Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 16 (1997), No. 4, 801-829

Weighted Norm Inequalities for Riemann-Liouville Fractional Integrals of Order Less than One

Y. Rakotondratsimba

Abstract. Necessary and sufficient condition on weight functions $u(\cdot)$ and $v()$ are derived in order that the Riemann-Liouville fractional integral operator R_α $(0 < \alpha < 1)$ is bounded from the weighted Lebesgue spaces $L^p((0,\infty), v(x)\,dx)$ into $L^q((0,\infty), u(x)\,dx)$ whenever $1 < p \leq q < \infty$ or $1 < q < p < \infty$. As a consequence for monotone weights then a simple characterization for this boundedness is given whenever $p \leq q$. Similar problems for convolution operators, acting on the whole real axis $(-\infty, \infty)$, are also solved.

Keywords: *Weighted inequalities, Riemann-Liouville operators, convolution operators*
AMS subject classification: 42 B 25, 26 A 33

1. Introduction

The Riemann-Liouville and Weyl fractional integral operators are defined, up to normalizing constants, respectively by

$$(R_\alpha f)(x) = \int_0^x (x-y)^{\alpha-1} f(y)\,dy \qquad (\alpha > 0)$$

and

$$(W_\alpha f)(x) = \int_x^\infty (y-x)^{\alpha-1} f(y)\,dy \qquad (\alpha > 0)$$

for all locally integrable functions $f(\cdot)$ on $(0,\infty)$. One of our purposes is to study weighted inequalities of the form

$$\left(\int_0^\infty (Tf)^q(x) u(x)\,dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p(x) v(x)\,dx\right)^{\frac{1}{p}} \qquad \text{for all } f(\cdot) \geq 0 \qquad (1.1)$$

where T is either R_α or W_α $(0 < \alpha < 1,\ 1 < p,q < \infty)$, $u(\cdot)$ and $v(\cdot)$ are non-negative weight functions, and $C > 0$ is a constant depending only on p, q, $u(\cdot)$ and $v(\cdot)$. For convenience (1.1) is also denoted by

$$T:\ L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big).$$

Y. Rakotondratsimba: Institut Polytechnique St.-Louis, E.P.M.I., 13 bd de l'Hautil, 95 092 Cergy-Pontoise cedex, France

ISSN 0232-2064 / $ 2.50

The boundedness (1.1) for R_α or W_α is very useful in Real Analysis. For instance, it can be used in order to derive analogous weighted inequalities for the Laplace transform and the Edélyi-Kober operators [1, 2]. Inequalities like (1.1) find also applications in studying boundedness of fractional maximal and integral operators on amalgam spaces with weights [3].

For the range $\alpha \geq 1$ and $1 < p \leq q < \infty$, a characterization of weights $u(\cdot)$ and $v(\cdot)$ for which (1.1) holds, was due to F. Martin-Reyes and E. Sawyer [8], and independently by Stepanov [12] who also solved the problem for $1 < q < p < \infty$.

So in this paper our study will be focused for the case $0 < \alpha < 1$ and with $1 < p, q < \infty$ which is from now assumed. In such a setting, problem (1.1) remains open in full generality. For a large class of weight functions, and particularly for monotone weights, we will completely solve this problem by using very simple characterizing conditions.

A necessary and sufficient condition for $W_\alpha : L^p((0,\infty), u(x)\,dx) \to L^{p^*}((0,\infty), u(x)\,dx)$, with $p < \frac{1}{\alpha}$ and $\frac{1}{p^*} = \frac{1}{p} - \alpha$, was found by K. Andersen and E. Sawyer [2]. For $u(\cdot) \neq v(\cdot)$ the boundedness $W_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ can be characterized whenever $u(\cdot) \in A^+_\infty$, i.e.

$$\left(\frac{1}{\varepsilon}\int_{b-\varepsilon}^{b} u(z)\,dz\right)^{\frac{1}{t}} \left(\frac{1}{\varepsilon}\int_{b}^{b+\varepsilon} u^{1-t'}(z)\,dz\right)^{\frac{1}{t'}} \leq A \qquad \text{for all } 0 < \varepsilon < b$$

for some fixed constants $t > 1$, $A > 0$ and with $t' = \frac{t}{t-1}$. Indeed, for such a weight $u(\cdot)$, it is known in [7] that $\int_0^\infty (W_\alpha f)^q(x) u(x)\,dx \approx \int_0^\infty (M^+_\alpha f)^q(x) u(x)\,dx$, where M^+_α is the right-sided fractional maximal operator studied by F. Martin-Reyes and A. de la Torre [9]. Thus (1.1) (with $T = W_\alpha$) becomes equivalent to $M^+_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ whose a characterization was also given by these authors. However, note that the characterizing condition is often difficult to use for explicit computations, since it is expressed in terms of the maximal operator itself and integrations over (special) arbitrary intervalls. Later M. Lorente and A. de la Torre [6] found a simpler characterizing condition for the range $p < q$. More details on their condition will be discussed in the next Section 2.

Without any further assumptions on $u(\cdot)$ and $v(\cdot)$ a result due to K. Andersen and H. Heinig [1] asserts that, for $p \leq q$ then $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ whenever, for some $\varepsilon \in [0,1]$ and $A > 0$,

$$\left(\int_R^\infty (y-R)^{(\alpha-1)\varepsilon q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^R (R-y)^{(\alpha-1)(1-\varepsilon)p'} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \leq A \qquad (1.2)$$

for all $R > 0$. Here and in the sequel $p' = \frac{p}{p-1}$. Condition (1.2) is only a sufficient one (generally not necessary) for (1.1) to hold. It will be seen in the next section that (1.2) cannot be used to treat the limiting case $\frac{1}{q} = \frac{1}{p} - \alpha$, and many weight functions $u(\cdot)$ are excluded. Also the case $q < p$ is not treated in [1]. These facts lead us to consider and study again inequality (1.1).

As we will see below, a necessary condition for $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty)$ $u(x)\,dx)$ when $p \le q$ is

$$\left(\int_{2R}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0. \tag{1.3}$$

So a main question, answered in this paper, is to find another companion condition [see Theorem 2.1] such that both of them are necessary and sufficient for R_α : $L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$. This companion condition is expressed in term of the boundedness of some restricted operator associated to R_α. Although a characterizing condition for this last boundedness remains open, surprisingly [see Proposition 2.3] a simple (pointwise) sufficient condition can be derived. This last is not far from a suitable necessary condition for $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$. As a consequence [in Proposition 2.5] we will see that (1.3) is a necessary and sufficient condition for (1.1) [with $T = R_\alpha$] for monotone weights. And really [see Corollary 2.6] this characterization remains true for a large class of weights. Similar results for the operator W_α as well as in the range $q < p$ are also obtained in Theorem 2.2, Proposition 2.4 and Corollary 2.7. Examples showing the computabilities of our conditions and also the gain over past results will be presented in Corollaries 2.8 – 2.10.

The second purpose of this paper is the generalization of results for R_α and W_α to the case of convolution operators (with decreasing kernel) which act on the whole real axis $(-\infty,\infty)$. These general results will be stated in Section 3. The last Section 4 is devoted to the proof of our results.

While this paper is typesetted, I receive a preprint from V. Kokilashvili [5] announcing a full characterization for $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ with $p < q$. So from his collaboration with I. Genebashvili and A. Gogatishvili [4] then it is known that this boundedness holds if and only if, for all $0 < \varepsilon < b$,

$$\left(\int_{b-\varepsilon}^{b+\varepsilon} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^{b-\varepsilon} (b-y)^{(\alpha-1)p'} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \le A \tag{1.4}$$

and

$$\left(\int_{b+\varepsilon}^{\infty} (y-b)^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_{b-\varepsilon}^{b+\varepsilon} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \le A \tag{1.5}$$

where $A > 0$ is a fixed constant. However, in [5] and [4] the reader would be aware of confusions in the range of integrations.

In comparison with their result, one of the contributions of the present paper is the treatement of the forbidden case $q \le p$. On the other hand, for the case $p < q$ the interest in our results can be found on the computabilities of the conditions introduced. The difficulties which appear in checking, for instance, condition (1.5) are alluded in the next Section 2.

2. Results for the Riemann-Liouville and Weyl operators

This section is devoted to the statement of our results (see Section 3) for the usal Riemann-Liouville and Weyl operators when they act on $(0,\infty)$. In this paper, it will be assumed that

$$0<\alpha<1, \qquad 1<p,q<\infty, \qquad p'=\frac{p}{p-1}, \qquad q'=\frac{q}{q-1},$$

and

$$u(\cdot), v^{1-p'}(\cdot) \qquad \text{are locally integrable and non-negative functions.}$$

First we give a necessary and sufficient condition for the boundednesses $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ and $W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ whenever $p \le q$ or $q<p$.

Theorem 2.1. *Suppose $p \le q$. The boundedness*

$$R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

holds if and only if for some constant $A>0$

$$\left(\int_{2R}^{\infty} x^{(\alpha-1)q}u(x)\,dx\right)^{\frac{1}{q}}\left(\int_0^R v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \le A \qquad \textit{for all } R>0 \tag{2.1}$$

and

$$\widetilde{R}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

where $\widetilde{R}_\alpha$ is the restricted operator given by

$$(\widetilde{R}_\alpha f)(x) = \int_{\frac{1}{2}x}^{x} (x-y)^{\alpha-1} f(y)\,dy.$$

Analogously,

$$W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

if and only if

$$\left(\int_{2R}^{\infty} x^{(\alpha-1)p'}v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}}\left(\int_0^R u(x)\,dx\right)^{\frac{1}{q}} \le A \qquad \textit{for all } R>0 \tag{2.2}$$

and

$$\widetilde{W}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

where

$$(\widetilde{W}_\alpha f)(x) = \int_{x}^{2x} (y-x)^{\alpha-1} f(y)\,dy.$$

Theorem 2.2. *Let $q < p$ and $r = \frac{qp}{p-q}$. The boundedness*

$$R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

holds if and only if for some constant $A > 0$:

$$\int_0^\infty \left[\left(\int_{2x}^\infty y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)\,dx < A^r \tag{2.3}$$

and

$$\widetilde{R}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big).$$

Similarly,

$$W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

if and only if

$$\int_0^\infty \left[\left(\int_{2x}^\infty y^{(\alpha-1)p'} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \left(\int_0^x u(y)\,dy\right)^{\frac{1}{p}}\right]^r u(x)\,dx < A^r \tag{2.4}$$

and

$$\widetilde{W}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big).$$

Therefore, the real difficulty to derive $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ is on getting the weighted inequality $\widetilde{R}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ for the restricted operator $\widetilde{R}_\alpha$. A characterization of weights $u(\cdot)$ and $v(\cdot)$ for which this last boundedness holds is an open problem. However, it is possible to give a sufficient condition as stated in the following two propositions.

Proposition 2.3. *Let $p \le q$. For $p < q$ and $p < \frac{1}{\alpha}$ it is also assumed that $q \le p^*$ with $\frac{1}{p^*} = \frac{1}{p} - \alpha$. The boundedness*

$$\widetilde{R}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

holds whenever, for a constant $A > 0$,

$$R^{\alpha+\frac{1}{q}-\frac{1}{p}} \left(\sup_{R<z<2R} u(z)\right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0. \tag{2.5}$$

Similarly,

$$\widetilde{W}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

whenever

$$R^{\alpha+\frac{1}{q}-\frac{1}{p}} \left(\sup_{\frac{1}{2}R<z<2R} u(z)\right)^{\frac{1}{q}} \left(\sup_{R<y<2R} v^{1-p'}(y)\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0. \tag{2.6}$$

The hypothesis on α, p, q and p^* in Proposition 2.3 is justified by the following fact:

A necessary condition for $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ *is* $\frac{1}{p} - \frac{1}{q} \le \alpha$.

Indeed, this boundedness implies

$$t^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{t}\int_{a+t}^{a+2t} u(z)dz\right)^{\frac{1}{q}}\left(\frac{1}{t}\int_{a}^{a+t} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \le A \quad \text{for all } a, t > 0$$

and with a fixed constant $A > 0$. Therefore, by the Lebesgue differentiation theorem, if $\alpha + \frac{1}{q} - \frac{1}{p} < 0$, then necessarily $u(\cdot) = 0$ or $v^{1-p'}(\cdot) = 0$ a.e.

The condition (2.5) is not too far from a necessary one for $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$. Indeed, this last boundedness implies that for a fixed constant $A > 0$,

$$R^{\alpha-1}\left(\int_{R}^{2R} u(z)dz\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}R}^{R} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0$$

and

$$\int_{R}^{2R} u(z)dz \le R \sup_{R<z<2R} u(z), \qquad \int_{\frac{1}{2}R}^{R} v^{1-p'}(y)\,dy \le R \sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y).$$

Of course, analogous observations can be made for condition (2.6).

Proposition 2.4. *Let* $q < p$ *and* $r = \frac{qp}{p-q}$. *The boundedness*

$$\widetilde{R}_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$$

holds whenever, for a constant $A > 0$ *and a sequence* $(\mathcal{B}(n))_{n\in\mathbb{Z}}$,

$$2^{n[\alpha+\frac{1}{q}-\frac{1}{p}]}\left(\sup_{2^n<y<2^{n+1}} u(y)\right)^{\frac{1}{q}}\left(\sup_{2^{n-1}<y<2^{n+1}} v^{1-p'}(y)\right)^{\frac{1}{p'}} \le \mathcal{B}(n) \quad \forall\, n \in \mathbb{Z} \tag{2.7}$$

and

$$\sum_{n\in\mathbb{Z}} [\mathcal{B}(n)]^r < A^r. \tag{2.8}$$

Conditions (2.7) and (2.8) are not too far from the necessary condition

$$\sum_{n\in\mathbb{Z}} [\mathcal{A}(n)]^r < c \tag{2.9}$$

for the boundedness $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$, where

$$\mathcal{A}(n) = 2^{n(\alpha-1)}\left(\int_{2^n}^{2^{n+1}} u(z)dz\right)^{\frac{1}{q}}\left(\int_{2^{n-1}}^{2^{n+1}} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}}.$$

The fact that (2.9) is a required condition can be seen by taking

$$f(x) = \sum_{k=-M}^{N} 2^{k(\alpha-1)\frac{r}{p}} \left(\int_{2^n}^{2^{n+1}} u(z)dz \right)^{\frac{r}{pq}} \times \left(\int_{2^{n-1}}^{x} v^{1-p'}(y)\,dy \right)^{\frac{r}{pq'}} v^{1-p'}(x) \mathbf{1}_{(2^{n-1},2^n)}(x)$$

in the corresponding inequality to $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ where N and M are arbitrary non-negative integers.

For monotone weight functions, Theorem 2.1 and Proposition 2.3 can be used to get easy characterizations for the above boundednesses.

Proposition 2.5. *Let $p \le q$, with $q \le p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Suppose that $u(\cdot)$ and $v^{1-p'}(\cdot)$ are monotone functions.*

Condition (2.1) *is a necessary and sufficient one for*

$$R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

whenever $u(\cdot)$ is an increasing function or whenever $v^{1-p'}(\cdot)$ is a decreasing function.

The above equivalence remains true if $u(\cdot)$ is decreasing and $v^{1-p'}(\cdot)$ increasing with $u(x) \le c\,u(2x)$ or $v^{1-p'}(2x) \le c\,v(x)$, for a fixed constant $c > 0$.

Condition (2.2) *is a necessary and sufficient one for*

$$W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

whenever $v^{1-p'}(\cdot)$ is an increasing function or whenever $u(\cdot)$ is a decreasing function. This last equivalence remains true if $v^{1-p'}(\cdot)$ is decreasing and $u(\cdot)$ increasing with $v^{1-p'}(x) \le c\,v(2x)$ or $u(2x) \le c\,u(x)$.

Each monotone weight function $w(\cdot)$ satisfies the growth condition

$$\sup_{\frac{1}{2}R<z<2R} w(z) \le C\frac{1}{R}\int_{2^{-N}R}^{2^N R} w(y)\,dy \qquad \text{for all } R > 0 \tag{$\mathcal{C}$}$$

where both $C > 0$ and N (an integer greater than 2) depend only on $w(\cdot)$. It will be denoted that $w(\cdot) \in \mathcal{C}$. For a monotone weight, then $w(\cdot) \in \mathcal{C}$ with the constant $N = 2$. There are also non-necessarily monotone weights for which this property is fulfilled. Indeed, it can be shown that $w(\cdot) \in \mathcal{C}$ whenever $w(\cdot) = w_0(\cdot)\mathbf{1}_{[0,1]}(\cdot) + w_1(\cdot)\mathbf{1}_{(1,\infty)}(\cdot)$ where $w_0(\cdot)$ or $w_1(\cdot)$ is an increasing or decreasing weight function, respectively.

For $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{C}$, by Proposition 2.3, then the boundedness $\widetilde{R}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ holds whenever

$$R^{\alpha-1}\left(\int_{2^{-2(N+1)}R}^{R} u(x)\,dx\right)^{\frac{1}{q}} \left(\int_{2^{-2(N+1)}R}^{R} v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \le A \quad \forall\, R > 0. \tag{2.10}$$

Also (2.10) is a sufficient condition which ensures $\widetilde{W}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$.

Corollary 2.6. *Let $p \leq q$, with $q \leq p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Suppose that $u(\cdot)$, $v^{1-p'}(\cdot) \in C$. Then* (2.1) *is a necessary and sufficient condition for*

$$R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

whenever (2.10) *is satisfied. Also under* (2.10), *then* (2.2) *is a necessary and sufficient condition for*

$$W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big).$$

In order to state an analogous result for the case $q < p$, it is convenient to introduce the condition

$$\begin{aligned} R^{\alpha-1}\left(\int_{2^{-(N+1)}R}^{2^{(N+1)}R} u(y)\,dy\right)^{\frac{1}{q}}\left(\int_{2^{-(N+1)}R}^{2^{(N+1)}R} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \\ \leq c\left(\int_{4R}^{\infty} y^{(\alpha-1)q}u(y)\,dy\right)^{\frac{1}{q}}\left(\int_0^R v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\left(\int_R^{2R} v^{1-p'}(y)\,dy\right)^{\frac{1}{r}}. \end{aligned} \tag{2.11}$$

Corollary 2.7. *Let $q < p$ and $r = \frac{qp}{p-q}$. Suppose that $u(\cdot)$, $v^{1-p'}(\cdot) \in C$. Then* (2.3) *is a necessary and sufficient condition for*

$$R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

whenever (2.11) *is satisfied.*

To illustrate these results, some examples are now given.

Corollary 2.8. *Let $p \leq q$, with $q \leq p^*$ for $p < q$ and $p < \frac{1}{\alpha}$. Define $u(x) = x^{\beta-1}$ and $v(x) = x^{\delta-1}$. The boundedness*

$$R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

holds if and only if both $\delta < p$ and $\beta < (1-\alpha)q$ and

$$\alpha + \frac{\beta}{q} = \frac{\delta}{p}. \tag{2.12}$$

Similarly,

$$W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

if and only if both $0 < \beta$ and $\alpha p < \delta$ and (2.12) *is satisfied.*

As mentioned in the introduction, according to K. Andersen and H. Heinig [1], then $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ whenever, for some $\varepsilon \in [0,1]$ and $A > 0$,

$$\left(\int_R^{\infty} (y-R)^{(\alpha-1)\varepsilon q}u(y)\,dy\right)^{\frac{1}{q}}\left(\int_0^R (R-y)^{(\alpha-1)(1-\varepsilon)p'}v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \leq A$$

for all $R > 0$. Taking $u(y) = y^{\beta-1}$ and $v(y) = y^{\delta-1}$ then

$$\begin{aligned}\int_R^\infty (y-R)^{(\alpha-1)\varepsilon q} u(y)\,dy &= \int_R^{2R} + \int_{2R}^\infty (y-R)^{(\alpha-1)\varepsilon q} y^{\beta-1}\,dy \\ &\approx R^{\beta-1}\int_0^R t^{(\alpha-1)\varepsilon q}dt + \int_{2R}^\infty y^{(\alpha-1)\varepsilon q+\beta-1}dt.\end{aligned}$$

Consequently, to continue the computations it is required that $\beta < (1-\alpha)\varepsilon q < 1$. And analogously

$$\begin{aligned}&\int_0^R (R-y)^{(\alpha-1)(1-\varepsilon)p'} v^{1-p'}(y)\,dy \\ &\quad = \int_0^{\frac{1}{2}R} + \int_{\frac{1}{2}R}^R (R-y)^{(\alpha-1)(1-\varepsilon)p'} v^{1-p'}(y)\,dy \\ &\quad \approx R^{(\alpha-1)(1-\varepsilon)p'} \int_0^{\frac{1}{2}R} y^{(1-p')(\delta-1)}dy + R^{(1-p')(\delta-1)} \int_0^{\frac{1}{2}R} t^{(\alpha-1)(1-\varepsilon)p'}dt\end{aligned}$$

and it requires that $\delta < p$ and $(1-\alpha)(1-\varepsilon)p' < 1$. So the real ε must satisfy $\frac{1}{p} - \alpha < \varepsilon(1-\alpha) < \frac{1}{q}$. Therefore, this Andersen-Heinig's result can be only applied whenever $\beta < 1$ and $\frac{1}{p} - \alpha < \frac{1}{q}$. In view of Corollary 2.8, these restrictions are not needed since it is necessary that $\beta < (1-\alpha)q$ and $\frac{1}{p} - \alpha \le \frac{1}{q}$.

As seen in the introduction, for $u(\cdot) \in A_\infty^+$ and $p < q$, by a result due to M. Lorente and A. de la Torre [6], the boundedness $W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ is equivalent to

$$\left(\int_{b-\varepsilon}^{b+\varepsilon} v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \left(\int_0^{b-\varepsilon} (b-y)^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \le A$$

for all b and ε with $0 < \varepsilon < b$.

Compared to (2.2) and (2.6) [see the proof of Corollary 2.8] this last condition is more delicate to check. To justify this claim consider again the case of power weights $u(y) = y^{\beta-1}$ and $v(y) = y^{\delta-1}$. The term $\int_{b-\varepsilon}^{b+\varepsilon} v^{1-p'}(y)\,dy$ is evaluated following $\frac{1}{2}b \le \varepsilon < b$ or $0 < \varepsilon < \frac{1}{2}b$. In the first case then $\int_{b-\varepsilon}^{b+\varepsilon} v^{1-p'}(y)\,dy \le \int_0^{2b} v^{1-p'}(y)\,dy$ and in the second case it is used that $v(y) \approx b^{\delta-1}$ for $b-\varepsilon < y < b+\varepsilon$. The term $\int_0^{b-\varepsilon}(b-y)^{(\alpha-1)q}u(y)\,dy$ is more subtle to estimate than the first one. For $\frac{1}{2}b \le \varepsilon < b$ the main point is $(b-y) \approx b$ for $0 < y < b-\varepsilon \le \frac{1}{2}b$ and then $\int_0^{b-\varepsilon}(b-y)^{(\alpha-1)q}u(y)\,dy \le c\, b^{(\alpha-1)q}\int_0^b u(y)\,dy$. And for $0 < \varepsilon < \frac{1}{2}b$ then

$$\begin{aligned}\int_0^{b-\varepsilon} (b-y)^{(\alpha-1)q} u(y)\,dy &= \int_0^{\frac{1}{2}b} + \int_{\frac{1}{2}b}^{b-\varepsilon} (b-y)^{(\alpha-1)q} u(y)\,dy \\ &\le c\, b^{(\alpha-1)q}\int_0^{\frac{1}{2}b} u(y)\,dy + b^{(\beta-1)}\int_\varepsilon^{\frac{1}{2}b} t^{(\alpha-1)q}dt.\end{aligned}$$

So the sequel of computations depends on the sign of $(\alpha-1)q+1$. For instance, if $(\alpha-1)q+1<0$, then $\int_\varepsilon^{\frac{1}{2}b} t^{(\alpha-1)q}dt \approx \varepsilon^{(\alpha-1)q+1}$ for $\varepsilon\to 0$.

All of these considerations lead to think that the conditions used in our results are quite easy to apply for explicit computations compared to known results.

Weights which are not necessarily of power type can be treated by the above results.

Corollary 2.9. *Let $p\le q$, with $q\le p^*$ for $p<q$ and $p<\frac{1}{\alpha}$. Define the weight functions*

$$u(x)=x^{\beta-1}\mathbf{1}_{(0,\frac{1}{2})}(x)+x^{\gamma-1}\mathbf{1}_{(\frac{1}{2},\infty)}(x) \qquad \text{with } (1-\alpha)q<\beta$$

and

$$v(x)=x^{p-1}\ln^p(x^{-1})\mathbf{1}_{(0,\frac{1}{2})}(x)+x^{\theta-1}\mathbf{1}_{(\frac{1}{2},\infty)}(x) \qquad \text{with } \theta<p.$$

The boundedness

$$R_\alpha:\ L^p\big((0,\infty),v(x)\,dx\big)\to L^q\big((0,\infty),u(x)\,dx\big)$$

holds if and only if $\gamma<(1-\alpha)q<\beta$ and

$$\alpha+\frac{\gamma}{q}\le\frac{\theta}{p}. \tag{2.13}$$

What is remarkable in this example is the fact that $\int_0^R v^{(1-p')\varepsilon}(y)dy=\infty$ for $R<\frac{1}{2}$ and $\varepsilon>1$. So this boundedness cannot be treated by using a bumping condition like

$$t^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{t}\int_{x_0-t}^{x_0+t}u^\varepsilon(z)dz\right)^{\frac{1}{\varepsilon q}}\left(\frac{1}{t}\int_{x_0-t}^{x_0+t}v^{(1-p')\varepsilon}(z)dz\right)^{\frac{1}{\varepsilon p'}}\le A \quad \forall\, t,x_0>0$$

as it is introduced and used in [11] to treat weighted inequalities for the two-sided operators $I_\alpha=R_\alpha+W_\alpha$.

Finally, we give an example for the case $q<p$ which is new since it seems there is no available papers which treats the problem for this case.

Corollary 2.10. *Let $q<p$ and $r=\frac{qp}{p-q}$. Let $u(x)=x^{\beta-1}\mathbf{1}_{(0,1)}(x)+x^{\gamma-1}\mathbf{1}_{(1,\infty)}(x)$ and $v(x)=x^{\delta-1}$. The boundedness*

$$R_\alpha:\ L^p\big((0,\infty),v(x)\,dx\big)\to L^q\big((0,\infty),u(x)\,dx\big)$$

holds whenever both $\delta<p$ and $\gamma<(1-\alpha)q$ and

$$\alpha+\frac{\gamma}{q}<\frac{\delta}{p}<\alpha+\frac{\beta}{q}. \tag{2.14}$$

Let $u^*(x)=x^{(1-q)(\delta-1)}$ and $v^*(x)=x^{(1-p)(\beta-1)}\mathbf{1}_{(0,1)}(x)+x^{(1-p)(\gamma-1)}\mathbf{1}_{(1,\infty)}(x)$. The boundedness

$$W_\alpha:\ L^p\big((0,\infty),v^*(x)\,dx\big)\to L^q\big((0,\infty),u^*(x)\,dx\big)$$

holds whenever both $\delta<q'$ and $\gamma<(1-\alpha)p'$ and

$$\alpha+\frac{\gamma}{p'}<\frac{\delta}{q'}<\alpha+\frac{\beta}{p'}. \tag{2.15}$$

For an explicit example suppose

$$0<\alpha<\frac{2}{8},\quad u(x)=x^{\beta-1}\mathbf{1}_{[0,1]}(x)+x^{\gamma-1}\mathbf{1}_{[1,\infty)}(x),\quad \gamma=\frac{1}{4},\quad \beta>2,\quad v(x)=x^{\frac{1}{2}}.$$

Then $R_\alpha:\ L^4\big((0,\infty),v(x)\,dx\big)\to L^2\big((0,\infty),u(x)\,dx\big)$.

3. Results for convolution operators

In this section the results in Section 2 are generalized for convolution operators like

$$(Tf)(x) = \int_{-\infty}^{x} K(x-y)f(y)\,dy$$

where K is a non-negative kernel quasi-decreasing, i.e.

$$K(R_2) \le c\,K(R_1) \qquad \text{for all } \; 0 < R_1 \le R_2 \tag{3.1}$$

and satisfying the growth condition

$$K(R) \le c\,K(2R) \qquad \text{for all } \; R > 0. \tag{3.2}$$

In (3.1) and (3.2) the constant $c > 0$ depends only on the kernel $K(\cdot)$. Without (3.2) our results remain true if in each occurence $K(R)$ or $K(x)$ is replaced by $K(CR)$ or $K(Cx)$, respectively, where $C > 0$ is a constant depending only on $K(\cdot)$, p and q.

Our purpose, in this section, is to study the boundedness

$$T:\; L^p\big((-\infty,\infty), v(x)\,dx\big) \to L^q\big((-\infty,\infty), u(x)\,dx\big)$$

which means

$$\left(\int_{-\infty}^{\infty} (Tf)^q(x)u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{-\infty}^{\infty} f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \quad \text{for all } f(\cdot) \ge 0. \tag{3.3}$$

Of course, here $C > 0$ is a fixed constant. For shortness, we will restrict to the range

$$p \le q.$$

The case $q < p$ can be also treated as it is done in Section 2 for the Riemann-Liouville and Weyl operators.

First a necessary and sufficient conditions for $T : L^p\big((-\infty,\infty), v(x)\,dx\big) \to L^q\big((-\infty,\infty), u(x)\,dx\big)$ is stated.

Theorem 3.1. *The boundedness*

$$T:\; L^p\big((-\infty,\infty), v(x)\,dx\big) \to L^q\big((-\infty,\infty), u(x)\,dx\big)$$

holds if and only if, for a constant $A > 0$, the three conditions

$$\left(\int_{2R}^{\infty} K^q(x)u(x)\,dx\right)^{\frac{1}{q}} \left(\int_{0}^{R} [v^{1-p'}(x) + v^{1-p'}(-x)]dx\right)^{\frac{1}{p'}} \le A \quad \forall\, R > 0 \tag{3.4}$$

$$\left(\int_{0}^{R} [u(x) + u(-x)]dx\right)^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x)v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A \quad \forall\, R > 0 \tag{3.5}$$

$$K(R)\left(\int_{\frac{1}{2}R}^{2R} u(x)\,dx\right)^{\frac{1}{q}} \left(\int_{0}^{R} v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A \quad \forall\, R > 0 \tag{3.6}$$

and

$$\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$
$$\widetilde{T}^* : L^p\big((0,\infty), v(-x)\,dx\big) \to L^q\big((0,\infty), u(-x)\,dx\big)$$

are satisfied, where

$$(\widetilde{T}f)(x) = \int_{\frac{1}{2}x}^{x} K(x-y)f(y)\,dy \quad \textit{and} \quad (\widetilde{T}^*f)(x) = \int_x^{2x} K(z-x)f(z)dz.$$

In general, the three conditions (3.4) - (3.6) do not overlap. Indeed, take for instance $K(x) = |x|^{\alpha-1}$ and $u(x) = |x|^{\beta-1}$. Then (3.4) can only be held whenever at least $\beta < q(1-\alpha)$. For (3.5) it is needed that $\beta > 0$ which is not a priori the case for (3.6).

Although a characterization of weights $u(\cdot)$ and $v(\cdot)$ for which $\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ is an open problem, it is not too difficult to derive a sufficient condition. This last one depends highly on further properties of the kernel K. So two results, going in this direction, are given.

Proposition 3.2. *Assume that, for some $\varepsilon \in [0,1]$ and $c > 0$,*

$$\int_0^R K^{\varepsilon q}(z)dz \le cR \times K^{\varepsilon q}(R), \qquad \int_0^R K^{(1-\varepsilon)p'}(z)dz \le cR \times K^{(1-\varepsilon)p'}(R). \tag{3.7}$$

The boundedness

$$\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

holds whenever for a constant $A > 0$

$$R^{\frac{1}{q}+\frac{1}{p'}} K(R)\Big(\sup_{R<z<2R} u(z)\Big)^{\frac{1}{q}} \Big(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\Big)^{\frac{1}{p'}} \le A \quad \forall\, R > 0. \tag{3.8}$$

Similarly,

$$\widetilde{T}^* : L^p\big((0,\infty), v(-x)\,dx\big) \to L^q\big((0,\infty), u(-x)\,dx\big)$$

holds whenever

$$R^{\frac{1}{q}+\frac{1}{p'}} K(R)\Big(\sup_{\frac{1}{2}R<z<2R} u(-z)\Big)^{\frac{1}{q}} \Big(\sup_{R<y<2R} v^{1-p'}(-y)\Big)^{\frac{1}{p'}} \le A \quad \forall\, R > 0. \tag{3.9}$$

For $K(x) = x^{\alpha-1}$ $(0 < \alpha < 1)$ then (3.7) can hold whenever

$$\Big(\frac{1}{p} - \alpha\Big) < (1-\alpha)\varepsilon < \frac{1}{q}. \tag{3.10}$$

Thus for $\frac{1}{\alpha} \le p$, condition (3.7) is always satisfied whenever $0 \le \varepsilon < \min(1, \frac{1}{1-\alpha}\frac{1}{q})$. And for $p < \frac{1}{\alpha}$, a necessary condition for (3.10) is $\frac{1}{p^*} = \frac{1}{p} - \alpha < \frac{1}{q}$ or $q < p^*$. Consequently, the boundedness $\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^{p^*}\big((0,\infty), u(x)\,dx\big)$ cannot be decided from Proposition 3.2, and another kind of criterion is needed.

Proposition 3.3. *Assume that for some* $\overline{p} \geq q$ *and* $c > 0$

$$T: L^p\big((0,\infty), dx\big) \to L^{\overline{p}}\big((0,\infty), dx\big) \tag{3.11}$$

and

$$1 \leq c\, R^{\frac{1}{p'}+\frac{1}{\overline{p}}} K(R) \qquad \textit{for all } R > 0. \tag{3.12}$$

Then condition (3.8) *implies the boundedness*

$$\widetilde{T}: L^p\big((0,\infty), v(x)\, dx\big) \to L^q\big((0,\infty), u(x)\, dx\big).$$

Similarly, condition (3.9) *yields*

$$\widetilde{T}^*: L^p\big((0,\infty), v(-x)\, dx\big) \to L^q\big((0,\infty), u(-x)\, dx\big)$$

whenever

$$T^*: L^p\big((0,\infty), dx\big) \to L^{\overline{p}}\big((0,\infty), dx\big) \tag{3.13}$$

and (3.12) *is satisfied.*

Hypothesis (3.12) is only introduced in order to have the same sufficient conditions in Propositions 3.2 and 3.3. Without (3.12) it will be seen in the proof that [with (3.11)] the boundedness $\widetilde{T}: L^p\big((0,\infty), v(x)\, dx\big) \to L^q\big((0,\infty), u(x)\, dx\big)$ holds whenever

$$R^{\frac{1}{q}-\frac{1}{\overline{p}}}\Big(\sup_{R<z<2R} u(z)\Big)^{\frac{1}{q}}\Big(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\Big)^{\frac{1}{p'}} \leq A \qquad \text{for all } R > 0.$$

Now these results are applied to the case of even and quasi-monotone weights. Here $w(\cdot)$ is said to be an even and quasi-monotone weight if $w(x) = w_0(x)$ for $x > 0$, $w(-x) = w_0(x)$ and where $w_0(\cdot)$ is quasi-monotone on $(0,\infty)$. Remind that the quasi-decrease is taken in the sense of (3.1) [and $\varphi(\cdot)$ is quasi-increasing if $\frac{1}{\varphi}(\cdot)$ is quasi-decreasing].

Proposition 3.4. *Assume that property* (3.7) *is fulfilled or all three conditions* (3.11) – (3.13) *are satisfied (so in this last case* $p \leq q \leq \overline{p}$*). Suppose that* $u(\cdot)$ *and* $v^{1-p'}(\cdot)$ *are even and quasi-monotone weight functions. The boundedness*

$$T: L^p\big((-\infty,\infty), v(x)\, dx\big) \to L^q\big((-\infty,\infty), u(x)\, dx\big)$$

holds if and only if the following three conditions are satisfied:

$$\Big(\int_{2R}^{\infty} K^q(x) u(x)\, dx\Big)^{\frac{1}{q}}\Big(\int_0^R v^{1-p'}(x)\, dx\Big)^{\frac{1}{p'}} \leq A \qquad \textit{for all } R > 0 \tag{3.14}$$

$$\Big(\int_0^R u(x)\, dx\Big)^{\frac{1}{q}}\Big(\int_{2R}^{\infty} K^{p'}(x) v^{1-p'}(x)\, dx\Big)^{\frac{1}{p'}} \leq A \qquad \textit{for all } R > 0 \tag{3.15}$$

$$K(R)\Big(\int_{\frac{1}{2}R}^{2R} u(x)\, dx\Big)^{\frac{1}{q}}\Big(\int_0^R v^{1-p'}(x)\, dx\Big)^{\frac{1}{p'}} \leq A \qquad \textit{for all } R > 0. \tag{3.16}$$

Better, if $u(\cdot)$ is quasi-increasing or $v^{1-p'}(\cdot)$ is quasi-decreasing, then

$$T: L^p((-\infty,\infty), v(x)\,dx) \to L^q((-\infty,\infty), u(x)\,dx)$$

if and only both (3.14) *and* (3.15) *are satisfied. This last equivalence remains true whenever both $u(\cdot)$ and $v^{1-p'}(\cdot)$ are quasi-decreasing with $u(x) \le c\,u(2x)$ or $v^{1-p'}(x) \le c\,v^{1-p'}(2x)$, respectively, for a fixed constant $c > 0$.*

Note that the conditions (3.14) and (3.16) can be combined as

$$\left(\int_{\frac{1}{2}R}^{\infty} K^q(x)u(x)\,dx\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \qquad \text{for all } R > 0.$$

Since the class $\mathcal{C}$ (see Section 2) is larger than that of quasi-monotone weights, it would be interesting to state results for weights belonging to this class.

Proposition 3.5. *Assume that property* (3.7) *is fulfilled or the conditions* (3.11) – (3.13) *are satisfied (so in this last case $p \le q \le \overline{p}$). Suppose that $u(\cdot)$, $v^{1-p'}(\cdot)$, $u(-\cdot)$, $v^{1-p'}(-\cdot) \in \mathcal{C}$ with the (integer) constant $N \ge 2$. Then* (3.4) – (3.6) *are necessary and sufficient conditions for the boundedness*

$$T: L^p((-\infty,\infty), v(x)\,dx) \to L^q((-\infty,\infty), u(x)\,dx)$$

to hold whenever

$$K(R)\left(\int_{2^{-2N}R}^{R} u(x)\,dx\right)^{\frac{1}{q}} \left(\int_{2^{-2N}R}^{R} v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \le A \quad \forall\, R > 0 \tag{3.17}$$

$$K(R)\left(\int_{2^{-2N}R}^{R} u(-x)\,dx\right)^{\frac{1}{q}} \left(\int_{2^{-2N}R}^{R} v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A \quad \forall\, R > 0. \tag{3.18}$$

For the dual operator T^* of T defined by

$$(T^*f)(x) = \int_x^{\infty} K(y-x)f(y)\,dy$$

similar results for the boundedness $T^*: L^p((-\infty,\infty), v(x)\,dx) \to L^q((-\infty,\infty), u(x)\,dx)$ could be also obtained, by using its equivalence with $T: L^{q'}((-\infty,\infty), u^{1-q'}(x)\,dx) \to L^{p'}((-\infty,\infty), v^{1-p'}(x)\,dx)$. Just the analogous of Theorem 3.1 is stated.

Theorem 3.6. *The boundedness*

$$T^*: L^p((-\infty,\infty), v(x)\,dx) \to L^q((-\infty,\infty), u(x)\,dx)$$

holds if and only if for a constant $A > 0$ the three conditions

$$\left(\int_{2R}^{\infty} K^{p'}(x)v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \left(\int_0^R [u(x)+u(-x)]dx\right)^{\frac{1}{q}} \le A \quad \forall\, R > 0 \tag{3.19}$$

$$\left(\int_0^R [v^{1-p'}(x)+v^{1-p'}(-x)]dx\right)^{\frac{1}{p'}} \left(\int_{2R}^{\infty} K^q(x)u(-x)\,dx\right)^{\frac{1}{q}} \le A \quad \forall\, R > 0 \tag{3.20}$$

$$K(R)\left(\int_{\frac{1}{2}R}^{2R} v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \left(\int_0^R u(-x)\,dx\right)^{\frac{1}{q}} \le A \quad \forall\, R > 0 \tag{3.21}$$

and

$$\widetilde{T} : L^{q'}\big((0,\infty), u^{1-q'}(x)\,dx\big) \to L^{p'}\big((0,\infty), v^{1-p'}(x)\,dx\big)$$

$$\widetilde{T}^* : L^{q'}\big((0,\infty), u^{1-q'}(-x)\,dx\big) \to L^{p'}\big((0,\infty), v^{1-p'}(-x)\,dx\big)$$

are satisfied.

4. Proofs of Results

First a useful lemma for the proofs is given. Next we will prove the results for convolutions operators stated in Section 3. The last place is devoted to the proofs of results in Section 2 which are not direct consequences of those in Section 3.

It is convenient to state the classical Hardy inequalities [10] in the appropriated forms as needed in the proofs.

Lemma. *Define the Hardy operators H and H^* by*

$$(Hf)(x) = \int_0^{\frac{1}{2}x} f(y)\,dy \qquad \textit{and} \qquad (H^*f)(x) = \int_{2x}^{\infty} f(y)\,dy.$$

Then:

(A) *For $p \le q$ or $q < p$,*

$$H : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), w(x)\,dx\big)$$

if and only if, for a constant $A > 0$ and all $R > 0$,

$$\left(\int_{2R}^{\infty} w(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \le A$$

or

$$\int_0^{\infty} \left[\left(\int_{2x}^{\infty} w(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)\,dx \le A^r,$$

respectively.

(B) *Similarly, for $p \le q$ or $q < p$,*

$$H^* : L^p\big((0,\infty), w(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

if and only if, for a constant $A > 0$ and all $R > 0$,

$$\left(\int_{2R}^{\infty} w^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \left(\int_0^R u(y)\,dy\right)^{\frac{1}{q}} \le A$$

or

$$\int_0^{\infty} \left[\left(\int_{2x}^{\infty} w^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \left(\int_0^x u(y)\,dy\right)^{\frac{1}{p}}\right]^r u(x)\,dx \le A^r,$$

respectively.

(C) *Analogously, for $p \le q$ and $(\mathcal{H}f)(x) = \int_0^{2x} f(y)\,dy$,*

$$\mathcal{H}: L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), w(x)\,dx\big)$$

if and only if, for a constant $A > 0$ and all $R > 0$,

$$\left(\int_{\frac{1}{2}R}^{\infty} w(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \le A$$

where $r = \frac{qp}{p-q}$ whenever $q < p$.

Proof of Theorem 3.1. To get

$$T: L^p\big((-\infty,\infty), v(x)\,dx\big) \to L^q\big((-\infty,\infty), u(x)\,dx\big)$$

[or (3.3)] it remains to estimate $\int_{-\infty}^{\infty}(T\varphi)^q(x)u(x)\,dx$ for any $\varphi(\cdot) \ge 0$. Since

$$\varphi(\cdot) = f(\cdot) + g(\cdot) \qquad \text{with } f(\cdot) = \varphi(\cdot)\mathbf{1}_{(0,\infty)}(\cdot) \text{ and } g(\cdot) = \varphi(\cdot)\mathbf{1}_{(-\infty,0)}(\cdot);$$

then

$$\int_{-\infty}^{\infty}(T\varphi)^q(x)u(x)\,dx \approx \Big\{\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4\Big\}$$

where

$$\mathcal{S}_1 = \int_{-\infty}^{0}(Tf)^q(x)u(x)\,dx, \qquad \mathcal{S}_2 = \int_0^{\infty}(Tf)^q(x)u(x)\,dx$$

$$\mathcal{S}_3 = \int_{-\infty}^{0}(Tg)^q(x)u(x)\,dx, \qquad \mathcal{S}_4 = \int_0^{\infty}(Tg)^q(x)u(x)\,dx.$$

So we have to bound each $\mathcal{S}_i$ $(i = 1,\dots,4)$ by $C\big(\int_{-\infty}^{\infty}\varphi^p(x)v(x)\,dx\big)^{\frac{q}{p}}$, where $C > 0$ is a constant which does not depend on the function $\varphi(\cdot)$.

Estimate of $\mathcal{S}_1$: For $x < 0$, by the definition of $f(\cdot)$,

$$(Tf)(x) = \int_{-\infty}^{x} K(x-y)f(y)\,dy = 0$$

and so $\mathcal{S}_1 = 0$.

Estimate of $\mathcal{S}_2$: The purpose is to get

$$\mathcal{S}_2 = \int_0^{\infty}(Tf)^q(x)u(x)\,dx \le C\left(\int_0^{\infty} f^p(x)v(x)\,dx\right)^{\frac{q}{p}}. \tag{4.1}$$

For each $x > 0$ then

$$\begin{aligned}(Tf)(x) &= \int_0^{\frac{1}{2}x} K(x-y)f(y)\,dy + \int_{\frac{1}{2}x}^{x} K(x-y)f(y)\,dy \\ &\approx K(x)\int_0^{\frac{1}{2}x} f(y)\,dy + \int_{\frac{1}{2}x}^{x} K(x-y)f(y)\,dy \\ &= K(x)(Hf)(x) + (\widetilde{T}f)(x).\end{aligned}$$

The equivalence is true since $\frac{1}{2}x < x - y < x$ for $0 < y < \frac{1}{2}x$ and the growth conditions (3.1) and (3.2) on $K(\cdot)$ lead to the conclusion. Consequently, inequality (4.1) holds if and only if

$$H : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), K^q(x)u(x)\,dx\big)$$

and

$$\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big).$$

By Part A of the Lemma [with $w(x) = K^q(x)u(x)$] the first boundedness is true if and only if, for a constant $A > 0$,

$$\left(\int_{2R}^{\infty} K^q(x)u(x)\,dx\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(x)\,dx\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0. \tag{4.2}$$

This inequality is one part of condition (3.4), whose other part is

$$\left(\int_{2R}^{\infty} K^q(x)u(x)\,dx\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0. \tag{4.3}$$

Note also that (4.1) is a necessary inequality for (3.3). So we have been proved that (4.2) and $\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ are necessary conditions for $T : L^p\big((-\infty,\infty), v(x)\,dx\big) \to L^q\big((-\infty,\infty), u(x)\,dx\big)$ to hold, and they are also sufficient to get (4.1).

Estimate of $\mathcal{S}_3$: Now the inequality under the consideration is

$$\mathcal{S}_3 = \int_{-\infty}^{0} (Tg)^q(x)u(x)\,dx \le C\left(\int_{-\infty}^{0} g^p(x)v(x)\,dx\right)^{\frac{q}{p}}. \tag{4.4}$$

For each $x < 0$ then

$$\begin{aligned}(Tg)(x) &= \int_{-\infty}^{2x} K(x-y)g(y)\,dy + \int_{2x}^{x} K(x-y)g(y)\,dy \\ &\approx \int_{-\infty}^{2x} K(-y)g(y)\,dy + \int_{2x}^{x} K(x-y)g(y)\,dy.\end{aligned}$$

Indeed, $-\frac{1}{2}y < x - y < -y$ for $y < 2x$ (< 0). So (4.4) becomes equivalent to

$$\int_{-\infty}^{0}\left[\int_{-\infty}^{2x} K(-y)g(y)\,dy\right]^q u(x)\,dx \le C\left(\int_{-\infty}^{0} g^p(x)v(x)\,dx\right)^{\frac{q}{p}} \tag{4.5}$$

and

$$\int_{-\infty}^{0}\left[\int_{2x}^{x} K(x-y)g(y)\,dy\right]^q u(x)\,dx \le C\left(\int_{-\infty}^{0} g^p(x)v(x)\,dx\right)^{\frac{q}{p}}. \tag{4.6}$$

Changes of variables yield

$$\int_{-\infty}^{0} g^p(x)v(x)\,dx = \int_0^{\infty} G(x)^p(x)v(-x)\,dx \qquad \text{with } G(x) = g(-x),$$

$$\int_{-\infty}^{0}\left[\int_{-\infty}^{2x} K(-y)g(y)\,dy\right]^{q} u(x)\,dx = \int_{0}^{\infty}\left[\int_{-\infty}^{-2x} K(-y)g(y)\,dy\right]^{q} u(-x)\,dx$$
$$= \int_{0}^{\infty}\left[\int_{2x}^{\infty} K(z)G(z)\,dz\right]^{q} u(-x)\,dx$$

and

$$\int_{-\infty}^{0}\left[\int_{2x}^{x} K(x-y)g(y)\,dy\right]^{q} u(x)\,dx = \int_{0}^{\infty}\left[\int_{-2x}^{-x} K(-x-y)g(y)\,dy\right]^{q} u(-x)\,dx$$
$$= \int_{0}^{\infty}\left[\int_{x}^{2x} K(z-x)G(z)dz\right]^{q} u(-x)\,dx$$
$$= \int_{0}^{\infty} (\widetilde{T}^{*}G)^{q}(x)u(-x)\,dx.$$

These computations show that (4.5) and (4.6) are respectively equivalent to

$$H^{*} : L^{p}\big((0,\infty), K^{-p}(x)v(-x)\,dx\big) \to L^{q}\big((0,\infty), u(-x)\,dx\big)$$

and

$$\widetilde{T}^{*} : L^{p}\big((0,\infty), v(-x)\,dx\big) \to L^{q}\big((0,\infty), u(-x)\,dx\big).$$

By Part B of the Lemma [with $w(x) = K^{-p}(x)v(-x)$] the first boundedness is true if and only if, for a constant $A > 0$,

$$\left(\int_{0}^{R} u(-x)\,dx\right)^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x)v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0. \tag{4.7}$$

This is one part of condition (3.5) whose other part is

$$\left(\int_{0}^{R} u(x)\,dx\right)^{\frac{1}{q}} \left(\int_{2R}^{\infty} K^{p'}(x)v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A \qquad \text{for all } R > 0. \tag{4.8}$$

Clearly, (4.4) is a necessary inequality for (3.3). So the conclusion is that (4.7) and $\widetilde{T}^{*} : L^{p}\big((0,\infty), v(-x)\,dx\big) \to L^{q}\big((0,\infty), u(-x)\,dx\big)$ are necessary conditions for $T : L^{p}\big((-\infty,\infty), v(x)\,dx\big) \to L^{q}\big((-\infty,\infty), u(x)\,dx\big)$ to be satisfied, and they are also sufficient to get (4.4).

Estimate of S_4: The aim is to prove

$$S_4 = \int_{0}^{\infty} (Tg)^{q}(x)u(x)\,dx \le C\left(\int_{-\infty}^{0} g^{p}(x)v(x)\,dx\right)^{\frac{q}{p}}. \tag{4.9}$$

For each $x > 0$ then

$$(Tg)(x) = \int_{-\infty}^{-2x} K(x-y)g(y)\,dy + \int_{-2x}^{0} K(x-y)g(y)\,dy$$
$$\approx \int_{-\infty}^{-2x} K(-y)g(y)\,dy + K(x)\int_{-2x}^{0} g(y)\,dy.$$

Indeed, $-\frac{1}{2}y < x - y < -2y$ for $y < -2x$ (< 0), and $x < x - y < 3x$ for $-2x < y < 0$. So (4.9) is equivalent both to

$$\int_0^\infty \left[\int_{-\infty}^{-2x} K(-y)g(y)\,dy\right]^q u(x)\,dx \le C\left(\int_{-\infty}^0 g^p(x)v(x)\,dx\right)^{\frac{q}{p}} \tag{4.10}$$

and

$$\int_0^\infty \left[\int_{-2x}^0 g(y)\,dy\right]^q K^q(x)u(x)\,dx \le C\left(\int_{-\infty}^0 g^p(x)v(x)\,dx\right)^{\frac{q}{p}}. \tag{4.11}$$

Again changes of variables is used with the function $G(x) = g(-x)$ in order that

$$\int_0^\infty \left[\int_{-\infty}^{-2x} K(-y)g(y)\,dy\right]^q u(x)\,dx = \int_0^\infty \left[\int_{2x}^\infty K(z)G(z)dz\right]^q u(x)\,dx$$

and

$$\int_0^\infty \left[\int_{-2x}^0 g(y)\,dy\right]^q K^q(x)u(x)\,dx = \int_0^\infty \left[\int_0^{2x} G(z)\,dz\right]^q K^q(x)u(x)\,dx.$$

Consequently, (4.10) and (4.11) are equivalent to

$$H^* : L^p\big((0,\infty), K^{-p}(x)v(-x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

and

$$\mathcal{H} : L^p\big((0,\infty), v(-x)\,dx\big) \to L^q\big((0,\infty), K^q(x)u(x)\,dx\big),$$

respectively. By Part B of the Lemma the first boundedness is equivalent to (4.8), and by Part C of the same Lemma, the second holds if and only if

$$\left(\int_{\frac{1}{2}R}^\infty K^q(x)u(x)\,dx\right)^{\frac{1}{q}}\left(\int_0^R v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0. \tag{4.12}$$

This last condition is both equivalent to (4.3) and

$$\left(\int_{\frac{1}{2}R}^{2R} K^q(x)u(x)\,dx\right)^{\frac{1}{q}}\left(\int_0^R v^{1-p'}(-x)\,dx\right)^{\frac{1}{p'}} \le A$$

which is nothing else than condition (3.6). Since (4.9) is a necessary condition for (3.3) to hold, then the boundedness $T : L^p\big((-\infty,\infty), v(x)\,dx\big) \to L^q\big((-\infty,\infty), u(x)\,dx\big)$ implies (4.8), (4.3) and (3.6). These conditions are also sufficient to obtain (4.9) ∎

Proof of Proposition 3.2. We only derive $\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ from (3.8), since similarly $\widetilde{T}^* : L^p\big((0,\infty), v(-x)\,d\big) \to L^q\big((0,\infty), u(-x)\,dx\big)$ can be obtained from (3.9). The main key is to see that (3.7) and (3.8) implies, for some positive constants $c, A > 0$ and all $R > 0$,

$$\int_R^{2R} K^{\epsilon q}(x-R)\left[\int_{\frac{1}{2}x}^x K^{(1-\epsilon)p'}(x-z)v^{1-p'}(z)\,dz\right]^{\frac{q}{p'}} u(x)\,dx < c\,A^q. \tag{4.13}$$

Once (4.13) is established, the fact that $\widetilde{T} : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ can be proved by using the usual Hölder inequality. Indeed, for $f(\cdot) \geq 0$, $x > 0$ and $\varepsilon \in [0,1]$ then

$$(\widetilde{T}f)(x) \leq \left(\int_{\frac{1}{2}x}^{x} K^{\varepsilon p}(x-y) f^p(y) v(y)\,dy\right)^{\frac{1}{p}} \times (\mathcal{V}(x))^{\frac{1}{p'}}$$

where $\mathcal{V}(x) = \int_{\frac{1}{2}x}^{x} K^{(1-\varepsilon)p'}(x-z) v^{1-p'}(z)\,dz$. So by the Minkowski inequality ($\frac{q}{p} \geq 1$) then (4.13) yields

$$\begin{aligned}
&\left(\int_0^\infty (\widetilde{T}f)^q(x) u(x)\,dx\right)^{\frac{p}{q}} \\
&\leq \left\{\int_0^\infty \left[\int_{\frac{1}{2}x}^{x} K^{\varepsilon p}(x-y) f^p(y) v(y)\,dy\right]^{\frac{q}{p}} \mathcal{V}^{\frac{q}{p'}}(x) u(x)\,dx\right\}^{\frac{p}{q}} \\
&\leq \int_0^\infty f^p(y) v(y) \left[\int_y^{2y} K^{\varepsilon q}(x-y) \mathcal{V}^{\frac{q}{p'}}(x) u(x)\,dx\right]^{\frac{p}{q}} dy \\
&\leq \left(\sup_{R>0} \int_R^{2R} K^{\varepsilon q}(x-R) \left[\int_{\frac{1}{2}x}^{x} K^{(1-\varepsilon)p'}(x-z) v^{1-p'}(z)\,dz\right]^{\frac{q}{p'}} u(x)\,dx\right)^{\frac{p}{q}} \\
&\quad \times \int_0^\infty f^p(y) v(y)\,dy \\
&\leq A^p \int_0^\infty f^p(y) v(y)\,dy.
\end{aligned}$$

Now to get (4.13) the following consequences of (3.7) are useful:

$$\int_R^{2R} K^{\varepsilon q}(x-R)\,dx = \int_0^R K^{\varepsilon q}(z)\,dz \leq cR \times K^{\varepsilon q}(R)$$

and

$$\begin{aligned}
\int_{\frac{1}{2}x}^{x} K^{(1-\varepsilon)p'}(x-z)\,dz &= \int_0^{\frac{1}{2}x} K^{(1-\varepsilon)p'}(z)\,dz \qquad (\text{for } 0<x<2R) \\
&\leq \int_0^R K^{(1-\varepsilon)p'}(z)\,dz \\
&\leq cR \times K^{(1-\varepsilon)p'}(R).
\end{aligned}$$

Indeed, using (3.8) then

$$\begin{aligned}
&\int_R^{2R} K^{\varepsilon q}(x-R) \left[\int_{\frac{1}{2}x}^{x} K^{(1-\varepsilon)p'}(x-z) v^{1-p'}(z)\,dz\right]^{\frac{q}{p'}} u(x)\,dx \\
&\leq \left(\sup_{R<z<2R} u(z)\right)\left(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\right)^{\frac{q}{p'}} \\
&\quad \times \int_R^{2R} K^{\varepsilon q}(x-R) \left[\int_{\frac{1}{2}x}^{x} K^{(1-\varepsilon)p'}(x-z)(z)\,dz\right]^{\frac{q}{p'}} dx
\end{aligned}$$

$$\leq \left[R^{\frac{1}{q}+\frac{1}{p'}} K(R)\Big(\sup_{R<z<2R} u(z)\Big)^{\frac{1}{q}} \Big(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\Big)^{\frac{1}{p'}}\right]^q$$

$$\leq A^q$$

and Proposition 3.2 is proved ∎

Proof of Proposition 3.3. For convenience set

$$\mathcal{U}(n) = \sup_{2^n<z<2^{n+1}} u(z) \qquad \text{and} \qquad \mathcal{V}(n) = \sup_{2^{n-1}<y<2^{n+1}} v^{1-p'}(y)$$

for each integer $n \in \mathbb{Z}$. A crucial key for the proof is

$$2^{n[\frac{1}{q}-\frac{1}{p}]}\mathcal{U}^{\frac{1}{q}}(n) \leq c\,A\big(v(y)\big)^{\frac{1}{p}} \qquad \text{for a.e. } y \text{ with } 2^{n-1} < y < 2^{n+1}. \tag{4.14}$$

Indeed, the chain of computations, which leads to the boundedness

$$\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$$

with $p \leq q \leq \overline{p}$, can be presented as follows:

$$\int_0^\infty (\widetilde{T}f)^q(x)u(x)\,dx = \sum_{n\in\mathbf{Z}} \int_{2^n<x<2^{n+1}} \left[\int_{\frac{1}{2}x}^{x} (x-y)^{\alpha-1} f(y)\,dy\right]^q u(x)\,dx$$

$$\leq \sum_{n\in\mathbf{Z}} \mathcal{U}_n \int_{2^n}^{2^{n+1}} (Tf\mathbf{1}_{[2^{n-1},2^{n+1}]})^q(x)\,dx$$

$$\leq \sum_{n\in\mathbf{Z}} 2^{n(1-\frac{q}{\overline{p}})}\mathcal{U}_n \left(\int_{2^n}^{2^{n+1}} (Tf\mathbf{1}_{[2^{n-1},2^{n+1}]})^{\overline{p}}(x)\,dx\right)^{\frac{q}{\overline{p}}}$$

(by the Hölder inequality if $\frac{\overline{p}}{q}>1$)

$$\leq c_1 \sum_{n\in\mathbf{Z}} 2^{n(1-\frac{q}{p})}\mathcal{U}_n \left(\int_{2^{n-1}}^{2^{n+1}} f^p(y)\,dy\right)^{\frac{q}{p}}$$

(since $T: L^p((0,\infty),dx) \to L^{\overline{p}}((0,\infty),dx)$)

$$= c_1 \sum_{n\in\mathbf{Z}} \left(\int_{2^{n-1}}^{2^{n+1}} f^p(y)\left[2^{n(\frac{1}{q}-\frac{1}{p})}\mathcal{U}^{\frac{1}{q}}(n)\right]^p dy\right)^{\frac{q}{p}}$$

$$\leq c_2 A^q \sum_{n\in\mathbf{Z}} \left(\int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y)\,dy\right)^{\frac{q}{p}} \qquad \text{(by using (4.14))}$$

$$\leq c_2 A^q \left(\sum_{n\in\mathbf{Z}} \int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y)\,dy\right)^{\frac{q}{p}} \qquad \text{(since } \tfrac{q}{p}\geq 1\text{)}$$

$$= c_2 A^q \left(\sum_{n\in\mathbf{Z}} \int_{2^{n-1}}^{2^n} + \int_{2^n}^{2^{n+1}} f^p(y)v(y)\,dy\right)^{\frac{q}{p}}$$

$$= c_3 A^q \left(\int_0^\infty f^p(x)v(x)\,dx\right)^{\frac{q}{p}}.$$

Observation (4.14) appears easily by using conditions (3.12) and (3.8). Indeed, if $2^{n-1} < y < 2^{n+1}$, then a.e.

$$\begin{aligned} 2^{n[\frac{1}{q}-\frac{1}{p}]}\mathcal{U}^{\frac{1}{q}}(n) &= 2^{n[\frac{1}{q}-\frac{1}{p}]}\mathcal{U}^{\frac{1}{q}}(n)\big(v^{1-p'}(y)\big)^{\frac{1}{p'}} \times \big(v(y)\big)^{\frac{1}{p}} \\ &\le 2^{n[\frac{1}{q}-\frac{1}{p}]}\mathcal{U}^{\frac{1}{q}}(n)\mathcal{V}^{\frac{1}{p'}}(n) \times \big(v(y)\big)^{\frac{1}{p}} \qquad \text{(by the definition of } \mathcal{V}(n)) \\ &\le c2^{n[\frac{1}{q}+\frac{1}{p'}]}K(2^n)\mathcal{U}^{\frac{1}{q}}(n)\mathcal{V}^{\frac{1}{p'}}(n) \times \big(v(y)\big)^{\frac{1}{p}} \qquad \text{(by property (3.12))} \\ &\le cA \times \big(v(y)\big)^{\frac{1}{p}} \qquad \text{(by condition (3.8))}. \end{aligned}$$

The proof for the boundedness $\widetilde{T}^* : L^p\big((0,\infty), v(-x)\,dx\big) \to L^q\big((0,\infty), u(-x)\,dx\big)$ can be also seen as above ∎

Proof of Proposition 3.4. Since $u(\cdot)$ and $v(\cdot)$ are even functions, then condition (3.4) and (3.5) is the same as (3.14) and (3.15), respectively, and (3.6) becomes (3.16). So following Theorem 3.1, then (3.14) – (3.16) are necessary conditions for $T : L^p\big((-\infty,\infty), v(x)\,dx\big) \to L^q\big((-\infty,\infty), u(x)\,dx\big)$.

Conversely, again by Theorem 3.1, to get this boundedness it remains to prove $\widetilde{T} : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ and $\widetilde{T}^* : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$. Since $u(-z) = u(z)$ and $v(y) = v(-y)$ then, following Proposition 3.2 or 3.3, it remains to check

$$\mathcal{A}(R) = R^{\frac{1}{q}+\frac{1}{p'}}K(R)\Big(\sup_{\frac{1}{2}R<z<2R} u(z)\Big)^{\frac{1}{q}}\Big(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\Big)^{\frac{1}{p'}} \le A \tag{4.15}$$

for all $R > 0$, where $A > 0$ is a fixed constant.
For $u(\cdot)\uparrow$ and $v^{1-p'}(\cdot)\downarrow$ (i.e. $u(\cdot)$ is quasi-increasing and $v^{1-p'}(\cdot)$ is quasi-decreasing) condition (3.14) is used to get

$$\mathcal{A}(R) \le c_1\Big(\int_{2R}^{\infty} K^q(z)u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{\frac{1}{4}R}^{\frac{1}{2}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_1A.$$

For $u(\cdot)\uparrow$ and $v^{1-p'}(\cdot)\uparrow$ condition (3.14) is also used to get

$$\mathcal{A}(R) \le c_2\Big(\int_{8R}^{\infty} K^q(z)u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{2R}^{4R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_2A.$$

For $u(\cdot)\downarrow$ and $v^{1-p'}(\cdot)\uparrow$, condition (3.15) is used to get

$$\mathcal{A}(R) \le c_3\Big(\int_{0}^{\frac{1}{2}R} u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{2R}^{\infty} K^{p'}(y)v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_3A.$$

Finally, for $u(\cdot)\downarrow$ and $v^{1-p'}(\cdot)\downarrow$, condition (3.16) is used to get

$$\mathcal{A}(R) \le c_4K(\frac{1}{2}R)\Big(\int_{\frac{1}{4}R}^{R} u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{0}^{\frac{1}{2}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_4A.$$

If moreover $u(x) \le c\,u(2x)$, then condition (3.14) is sufficient to conclude since

$$\mathcal{A}(R) \le c_5\Big(\int_{R}^{\infty} K^q(z)u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{0}^{\frac{1}{2}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_5A.$$

Similarly, condition (3.15) leads to the conclusion if moreover $v^{1-p'}(x) \le c\,v^{1-p'}(2x)$ ∎

Proof of Proposition 3.5. By Theorem 3.1, conditions (3.4) – (3.6) are necessary ones for $T: L^p((-\infty,\infty), v(x)\,dx) \to L^q((-\infty,\infty), u(x)\,dx)$.

Conversely, again by Theorem 3.1, it remains to get $\widetilde{T}: L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ and $\widetilde{T}^*: L^p((0,\infty), v(-x)\,dx) \to L^q((0,\infty), u(-x)\,dx)$. And, by Proposition 3.2 or 3.3, it is sufficient to prove inequalities (3.8) and (3.9). The first inequality appears now by using the fact that $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{C}$ and (3.17) since

$$\begin{aligned}
&R^{\frac{1}{q}+\frac{1}{p'}} K(R)\Big(\sup_{R<z<2R} u(z)\Big)^{\frac{1}{q}} \Big(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\Big)^{\frac{1}{p'}}\\
&\le c_1 K(R)\Big(\int_{2^{-N}R<z<2^N R} u(z)\,dz\Big)^{\frac{1}{q}} \Big(\int_{2^{-N}R<y<2^N R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}}\\
&\le c_1 A.
\end{aligned}$$

Similarly, inequality (3.9) also appears by using the fact that $u(-\cdot)$, $v^{1-p'}(-\cdot) \in \mathcal{C}$ and (3.18) ∎

Proof of Theorem 2.1. This results is an immediate consequence of Theorem 3.1. Indeed, for instance, the boundedness $R_\alpha: L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ can be seen as $T: L^p((-\infty,\infty), v(x)\,dx) \to L^q((-\infty,\infty), u(x)\,dx)$ with $K(x) = x^{\alpha-1}$, $K(-x) = 0$, $v(-x) = 0$ and $u(-x) = 0$ for $x > 0$ ∎

Proof of Theorem 2.2. With $K(x) = x^{\alpha-1}$, $v(-x) = 0$ and $u(-x) = 0$ for $x > 0$, by the proof of Theorem 3.1, the boundedness $R_\alpha: L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ is equivalent to $\widetilde{R}_\alpha: L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ and $H: L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), x^{(\alpha-1)q} u(x)\,dx)$. This last boundedness is equivalent to condition (2.3) because of Part (A) of the Lemma.

The result for $W_\alpha: L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ can be immediatly deduced from the first part since this boundedness is equivalent to $R_\alpha: L^{q'}((0,\infty), u^{1-q'}(x)\,dx) \to L^{p'}((0,\infty), v^{1-p'}(x)\,dx)$. Thus Theorem 2.2 is proved ∎

Proof of Proposition 2.3. The boundedness $\widetilde{R}_\alpha: L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ will be obtained from Proposition 3.2 or Proposition 3.3.

As it is seen in Section 3, Proposition 3.2 can be applied under one of the following conditions:

$p = q$ for which $\varepsilon = \frac{1}{p}$.

$\frac{1}{\alpha} \le p < q$ for which ε is taken such that $0 \le \varepsilon < \min(1, \frac{1}{1-\alpha}\,\frac{1}{q})$.

$p < q$, $p < \frac{1}{\alpha}$ and $p < q < p^*$ for which $\varepsilon \in (0,1]$ is taken as in (3.10).

Proposition 3.3 is really needed when $p < q$, $p < \frac{1}{\alpha}$ and $p < q = p^*$. The boundedness in (3.11) [with $\overline{p} = p^*$] is satisfied since it is well-known that $R_\alpha: L^p((0,\infty), dx) \to L^{p^*}((0,\infty), dx)$ (see, for instance, [2]) and (3.12) is satisfied since $\frac{1}{p'} + \frac{1}{p^*} + \alpha - 1 = 0$ ∎

Proof of Proposition 2.4. The crucial key for the proof is

$$\int_{2^n}^{2^{n+1}} (\widetilde{R}_\alpha f)^q(x)u(x)\,dx \le c\,\mathcal{B}^q(n)\Big(\int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y)\,dy\Big)^{\frac{q}{p}} \tag{4.16}$$

where $\mathcal{B}(n)$ is given as in (2.7) and $c > 0$ is a constant which does not depend on n. Indeed the chain of computations, which leads to $\widetilde{R}_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$, is as follows:

$$\begin{aligned}\int_0^\infty (\widetilde{R}_\alpha f)^q(x)u(x)\,dx &\le c_1 \sum_{n\in\mathbf{Z}} \mathcal{B}^q(n)\Big(\int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y)\,dy\Big)^{\frac{q}{p}} \quad \text{(by (4.16))}\\ &\le c_1\Big(\sum_{m\in\mathbf{Z}} \mathcal{B}^r(m)\Big)^{1-\frac{q}{p}}\Big(\sum_{n\in\mathbf{Z}}\int_{2^{n-1}}^{2^{n+1}} f^p(y)v(y)\,dy\Big)^{\frac{q}{p}}\\ &\le c_1 A^q\Big(\sum_{n\in\mathbf{Z}}\int_{2^{n-1}}^{2^n} + \int_{2^n}^{2^{n+1}} f^p(y)v(y)\,dy\Big)^{\frac{q}{p}} \quad \text{(by (2.8))}\\ &= c_2 A^q\Big(\int_0^\infty f^p(x)v(x)\,dx\Big)^{\frac{q}{p}}.\end{aligned}$$

It remains to prove (4.16). For this purpose define $\mathcal{U}(n)$ and $\mathcal{V}(n)$ as in the proof of Proposition 3.3 and observe that, for $2^n < x < 2^{n+1}$, then

$$\begin{aligned}(\widetilde{R}_\alpha f)(x) &= \int_{\frac{1}{2}x}^{x} (x-y)^{\alpha-1} f(y)\,dy\\ &\le \Big(\int_{\frac{1}{2}x}^{x} (x-y)^{\alpha-1} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}}\Big(\int_{\frac{1}{2}x}^{x} f^p(z)v(z)(x-z)^{\alpha-1}\,dz\Big)^{\frac{1}{p}}\\ &\le c_3 2^{n\alpha\frac{1}{p'}}\mathcal{V}^{\frac{1}{p'}}(n)\Big(\int_{\frac{1}{2}x}^{x} f^p(z)v(z)(x-z)^{\alpha-1}\,dz\Big)^{\frac{1}{p}}.\end{aligned}$$

Consequently, (4.16) appears since

$$\begin{aligned}&\int_{2^n}^{2^{n+1}} (\widetilde{R}_\alpha f)^q(x)u(x)\,dx\\ &\le c_3 2^{n\alpha\frac{q}{p'}}\mathcal{V}^{\frac{q}{p'}}(n)\mathcal{U}(n)\int_{2^n}^{2^{n+1}}\Big[\int_{\frac{1}{2}x}^{x} f^p(z)v(z)(x-z)^{\alpha-1}\,dz\Big]^{\frac{q}{p}}dx\\ &\le c_4 2^{n[\alpha\frac{q}{p'}+1-\frac{q}{p}]}\mathcal{V}^{\frac{q}{p'}}(n)\mathcal{U}(n)\Big(\int_{2^{n-1}}^{2^{n+1}} f^p(z)v(z)\Big[\int_z^{2z}(x-z)^{\alpha-1}\,dx\Big]dz\Big)^{\frac{q}{p}}\\ &\le c_5 2^{n[\alpha\frac{q}{p'}+1-\frac{q}{p}+\alpha\frac{q}{p}]}\mathcal{V}^{\frac{q}{p'}}(n)\mathcal{U}(n)\Big(\int_{2^{n-1}}^{2^{n+1}} f^p(z)v(z)\,dz\Big)^{\frac{q}{p}}\\ &= c_5\Big[2^{n[\alpha+\frac{1}{q}-\frac{1}{p}]}\mathcal{V}^{\frac{1}{p'}}(n)\mathcal{U}^{\frac{1}{q}}(n)\Big]^q\Big(\int_{2^{n-1}}^{2^{n+1}} f^p(z)v(z)\,dz\Big)^{\frac{q}{p}}\\ &\le c_6\mathcal{B}^q(n)\Big(\int_{2^{n-1}}^{2^{n+1}} f^p(z)v(z)dz\Big)^{\frac{q}{p}}\end{aligned}$$

and Proposition 2.4 is proved ∎

Proof of Proposition 2.5. In view of Theorem 2.1, the main problem is to prove $\widetilde{R}_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ which by Proposition 2.3 remains to check (2.5), i.e. for all $R > 0$

$$\mathcal{A}(R) = R^{\alpha+\frac{1}{q}-\frac{1}{p}}\Big(\sup_{R<z<2R} u(z)\Big)^{\frac{1}{q}}\Big(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\Big)^{\frac{1}{p'}} \le cA.$$

where $c > 0$ is a fixed constant and $A > 0$ will come from the condition (2.1) which is used in each of the following cases.

For $u(\cdot)\uparrow$ and $v^{1-p'}(\cdot)\downarrow$ then

$$\mathcal{A}(R) \le c_1\Big(\int_{2R}^{\infty} z^{(\alpha-1)q}u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{\frac{1}{4}R}^{\frac{1}{2}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_1A.$$

For $u(\cdot)\uparrow$ and $v^{1-p'}(\cdot)\uparrow$ then

$$\mathcal{A}(R) \le c_2\Big(\int_{8R}^{\infty} z^{(\alpha-1)q}u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{2R}^{4R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_2A.$$

For $u(\cdot)\downarrow$ and $v^{1-p'}(\cdot)\downarrow$ then

$$\mathcal{A}(R) \le c_3\Big(\int_{\frac{1}{2}R}^{\infty} z^{(\alpha-1)q}u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{\frac{1}{8}R}^{\frac{1}{4}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_3A.$$

Finally, for $u(\cdot)\downarrow$ and $v^{1-p'}(\cdot)\uparrow$, the extra-assumption for $u(\cdot)$ or $v^{1-p'}(\cdot)$ is useful. For instance, when $u(x) \le c\,u(2x)$, then

$$\mathcal{A}(R) \le c_4\Big(\int_{8R}^{\infty} z^{(\alpha-1)q}u(z)dz\Big)^{\frac{1}{q}}\Big(\int_{2R}^{4R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_4A.$$

The same conclusion is also satisfied if $v^{1-p'}(2x) \le c\,v^{1-p'}(x)$, since

$$\mathcal{A}(R) \le c_5\Big(\int_{\frac{1}{2}R}^{\infty} z^{(\alpha-1)q}u(z)\,dz\Big)^{\frac{1}{q}}\Big(\int_{\frac{1}{8}R}^{\frac{1}{4}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le c_5A.$$

The result for $W_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ can be obtained by duality arguments ∎

Since Corollary 2.6 can be seen as Proposition 2.5, we can only focuse on the

Proof of Corollary 2.7. In view of Theorem 2.2 and Proposition 2.4, to get $R_\alpha : L^p((0,\infty), v(x)\,dx) \to L^q((0,\infty), u(x)\,dx)$ then it is sufficient to check conditions (2.7) and (2.8) from (2.11) and (2.3). By the growth condition (C) then

$$\sup_{2^n<y<2^{n+1}} u(y) \le c_1(2^{-n})\int_{2^{-N+n}}^{2^{N+n}} u(z)\,dz \le c_2(2^{-(n+1)})\int_{2^{-(N+1+n)}}^{2^{N+1+n}} u(z)\,dz$$

and

$$\sup_{2^{n-1}<y<2^{n+1}} v^{1-p'}(y) \le c_3(2^{-(n+1)}) \int_{2^{-(N+1+n)}}^{2^{N+1+n}} v^{1-p'}(z)\,dz.$$

Calling $\mathcal{A}(n)$ the left member of (2.7) and taking $R = 2^n$ in (2.11) then

$$\mathcal{A}(n) \le c_4 \left(\int_{4(2^n)}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^{2^n} v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}} \left(\int_{2^n}^{2^{n+1}} v^{1-p'}(x)\,dx\right)^{\frac{1}{r}} = c_4 \mathcal{B}(n)$$

which is nothing else than (2.7). Condition (2.8) can be deduced from (2.3) as follows:

$$\begin{aligned}
&\sum_{n\in\mathbf{Z}} \mathcal{B}^r(n)\\
&\le c_5 \sum_{n\in\mathbf{Z}} \left(\int_{4(2^n)}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{r}{q}} \left(\int_0^{2^n} v^{1-p'}(y)\,dy\right)^{\frac{r}{q'}} \left(\int_{2^n}^{2^{n+1}} v^{1-p'}(x)\,dx\right)\\
&= c_5 \sum_{n\in\mathbf{Z}} \int_{2^n}^{2^{n+1}} \left[\left(\int_{4(2^n)}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^{2^n} v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)\,dx\\
&\le c_5 \sum_{n\in\mathbf{Z}} \int_{2^n}^{2^{n+1}} \left[\left(\int_{2x}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^{x} v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)\,dx\\
&\le c_5 \int_0^{\infty} \left[\left(\int_{2x}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^{x} v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)\,dx\\
&\le c_5 A^r
\end{aligned}$$

and Corollary 2.7 is proved ∎

Proof of Corollary 2.8. In view of Theorem 2.1 and Proposition 2.3 to get $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ it is sufficient to check conditions (2.1) and (2.5). For this purpose observe that, for all $R > 0$,

$$\int_0^R v^{1-p'}(y)\,dy = \int_0^R y^{[(1-p')(\delta-1)+1]-1}\,dy \approx R^{p'[1-\frac{\delta}{p}]} \quad \text{whenever } \delta < p \tag{4.17}$$

and

$$\int_{2R}^{\infty} z^{(\alpha-1)q} u(z)\,dz = \int_{2R}^{\infty} z^{[(\alpha-1)q+\beta]-1}\,dz \approx R^{q[(\alpha-1)+\frac{\beta}{q}]} \quad \text{for } \beta < (1-\alpha)q. \tag{4.18}$$

Consequently, for $\delta < p$ and $\beta < (1-\alpha)p$,

$$\mathcal{H}(R) = \left(\int_{2R}^{\infty} z^{(\alpha-1)q} u(z)\,dz\right)^{\frac{1}{q}} \left(\int_0^R v^{1-p'}(y)\,dy\right)^{\frac{1}{p'}} \approx R^{(\alpha-1)+\frac{\beta}{q}} R^{1-\frac{\delta}{p}} = R^{\alpha+\frac{\beta}{q}-\frac{\delta}{p}}$$

and condition (2.1) is satisfied whenever $\alpha + \frac{\beta}{q} - \frac{\delta}{p} = 0$ [which is (2.12)]. On the other hand,

$$\mathcal{A}(R) = R^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\sup_{R<z<2R} u(z)\right)^{\frac{1}{q}}\left(\sup_{\frac{1}{2}R<y<2R} v^{1-p'}(y)\right)^{\frac{1}{p'}} \approx R^{\alpha+\frac{\beta}{q}-\frac{\delta}{p}}.$$

Then condition (2.5) is reduced to (2.12).

Since $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ implies (2.1), then condition (2.12) appears immediatly. Also, in view of (4.17) and (4.18), if (2.1) holds then necessarily $\delta < p$ and $\beta < (1-\alpha)q$.

Results for $W_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ can be easily deduced from the above, since this boundedness is equivalent to $R_\alpha : L^{p_1}\big((0,\infty), v_1(x)\,dx\big) \to L^{q_1}\big((0,\infty), u_1(x)\,dx\big)$ with $p_1 = q'$, $q_1 = p'$, $v_1(x) = u^{1-q'}(x)$, $u_1(x) = v^{1-p'}(x)$ ∎

Proof of Corollary 2.9. Again, to get $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ it is sufficient to check conditions (2.1) and (2.5). First observe that

$$\int_0^R v^{1-p'}(y)\,dy = \int_0^R [\ln^{-p'}(y^{-1})]y^{-1}\,dy \approx \ln^{-\frac{p'}{p}}(R^{-1}) \qquad \text{for } R < \frac{1}{2}$$

and, for $R > \frac{1}{2}$,

$$\begin{aligned}\int_0^R v^{1-p'}(y)\,dy &= \int_0^{\frac{1}{2}} [\ln^{-p'}(y^{-1})]y^{-1}\,dy + \int_{\frac{1}{2}}^R y^{[(1-p')(\theta-1)+1]-1}\,dy\\ &\le c_1 + R^{p'[1-\frac{\theta}{p}]}\\ &\le c_2 R^{p'[1-\frac{\theta}{p}]} \qquad \text{(whenever } \theta < p).\end{aligned}$$

On the other hand, for $R > \frac{1}{4}$ [or $2R > \frac{1}{2}$] then

$$\int_{2R}^\infty z^{(\alpha-1)q} u(z)\,dz = \int_{2R}^\infty z^{[(\alpha-1)q+\gamma]-1}\,dz \approx R^{q[(\alpha-1)+\frac{\gamma}{q}]} \quad \text{for } \gamma < (1-\alpha)q \tag{4.19}$$

and, for $R < \frac{1}{4}$,

$$\int_{2R}^\infty z^{(\alpha-1)q} u(z)\,dz = \int_{2R}^{\frac{1}{2}} z^{(\alpha-1)q+\beta-1}\,dz + \int_{\frac{1}{2}}^\infty z^{(\alpha-1)q+\gamma-1}\,dz \le c_3 + c_4$$

whenever $\gamma < (1-\alpha)q < \beta$. With $\mathcal{H}(R)$ defined as in the proof of Corollary 2.8 then

$$\mathcal{H}(R) \le c_5 \ln^{-\frac{1}{p}}(R^{-1}) \le c_6 \qquad \text{for } R < \tfrac{1}{4},$$

also for $\theta < p$ and $\gamma < (1-\alpha)q$ then

$$\mathcal{H}(R) \le c_7 R^{\alpha+\frac{\gamma}{q}-\frac{\theta}{p}} \qquad \text{for } R > \tfrac{1}{2}.$$

So condition (2.1) is satisfied whenever $\alpha + \frac{\gamma}{q} - \frac{\theta}{p} \le 0$ [which is (2.13)]. Also, with $\mathcal{A}(R)$ defined as in the proof of Corollary 2.8, then

$$\mathcal{A}(R) \le c_8 R^{\alpha+\frac{\beta}{q}-1}\ln^{-1}(R^{-1}) \le c_8 R^{\alpha+\frac{\beta}{q}-1} \qquad \text{for } R < \tfrac{1}{4}$$

and

$$\mathcal{A}(R) \le c_9 R^{\alpha+\frac{\gamma}{q}-\frac{\theta}{p}} \qquad \text{for } R > \tfrac{1}{2}.$$

Consequently, condition (2.5) is satisfied whenever $(1-\alpha)q \le \beta$ and $\alpha + \frac{\gamma}{q} - \frac{\theta}{p} \le 0$.

Since $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ implies (2.1), and in view of (4.19), then $\gamma < (1-\alpha)q$. The necessity of (2.13) can be also derived from condition (2.1), since for $R > \frac{1}{2}$ then $\mathcal{H}(R) \ge R^{\alpha+\frac{\gamma}{q}-\frac{\theta}{p}}$ ∎

Proof of Corollary 2.10. In view of Theorem 2.2 and Proposition 2.4, to get $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ it is sufficient to check conditions (2.3), (2.7) and (2.8). Condition (2.3) is equivalent to the finitness of

$$I_1 = \int_0^{\frac{1}{2}} \left[\left(\int_{2x}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)\,dx$$

and

$$I_2 = \int_{\frac{1}{2}}^{\infty} \left[\left(\int_{2x}^{\infty} y^{(\alpha-1)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(y)\,dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)\,dx.$$

Using $\delta < p$, $\beta < (1-\alpha)q$, (4.17) and (4.18) then

$$I_1 \approx c_1 \int_0^{\frac{1}{2}} x^{[(\alpha-1)+\frac{\beta}{q}+\frac{p'}{q'}(1-\frac{\delta}{p})]r} x^{p'[1-\frac{\delta}{p}]-1}\,dx = \int_0^{\frac{1}{2}} x^{[\alpha+\frac{\beta}{q}-\frac{\delta}{p}]r-1}\,dx = A_1^r$$

whenever $\alpha + \frac{\beta}{q} - \frac{\delta}{p} > 0$. On the otherhand, the inequality $\alpha + \frac{\gamma}{q} - \frac{\delta}{p} < 0$ leads to

$$I_2 \approx \int_{\frac{1}{2}}^{\infty} x^{[(\alpha-1)+\frac{\gamma}{q}+\frac{p'}{q'}(1-\frac{\delta}{p})]r} x^{p'[1-\frac{\delta}{p}]-1}\,dx = \int_{\frac{1}{2}}^{\infty} x^{[\alpha+\frac{\gamma}{q}-\frac{\delta}{p}]r-1}\,dx = A_2^r.$$

To check conditions (2.7) and (2.8), it is convenient to take $\mathcal{B}(n) = \mathcal{A}(n)$ where $\mathcal{A}(n)$ is defined as above. For $n+1 < 0$ or $n < -1$ then

$$\mathcal{A}(n) \approx 2^{n[\alpha+\frac{1}{q}-\frac{1}{p}]} \times 2^{n\frac{1}{q}(\beta-1)} \times 2^{-n\frac{1}{p}(\delta-1)} = 2^{n[\alpha+\frac{\beta}{q}-\frac{\delta}{p}]}$$

and consequently $\sum_{-\infty}^{-2} \mathcal{B}^r(n) < \infty$ whenever $\alpha + \frac{\beta}{q} - \frac{\delta}{p} > 0$. For $0 < n-1$ or $1 < n$ then

$$\mathcal{A}(n) \approx 2^{n[\alpha+\frac{1}{q}-\frac{1}{p}]} \times 2^{n\frac{1}{q}(\gamma-1)} \times 2^{-n\frac{1}{p}(\delta-1)} = 2^{n[\alpha+\frac{\gamma}{q}-\frac{\delta}{p}]}$$

and so $\sum_{k=2}^{\infty} \mathcal{B}^r(n) < \infty$ whenever $\alpha + \frac{\gamma}{q} - \frac{\delta}{p} < 0$.

The boundedness $R_\alpha : L^p\big((0,\infty), v(x)\,dx\big) \to L^q\big((0,\infty), u(x)\,dx\big)$ implies the condition (2.3). And this last one is equivalent to $I_1 < \infty$ and $I_2 < \infty$. So the above computations lead to the inequalities $\alpha + \frac{\gamma}{q} - \frac{\delta}{p} < 0 < \alpha + \frac{\beta}{q} - \frac{\delta}{p}$ which are nothing else than condition (2.14). The finitness of I_1 can only be held whenever $\delta < p$ and similarly $I_2 < \infty$ implies necessarily $\gamma < (1-\alpha)q$.

The result for $W_\alpha : L^p\big((0,\infty), v^*(x)\,dx\big) \to L^q\big((0,\infty), u^*(x)\,dx\big)$ can be deduced from the above one by duality. Indeed, by this boundedness is equivalent to R_α : $L^{p_1}\big((0,\infty), v_1(x)\,dx\big) \to L^{q_1}\big((0,\infty), u_1(x)\,dx\big)$ where $p_1 = q'$, $q_1 = p'$, $v_1(x) = (u^*)^{(1-q_1)}(x) = x^{\delta-1}$ and $u_1(x) = (v^*)^{(1-p_1)}(x) = x^{\beta-1}\mathbf{1}_{(0,1)}(x) + x^{\delta-1}\mathbf{1}_{(1,\infty)}(x)$ ∎

Acknowledgement. I would like to thank the referee for his helpful comments and suggestions. Also thanks to Professor V. Kokilashvili for sending me his paper [5] and for pointing some confusions in the statements in [4] and [5].

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Received 10.05.1996; in revised form 27.05.1997