}$
 \ge (integer) constant $N \ge 2$.
 Cor the boundedness
 $L^p((-\infty,\infty), v(x) dx) \to L^q$
 $\left(\frac{R}{2^{-2NR}}v^{1-\frac{1}{2}}\right)$
 $\frac{u(x) dx}{\left(\int_{2^{-2NR}}^R v^{1-p'}\right)}$
 $\frac{u(-x) dx}{\left(\int_{2^{-2NR}}^R v^{1-p'}\right)}$ 2) is larger than that of ϵ
 I is for weights belonging to
 It property (3.7) is fulfilled
 Case $p \le q \le \overline{p}$). Suppose
 nstant $N \ge 2$. Then (3.4)
 I I I I I I I I ((- ∞ , ∞),
 \int_{0}^{\frac

$$
T: L^p((-\infty,\infty),v(x)\,dx)\to L^q((-\infty,\infty),u(x)\,dx)
$$

to hold whenever

e satisfied (so in this last case
$$
p \le q \le \overline{p}
$$
). Suppose that $u(\cdot)$, $v^{1-p}(\cdot)$, $u(-\cdot)$,
\n $\in C$ with the (integer) constant $N \ge 2$. Then (3.4) – (3.6) are necessary and
conditions for the boundedness
\n $T: L^p((-\infty, \infty), v(x) dx) \to L^q((-\infty, \infty), u(x) dx)$
\nhenever
\n $K(R) \Biggl(\int_{2^{-2N}R}^{R} u(x) dx \Biggr)^{\frac{1}{q}} \Biggl(\int_{2^{-2N}R}^{R} v^{1-p'}(x) dx \Biggr)^{\frac{1}{p'}} \le A \quad \forall R > 0 \quad (3.17)$
\n $K(R) \Biggl(\int_{2^{-2N}R}^{R} u(-x) dx \Biggr)^{\frac{1}{q}} \Biggl(\int_{2^{-2N}R}^{R} v^{1-p'}(-x) dx \Biggr)^{\frac{1}{p'}} \le A \quad \forall R > 0. \quad (3.18)$
\nthe dual operator T^* of T defined by

$$
K(R)\left(\int_{2^{-2N}R}^{\Lambda}u(-x)\,dx\right)^{\frac{1}{q}}\left(\int_{2^{-2N}R}^{\Lambda}v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}}\leq A \quad \forall R>0. \tag{3.18}
$$

For the dual operator *T** of *T* defined by

$$
(T^*f)(x) = \int_x^{\infty} K(y-x)f(y) dy
$$

similar results for the boundedness $T^*: L^p((-\infty,\infty), v(x) dx) \to L^q((-\infty,\infty), u(x) dx)$ could be also obtained, by using its equivalence with $T: L^{q'}((-\infty, \infty), u^{1-q'}(x) dx) \rightarrow$ $L^{p'}((-\infty,\infty),v^{1-p'}(x)dx)$. Just the analogous of Theorem 3.1 is stated.

Theorem 3.6. *The boundedness*

$$
T^*: L^p((-\infty,\infty),v(x)\,dx)\to L^q((-\infty,\infty),u(x)\,dx)
$$

holds if and only if for a constant $A > 0$ *the three conditions*

results for the boundedness
$$
T^*: L^p((-\infty, \infty), v(x) dx) \to L^q((-\infty, \infty), u(x) dx)
$$

\ne also obtained, by using its equivalence with $T: L^{q'}((-\infty, \infty), u^{1-q'}(x) dx) \to$
\n $\infty, \infty), v^{1-p'}(x) dx$. Just the analogous of Theorem 3.1 is stated.
\neorem 3.6. The boundedness
\n $T^*: L^p((-\infty, \infty), v(x) dx) \to L^q((-\infty, \infty), u(x) dx)$
\nand only if for a constant $A > 0$ the three conditions
\n
$$
\left(\int_{2R}^{\infty} K^{p'}(x) v^{1-p'}(x) dx\right)^{\frac{1}{p'}} \left(\int_{0}^{R} [u(x) + u(-x)] dx\right)^{\frac{1}{q}} \leq A \quad \forall R > 0 \quad (3.19)
$$
\n
$$
\int_{0}^{R} [v^{1-p'}(x) + v^{1-p'}(-x)] dx \right)^{\frac{1}{p'}} \left(\int_{0}^{\infty} K^q(x) u(-x) dx\right)^{\frac{1}{q}} \leq A \quad \forall R > 0 \quad (3.20)
$$

$$
\left(\int_0^{\Lambda} [v^{1-p'}(x) + v^{1-p'}(-x)] dx\right)^{p'} \left(\int_{2R}^{\infty} K^q(x) u(-x) dx\right)^{q} \leq A \quad \forall R > 0 \quad (3.20)
$$

$$
\begin{aligned}\n\binom{1}{x} \binom{1}{y} \binom{1}{x} \binom{1}{y} \bin
$$

and

$$
\widetilde{T}: L^{q'}((0,\infty), u^{1-q'}(x) dx) \to L^{p'}((0,\infty), v^{1-p'}(x) dx)
$$

$$
\widetilde{T}^*: L^{q'}((0,\infty), u^{1-q'}(-x) dx) \to L^{p'}((0,\infty), v^{1-p'}(-x) dx)
$$

are satisfied.

4. Proofs of Results

First a useful lemma for the proofs is given. Next we will prove the results for convolutions operators stated in Section 3. The last place is devoted to the proofs of results in Section 2 which are not direct consequences of those in Section 3. **Its**
the proofs is given the proofs is given 3. The direct consequent
tate the classion coofs.
Paray operato
 $\int_{0}^{\frac{1}{2}x} f(y) dy$
 $\leq p$, **s**
 s
 e proofs is given. Next we will prove the

ection 3. The last place is devoted to the

rect consequences of those in Section 3.

the the classical Hardy inequalities [10]

pofs.
 Hardy operators H and H by*

forms as needed in the proofs.

Lemma. *Define the Hardy operators H and H^{*} by*

It is convenient to state the classical Hardy inequalities [10] in the appropriate
is as needed in the proofs.
Lemma. *Define the Hardy operators H and H*^{*} by

$$
(Hf)(x) = \int_0^{\frac{1}{2}x} f(y) dy \quad and \quad (H^*f)(x) = \int_{2x}^{\infty} f(y) dy.
$$

Then:

(A) For $p \leq q$ or $q < p$,

$$
H: L^p((0,\infty),v(x)\,dx)\to L^q\big((0,\infty),w(x)\,dx\big)
$$

if and only if, for a constant $A > 0$ *and all* $R > 0$,

$$
L^p((0,\infty), v(x) dx) \to L^q((0,\infty), w(x) dx)
$$

\n
$$
m\tanh A > 0 \text{ and all } R > 0,
$$

\n
$$
\left(\int_{2R}^{\infty} w(y) dy\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \leq A
$$

\n
$$
\int_{x}^{\infty} w(y) dy \right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y) dy\right)^{\frac{1}{q'}} \Big|_{v=1-p'}^r(x)
$$

or

 \cdot

$$
H: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), w(x) dx)
$$

if, for a constant $A > 0$ and all $R > 0$,

$$
\left(\int_{2R}^{\infty} w(y) dy\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \leq A
$$

$$
\int_0^{\infty} \left[\left(\int_{2x}^{\infty} w(y) dy\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y) dy\right)^{\frac{1}{q'}}\right]^{r} v^{1-p'}(x) dx \leq A^r,
$$

respectively.

(B) *Similarly, for* $p \le q$ *or* $q < p$,

$$
H^*: L^p((0,\infty),w(x) dx) \to L^q((0,\infty),u(x) dx)
$$

if and only if, for a constant $A > 0$ *and all* $R > 0$

$$
w(y) dy \int \int_0^{\infty} v^{k-r} (y) dy \int v^{k-r} (x)
$$

\n
$$
p \le q \text{ or } q < p,
$$

\n
$$
L^p((0, \infty), w(x) dx) \to L^q((0, \infty), u(x) dx)
$$

\n
$$
m \int_{2R}^{\infty} w^{1-p'} (y) dy \int_0^{\frac{1}{p'}} \left(\int_0^R u(y) dy \right)^{\frac{1}{q}} \le A
$$

\n
$$
\int_0^{\infty} w^{1-p'} (y) dy \int_0^{\frac{1}{p'}} \left(\int_0^x u(y) dy \right)^{\frac{1}{p}} \left[\int_0^x u(x) dy \right]^{\frac{1}{p}} dx
$$

or

$$
H^* : L^p((0,\infty), w(x) dx) \to L^q((0,\infty), u(x) dx)
$$

for a constant $A > 0$ and all $R > 0$,

$$
\left(\int_{2R}^{\infty} w^{1-p'}(y) dy\right)^{\frac{1}{p'}} \left(\int_0^R u(y) dy\right)^{\frac{1}{q}} \leq A
$$

$$
\int_0^{\infty} \left[\left(\int_{2x}^{\infty} w^{1-p'}(y) dy\right)^{\frac{1}{p'}} \left(\int_0^x u(y) dy\right)^{\frac{1}{p}}\right]^r u(x) dx \leq A^r,
$$

respectively.

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ectively.
(C) Analogously, for
$$
p \le q
$$
 and $(\mathcal{H}f)(x) = \int_0^{2x} f(y) dy$,
 $\mathcal{H}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), w(x) dx)$
and only if, for a constant $A > 0$ and all $R > 0$,

$$
\left(\int_{\frac{1}{2}R}^{\infty} w(y) dy\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \leq A
$$

if and only if, for a constant $A > 0$ *and all* $R > 0$ *,*
 \int_{0}^{∞} \int_{0}^{1} \int_{0}^{R} \int_{0}^{R} \int_{0}^{R}

$$
\left(\int_{\frac{1}{2}R}^{\infty} w(y) \, dy\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(y) \, dy\right)^{\frac{1}{p'}} \leq A
$$

 $\bigcup_{\frac{1}{2}R}w(a)$
where $r=\frac{4P}{p-q}$ whenever $q < p$.

Proof of Theorem 3.1. To *get*

$$
\bigcup_{\substack{1 \to \infty}} \bigcup_{\mathfrak{p}} \bigcap_{\mathfrak{p}} \bigcap_{\mathfrak
$$

$$
\varphi(\cdot) = f(\cdot) + g(\cdot) \quad \text{with } f(\cdot) = \varphi(\cdot) \mathbf{1}_{(0,\infty)}(\cdot) \text{ and } g(\cdot) = \varphi(\cdot) \mathbf{1}_{(-\infty,0)}(\cdot),
$$

then

$$
\text{so estimate } \int_{-\infty}^{\infty} (T\varphi)^q(x) u(x) \, dx \text{ for any } \varphi(\cdot) \geq
$$
\n
$$
(\cdot) \qquad \text{with } f(\cdot) = \varphi(\cdot) \mathbf{1}_{(0,\infty)}(\cdot) \text{ and } g(\cdot) = \varphi(\cdot)
$$
\n
$$
\int_{-\infty}^{\infty} (T\varphi)^q(x) u(x) \, dx \approx \left\{ \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 \right\}
$$

where

$$
\varphi(\cdot) = f(\cdot) + g(\cdot) \quad \text{with } f(\cdot) = \varphi(\cdot) \mathbf{1}_{(0,\infty)}(\cdot) \text{ and } g(\cdot) = \varphi(\cdot) \mathbf{1}_{(-\infty,0)}(\cdot);
$$

then

$$
\int_{-\infty}^{\infty} (T\varphi)^q(x) u(x) dx \approx \left\{ S_1 + S_2 + S_3 + S_4 \right\}
$$

where

$$
S_1 = \int_{-\infty}^0 (Tf)^q(x) u(x) dx, \qquad S_2 = \int_0^{\infty} (Tf)^q(x) u(x) dx
$$

$$
S_3 = \int_{-\infty}^0 (Tg)^q(x) u(x) dx, \qquad S_4 = \int_0^{\infty} (Tg)^q(x) u(x) dx.
$$

So we have to bound each S_i $(i = 1, ..., 4)$ by $C(\int_{-\infty}^{\infty} \varphi^p(x) v(x) dx)^{\frac{q}{p}}$, where $C > 0$ is

a constant which does not depend on the function $\varphi(\cdot)$.

Estimate of S_1 : For $x < 0$, by the definition of $f(.)$,

$$
x < 0, \text{ by the definition of } f(\cdot),
$$
\n
$$
(Tf)(x) = \int_{-\infty}^{x} K(x - y)f(y) \, dy = 0
$$

and so $S_1 = 0$.

Estimate of S_2 **:** The purpose is to get

$$
= \int_{-\infty}^{1} (Tg)^{q}(x)u(x) dx, \qquad S_{4} = \int_{0}^{1} (Tg)^{q}(x)u(x) dx.
$$

\n
$$
= \int_{-\infty}^{1} (Tg)^{q}(x)u(x) dx, \qquad S_{4} = \int_{0}^{1} (Tg)^{q}(x)u(x) dx.
$$

\n
$$
= \int_{-\infty}^{1} K(i)u(x) dx, \qquad S_{4} = \int_{0}^{1} (Tg)^{q}(x)u(x) dx.
$$

\n
$$
= \int_{-\infty}^{1} K(x - y)f(y) dy = 0
$$

\n
$$
= \int_{-\infty}^{1} K(x - y)f(y) dy = 0
$$

\n
$$
= \int_{0}^{1} (Tf)^{q}(x)u(x) dx \leq C \left(\int_{0}^{1} f^{p}(x)v(x) dx\right)^{\frac{q}{p}}.
$$

\n
$$
= \int_{0}^{1} (Tf)^{q}(x)u(x) dx \leq C \left(\int_{0}^{1} f^{p}(x)v(x) dx\right)^{\frac{q}{p}}.
$$

\n
$$
= \int_{0}^{1} (1 + f^{q}(x))u(x) dx \leq C \left(\int_{0}^{1} (1 + f^{q}(x))u(x) dx\right)^{\frac{q}{p}}.
$$

\n
$$
= \int_{0}^{1} (1 + f^{q}(x))u(x) dx \leq C \left(\int_{0}^{1} (1 + f^{q}(x))u(x) dx\right)^{\frac{q}{p}}.
$$

\n(4.1)

= ¹⁰ *K(: y)f(y) dy + J 1t(x y)f(y) dy*

For each $x>0$ then

$$
(Tf)(x) = \int_0^{\frac{1}{2}x} K(x - y)f(y) dy + \int_{\frac{1}{2}x}^x K(x - y)f(y) dy
$$

$$
\approx K(x) \int_0^{\frac{1}{2}x} f(y) dy + \int_{\frac{1}{2}x}^x K(x - y)f(y) dy
$$

$$
= K(x)(Hf)(x) + (\tilde{T}f)(x).
$$

The equivalence is true since $\frac{1}{2}x < x - y < x$ for $0 < y < \frac{1}{2}x$ and the growth conditions (3.1) and (3.2) on $K(\cdot)$ lead to the conclusion. Consequently, inequality (4.1) holds if and only if

$$
H:L^p((0,\infty),v(x)\,dx\big)\to L^q((0,\infty),K^q(x)u(x)\,dx\big)
$$

and

 $\ddot{}$

 $\widetilde{T}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx).$

By Part A of the Lemma [with $w(x) = K^q(x)u(x)$] the first boundedness is true if and only if, for a constant $A > 0$,

$$
\widetilde{T}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx).
$$

A of the Lemma [with $w(x) = K^q(x)u(x)$] the first boundedness is true if and
or a constant $A > 0$,

$$
\left(\int_{2R}^{\infty} K^q(x)u(x) dx\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(x) dx\right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0. \qquad (4.2)
$$

quality is one part of condition (3.4), whose other part is

$$
\int_{2R}^{\infty} K^q(x)u(x) dx \right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(-x) dx\right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0. \qquad (4.3)
$$

This inequality is one part of condition (3.4), whose other part is

for a constant
$$
A > 0
$$
,
\n
$$
\left(\int_{2R}^{\infty} K^{q}(x)u(x) dx\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(x) dx\right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \quad (4.2)
$$
\n
$$
\text{nequality is one part of condition (3.4), whose other part is}
$$
\n
$$
\left(\int_{2R}^{\infty} K^{q}(x)u(x) dx\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(-x) dx\right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \quad (4.3)
$$
\n
$$
\text{where } \text{the series is a constant.}
$$

Note also that (4.1) is a necessary inequality for (3.3). So we have been proved that (4.2) and \widetilde{T} : $L^p((0,\infty),v(x)dx) \to L^q((0,\infty),u(x)dx)$ are necessary conditions for $T: L^p((-\infty,\infty),v(x)dx) \to L^q((-\infty,\infty),u(x)dx)$ to hold, and they are also sufficient to get (4.1).

Estimate of S_3 **:** Now the inequality under the consideration is

f
$$
S_3
$$
: Now the inequality under the consideration is
\n
$$
S_3 = \int_{-\infty}^0 (Tg)^q(x)u(x) dx \leq C \left(\int_{-\infty}^0 g^p(x)v(x) dx \right)^{\frac{q}{p}}.
$$
\n(4.4)

For each $x < 0$ then

Estimate of
$$
S_3
$$
: Now the inequality under the consideration is
\n
$$
S_3 = \int_{-\infty}^0 (Tg)^q(x)u(x) dx \leq C \left(\int_{-\infty}^0 g^p(x)v(x) dx \right)^{\frac{q}{p}}.
$$
\n(4.4)
\nFor each $x < 0$ then
\n
$$
(Tg)(x) = \int_{-\infty}^{2x} K(x - y)g(y) dy + \int_{2x}^x K(x - y)g(y) dy
$$
\n
$$
\approx \int_{-\infty}^{2x} K(-y)g(y) dy + \int_{2x}^x K(x - y)g(y) dy.
$$
\nIndeed, $-\frac{1}{2}y < x - y < -y$ for $y < 2x$ ($\lt 0$). So (4.4) becomes equivalent to
\n
$$
\int_{-\infty}^0 \left[\int_{-\infty}^{2x} K(-y)g(y) dy \right]^q u(x) dx \leq C \left(\int_{-\infty}^0 g^p(x)v(x) dx \right)^{\frac{q}{p}}.
$$
\n(4.5)
\nand
\n
$$
\int_{-\infty}^0 \left[\int_{2x}^x K(x - y)g(y) dy \right]^q u(x) dx \leq C \left(\int_{-\infty}^0 g^p(x)v(x) dx \right)^{\frac{q}{p}}.
$$
\n(4.6)
\nChanges of variables yield

$$
J_{-\infty} \qquad J_{2x}
$$
\n
$$
\langle x - y \langle -y \text{ for } y \rangle \langle 2x \rangle \langle 0 \rangle. \text{ So (4.4) becomes equivalent to}
$$
\n
$$
\int_{-\infty}^{0} \left[\int_{-\infty}^{2x} K(-y)g(y) \, dy \right]^q u(x) \, dx \leq C \left(\int_{-\infty}^{0} g^p(x) v(x) \, dx \right)^{\frac{q}{p}} \qquad (4.5)
$$
\n
$$
\int_{-\infty}^{0} \left[\int_{2x}^{x} K(x - y)g(y) \, dy \right]^q u(x) \, dx \leq C \left(\int_{-\infty}^{0} g^p(x) v(x) \, dx \right)^{\frac{q}{p}}. \qquad (4.6)
$$
\n
$$
\text{variables yield}
$$

and

$$
\int_{-\infty}^{\infty} \left[\int_{-\infty}^{2\pi} K(-y)g(y) \, dy \right]' u(x) \, dx \le C \left(\int_{-\infty}^{\infty} g^p(x)v(x) \, dx \right)'
$$
\n
$$
(4.5)
$$
\n
$$
\int_{-\infty}^{0} \left[\int_{2x}^{2\pi} K(x-y)g(y) \, dy \right]^{q} u(x) \, dx \le C \left(\int_{-\infty}^{0} g^p(x)v(x) \, dx \right)^{\frac{q}{p}}.
$$
\n
$$
\text{variables yield}
$$
\n
$$
\int_{-\infty}^{\infty} g^p(x)v(x) \, dx = \int_{0}^{\infty} G(x)^p(x)v(-x) \, dx \quad \text{with } G(x) = g(-x),
$$
\n(4.6)

Changes of variables yield

$$
\int_{-\infty}^{0} \left[\int_{2x}^{x} K(x - y)g(y) dy \right]^{q} u(x) dx \le C \left(\int_{-\infty}^{0} g^{p}(x)u(x) dx \right)^{\frac{q}{p}}.
$$

of variables yield

$$
\int_{-\infty}^{0} g^{p}(x)u(x) dx = \int_{0}^{\infty} G(x)^{p}(x)u(-x) dx \quad \text{with} \quad G(x) = g(-x),
$$

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\n
$$
\int_{-\infty}^{0} \left[\int_{-\infty}^{2x} K(-y)g(y) dy \right]^q u(x) dx = \int_{0}^{\infty} \left[\int_{-\infty}^{-2x} K(-y)g(y) dy \right]^q u(-x) dx
$$
\n
$$
= \int_{0}^{\infty} \left[\int_{2x}^{\infty} K(z)G(z) dz \right]^q u(-x) dx
$$
\n
$$
\int_{-\infty}^{0} \left[\int_{2x}^x K(x-y)g(y) dy \right]^q u(x) dx = \int_{0}^{\infty} \left[\int_{-2x}^{-x} K(-x-y)g(y) dy \right]^q u(-x) dx
$$
\n
$$
= \int_{0}^{\infty} \left[\int_{x}^{2x} K(z-x)G(z) dz \right]^q u(-x) dx
$$

and

$$
= \int_0^{\infty} \left[\int_{2x}^x K(z)G(z) dz \right] u(-x) dx
$$

$$
\int_{-\infty}^0 \left[\int_{2x}^x K(x-y)g(y) dy \right]^q u(x) dx = \int_0^{\infty} \left[\int_{-2x}^{-x} K(-x-y)g(y) dy \right]^q u(-x) dx
$$

$$
= \int_0^{\infty} \left[\int_x^{2x} K(z-x)G(z) dz \right]^q u(-x) dx
$$

$$
= \int_0^{\infty} (\tilde{T}^* G)^q(x) u(-x) dx.
$$

These computations show that (4.5) and (4.6) are respectively equivalent to

$$
H^*: L^p((0,\infty),K^{-p}(x)v(-x)\,dx)\to L^q((0,\infty),u(-x)\,dx)
$$

and

$$
: L^{p}((0, \infty), K^{-p}(x)v(-x) dx) \rightarrow L^{q}((0, \infty), u(-x) dx)
$$

$$
\widetilde{T}^{*}: L^{p}((0, \infty), v(-x) dx) \rightarrow L^{q}((0, \infty), u(-x) dx).
$$

By Part B of the Lemma [with $w(x) = K^{-p}(x)v(-x)$] the first boundedness is true if and only if, for a constant $A > 0$,

$$
= \int_{0}^{R} (T^{*}G)^{q}(x)u(-x) dx.
$$

\n
$$
= \int_{0}^{R} (T^{*}G)^{q}(x)u(-x) dx
$$

\n
$$
H^{*}: L^{p}((0, \infty), K^{-p}(x)v(-x) dx) \to L^{q}((0, \infty), u(-x) dx)
$$

\n
$$
\tilde{T}^{*}: L^{p}((0, \infty), v(-x) dx) \to L^{q}((0, \infty), u(-x) dx).
$$

\n
$$
\text{and} \quad \text{B of the Lemma [with } w(x) = K^{-p}(x)v(-x) \text{] the first boundedness is true if}
$$

\n
$$
\int_{0}^{R} u(-x) dx \int_{0}^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x)v^{1-p'}(-x) dx \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \quad (4.7)
$$

\nis one part of condition (3.5) whose other part is
\n
$$
\int_{0}^{R} u(x) dx \int_{0}^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x)v^{1-p'}(-x) dx \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \quad (4.8)
$$

\n
$$
\text{d} u(x) dx \int_{0}^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x)v^{1-p'}(-x) dx \right)^{\frac{1}{p'}} \leq A \quad \text{for all } R > 0. \quad (4.8)
$$

This is one part of condition (3.5) whose other part is

$$
\left(\int_0^R u(x) \, dx\right)^{\frac{1}{q}} \left(\int_{2R}^\infty K^{p'}(x) v^{1-p'}(-x) \, dx\right)^{\frac{1}{p'}} \le A \qquad \text{for all} \ \ R > 0. \tag{4.8}
$$

Clearly, (4.4) is a necessary inequality for (3.3). So the conclusion is that (4.7) and \widetilde{T} : $L^p((0,\infty),v(-x)dx) \to L^q((0,\infty),u(-x)dx)$ are necessary conditions for *T*: $L^p((-\infty,\infty),v(x) dx) \to L^q((-\infty,\infty),u(x) dx)$ to be satisfied, and they are also sufficient to get (4.4). on (3.5) whose other part is
 $\int_{R}^{R} K^{p'}(x)v^{1-p'}(-x)dx \Big)^{\frac{1}{p'}} \leq A$ for all $R > 0$. (4.8)
 $\int_{R}^{R} K^{p'}(x)v^{1-p'}(-x)dx \Big)$ is the conclusion is that (4.7) and
 $\int_{R}^{R} \frac{1}{\sqrt{2}} \left((-\infty, \infty), u(x) dx \right)$ are necessary conditio

Estimate of S_4 **: The aim is to prove**

$$
S_4 = \int_0^\infty (Tg)^q(x)u(x) \, dx \le C \left(\int_{-\infty}^0 g^p(x)v(x) \, dx \right)^{\frac{q}{p}}.
$$
 (4.9)

 \int For each $x > 0$ then

$$
S(x,y) = \int_{0}^{\infty} f(x,y) \, dy \, dx
$$
\n
$$
V(x) \, dx \to L^{q}((0,\infty), u(-x)) \, dx
$$
\n
$$
V(x) \, dx \to L^{q}((- \infty, \infty), u(x) \, dx)
$$
\n
$$
V(x) \, dx \to L^{q}((- \infty, \infty), u(x) \, dx)
$$
\n
$$
S_{4} = \int_{0}^{\infty} (Tg)^{q}(x) u(x) \, dx \leq C \left(\int_{-\infty}^{0} g^{p}(x) v(x) \, dx \right)^{\frac{q}{p}}.
$$
\n
$$
S_{4} = \int_{0}^{\infty} (Tg)^{q}(x) u(x) \, dx \leq C \left(\int_{-\infty}^{0} g^{p}(x) v(x) \, dx \right)^{\frac{q}{p}}.
$$
\n
$$
S_{4} = \int_{0}^{\infty} (Tg)^{q}(x) u(x) \, dx \leq C \left(\int_{-\infty}^{0} g^{p}(x) v(x) \, dx \right)^{\frac{q}{p}}.
$$
\n
$$
S_{4} = \int_{0}^{\infty} (Tg)^{q}(x) u(x) \, dx \leq C \left(\int_{-\infty}^{0} g^{p}(x) v(x) \, dx \right)^{\frac{q}{p}}.
$$
\n
$$
S_{4} = \int_{0}^{\infty} (Tg)^{q}(x) u(x) \, dx \leq C \left(\int_{-\infty}^{0} g^{p}(x) v(x) \, dx \right)^{\frac{q}{p}}.
$$

Indeed, $-\frac{1}{2}y < x - y < -2y$ for $y < -2x$ (< 0), and $x < x - y < 3x$ for $-2x < y < 0$. So (4.9) is equivalent both to

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\n
$$
\langle x - y \langle -2y \text{ for } y \langle -2x \rangle \langle 0 \rangle, \text{ and } x \langle x - y \langle 3x \text{ for } -2x \langle y \rangle \langle 0 \rangle.
$$
\nquivalent both to
\n
$$
\int_0^\infty \left[\int_{-\infty}^{-2x} K(-y)g(y) dy \right]^q u(x) dx \leq C \left(\int_{-\infty}^0 g^p(x)v(x) dx \right)^{\frac{q}{p}} \qquad (4.10)
$$
\n
$$
\int_0^\infty \left[\int_{-2x}^0 g(y) dy \right]^q K^q(x) u(x) dx \leq C \left(\int_{-\infty}^0 g^p(x)v(x) dx \right)^{\frac{q}{p}} \qquad (4.11)
$$
\n
$$
\text{es of variables is used with the function } G(x) = g(-x) \text{ in order that}
$$
\n
$$
\left[\int_{-\infty}^{-2x} K(-y)g(y) dy \right]^q u(x) dx = \int_0^\infty \left[\int_{2x}^\infty K(z)G(z) dz \right]^q u(x) dx
$$

and

$$
\int_0^{\infty} \left[\int_{-2x}^0 g(y) \, dy \right]^q K^q(x) u(x) \, dx \le C \left(\int_{-\infty}^0 g^p(x) v(x) \, dx \right)^{\frac{q}{p}}.
$$
 (4.11)

Again changes of variables is used with the function
$$
G(x) = g(-x)
$$
 in order that
\n
$$
\int_0^\infty \left[\int_{-\infty}^{-2x} K(-y)g(y) dy \right]^q u(x) dx = \int_0^\infty \left[\int_{2x}^\infty K(z)G(z) dz \right]^q u(x) dx
$$
\nand

and

$$
\int_0^\infty \left[\int_{-2x}^0 g(y) \, dy \right]^q K^q(x) u(x) \, dx = \int_0^\infty \left[\int_0^{2x} G(z) \, dz \right]^q K^q(x) u(x) \, dx.
$$

Consequently, (4.10) and (4.11) are equivalent to

$$
H^*: L^p((0,\infty),K^{-p}(x)v(-x) dx) \rightarrow L^q((0,\infty),u(x) dx)
$$

and

$$
\mathcal{H}:\,L^p\big((0,\infty),v(-x)\,dx\big)\rightarrow L^q\big((0,\infty),K^q(x)u(x)\,dx\big),
$$

respectively. By Part B of the Lemma the first boundedness is equivalent to (4.8), and by Part C of the same Lemma, the second holds if and only if

$$
H^*: L^p((0,\infty), K^{-p}(x)v(-x) dx) \to L^q((0,\infty), u(x) dx)
$$

\n
$$
\mathcal{H}: L^p((0,\infty), v(-x) dx) \to L^q((0,\infty), K^q(x)u(x) dx),
$$

\nively. By Part B of the Lemma the first boundedness is equivalent to (4.8), and
\nt C of the same Lemma, the second holds if and only if
\n
$$
\left(\int_{\frac{1}{2}R}^{\infty} K^q(x)u(x) dx\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(-x) dx\right)^{\frac{1}{p'}} \leq A \text{ for all } R > 0.
$$
\n(4.12)
\nst condition is both equivalent to (4.3) and
\n
$$
\left(\int_{\frac{1}{2}R}^{2R} K^q(x)u(x) dx\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(-x) dx\right)^{\frac{1}{p'}} \leq A
$$

This last condition is both equivalent to (4.3) and

$$
\left(\int_0^{2R} K^q(x)u(x) dx\right)^{\frac{1}{q}} \left(\int_0^{R} v^{1-p'}(-x) dx\right)^{\frac{1}{p'}} \leq A
$$

which is nothing else than condition (3.6) . Since (4.9) is a necessary condition for (3.3) to hold, then the boundedness $T : L^p((-\infty,\infty), v(x) dx) \to L^q((-\infty,\infty), u(x) dx)$ implies (4.8), (4.3) and (3.6). These conditions are also sufficient to obtain (4.9)

Proof of Proposition 3.2. We only derive $\widetilde{T}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty),$ $u(x) dx$) from (3.8), since similarly \widetilde{T}^* : $L^p((0,\infty), v(-x) d) \to L^q((0,\infty), u(-x) dx)$ can be obtained from (3.9). The main key is to see that (3.7) and (3.8) implies, for some positive constants $c, A > 0$ and all $R > 0$, nothing else than cothen the bounded
i.8), (4.3) and (3.6)
f of Proposition
from (3.8), since si
tained from (3.9).
itive constants c, A
 $2R$
 $K^{\epsilon q}(x - R) \left[\int_{\frac{1}{2}i}^{x}$ $\left(\int_{\frac{1}{2}R}^{2R} K^{q}(x)u(x) dx\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(-x) dx\right)^{\frac{1}{p'}} \leq A$
hing else than condition (3.6). Since (4.9) is a necessary condition for (3.3)
in the boundedness $T : L^{p}((-\infty, \infty), v(x) dx) \to L^{q}((-\infty, \infty), u(x) dx)$,
(4.3)

$$
\int_{R}^{2R} K^{\epsilon q}(x-R) \left[\int_{\frac{1}{2}x}^{x} K^{(1-\epsilon)p'}(x-z)v^{1-p'}(z) dz \right]^{\frac{q}{p'}} u(x) dx < c A^{q}.
$$
 (4.13)

 \mathcal{A}

Once (4.13) is established, the fact that \widetilde{T} : $L^p((0,\infty),v(x)\,dx)\to L^q((0,\infty),u(x)\,dx)$ can be proved by using the usual Hölder inequality. Indeed, for $f(.) \geq 0$, $x > 0$ and $\varepsilon \in [0,1]$ then \mathbb{R}^2

$$
(\widetilde{T}f)(x) \leq \left(\int_{\frac{1}{2}x}^{x} K^{\epsilon p}(x-y) f^{p}(y) v(y) dy\right)^{\frac{1}{p}} \times (\mathcal{V}(x))^{\frac{1}{p'}}
$$

where $V(x) = \int_{\frac{1}{2}x}^{x} K^{(1-\epsilon)p'}(x-z)v^{1-p'}(z) dz$. So by the Minkowski inequality $(\frac{q}{p} \ge 1)$ then (4.13) yields

$$
\left(\int_0^\infty (\widetilde{T}f)^q(x)u(x)\,dx\right)^{\frac{p}{q}}
$$
\n
$$
\leq \left\{\int_0^\infty \left[\int_{\frac{1}{2}x}^x K^{\epsilon p}(x-y)f^p(y)v(y)\,dy\right]^{\frac{q}{p}}V^{\frac{q}{p'}}(x)u(x)\,dx\right]^{\frac{p}{q}}
$$
\n
$$
\leq \int_0^\infty f^p(y)v(y)\left[\int_y^{2y} K^{\epsilon q}(x-y)V^{\frac{q}{p'}}(x)u(x)\,dx\right]^{\frac{p}{q}}dy
$$
\n
$$
\leq \left(\sup_{R>0}\int_R^{2R} K^{\epsilon q}(x-R)\left[\int_{\frac{1}{2}x}^x K^{(1-\epsilon)p'}(x-z)v^{1-p'}(z)\,dz\right]^{\frac{q}{p'}}u(x)\,dx\right)^{\frac{p}{q}}
$$
\n
$$
\times \int_0^\infty f^p(y)v(y)\,dy
$$
\n
$$
\leq A^p \int_0^\infty f^p(y)v(y)\,dy.
$$

Now to get (4.13) the following consequences of (3.7) are useful

the following consequences of (3.7) are useful:
\n
$$
\int_{R}^{2R} K^{\epsilon q}(x-R) dx = \int_{0}^{R} K^{\epsilon q}(z) dz \leq cR \times K^{\epsilon q}(R)
$$

and

 ~ 100 km s $^{-1}$

 $\bar{\gamma}$

$$
A^{p} \int_{0}^{\infty} f^{p}(y)v(y) dy.
$$

\n1.13) the following consequences of (3.7) are useful:
\n
$$
\int_{R}^{2R} K^{\epsilon q}(x-R) dx = \int_{0}^{R} K^{\epsilon q}(z) dz \le cR \times K^{\epsilon q}(R)
$$

\n
$$
\int_{\frac{1}{2}x}^{x} K^{(1-\epsilon)p'}(x-z) dz = \int_{0}^{\frac{1}{2}x} K^{(1-\epsilon)p'}(z) dz \qquad (for \ 0 < x < 2R)
$$

\n
$$
\le \int_{0}^{R} K^{(1-\epsilon)p'}(z) dz
$$

\n
$$
\le cR \times K^{(1-\epsilon)p'}(R).
$$

\n(3.8) then
\n
$$
\int_{R}^{2R} K^{\epsilon q}(x-R) \left[\int_{\frac{1}{2}x}^{x} K^{(1-\epsilon)p'}(x-z)v^{1-p'}(z) dz \right]^{\frac{q}{p'}} u(x) dx
$$

Indeed, using (3.8) then

$$
\leq cR \times K^{(1-\epsilon)p'}(R).
$$
\n(3.8) then\n
$$
\int_{R}^{2R} K^{\epsilon q}(x-R) \left[\int_{\frac{1}{2}x}^{x} K^{(1-\epsilon)p'}(x-z)v^{1-p'}(z) dz \right]^{\frac{q}{p'}} u(x) dx
$$
\n
$$
\leq \left(\sup_{R < z < 2R} u(z) \right) \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \right)^{\frac{q}{p'}} \times \int_{R}^{2R} K^{\epsilon q}(x-R) \left[\int_{\frac{1}{2}x}^{x} K^{(1-\epsilon)p'}(x-z)(z) dz \right]^{\frac{q}{p'}} dx
$$

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\n
$$
\leq \left[R^{\frac{1}{q} + \frac{1}{p'}} K(R) \left(\sup_{R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \right]^q
$$
\n
$$
\leq A^q
$$
\nposition 3.2 is proved **1**
\nof of Proposition 3.3. For convenience set
\n
$$
\mathcal{U}(n) = \sup_{2^n < z < 2^{n+1}} u(z) \quad \text{and} \quad \mathcal{V}(n) = \sup_{2^{n-1} < y < 2^{n+1}} v^{1-p'}(y)
$$
\ninteger $n \in \mathbb{Z}$. A crucial key for the proof is
\n
$$
2^{n[\frac{1}{q} - \frac{1}{p}]} \mathcal{U}^{\frac{1}{q}}(n) \leq c A(v(y))^{\frac{1}{p}}
$$
 for a.e. y with $2^{n-1} < y < 2^{n+1}$. (4.14)
\nthe chain of computations, which leads to the boundedness
\n $\widetilde{T}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$

 $\leq A^q$

and Proposition 3.2 is proved \blacksquare

Proof of Proposition 3.3. For convenience set

$$
\leq A^{q}
$$

stition 3.2 is proved
Example 3.3. For convenience set

$$
\mathcal{U}(n) = \sup_{2^{n} \leq x \leq 2^{n+1}} u(z) \quad \text{and} \quad \mathcal{V}(n) = \sup_{2^{n-1} \leq y \leq 2^{n+1}} v^{1-p'}(y)
$$

for each integer $n \in \mathbb{Z}$. A crucial key for the proof is

$$
2^{n\left[\frac{1}{q}-\frac{1}{p}\right]}\mathcal{U}^{\frac{1}{q}}(n)\leq c\,A\big(v(y)\big)^{\frac{1}{p}}\qquad\text{for a.e. }y\text{ with }2^{n-1}
$$

Indeed, the chain of computations, which leads to the boundedness

$$
\widetilde{T}: L^p((0,\infty),v(x)\,dx)\to L^q\big((0,\infty),u(x)\,dx\big)
$$

with $p \le q \le \overline{p}$, can be presented as follows:

h integer
$$
n \in \mathbb{Z}
$$
. A crucial key for the proof is
\n
$$
2^{n[\frac{1}{q} - \frac{1}{p}]} U^{\frac{1}{q}}(n) \le c A(v(y))^{\frac{1}{p}}
$$
 for a.e. y with $2^{n-1} < y < 2^{n+1}$. (4.
\n, the chain of computations, which leads to the boundedness
\n $\tilde{T}: L^{p}((0, \infty), v(x) dx) \to L^{q}((0, \infty), u(x) dx)$
\n
$$
\le q \le \overline{p},
$$
 can be presented as follows:
\n
$$
\int_{0}^{\infty} (\widetilde{T}f)^{q}(x)u(x) dx = \sum_{n \in \mathbb{Z}} \int_{2^{n} < z < 2^{n+1}} \left[\int_{\frac{1}{2}z}^{z} (x - y)^{\alpha-1} f(y) dy \right]^{q} u(x) dx
$$

\n
$$
\le \sum_{n \in \mathbb{Z}} 2^{n} \int_{2^{n}}^{2^{n+1}} (Tf1_{[2^{n-1}, 2^{n+1}]})^{q}(x) dx
$$

\n
$$
\le \sum_{n \in \mathbb{Z}} 2^{n(1-\frac{z}{p})} U_{n} \left(\int_{2^{n}}^{2^{n+1}} (Tf1_{[2^{n-1}, 2^{n+1}]})^{\overline{p}}(x) dx \right)^{q\overline{p}}
$$

\n
$$
\le c_{1} \sum_{n \in \mathbb{Z}} 2^{n(1-\frac{z}{p})} U_{n} \left(\int_{2^{n-1}}^{2^{n+1}} f^{p}(y) dy \right)^{\frac{z}{p}}
$$

\n
$$
\le c_{2} \sum_{n \in \mathbb{Z}} 2^{n(1-\frac{z}{p})} U_{n} \left(\int_{2^{n-1}}^{2^{n+1}} f^{p}(y) dy \right)^{\frac{z}{p}}
$$

\n
$$
= c_{1} \sum_{n \in \mathbb{Z}} \left(\int_{2^{n-1}}^{2^{n+1}} f^{p}(y) v(y) dy \right)^{\frac{z}{p}}
$$

\n
$$
\le c_{2} A^{q} \sum_{n \in \mathbb{Z}} \left(\int_{2^{n-1}}^{2^{n+1}} f
$$

Observation (4.14) appears easily by using conditions (3.12) and (3.8) . Indeed, if 2^{n-1} < $y < 2^{n+1}$, then a.e. $\frac{1}{n+1}, \text{ then } \frac{1}{n}\n\frac{1}{q} - \frac{1}{p} \frac{1}{\mathcal{U}} \frac{1}{q}$

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\nObservation (4.14) appears easily by using conditions (3.12) and (3.8). Indeed, if
$$
2^{n-1} < y < 2^{n+1}
$$
, then a.e.
\n
$$
2^{n[\frac{1}{q} - \frac{1}{p}]} \mathcal{U}^{\frac{1}{q}}(n) = 2^{n[\frac{1}{q} - \frac{1}{p}]} \mathcal{U}^{\frac{1}{q}}(n) (v^{1-p'}(y))^{\frac{1}{p'}} \times (v(y))^{\frac{1}{p}}
$$
\n
$$
\leq 2^{n[\frac{1}{q} - \frac{1}{p}]} \mathcal{U}^{\frac{1}{q}}(n) \mathcal{V}^{\frac{1}{p'}}(n) \times (v(y))^{\frac{1}{p}}
$$
\n
$$
\leq c2^{n[\frac{1}{q} + \frac{1}{p}]} K(2^n) \mathcal{U}^{\frac{1}{q}}(n) \mathcal{V}^{\frac{1}{p'}}(n) \times (v(y))^{\frac{1}{p}}
$$
\n
$$
\leq cA \times (v(y))^{\frac{1}{p}}
$$
\n
$$
\leq cA \times (v(y))^{\frac{1}{p}}
$$
\n
$$
\text{We condition (3.8).}
$$
\nThe proof for the boundedness $\tilde{T}^* : L^p((0, \infty), v(-x) dx) \to L^q((0, \infty), u(-x) dx) \text{ can}$

be also seen as above \blacksquare

Proof of Proposition 3.4. Since $u(\cdot)$ and $v(\cdot)$ are even functions, then condition (3.4) and (3.5) *is* the same as *(3.14)* and (3.15), respectively, and (3.6) becomes (3.16). So following Theorem 3.1, then $(3.14) - (3.16)$ are necessary conditions for $T: L^p((-\infty,\infty),v(x) dx) \to L^q((-\infty,\infty),u(x) dx).$

Conversely, again by Theorem *3.1,* to get this boundedness it remains to prove *T:* $L^p((0,\infty),v(x)\,dx)\,\rightarrow\,L^q\big((0,\infty),u(x)\,dx\big)\,\text{ and }\,\widetilde{T}^*\,:\,\,L^p\big((0,\infty),v(x)\,dx\big)\,\rightarrow\,L^q\big((0,\infty),v(x)\,dx\big)$ $u(x) dx$). Since $u(-z) = u(z)$ and $v(y) = v(-y)$ then, following Proposition 3.2 or 3.3, it remains to check *1 1 sup u(Z))(sup v I_P'()) A (4.15)* y Theorem 3.1
 $L^q((0, \infty), u(x))$
 $= u(z)$ and $v(y)$
 $\langle K(R) \begin{pmatrix} \text{sup} \\ \frac{1}{2}R < i \leq 2R \end{pmatrix}$
 > 0 is a fixed c
 \downarrow (i.e. $u(\cdot)$ is d

to get
 $\left(\int_{2R}^{\infty} K^q(z)u(z)\right)$ Δx) \rightarrow $L^q((-\infty, \infty), u(x) dx)$.
 y Theorem 3.1, to get this boundedness i
 $L^q((0, \infty), u(x) dx)$ and \widetilde{T}^* : $L^p((0, \infty), u(x) dx)$
 $= u(z)$ and $v(y) = v(-y)$ then, following
 $\angle (R) \begin{pmatrix} \text{sup} & u(z) \\ \frac{1}{2}R < i < 2R \end{pmatrix}^{\frac{1$

$$
\mathcal{A}(R) = R^{\frac{1}{q} + \frac{1}{p'}} K(R) \bigg(\sup_{\frac{1}{2}R < z < 2R} u(z) \bigg)^{\frac{1}{q}} \bigg(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \bigg)^{\frac{1}{p'}} \leq A \tag{4.15}
$$

for all $R > 0$, where $A > 0$ is a fixed constant.

condition *(3.14) is* used to get

it remains to check
\n
$$
\mathcal{A}(R) = R^{\frac{1}{q} + \frac{1}{p'}} K(R) \left(\sup_{\frac{1}{2}R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A \quad (4.15)
$$
\nfor all $R > 0$, where $A > 0$ is a fixed constant.
\nFor $u(\cdot) \uparrow$ and $v^{1-p'}(\cdot) \downarrow$ (i.e. $u(\cdot)$ is quasi-increasing and $v^{1-p'}(\cdot)$ is quasi-decreasing) condition (3.14) is used to get
\n
$$
\mathcal{A}(R) \leq c_1 \left(\int_{2R}^{\infty} K^q(z) u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{4}R}^{\frac{1}{2}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_1 A.
$$
\nFor $u(\cdot) \uparrow$ and $v^{1-p'}(\cdot) \uparrow$ condition (3.14) is also used to get
\n
$$
\mathcal{A}(R) \leq c_2 \left(\int_{8R}^{\infty} K^q(z) u(z) dz \right)^{\frac{1}{q}} \left(\int_{2R}^{4R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_2 A.
$$
\nFor $u(\cdot) \downarrow$ and $v^{1-p'}(\cdot) \uparrow$, condition (3.15) is used to get
\n
$$
\mathcal{A}(R) \leq c_3 \left(\int_{8R}^{1} u(z) dz \right)^{\frac{1}{q}} \left(\int_{8R}^{\infty} K^p(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_3 A.
$$

For $u(\cdot)$ \uparrow and $v^{1-p'}(\cdot)$ \uparrow condition (3.14) is also used to get

and
$$
v^{1-p'}(.)
$$
 \uparrow condition (3.14) is also used to get
\n
$$
\mathcal{A}(R) \le c_2 \bigg(\int_{8R}^{\infty} K^q(z) u(z) dz \bigg)^{\frac{1}{q}} \bigg(\int_{2R}^{4R} v^{1-p'}(y) dy \bigg)^{\frac{1}{p'}} \le c_2 A.
$$
\nand $v^{1-p'}(.)$ \uparrow , condition (3.15) is used to get
\n
$$
\mathcal{A}(R) \le c_3 \bigg(\int_0^{\frac{1}{2}R} u(z) dz \bigg)^{\frac{1}{q}} \bigg(\int_{2R}^{\infty} K^{p'}(y) v^{1-p'}(y) dy \bigg)^{\frac{1}{p'}} \le c_3 A.
$$
\n
$$
u(.) + 3Rd u^{1-p'}(.) + condition (3.16) is used to get
$$

For $u(\cdot) \downarrow$ and $v^{1-p'}(\cdot) \uparrow$, condition (3.15) is used to get

$$
\mathcal{A}(R) \leq c_2 \bigg(\int_{8R}^{\infty} K^q(z) u(z) dz \bigg)^* \bigg(\int_{2R}^{4R} v^{1-p'}(y) dy \bigg)^{p'} \leq c_2 A.
$$

and $v^{1-p'}(\cdot)$ \uparrow , condition (3.15) is used to get

$$
\mathcal{A}(R) \leq c_3 \bigg(\int_0^{\frac{1}{2}R} u(z) dz \bigg)^{\frac{1}{q}} \bigg(\int_{2R}^{\infty} K^{p'}(y) v^{1-p'}(y) dy \bigg)^{\frac{1}{p'}} \leq c_3 A.
$$

$$
u(\cdot) \downarrow \text{ and } v^{1-p'}(\cdot) \downarrow, \text{ condition (3.16) is used to get}
$$

$$
\mathcal{A}(R) \leq c_4 K(\frac{1}{2}R) \bigg(\int_{\frac{1}{4}R}^R u(z) dz \bigg)^{\frac{1}{q}} \bigg(\int_0^{\frac{1}{2}R} v^{1-p'}(y) dy \bigg)^{\frac{1}{p'}} \leq c_4 A.
$$

Finally, for $u(\cdot) \downarrow$ and $v^{1-p'}(\cdot) \downarrow$, condition (3.16) is used to get

$$
\mathcal{A}(R) \le c_4 K(\frac{1}{2}R) \bigg(\int_{\frac{1}{4}R}^{R} u(z) dz \bigg)^{\frac{1}{q}} \bigg(\int_{0}^{\frac{1}{2}R} v^{1-p'}(y) dy \bigg)^{\frac{1}{p'}} \le c_4 A.
$$

If moreover $u(x) \leq c u(2x)$, then condition (3.14) is sufficient to conclude since

$$
u(\cdot) \downarrow \text{ and } v^{1-p}(\cdot) \downarrow, \text{ condition (3.16) is used to get}
$$
\n
$$
\mathcal{A}(R) \le c_4 K(\frac{1}{2}R) \bigg(\int_{\frac{1}{4}R}^{R} u(z) \, dz \bigg)^{\frac{1}{q}} \bigg(\int_0^{\frac{1}{2}R} v^{1-p'}(y) \, dy \bigg)^{\frac{1}{p'}} \le c_4 A
$$
\n
$$
u(x) \le c u(2x), \text{ then condition (3.14) is sufficient to conclude that}
$$
\n
$$
\mathcal{A}(R) \le c_5 \bigg(\int_R^{\infty} K^q(z) u(z) \, dz \bigg)^{\frac{1}{q}} \bigg(\int_0^{\frac{1}{2}R} v^{1-p'}(y) \, dy \bigg)^{\frac{1}{p'}} \le c_5 A.
$$
\n
$$
\text{condition (3.15) leads to the conclusion if the process are $u^{1-p'}(x) < c$.
$$

Similarly, condition (3.15) leads to the conclusion if moreover $v^{1-p'}(x) \leq cv^{1-p'}(2x)$

Proof of Proposition 3.5. By Theorem 3.1, conditions (3.4)—(3.6) are necessary ones for $T: L^p((-\infty,\infty),v(x) dx) \to L^q((-\infty,\infty),u(x) dx).$

Conversely, again by Theorem 3.1, it remains to get \tilde{T} : $L^p((0,\infty),v(x) dx) \rightarrow$ $L^q((0,\infty),u(x) dx)$ and \widetilde{T}^* : $L^p((0,\infty),v(-x) dx) \rightarrow L^q((0,\infty),u(-x) dx)$. And, by Proposition 3.2 or 3.3, it is sufficient to prove inequalities (3.8) and (3.9). The first inequality appears now by using the fact that $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{C}$ and (3.17) since

Weighted Norm Inequalities
\nof of Proposition 3.5. By Theorem 3.1, conditions (3.4) – (3.6) are no-
\n
$$
T: L^{p}((-\infty, \infty), v(x) dx) \rightarrow L^{q}((-\infty, \infty), u(x) dx).
$$

\nwersely, again by Theorem 3.1, it remains to get $\tilde{T}: L^{p}((0, \infty), v(x)$
\n $\infty), u(x) dx$ and $\tilde{T}^{*}: L^{p}((0, \infty), v(-x) dx) \rightarrow L^{q}((0, \infty), u(-x) dx).$
\nition 3.2 or 3.3, it is sufficient to prove inequalities (3.8) and (3.9). T
\nity appears now by using the fact that $u(\cdot), v^{1-p'}(\cdot) \in C$ and (3.17) since
\n $R^{\frac{1}{q}+\frac{1}{p'}}K(R)\left(\sup_{R\leq z\leq R} u(z)\right)^{\frac{1}{q}}\left(\sup_{\frac{1}{2}R
\n $\leq c_1K(R)\left(\int_{2^{-N}R< z\leq 2^NR} u(z)dz\right)^{\frac{1}{q}}\left(\int_{2^{-N}R< y\leq 2^NR} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}$
\n $\leq c_1A.$$

Similarly, inequality (3.9) also appears by using the fact that $u(-)$, $v^{1-p'}(-) \in \mathcal{C}$ and (3.18)

Proof of Theorem 2.1. This results is.an immediate consequence of Theorem 3.1. Indeed, for instance, the boundedness $R_\alpha : L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)$ can be seen as $T: L^p((-\infty,\infty),v(x) dx) \to L^q((-\infty,\infty),u(x) dx)$ with $K(x) = x^{\alpha-1}$, $K(-x) = 0$, $v(-x) = 0$ and $u(-x) = 0$ for $x > 0$.

Proof of Theorem 2.2. With $K(x) = x^{\alpha-1}$, $v(-x) = 0$ and $u(-x) = 0$ for $x > 0$, by the proof of Theorem 3.1, the boundedness $R_{\alpha} : L^p((0,\infty),v(x) dx) \rightarrow$ $L^q((0,\infty),u(x) dx)$ is equivalent to $\widetilde{R}_{\alpha}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)$
and $H: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),x^{(\alpha-1)q}u(x) dx)$. This last boundedness is Similarly, inequality (3.9) also appears by using the fact that $u(-)$, $v^{1-p'}(-) \in C$ and
 (3.18) **E**
 Proof of Theorem 2.1. This results is an immediate consequence of Theorem 3.1.

Indeed, for instance, the boundedne equivalent to condition (2.3) because of Part (A) of the Lemma.

The result for $W_{\alpha}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$ can be immediatly deduced from the first part since this boundedness is equivalent to R_α : $L^{q'}\big((0,\infty),u^{1-q'}\big)$ For the result for $W_{\alpha}: L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)$
deduced from the first part since this boundedness is equivalent to $I(x) dx$. $\to L^{p'}((0,\infty),v^{1-p'}(x) dx)$. Thus Theorem 2.2 is proved \blacksquare

Proof of Proposition 2.3. The boundedness \widetilde{R}_{α} : $L^p((0,\infty),v(x) dx) \to L^q((0,\infty))$ ∞), $u(x) dx$) will be obtained from Proposition 3.2 or Proposition 3.3.

As it is seen in Section 3, Proposition 3.2 can be applied under one of the following conditions:

 $p = q$ for which $\varepsilon = \frac{1}{p}$.

 $\frac{1}{\alpha} \leq p < q$ for which ε is taken such that $0 \leq \varepsilon < \min(1, \frac{1}{1-\alpha} \frac{1}{q}).$

 $p < q$, $p < \frac{1}{\alpha}$ and $p < q < p^*$ for which $\varepsilon \in (0,1]$ is taken as in (3.10).

Proposition 3.3 is really needed when $p < q$, $p < \frac{1}{\alpha}$ and $p < q = p^*$. The boundedness in (3.11) [with $\bar{p} = p^*$] is satisfied since it is well-known that $R_{\alpha} : L^p((0, \infty), dx) \rightarrow$ $L^{p^*}((0,\infty), dx)$ (see, for instance, [2]) and (3.12) is satisfied since $\frac{1}{p'} + \frac{1}{p^*} + \alpha - 1 = 0$ **Proof of Proposition 2.4.** The crucial key for the proof is

cotondratsimba
\nProposition 2.4. The crucial key for the proof is
\n
$$
\int_{2^n}^{2^{n+1}} (\widetilde{R}_{\alpha}f)^q(x)u(x) dx \leq c B^q(n) \left(\int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y) dy \right)^{\frac{q}{p}}
$$
\n(4.16)
\ngiven as in (2.7) and $c > 0$ is a constant which does not depend on
\ne chain of computations, which leads to \widetilde{R}_{α} : $L^p((0, \infty), v(x) dx) \rightarrow$

where $B(n)$ is given as in (2.7) and $c > 0$ is a constant which does not depend on where $D(n)$ is given as in (2.1) and $c > 0$ is a constant which does not depend on
n. Indeed the chain of computations, which leads to \tilde{R}_{α} : $L^p((0,\infty),v(x) dx)$ – $L^q((0, \infty), u(x) \, dx)$, is as follows:
 $\int_0^\infty dx$

Y. Rakotondratsimba
\n**Proof of Proposition 2.4.** The crucial key for the proof is
\n
$$
\int_{2^n}^{2^{n+1}} (\widetilde{R}_{\alpha}f)^q(x)u(x) dx \leq c B^q(n) \left(\int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y) dy \right)^{\frac{q}{p}}
$$
\nB(n) is given as in (2.7) and $c > 0$ is a constant which does not depend
\ndeed the chain of computations, which leads to \widetilde{R}_{α} : $L^p((0, \infty), v(x) d, \infty), u(x) dx$), is as follows:
\n
$$
\int_0^{\infty} (\widetilde{R}_{\alpha}f)^q(x)u(x) dx \leq c_1 \sum_{n \in \mathbb{Z}} B^q(n) \left(\int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y) dy \right)^{\frac{q}{p}}
$$
\n
$$
\leq c_1 \left(\sum_{m \in \mathbb{Z}} B^r(m) \right)^{1-\frac{q}{p}} \left(\sum_{n \in \mathbb{Z}} \int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y) dy \right)^{\frac{q}{p}}
$$
\n
$$
\leq c_1 A^q \left(\sum_{n \in \mathbb{Z}} \int_{2^{n-1}}^{2^n} + \int_{2^n}^{2^{n+1}} f^p(y)v(y) dy \right)^{\frac{q}{p}}
$$
\n
$$
= c_2 A^q \left(\int_0^{\infty} f^p(x)v(x) dx \right)^{\frac{q}{p}}.
$$

It remains to prove (4.16). For this purpose define $\mathcal{U}(n)$ and $\mathcal{V}(n)$ as in the proof of Proposition 3.3 and observe that, for $2^n < x < 2^{n+1}$, then

$$
\begin{split}\n\text{max to prove (4.16). For this purpose define } \mathcal{U}(n) \text{ and } \mathcal{V}(n) \text{ as in the p:} \\
&= c_2 A^q \left(\int_0^\infty f^p(x)v(x) \, dx \right)^{\frac{q}{p}}. \\
\text{sum of a 3 and observe that, for } 2^n < x < 2^{n+1}, \text{ then} \\
(\widetilde{R}_{\alpha}f)(x) &= \int_{\frac{1}{2}x}^x (x-y)^{\alpha-1} f(y) \, dy \\
&\leq \left(\int_{\frac{1}{2}x}^x (x-y)^{\alpha-1} v^{1-p'}(y) \, dy \right)^{\frac{1}{p'}} \left(\int_{\frac{1}{2}x}^x f^p(z)v(z)(x-z)^{\alpha-1} \, dz \right)^{\frac{1}{p}} \\
&\leq c_3 2^{n\alpha \frac{1}{p'}} \mathcal{V}_{p'}^{1}(n) \left(\int_{\frac{1}{2}x}^x f^p(z)v(z)(x-z)^{\alpha-1} \, dz \right)^{\frac{1}{p}}.\n\end{split}
$$

Consequently, *(4.16)* appears since 24.1

$$
\leq \left(\int_{\frac{1}{2}x}^{x} (x - y)^{\alpha - 1} v^{1 - p'}(y) dy \right)^{\frac{1}{p'}} \left(\int_{\frac{1}{2}x}^{x} f^{p}(z) v(z) (x - z)^{\alpha - 1} dz \right)^{\frac{1}{p}}
$$
\n
$$
\leq c_3 2^{n\alpha \frac{1}{p'}} \mathcal{V}^{\frac{1}{p'}}(n) \left(\int_{\frac{1}{2}x}^{x} f^{p}(z) v(z) (x - z)^{\alpha - 1} dz \right)^{\frac{1}{p}}
$$
\nequently, (4.16) appears since\n
$$
\int_{2^{n}}^{2^{n+1}} (\widetilde{R}_{\alpha}f)^{q}(x) u(x) dx
$$
\n
$$
\leq c_3 2^{n\alpha \frac{q}{p'}} \mathcal{V}^{\frac{q}{p'}}(n) \mathcal{U}(n) \int_{2^{n}}^{2^{n+1}} \left[\int_{\frac{1}{2}x}^{x} f^{p}(z) v(z) (x - z)^{\alpha - 1} dz \right]^{\frac{q}{p}} dx
$$
\n
$$
\leq c_4 2^{n[\alpha \frac{q}{p'} + 1 - \frac{q}{p}]} \mathcal{V}^{\frac{q}{p'}}(n) \mathcal{U}(n) \left(\int_{2^{n-1}}^{2^{n+1}} f^{p}(z) v(z) \left[\int_{z}^{2z} (x - z)^{\alpha - 1} dx \right] dz \right)^{\frac{q}{p}}
$$
\n
$$
\leq c_5 2^{n[\alpha \frac{q}{p'} + 1 - \frac{q}{p} + \alpha \frac{q}{p}]} \mathcal{V}^{\frac{q}{p'}}(n) \mathcal{U}(n) \left(\int_{2^{n-1}}^{2^{n+1}} f^{p}(z) v(z) dz \right)^{\frac{q}{p}}
$$
\n
$$
= c_5 \left[2^{n[\alpha + \frac{1}{q} - \frac{1}{p}]} \mathcal{V}^{\frac{1}{p'}}(n) \mathcal{U}^{\frac{1}{q}}(n) \right]^{\frac{q}{q}} \left(\int_{2^{n-1}}^{2^{n+1}} f^{p}(z) v(z) dz \right)^{\frac{q}{p}}
$$
\n
$$
\leq c_6 \mathcal{B}^q(n) \left(\
$$

and Proposition *2.4 is* proved I

 $\ddot{\cdot}$

Proof of Proposition **2.5.** In view of Theorem 2.1, the main problem is to prove \widetilde{R}_{α} : $L^p((0,\infty),v(x)dx) \rightarrow L^q((0,\infty),u(x)dx)$ which by Proposition 2.3 remains to check (2.5) , i.e. for all $R>0$ **At Proposition 2**
 (∞) , $v(x) dx$ \rightarrow

i.e. for all $R > 0$
 $\mathcal{A}(R) = R^{\alpha + \frac{1}{q} - \frac{1}{p}}$ **(5.** In view of Theorem 2.1, the main problem
 $L^q((0, \infty), u(x) dx)$ which by Proposition 2.3
 $\left(\sup_{R < z < 2R} u(z)\right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y)\right)^{\frac{1}{p'}} \le cA.$

Int and $A > 0$ will come from the condition (2) **n** 2.5. In vi
 $\rightarrow L^q((0, \infty)$
 >0
 $-\frac{1}{p} \left(\sup_{R < z < 2R} R \right)$

astant and A

ing cases.
 $\rightarrow \downarrow$ then
 $\int_{2R}^{\infty} z^{(\alpha-1)q} u$

then Weighted Norm

orem 2.1, the main

which by Propo

sup $v^{1-p'}(y)$
 $\frac{1}{2}R < y < 2R$

come from the co
 $\int_{\frac{1}{4}R}^{1} v^{1-p'}(y) dy$ **2.5.** In view $L^q((0, \infty))$
 \rightarrow $L^q((0, \infty))$
 \rightarrow $\frac{1}{p} \left(\sup_{R < z < 2R} 1 \right)$

stant and A
 \rightarrow stand \rightarrow $\frac{1}{q}$
 \rightarrow $z^{(\alpha - 1)q}u(z)$

$$
\mathcal{A}(R)=R^{\alpha+\frac{1}{q}-\frac{1}{p}}\bigg(\sup_{R
$$

where $c > 0$ is a fixed constant and $A > 0$ will come from the condition (2.1) which is used in each of the following cases.

For $u(\cdot)$ \uparrow and $v^{1-p'}(\cdot)$ \downarrow then

$$
(\infty), v(x) dx \rightarrow L^{q}((0, \infty), u(x) dx) \text{ which by Proposition 2.3}
$$

i.e. for all $R > 0$

$$
\mathcal{A}(R) = R^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\sup_{R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq cA.
$$

Q is a fixed constant and $A > 0$ will come from the condition (2)
to the following cases.
 \uparrow and $v^{1-p'}(\cdot) \downarrow$ then

$$
\mathcal{A}(R) \leq c_1 \left(\int_{2R}^{\infty} z^{(\alpha - 1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{4}R}^{\frac{1}{2}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_1 A.
$$

and $v^{1-p'}(\cdot) \uparrow$ then

For $u(\cdot)$ \uparrow and $v^{1-p'}(\cdot)$ \uparrow then

is a fixed constant and
$$
A > 0
$$
 with the form the definition $(\pm \sqrt{2})$
\nof the following cases.
\n
$$
\int \text{ and } v^{1-p'}(\cdot) \downarrow \text{ then}
$$
\n
$$
\mathcal{A}(R) \leq c_1 \left(\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{4}R}^{\frac{1}{2}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_1 A.
$$
\n
$$
\text{and } v^{1-p'}(\cdot) \uparrow \text{ then}
$$
\n
$$
\mathcal{A}(R) \leq c_2 \left(\int_{8R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{2R}^{4R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_2 A.
$$
\n
$$
\text{and } v^{1-p'}(\cdot) \downarrow \text{ then}
$$
\n
$$
\mathcal{A}(R) \leq c_3 \left(\int_{\frac{1}{2}R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{8}R}^{\frac{1}{4}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_3 A
$$
\n
$$
u(\cdot) \downarrow \text{ and } v^{1-p'}(\cdot) \uparrow, \text{ the extra-assumption for } u(\cdot) \text{ or } v^{1-p'}(\cdot) \text{ i}
$$
\n
$$
\text{then } u(x) \leq c_4 \left(\int_{8R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{2R}^{4R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_4 A
$$
\n
$$
\text{and } v^{1-p'}(\cdot) \uparrow \text{ then}
$$
\n
$$
\mathcal{A}(R) \leq c_4 \left(\int_{8R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{2R}^{4R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_4 A
$$

For $u(\cdot) \perp$ and $v^{1-p'}(\cdot) \perp$ then

$$
\mathcal{A}(R) \le c_2 \left(\int_{8R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{2R}^{4R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \le c_2 A.
$$

and $v^{1-p'}(\cdot) \downarrow$ then

$$
\mathcal{A}(R) \le c_3 \left(\int_{\frac{1}{2}R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{8}R}^{\frac{1}{4}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \le c_3 A.
$$

Finally, for $u(\cdot) \downarrow$ and $v^{1-p'}(\cdot) \uparrow$, the extra-assumption for $u(\cdot)$ or $v^{1-p'}(\cdot)$ is useful. For instance, when $u(x) \leq c u(2x)$, then

$$
\mathcal{A}(R) \le c_4 \left(\int_{8R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{2R}^{4R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \le c_4 A.
$$

The same conclusion is also satisfied if $v^{1-p'}(2x) \le c v^{1-p'}(x)$, since

$$
\mathcal{A}(R) \leq c_3 \left(\int_{\frac{1}{2}R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{8}R}^{\frac{1}{4}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_3 A.
$$
\n
$$
u(\cdot) \downarrow \text{ and } v^{1-p'}(\cdot) \uparrow, \text{ the extra-assumption for } u(\cdot) \text{ or } v^{1-p'}(\cdot) \text{ is}
$$
\n
$$
\mathcal{A}(R) \leq c_4 \left(\int_{8R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{2R}^{4R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_4 A.
$$
\nonclusion is also satisfied if $v^{1-p'}(2x) \leq c v^{1-p'}(x)$, since\n
$$
\mathcal{A}(R) \leq c_5 \left(\int_{\frac{1}{2}R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{\frac{1}{8}R}^{\frac{1}{4}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_5 A.
$$
\nfor $W_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$ can be u and u must \blacksquare \n
$$
\text{orollary 2.6 can be seen as Proposition 2.5, we can only focus of Corollary 2.7. In view of Theorem 2.2 and Proposition, ∞), $v(x) dx \to L^q((0, \infty), u(x) dx)$ then it is sufficient to check 2.8) from (2.11) and (2.3) . By the growth condition (C) then\n
$$
u_1(x) = \frac{u_1(x) + u_2(x) + u_3(x) + u_4(x)}{u_2(x) + u_3(x) + u_4(x)} \int_{2-(N+1+n)}^{2^{N+1}n} u(x) \leq c_1(2^{-n}) \int_{2-(N+1+n)}^{2^{N+1}n} u(x) dx \leq c_2(2^{-(n+1)}) \int_{2
$$
$$

The result for W_{α} : $L^p((0,\infty),v(x)dx) \to L^q((0,\infty),u(x)dx)$ can be obtained by duality arguments

Since Corollary 2.6 can be seen as Proposition 2.5, we can only focuse on the

Proof of Corollary 2.7. In view of Theorem 2.2 and Proposition 2.4, to get $R_{\alpha}: L^p((0,\infty),v(x)dx) \to L^q((0,\infty),u(x)dx)$ then it is sufficient to check conditions (2.7) and (2.8) from (2.11) and (2.3) . By the growth condition (C) then Supper W_{α} : $L^p((0, \infty), v(x) dx) \rightarrow L^q((0, \infty), u(x) dx)$ can be obtainuments

Supper $L^p((0, \infty), v(x) dx) \rightarrow L^q((0, \infty), u(x) dx)$ can be obtainuments

Supper 2.5 , the vector of Theorem 2.2 and Proposition 2.4,
 (2.8) from $(2.11$

$$
\sup_{2^{n} \leq y \leq 2^{n+1}} u(y) \leq c_1 (2^{-n}) \int_{2^{-N+n}}^{2^{N+n}} u(z) dz \leq c_2 (2^{-(n+1)}) \int_{2^{-(N+1+n)}}^{2^{N+1+n}} u(z) dz
$$

and

$$
\begin{aligned}\n\text{otondratsimba} \\
\sup_{2^{n-1} < y < 2^{n+1}} v^{1-p'}(y) \le c_3(2^{-(n+1)}) \int_{2^{-(N+1+n)}}^{2^{N+1+n}} v^{1-p'}(z) \, dz. \\
\text{the left member of (2.7) and taking } R &= 2^n \text{ in (2.11) then}\n\end{aligned}
$$

Calling
$$
A(n)
$$
 the left member of (2.7) and taking $R = 2^n$ in (2.11) then
\n
$$
A(n) \le c_4 \left(\int_{4(2^n)}^{\infty} y^{(\alpha-1)q} u(y) dy \right)^{\frac{1}{q}}
$$
\n
$$
\left(\int_0^{2^n} v^{1-p'}(y) dy \right)^{\frac{1}{q'}} \left(\int_{2^n}^{2^{n+1}} v^{1-p'}(x) dx \right)^{\frac{1}{r}} = c_4 \mathcal{B}(n)
$$

and
\n
$$
\sup_{2^{n-1} < y < 2^{n+1}} v^{1-p'}(y) \le c_3(2^{-(n+1)}) \int_{2^{-(N+1+n)}}^{2^{N+1+n}} v^{1-p'}(z) dz.
$$
\nCalling $A(n)$ the left member of (2.7) and taking $R = 2^n$ in (2.11) then
\n
$$
A(n) \le c_4 \left(\int_{4(2^n)}^{\infty} y^{(\alpha-1)q} u(y) dy \right)^{\frac{1}{q}}
$$
\n
$$
\left(\int_0^{2^n} v^{1-p'}(y) dy \right)^{\frac{1}{q'}} \left(\int_{2^n}^{2^{n+1}} v^{1-p'}(x) dx \right)^{\frac{1}{r}} = c_4 B(n)
$$
\nwhich is nothing else than (2.7). Condition (2.8) can be deduced from (2.3) as follows\n
$$
\sum_{n \in \mathbb{Z}} B^r(n)
$$
\n
$$
\le c_5 \sum_{n \in \mathbb{Z}} \left(\int_{4(2^n)}^{\infty} y^{(\alpha-1)q} u(y) dy \right)^{\frac{r}{q}} \left(\int_0^{2^n} v^{1-p'}(y) dy \right)^{\frac{1}{q'}} \left(\int_{2^n}^{2^{n+1}} v^{1-p'}(x) dx \right)
$$
\n
$$
= c_5 \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} \left[\left(\int_{4(2^n)}^{\infty} y^{(\alpha-1)q} u(y) dy \right)^{\frac{1}{q}} \left(\int_0^{2^n} v^{1-p'}(y) dy \right)^{\frac{1}{q'}} \right]^{r} v^{1-p'}(x) dx
$$
\n
$$
\le c_5 \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} \left[\left(\int_{2x}^{\infty} y^{(\alpha-1)q} u(y) dy \right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y) dy \right)^{\frac{1}{q'}} \right]^{r} v^{1-p'}(x) dx
$$
\n
$$
\le c_5 \int_0^{\infty} \left[\left(\int_{2x}^{\infty} y^{(\alpha-1)q} u(y) dy \right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y) dy \right)^{\frac{1
$$

and Corollary 2.7 is proved \blacksquare

Proof of Corollary 2.8. In view of Theorem 2.1 and Proposition 2.3 to get R_{α} $L^p((0,\infty),v(x)dx) \to L^q((0,\infty),u(x)dx)$ it is sufficient to check conditions (2.1) and (2.5). For this purpose observe that, for all $R > 0$, $\leq c_5 A^r$
 R collary 2.7 is proved
 A Corollary 2
 ∞ , $v(x) dx$ $\to L^q$
 $\left(\int_0^R v^{1-p'}(y) dy\right) = \int_0^R$ *f* **of Corollary 2.8.** In view of Theorem 2.1 and Proposition 2.3 to get R_{α} :
 y, $v(x) dx$ $\rightarrow L^{q}((0, \infty), u(x) dx)$ it is sufficient to check conditions (2.1) and this purpose observe that, for all $R > 0$,
 $v^{1-p'}(y) dy = \int$ \leq c₅ A^r
 \leq c₅ A^r

orollary 2.7 is prove
 oof of Corollary
 ∞ , $v(x) dx$) $\rightarrow L^q$

For this purpose ob
 $\int_0^R v^{1-p'}(y) dy = \int_0^R$ **8.** In view of Theorem 2.
 (3), ∞), $u(x) dx$) it is sufficity

rve that, for all $R > 0$,
 $y^{[(1-p')(6-1)+1]-1} dy \approx R^{p'}$
 $z^{[(\alpha-1)q+\beta]-1} dz \approx R^{q[(\alpha-\beta)]}$
 $\beta < (1-\alpha)p$,
 $\frac{1}{2} \int_{0}^{1} \int_{0}^{R} f(x^{1-p'}(x) dx) dx$

\n- \n For this purpose observe that, for all
$$
R > 0
$$
,\n
$$
\int_0^R v^{1-p'}(y) \, dy = \int_0^R y^{[(1-p')(\delta - 1) + 1] - 1} \, dy \approx R^{p'[1 - \frac{\delta}{p}]} \quad \text{whenever} \quad \delta < p \qquad (4.17)
$$
\n
\n- \n
$$
\int_{2R}^\infty z^{(\alpha - 1)q} u(z) \, dz = \int_{2R}^\infty z^{[(\alpha - 1)q + \beta] - 1} \, dz \approx R^{q[(\alpha - 1) + \frac{\beta}{q}]} \quad \text{for} \quad \beta < (1 - \alpha)q. \tag{4.18}
$$
\n sequentially, for $\delta < p$ and $\beta < (1 - \alpha)p$,\n
\n

and

$$
J_0 \t J_0
$$
\nd\n
$$
\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz = \int_{2R}^{\infty} z^{[(\alpha-1)q+\beta]-1} dz \approx R^{q[(\alpha-1)+\frac{\beta}{q}]} \text{ for } \beta < (1-\alpha)q. \quad (4.18)
$$
\n\nnsequently, for $\delta < p$ and $\beta < (1-\alpha)p$,\n\n
$$
\mathcal{H}(R) = \left(\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz \right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \approx R^{(\alpha-1)+\frac{\beta}{q}} R^{1-\frac{\delta}{p}} = R^{\alpha+\frac{\beta}{q}-\frac{\delta}{p}}
$$

Consequently, for $\delta < p$ and $\beta < (1 - \alpha)p$,

$$
\mathcal{H}(R) = \left(\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \approx R^{(\alpha-1)+\frac{\beta}{q}} R^{1-\frac{\delta}{p}} = R^{\alpha+\frac{\beta}{q}-\frac{\delta}{p}}
$$

and condition (2.1) is satisfied whenever $\alpha + \frac{\beta}{q} - \frac{\delta}{p} = 0$ [which is (2.12)]. On the other hand,

Weighted Norm Inequalities
\n
$$
\text{Riemannian} \begin{aligned}\n\text{Weighted Norm Inequalities} \\
\text{Riemannian} \end{aligned}
$$
\n
$$
\mathcal{A}(R) = R^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\sup_{R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \approx R^{\alpha + \frac{\beta}{q} - \frac{\delta}{p}}.
$$
\n
$$
\text{Riemannian} \begin{aligned}\n\text{Riemannian} \end{aligned}
$$
\n
$$
\text{Riemannian} \begin{aligned}\n\text{Riemannian} \end{aligned}
$$
\n
$$
\mathcal{A}(R) = R^{\alpha + \frac{1}{q} - \frac{1}{p}} \left(\sup_{R < z < 2R} u(z) \right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R < y < 2R} v^{1-p'}(y) \right)^{\frac{1}{p'}} \approx R^{\alpha + \frac{\beta}{q} - \frac{\delta}{p}}.
$$

Then condition (2.5) is reduced to (2.12).

Since R_{α} : $L^p((0,\infty),v(x) dx) \to L^q((0,\infty),u(x) dx)$ implies (2.1), then condition *(2.12)* appears immediatly. Also, in view of *(4.17)* and (4.18), if (2.1) holds then necessarily $\delta < p$ and $\beta < (1 - \alpha)q$.

Results for W_{α} : $L^p((0,\infty),v(x)dx) \to L^q((0,\infty),u(x)dx)$ can be easily deduced from the above, since this boundedness is equivalent to $R_{\alpha}: L^{p_1}((0,\infty), v_1(x) dx) \rightarrow$ *f* **(1, 2, 1, 1, 1, 1, 1)** *(i)* \mathcal{L} *P* $((0, \infty), v(x) dx)$
 i the above, since this boundedness
 $((0, \infty), u_1(x) dx)$ with $p_1 = q'$, $q_1 =$ $p', v_1(x) = u^{1-q'}(x), u_1(x) = v^{1-p'}(x)$ $v \leq p$ and $p \leq (1 -$

for W_{α} : $L^p((0, \infty)$
 ∞ *u*₁ $(x) dx$) with $p_1 =$

of Corollary 2.9.

fficient to check con
 $\binom{R}{v} v^{1-p'}(y) dy = \int_0^R$

dx) it is sufficient to check conditions (2.1) and (2.5). First observe that

$$
(0, \infty), u_1(x) dx
$$
 with $p_1 = q', q_1 = p', v_1(x) = u^{1-q}(x), u_1(x) = v^{1-p}(x) = 0$
\nProof of Corollary 2.9. Again, to get $R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x))$
\nit is sufficient to check conditions (2.1) and (2.5). First observe that
\n
$$
\int_0^R v^{1-p'}(y) dy = \int_0^R [\ln^{-p'}(y^{-1})]y^{-1} dy \approx \ln^{-\frac{p'}{p}}(R^{-1}) \quad \text{for } R < \frac{1}{2}
$$

and, for $R > \frac{1}{2}$,

$$
J_0 \t J_0
$$

\nfor $R > \frac{1}{2}$,
\n
$$
\int_0^R v^{1-p'}(y) dy = \int_0^{\frac{1}{2}} [\ln^{-p'}(y^{-1})] y^{-1} dy + \int_{\frac{1}{2}}^R y^{[(1-p')((\theta-1)+1]-1]} dy
$$
\n
$$
\leq c_1 + R^{p'[1-\frac{\theta}{p}]}
$$
\n
$$
\leq c_2 R^{p'[1-\frac{\theta}{p}]}
$$
 (whenever $\theta < \rho$).
\nthe other hand, for $R > \frac{1}{4}$ [or $2R > \frac{1}{2}$] then
\n
$$
\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz = \int_{2R}^{\infty} z^{[(\alpha-1)q+\gamma]-1} dz \approx R^{q[(\alpha-1)+\frac{\gamma}{q}]} \text{ for } \gamma < (1-\alpha)
$$

\n, for $R < \frac{1}{4}$,

On the other hand, for $R > \frac{1}{4}$ [or $2R > \frac{1}{2}$] then

On the other hand, for
$$
R > \frac{1}{4}
$$
 [or $2R > \frac{1}{2}$] then
\n
$$
\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz = \int_{2R}^{\infty} z^{[(\alpha-1)q+\gamma]-1} dz \approx R^{q[(\alpha-1)+\frac{\gamma}{q}]} \quad \text{for } \gamma < (1-\alpha)q \quad (4.19)
$$
\nand, for $R < \frac{1}{4}$,
\n
$$
\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz = \int_{2R}^{\frac{1}{2}} z^{(\alpha-1)q+\beta-1} dz + \int_{\frac{1}{2}}^{\infty} z^{(\alpha-1)q+\gamma-1} dz \leq c_3 + c_4
$$
\nwhenever $\gamma < (1-\alpha)q < \beta$. With $\mathcal{H}(R)$ defined as in the proof of Corollary 2.8 then
\n
$$
\mathcal{H}(R) \leq c_5 \ln^{-\frac{1}{p}}(R^{-1}) \leq c_6 \quad \text{for } R < \frac{1}{4},
$$
\nalso for $\theta < p$ and $\gamma < (1-\alpha)q$ then
\n
$$
\mathcal{H}(R) \leq c_7 R^{\alpha+\frac{7}{q}-\frac{\beta}{p}} \quad \text{for } R > \frac{1}{2}.
$$

and, for $R < \frac{1}{4}$,

$$
R < \frac{1}{4},
$$
\n
$$
\int_{2R}^{\infty} z^{(\alpha-1)q} u(z) dz = \int_{2R}^{\frac{1}{2}} z^{(\alpha-1)q + \beta - 1} dz + \int_{\frac{1}{2}}^{\infty} z^{(\alpha-1)q + \gamma - 1} dz \leq c_3 + c_4
$$
\n
$$
\text{For } \gamma < (1 - \alpha)q < \beta. \text{ With } \mathcal{H}(R) \text{ defined as in the proof of Corollary 2.8}
$$
\n
$$
\mathcal{H}(R) \leq c_5 \ln^{-\frac{1}{p}} (R^{-1}) \leq c_6 \qquad \text{for } R < \frac{1}{4},
$$
\n
$$
\text{or } \theta < p \text{ and } \gamma < (1 - \alpha)q \text{ then}
$$
\n
$$
\mathcal{H}(R) \leq c_7 R^{\alpha + \frac{7}{q} - \frac{\theta}{p}} \qquad \text{for } R > \frac{1}{2}.
$$

whenever $\gamma < (1 - \alpha)q < \beta$. With $\mathcal{H}(R)$ defined as in the proof of Corollary 2.8 then

$$
\mathcal{H}(R) \leq c_5 \ln^{-\frac{1}{p}}(R^{-1}) \leq c_6 \quad \text{for } R < \frac{1}{4},
$$

$$
\mathcal{H}(R) \le c_7 R^{\alpha + \frac{7}{q} - \frac{\theta}{p}} \qquad \text{for } R > \frac{1}{2}.
$$

So condition (2.1) is satisfied whenever $\alpha + \frac{\gamma}{q} - \frac{\theta}{p} \le 0$ [which is (2.13)]. Also, with $\mathcal{A}(R)$
defined as in the proof of Corollary 2.8, then defined as in the proof of Corollary 2.8, then otondratsimba

.1) is satisfied whenever $\alpha + \frac{\gamma}{q} - \frac{\theta}{p} \le 0$ (which is

ne proof of Corollary 2.8, then
 $\mathcal{A}(R) \le c_8 R^{\alpha + \frac{\theta}{q} - 1} \ln^{-1}(R^{-1}) \le c_8 R^{\alpha + \frac{\theta}{q} - 1}$
 $\mathcal{A}(R) \le c_9 R^{\alpha + \frac{\gamma}{q} - \frac{\theta}{p}}$ for $R > \frac{1}{$ a

ed whenever $\alpha + \frac{\gamma}{q} - \frac{\theta}{p} \le 0$ [which if

Corollary 2.8, then
 $2^{\alpha + \frac{\theta}{q} - 1} \ln^{-1}(R^{-1}) \le c_8 R^{\alpha + \frac{\theta}{q} - 1}$
 $\mathcal{A}(R) \le c_9 R^{\alpha + \frac{\gamma}{q} - \frac{\theta}{p}}$ for $R > \frac{1}{2}$.

(b) is satisfied whenever $(1 - \alpha)q$:
 $v(x) dx)$

$$
\mathcal{A}(R) \leq c_8 R^{\alpha + \frac{\beta}{q} - 1} \ln^{-1}(R^{-1}) \leq c_8 R^{\alpha + \frac{\beta}{q} - 1} \qquad \text{for } R < \frac{1}{4}
$$

and

$$
\mathcal{A}(R) \le c_9 R^{\alpha + \frac{\gamma}{q} - \frac{\theta}{p}} \quad \text{for } R > \frac{1}{2}.
$$

Consequently, condition (2.5) is satisfied whenever $(1 - \alpha)q \leq \beta$ and $\alpha + \frac{\gamma}{q} - \frac{\theta}{p} \leq 0$.

Since $R_{\alpha}: L^p((0,\infty),v(x)dx) \to L^q((0,\infty),u(x)dx)$ implies (2.1), and in view of (4.19), then $\gamma < (1 - \alpha)q$. The necessity of (2.13) can be also derived from condition (2.1), since for $R > \frac{1}{2}$ then $\mathcal{H}(R) \geq R^{\alpha + \frac{7}{4} - \frac{\beta}{p}} \blacksquare$ So condition (2.1) is satisfied whenever $\alpha + \frac{\gamma}{q}$

defined as in the proof of Corollary 2.8, then
 $A(R) \leq c_8 R^{\alpha + \frac{\beta}{q} - 1} \ln^{-1}(R^{-1}) \leq$

and
 $A(R) \leq c_9 R^{\alpha + \frac{\gamma}{q} - \frac{\beta}{p}}$

Consequently, condition (2.5) is satisf *I* Americal $(1 - \alpha)q \le$
 $(0, \infty), u(x) dx)$ implof (2.13) can be also
 $-\frac{e}{r}$
 I of Theorem 2.2 and
 dx) it is sufficient to the finitness of
 $\frac{1}{r}$
 $\left(\int_0^x v^{1-p'}(y) dy\right)^{\frac{1}{r}}$

Proof of Corollary 2.10. In view of Theorem 2.2 and Proposition 2.4, to get $R_{\alpha}: L^p((0,\infty), v(x) dx) \to L^q((0,\infty), u(x) dx)$ it is sufficient to check conditions (2.3), (2.7) and (2.8). Condition (2.3) is equivalent to the finitness of $\begin{align} \n\begin{cases}\n\frac{\partial}{\partial t} & \text{if } t \in \mathbb{R} \\
\frac{\partial}{\partial t} & \text{if } t \in \mathbb{R}\n\end{cases} \\
\frac{\partial}{\partial t} & \text{if } t \in \mathbb{R}.\n\end{align}$ of Theorem 2.2 and P
 dx) it is sufficient to dent to the finitness of
 $\int_{a}^{\frac{1}{q}} \left(\int_{a}^{x} v^{1-p'}(y) dy \right)^{\frac{1}{q'}}$

$$
\mathcal{A}(R) \leq c_8 R^{\alpha + \frac{\beta}{q} - 1} \ln^{-1}(R^{-1}) \leq c_8 R^{\alpha + \frac{\beta}{q} - 1} \qquad \text{for } R < \frac{1}{4}
$$
\n
$$
\mathcal{A}(R) \leq c_9 R^{\alpha + \frac{7}{q} - \frac{\beta}{p}} \qquad \text{for } R > \frac{1}{2}.
$$
\nently, condition (2.5) is satisfied whenever $(1 - \alpha)q \leq \beta$ and $\alpha + \frac{\gamma}{q} - R_{\alpha}$: $L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx)$ implies (2.1), and\nen $\gamma < (1 - \alpha)q$. The necessity of (2.13) can be also derived from\ne for $R > \frac{1}{2}$ then $\mathcal{H}(R) \geq R^{\alpha + \frac{\gamma}{q} - \frac{\beta}{p}}$ \blacksquare \n**f of Corollary 2.10.** In view of Theorem 2.2 and Proposition 2\n0, ∞), $v(x) dx$) $\to L^q((0, \infty), u(x) dx)$ it is sufficient to check condition (2.8). Condition (2.3) is equivalent to the finitness of\n
$$
I_1 = \int_0^{\frac{1}{2}} \left[\left(\int_{2x}^{\infty} y^{(\alpha - 1)q} u(y) dy \right)^{\frac{1}{q}} \left(\int_0^x v^{1 - p'}(y) dy \right)^{\frac{1}{q'}} \right]_0^r v^{1 - p'}(x) dx
$$
\n
$$
I_2 = \int_0^{\infty} \left[\left(\int_{2x}^{\infty} y^{(\alpha - 1)q} u(y) dy \right)^{\frac{1}{q}} \left(\int_0^x v^{1 - p'}(y) dy \right)^{\frac{1}{q'}} \right]_0^r \left[\int_{-1}^1 v^{1 - p'}(x) dx \right]
$$

and

So condition (2.1) is satisfied whenever
$$
\alpha + \frac{7}{q} - \frac{\theta}{p} \le 0
$$
 [which is (2.13)]. Also, with *A* defined as in the proof of Corollary 2.8, then
\n
$$
\mathcal{A}(R) \le c_8 R^{\alpha + \frac{\theta}{q} - 1} \ln^{-1}(R^{-1}) \le c_8 R^{\alpha + \frac{\theta}{q} - 1} \quad \text{for } R < \frac{1}{4}
$$
\nand
\n
$$
\mathcal{A}(R) \le c_8 R^{\alpha + \frac{\theta}{q} - 1} \ln^{-1}(R^{-1}) \le c_8 R^{\alpha + \frac{\theta}{q} - 1} \quad \text{for } R < \frac{1}{4}
$$
\nand
\n
$$
\mathcal{A}(R) \le c_8 R^{\alpha + \frac{\theta}{q} - 1} \quad \text{for } R > \frac{1}{2}.
$$
\nConsequently, condition (2.5) is satisfied whenever $(1 - \alpha)q \le \beta$ and $\alpha + \frac{7}{q} - \frac{\theta}{p} \le$
\nSince $R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx) \text{ implies (2.1), and in vie}$
\n(2.1), since for $R > \frac{1}{2}$ then $\mathcal{H}(R) \ge R^{\alpha + \frac{7}{4} - \frac{\theta}{p}} \blacksquare$
\nProof of Corollary 2.10. In view of Theorem 2.2 and Proposition 2.4, t
\n $R_{\alpha}: L^p((0, \infty), v(x) dx) \to L^q((0, \infty), u(x) dx) \text{ it is sufficient to check conditions (}$
\n(2.7) and (2.8). Condition (2.3) is equivalent to the finitness of
\n
$$
I_1 = \int_0^{\frac{1}{2}} \left[\left(\int_{2x}^{\infty} y^{(\alpha - 1)q} u(y) dy \right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y) dy \right)^{\frac{1}{q'}} \right] v^{1-p'}(x) dx.
$$
\nand
\n
$$
I_2 = \int_{\frac{1}{2}}^{\infty} \left[\left(\int_{2x}^{\infty} y^{(\alpha - 1)q} u(y) dy \
$$

Using $\delta < p, \beta < (1 - \alpha)q, (4.17)$ and (4.18) then

$$
\begin{array}{l}\n\delta < p, \ \beta < (1 - \alpha)q, \ (4.17) \text{ and } (4.18) \text{ then} \\
I_1 \approx c_1 \int_0^{\frac{1}{2}} x^{[(\alpha - 1) + \frac{\beta}{q} + \frac{p'}{q'}(1 - \frac{\delta}{p})]r} x^{p'[1 - \frac{\delta}{p}] - 1} dx = \int_0^{\frac{1}{2}} x^{[\alpha + \frac{\beta}{q} - \frac{\delta}{p}]r - 1} dx = A_1^r \\
\text{ver } \alpha + \frac{\beta}{q} - \frac{\delta}{p} > 0. \text{ On the other hand, the inequality } \alpha + \frac{\gamma}{q} - \frac{\delta}{p} < 0 \text{ leads} \\
I_2 \approx \int_{\frac{1}{2}}^{\infty} x^{[(\alpha - 1) + \frac{\gamma}{q} + \frac{p'}{q'}(1 - \frac{\delta}{p})]r} x^{p'[1 - \frac{\delta}{p}] - 1} dx = \int_{\frac{1}{2}}^{\infty} x^{[\alpha + \frac{\gamma}{q} - \frac{\delta}{p}]r - 1} dx = A_2^r.\n\end{array}
$$
\nck conditions (2.7) and (2.8), it is convenient to take $\mathcal{B}(n) = \mathcal{A}(n)$ where .

\n1 as above. For $n + 1 < 0$ or $n < -1$ then

\n
$$
\mathcal{A}(n) \approx 2^{n[\alpha + \frac{1}{q} - \frac{1}{p}]} \times 2^{n\frac{1}{q}(\beta - 1)} \times 2^{-n\frac{1}{p}(\delta - 1)} = 2^{n[\alpha + \frac{\beta}{q} - \frac{\delta}{p}]}
$$
\nnsequently $\sum_{-\infty}^{-2} B^r(n) < \infty$ whenever $\alpha + \frac{\beta}{q} - \frac{\delta}{p} > 0$. For $0 < n - 1$ or

\n
$$
\mathcal{A}(n) \approx 2^{n[\alpha + \frac{1}{q} - \frac{1}{p}]} \times 2^{n\frac{1}{q}(\gamma - 1)} \times 2^{-n\frac{1}{p}(\delta - 1)} = 2^{n[\alpha + \frac{\gamma}{q} - \frac{\delta}{p}]}
$$
\n
$$
\sum_{k=2
$$

$$
I_2 \approx \int_{\frac{1}{2}}^{\infty} x^{[(\alpha-1)+\frac{7}{q}+\frac{p'}{q'}(1-\frac{\delta}{p})]r} x^{p'[1-\frac{\delta}{p}]-1} dx = \int_{\frac{1}{2}}^{\infty} x^{[\alpha+\frac{7}{q}-\frac{\delta}{p}]r-1} dx = A_2^r.
$$

To check conditions (2.7) and (2.8), it is convenient to take $\mathcal{B}(n) = \mathcal{A}(n)$ where $\mathcal{A}(n)$ is defined as above. For $n+1 < 0$ or $n < -1$ then

$$
\mathcal{A}(n) \approx 2^{n[\alpha+\frac{1}{q}-\frac{1}{p}]} \times 2^{n\frac{1}{q}(\beta-1)} \times 2^{-n\frac{1}{p}(\delta-1)} = 2^{n[\alpha+\frac{\beta}{q}-\frac{\delta}{p}]}
$$

whenever $\alpha + \frac{\beta}{q} - \frac{\delta}{p} > 0$. On the otherhand, the inequali
 $I_2 \approx \int_{\frac{1}{2}}^{\infty} x^{[(\alpha-1)+\frac{7}{q}+\frac{p'}{q'}(1-\frac{\delta}{p})]r} x^{p'[1-\frac{\delta}{p}]-1} dx = \int_{\frac{1}{2}}^{\infty}$

To check conditions (2.7) and (2.8), it is convenient to tal

de $B^{r}(n) < \infty$ whenever $\alpha + \frac{\beta}{q} - \frac{\delta}{p} > 0$. For $0 < n - 1$ or $1 < n$ then To check conditions (2.7) and (2.8), it is convenien
defined as above. For $n + 1 < 0$ or $n < -1$ then
 $\mathcal{A}(n) \approx 2^{n[\alpha + \frac{1}{q} - \frac{1}{p}]} \times 2^{n\frac{1}{q}(\beta - 1)} \times 2^{-}$
and consequently $\sum_{-\infty}^{-2} B^{r}(n) < \infty$ whenever α +
then

$$
\mathcal{A}(n) \approx 2^{n[\alpha + \frac{1}{q} - \frac{1}{p}]} \times 2^{n\frac{1}{q}(\gamma - 1)} \times 2^{-n\frac{1}{p}(\delta - 1)} = 2^{n[\alpha + \frac{\gamma}{q} - \frac{\delta}{p}]}
$$

and so $\sum_{k=2}^{\infty} B^{r}(n) < \infty$ whenever $\alpha + \frac{\gamma}{q} - \frac{\delta}{p} < 0$.
The boundedness $R_{\alpha} : L^{p}((0, \infty), v(x) dx) \rightarrow L^{q}((0, \infty), u(x) dx)$ implies the condition (2.3). And this last one is equivalent to $I_1 < \infty$ and $I_2 < \infty$. So the above computations lead to the inequalities $\alpha + \frac{\gamma}{q} - \frac{\delta}{p} < 0 < \alpha + \frac{\beta}{q} - \frac{\delta}{p}$ which are nothing else than condition (2.14). The finitness of I_1 can only be held whenever $\delta < p$ and similarly $I_2 < \infty$ implies necessarily $\gamma < (1 - \alpha)q$.

The result for $W_{\alpha}: L^p((0,\infty),v^*(x)dx) \to L^q((0,\infty),u^*(x)dx)$ can be deduced from the above one by duality. Indeed, by this boundedness is equivalent to R_{α} Let us then the interpretation L_1 is the minimizer δ of p and similarly $I_2 < \infty$ implies necessarily $\gamma < (1 - \alpha)q$.

The result for $W_{\alpha}: L^p((0, \infty), v^*(x) dx) \to L^q((0, \infty), u^*(x) dx)$ can be deduced

from the above one by from the above one by duality. Indeed, by this boundedness is equinople $L^{p_1}((0, \infty), v_1(x) dx) \to L^{q_1}((0, \infty), u_1(x) dx)$ where $p_1 = q', q_1 = p', v_1(x)$
 $(x) = x^{\delta - 1}$ and $u_1(x) = (v^*)^{(1-p_1)}(x) = x^{\beta - 1} \mathbf{1}_{(0,1)}(x) + x^{\delta - 1} \mathbf{1}_{$

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for pointing some confusions in the statements in [4] and [5].
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