Nonlinear Geometric Optics for Shock Waves
Part II: System Case

Ya-Guang Wang

Abstract. In this paper we investigate the nonlinear geometric optics of a stable shock wave perturbed by high frequency oscillations for quasilinear hyperbolic conservation laws in one space variable. We obtain the existence of the oscillatory shock wave and its leading profiles, which are solutions to a boundary value problem of integro-differential systems. Furthermore, the asymptotic properties of the oscillatory shock wave as well as the shock front are justified.

Keywords: *Nonlinear geometric optics, conservation laws, shock waves*

AMS subject classification: 35 B 05, 35 L 65, 35 L 67

1. Introduction

This paper is devoted to the study of rapidly oscillatory waves for quasilinear hyperbolic conservation laws in one space variable

$$\left.\begin{aligned} \partial_t u + \partial_x f(u) = 0 \quad (x \in \mathbb{R}, t > 0) \\ u|_{t=0} = u_0(x) \end{aligned}\right\} \tag{1.1}$$

when the initial data $u_0(x)$ are the perturbation of a shock wave by small amplitude, high frequency oscillations. Usually, the interesting points of this problem are focused on the formal analysis of leading profiles for oscillations, the proof of existence for leading profiles and the exact oscillatory solutions in a domain independent of high frequencies, and the rigorous justification for the asymptotics of oscillations. The problem (1.1) was investigated by A. Majda and M. Artola [14] in the formal analysis even for the case of several space variables.

Under the assumption of the unperturbed shock wave being stable in the sense of A. Majda [12], we study the rigorous justification of asymptotic expansions for the rapidly oscillatory shock wave for the problem (1.1). After the free boundary problem of oscillatory shock wave and shock front is transformed into a fixed boundary problem, we prove the existence of the rapidly oscillatory shock wave in a domain independent of frequencies of oscillations by extending the classical theory of Cauchy problems for one space dimensional quasilinear hyperbolic systems into the case of boundary value

Ya-Guang Wang: Shanghai Jiao Tong University, Dept. Appl. Math., Shanghai 200030, P.R. China

problems. Similar to A. Majda and R. R. Rosales in [15] and other references in the case of Cauchy problems, the leading profiles of the oscillatory shock wave satisfy a mixed problem of integro-differential systems. We discuss the asymptotics of the shock front as well as the shock speed. The justification of the asymptotics is established by the idea of the simultaneous Picard iteration, which has been adopted in some references (see, e.g., [8]). That means to estimate

$$u^{\varepsilon,\nu}(t,x) - U^{\nu}\big(t,x; \tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon}\big)$$

for each ν with $u^{\varepsilon,\nu}$ and U^{ν} being the convergent sequences to exact solutions u^{ε} and profiles U, respectively. We use the usual Picard iteration for the boundary value problem of profiles to construct the solution sequence U^{ν}. As A. Majda in [13], the problem of the shock wave is studied by using the Picard iteration for the nonlinear equation, and the Newton iteration for the nonlinear boundary condition to construct the solution sequence $u^{\varepsilon,\nu}$. Moreover, in order to make the above estimate of the asymptotics valid for the case $\nu = 0$, we must properly construct the zero-th order approximate solutions of u^{ε} as well as U for boundary value problems, which is different from the case of Cauchy problems.

The present paper is a continuation of the study in [18], where the scalar conservation law was investigated. In the scalar case, the coupled problem of the shock state and shock front can be decoupled into two problems, which makes us possible to use some existing results directly, the rigorous justification of the asymptotics for the rapidly oscillatory shock wave in the scalar conservation law was established there. As in [18], it is observed that the leading term of the shock front does not oscillate, and oscillations are only appeared in the leading term of the shock speed. For the motivation of this problem we refer to the introduction of [18].

There is a rich literature devoted to the constructions and applications of weakly nonlinear asymptotic expansions for rapidly oscillatory waves. Most of the rigorous justifications are given in the setting of smooth solutions. See papers of J. L. Joly, G. Métivier and J. Rauch [8, 9] and references therein for Cauchy problems, and those of J. Chikhi [2] and M. Williams [19, 20] for mixed value problems with fixed boundaries. In recent years, there also have been a lot of works devoted to the rigorous study of the formal analysis in the setting of bounded variation solutions. We mention the interesting works of C. Cheverry [1], R. DiPerna and A. Majda [5], S. Schochet [17] for initial value problems, and that of M. Sablé-Tougeron [16] for the boundary value problem. The asymptotic analysis of nonlinear hyperbolic waves had also been investigated by Y. He and T. B. Moodie in [7] and references quoted there.

The remainder of this paper is arranged as follows. In Subsection 2.1, we formulate the problem of the oscillatory shock wave as well as that of leading profiles by using the method of multiple scales. In Subsection 2.2, we discuss the compatibility conditions of these boundary problems, and state the main results. The problem of the oscillatory shock wave is studied in Section 3, which contains three subsections in the construction of the zero-th order approximate solutions, the analysis of linear problems, and the proof of the convergence of solution sequences. Section 4 is devoted to the study of the problem of leading profiles by a procedure similar to Section 3. Finally, in Section

5 we prove the asymptotic properties of oscillations, which concludes the result of the nonlinear geometric optics.

2. Formulations of problems and main results

2.1 Formulation of problems. For the $m \times m$ conservation laws in one space variable

$$\partial_t u + \partial_x f(u) = 0 \qquad (2.1.1)$$

we assume that they are strictly hyperbolic in t for each $u \in \mathbb{R}^m$, i.e. if $A(u) = f'(u)$ is the Jacobian of $f(u)$, then the algebraic equation

$$\det |\lambda I - A(u)| = 0 \qquad (2.1.2)$$

admits m distinct real roots

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_m(u).$$

Let $r_k(u)$ and $l_k(u)$ be the corresponding right and left eigenvectors of $A(u)$ with respect to $\lambda_k(u)$ for each $k \in \{1, \ldots, m\}$,

$$(\lambda_k I - A(u))r_k(u) = 0 \qquad \text{and} \qquad l_k(u)(\lambda_k I - A(u)) = 0, \qquad (2.1.3)$$

with normalization

$$l_i(u) \cdot r_k(u) = \delta_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases} \qquad (2.1.4)$$

for any $u \in \mathbb{R}^m$ and $i, k \in \{1, \ldots, m\}$.

For a fixed $j \in \{1, \ldots, m\}$ we assume that $\lambda_j(u)$ is genuinely nonlinear,

$$\nabla \lambda_j(u) \cdot r_j(u) \neq 0 \qquad (2.1.5)$$

for any $u \in \mathbb{R}^m$, and let u_+, u_- and σ be constants such that

$$u = \begin{cases} u_+ & \text{if } x > \sigma t \\ u_- & \text{if } x < \sigma t \end{cases} \qquad (2.1.6)$$

is the j-th shock wave solution of (2.1.1), which satisfies the Lax entropy condition (see [11])

$$\lambda_j^+ < \sigma < \lambda_j^- \qquad \text{and} \qquad \lambda_{j-1}^- < \sigma < \lambda_{j+1}^+ \qquad (2.1.7)$$

and the Rankine-Hugoniot condition

$$\sigma[u] = [f(u)] \qquad (2.1.8)$$

where we denote $\lambda_i^+ = \lambda_i(u_+)$ and $\lambda_i^- = \lambda_i(u_-)$ for each i, $\lambda_{j-1}^- = -\infty$ when $j = 1$, $\lambda_{j+1}^+ = +\infty$ when $j = m$, and denote by $[u] = u_+ - u_-$ the jump across the shock front.

We assume that

<div align="center"><i>the plane shock (2.1.6) is stable</i></div>   (2.1.9)

in Majda's sense [12], which implies that the matrix (see [12: Proposition 3.1])

$$\left(r_1^-, \ldots, r_{j-1}^-, [u], r_{j+1}^+, \ldots, r_m^+\right)^T \quad \text{is invertible} \qquad (2.1.10)$$

where $r_i^- = r_i(u_-)$ and $r_i^+ = r_i(u_+)$. As A. Majda in [12], it is easy to see that, under the genuinely nonlinear assumption (2.1.5), the stability condition (2.1.10) is satisfied when the shock (2.1.6) is weak enough.

Let us study the following Cauchy problem for (2.1.1) with initial data being a small perturbation of the plane j-th shock (2.1.6):

$$\left. \begin{array}{l} \partial_t U^\varepsilon + \partial_x f(U^\varepsilon) = 0 \quad (t > 0, \ x \in \mathbb{R}) \\[2mm] U^\varepsilon(0, x) = \left\{ \begin{array}{ll} u_+ + \varepsilon u_{+,0}^\varepsilon(x) & \text{if } x > 0 \\ u_- + \varepsilon u_{-,0}^\varepsilon(x) & \text{if } x < 0 \end{array} \right. \end{array} \right\} \qquad (2.1.11)$$

where $\varepsilon > 0$ is small enough, and $u_{\pm,0}^\varepsilon \in C^1$. Since the shock (u_+, u_-, σ) is stable, as A. Corli and M. Sablé-Tougeron in [4], we can assume that when $\varepsilon \in (0, \varepsilon_0]$ is small enough, and $u_{\pm,0}^\varepsilon(x)$ satisfy some compatibility conditions, which will be given precisely, the initial value problem (2.1.11) determines a local shock around the origin.

Before giving assumptions on the problem (2.1.11), let us first introduce some notations. Given a small closed neighborhood $\omega \subset \{t = 0\}$ of the origin, suppose Ω is the closure of a determinacy domain of ω for the Cauchy problem (2.1.11) when $|U^\varepsilon - u_\pm| < \eta$. The space $C^k(\Omega)$ is the usual one of functions whose derivatives of order less k are continuous in Ω. Equip this space with a family of norms

$$\|u\|_{\varepsilon,k,\Omega} = \sum_{|\alpha| \leq k} \varepsilon^{|\alpha|} \|\partial^\alpha u\|_{L^\infty(\Omega)}.$$

A family $u^\varepsilon \in C^k(\Omega)$ $(\varepsilon > 0)$ is *bounded* in $C_\varepsilon^k(\Omega)$ if the family of norms $\|u^\varepsilon\|_{\varepsilon,k,\Omega}$ $(\varepsilon > 0)$ is bounded, and a family $\{\phi^\varepsilon\}_\varepsilon$ is *bounded* in $\widetilde{C}_\varepsilon^k[0,T]$ if $\phi^\varepsilon \in C^k[0,T]$ and the family of norms $\|d_t \phi^\varepsilon\|_{\varepsilon,k-1,[0,T]}$ $(\varepsilon > 0)$ is bounded for $k \geq 1$. Obviously, we have $C_\varepsilon^0(\Omega) = C^0(\Omega) \cap L^\infty(\Omega)$.

Let $C_p^0(\mathbb{R}^q)$ be the space of continuous almost periodic functions in $\theta \in \mathbb{R}^q$ (see Y. Katznelson [10]). Denote by $\mathcal{C}^0(\Omega : \mathbb{R}^q) = C^0(\Omega : C_p^0(\mathbb{R}^q))$ the space of continuous functions from Ω into $C_p^0(\mathbb{R}^q)$. For $k \in \mathbb{N}$, define the space $\mathcal{C}^k(\Omega : \mathbb{R}^q)$ of those functions $U \in \mathcal{C}^0(\Omega : \mathbb{R}^q)$ whose derivatives $\partial_{(t,x;\theta)}^\alpha U$ belong to $\mathcal{C}^0(\Omega : \mathbb{R}^q)$ for any $|\alpha| \leq k$.

For the problem (2.1.11), we assume that there are $U_{\pm,0}(x, \theta) \in \mathcal{C}^1(\omega^\pm : \mathbb{R})$ such that

$$\left\| u_{\pm,0}^\varepsilon(x) - U_{\pm,0}\left(x, \tfrac{x}{\varepsilon}\right) \right\|_{\varepsilon,1,\omega^\pm} = o(1) \qquad (2.1.12)$$

when $\varepsilon \to 0$, where $\omega^+ = \omega \cap \{x \geq 0\}$ and $\omega^- = \omega \cap \{x \leq 0\}$ (we use the notation "$\pm$" to mean two cases according to the upper and lower signs, and it will be used in this whole paper). Obviously, from (2.1.12), we have that $u_{\pm,0}^\varepsilon$ is bounded in $C_\varepsilon^1(\omega^\pm)$.

The aim of the present paper is to study the local existence of the shock wave solution U^ϵ to the problem (2.1.11), and the asymptotic expansions of U^ϵ and its shock front $\{x = \Psi_\epsilon(t)\}$ with respect to ϵ.

Now, let us simplify the problem (2.1.11), and deduce the problem of leading profiles of U^ϵ as well as the shock front $\{x = \Psi_\epsilon(t)\}$.

As in [18], let

$$U^\epsilon(t,x) = \begin{cases} u_+ + \epsilon u_+^\epsilon(t,x) & \text{if } x > \sigma t + \epsilon \phi^\epsilon(t) \\ u_- + \epsilon u_-^\epsilon(t,x) & \text{if } x < \sigma t + \epsilon \phi^\epsilon(t) \end{cases} \qquad (2.1.13)$$

be the shock wave solution of the problem (2.1.11), i.e. $(u_\pm^\epsilon, \phi^\epsilon)$ satisfy

$$\left. \begin{array}{ll} \partial_t(u_+ + \epsilon u_+^\epsilon) + \partial_x f(u_+ + \epsilon u_+^\epsilon) = 0 & \text{if } x > \sigma t + \epsilon \phi^\epsilon(t) \\ \partial_t(u_- + \epsilon u_-^\epsilon) + \partial_x f(u_- + \epsilon u_-^\epsilon) = 0 & \text{if } x < \sigma t + \epsilon \phi^\epsilon(t) \end{array} \right\} \qquad (2.1.14)$$

which is equivalent to

$$\left. \begin{array}{ll} \partial_t u_+^\epsilon + A(u_+ + \epsilon u_+^\epsilon)\partial_x u_+^\epsilon = 0 & \text{if } x > \sigma t + \epsilon \phi^\epsilon(t) \\ \partial_t u_-^\epsilon + A(u_- + \epsilon u_-^\epsilon)\partial_x u_-^\epsilon = 0 & \text{if } x < \sigma t + \epsilon \phi^\epsilon(t) \end{array} \right\} \qquad (2.1.15)$$

and satisfy the Rankine-Hugoniot condition

$$\left(\sigma + \epsilon \frac{d\phi^\epsilon}{dt}\right)(\epsilon[u^\epsilon] + [u]) = [f(u + \epsilon u^\epsilon)] \qquad (2.1.16)$$

on $\{x = \sigma t + \epsilon \phi^\epsilon(t)\}$ with $[u^\epsilon] = (u_+^\epsilon - u_-^\epsilon)(t, \sigma t + \epsilon \phi^\epsilon(t))$.

At this stage, both of functions $u_\pm^\epsilon$ and ϕ^ϵ are unknown. Thus, (2.1.15) - (2.1.16) is a free boundary value problem. In order to transform this problem into the fixed boundary case, we perform the tranformation

$$\begin{cases} \tilde{t} = t \\ \tilde{x} = x - \sigma t - \epsilon \phi^\epsilon(t) \end{cases}$$

in (2.1.15), and obtain that $\tilde{u}_\pm^\epsilon(\tilde{t}, \tilde{x}) = u_\pm^\epsilon(t, x)$ satisfy

$$\left. \begin{array}{ll} \partial_{\tilde{t}} \tilde{u}_+^\epsilon + (A(u_+ + \epsilon \tilde{u}_+^\epsilon) - (\sigma + \epsilon d_t \phi^\epsilon)I)\partial_{\tilde{x}} \tilde{u}_+^\epsilon = 0 & \text{if } \tilde{x} > 0 \\ \partial_{\tilde{t}} \tilde{u}_-^\epsilon + (A(u_- + \epsilon \tilde{u}_-^\epsilon) - (\sigma + \epsilon d_t \phi^\epsilon)I)\partial_{\tilde{x}} \tilde{u}_-^\epsilon = 0 & \text{if } \tilde{x} < 0. \end{array} \right\} \qquad (2.1.17)$$

By taking the transformation
$$\begin{cases} \bar{t} = \tilde{t} \\ \bar{x} = -\tilde{x} \end{cases}$$

in the second line of (2.1.17), we know that $(u_{\pm}^{\epsilon}, \phi^{\epsilon})$ satisfy the coupled problem with a fixed boundary

$$
\left.
\begin{aligned}
\partial_t u_{\pm}^{\epsilon} \pm \big(A(u_{\pm} + \varepsilon u_{\pm}^{\epsilon}) - (\sigma + \varepsilon d_t \phi^{\epsilon})I\big)\partial_x u_{\pm}^{\epsilon} &= 0 \quad ((t,x) \in \Omega^+) \\
(\sigma + \varepsilon d_t \phi^{\epsilon})(\varepsilon[u^{\epsilon}] + [u]) &= [f(u + \varepsilon u^{\epsilon})] \quad (x = 0) \\
\phi^{\epsilon}(0) &= 0 \\
u_{\pm}^{\epsilon}(0, x) &= u_{\pm,0}^{\epsilon}(x)
\end{aligned}
\right\} \tag{2.1.18}
$$

where the tilde and bar of notations are dropped for simplicity, $\Omega^+ = \Omega \cap \{x > 0\}$, and $[f] = f_+ - f_-$ is the jump on $\{x = 0\}$ for any function f.

Suppose that the solutions $(u_{\pm}^{\epsilon}, \phi^{\epsilon})$ of the problem (2.1.18) have the forms

$$
\left.
\begin{aligned}
u_{\pm}^{\epsilon}(t, x) &= U_{\pm}(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}) + \varepsilon V_{\pm}(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}) + O(\varepsilon^2) \\
\phi^{\epsilon}(t) &= \phi(t, \tfrac{t}{\epsilon}) + \varepsilon \varphi(t, \tfrac{t}{\epsilon}) + O(\varepsilon^2)
\end{aligned}
\right\} \tag{2.1.19}
$$

where $U_{\pm}(t, x; \tau, \theta)$, $V_{\pm}(t, x; \tau, \theta)$, $\phi(t, \tau)$ and $\varphi(t, \tau)$ are almost periodic in $(\tau, \theta) \in \mathbb{R}^2$. Let us formally deduce the problem of $(U_{\pm}, \phi)$ from (2.1.18).

Set $\tau = \frac{t}{\epsilon}$ and $\theta = \frac{x}{\epsilon}$. Plugging the formal expressions (2.1.19) into the equations in (2.1.18), expanding $A(u_{\pm} + \varepsilon u_{\pm}^{\epsilon})$ by Taylor's formula at $u_{\pm}$ and grouping each power of ε, it follows that the term of "ε^{-1}" is

$$
\partial_\tau U_{\pm} \pm \big(A_{\pm} - (\sigma + \partial_\tau \phi)I\big)\partial_\theta U_{\pm} = 0 \tag{2.1.20}
$$

with $A_{\pm} = A(u_{\pm})$, and the term of "ε^0" is

$$
\begin{aligned}
\partial_t U_{\pm} + \partial_\tau V_{\pm} &\pm \big(A_{\pm} - (\sigma + \partial_\tau \phi)I\big)(\partial_x U_{\pm} + \partial_\theta V_{\pm}) \\
&\pm B_{\pm}(\partial_\theta U_{\pm}, U_{\pm}) \mp (\partial_t \phi + \partial_\tau \varphi)\partial_\theta U_{\pm} = 0
\end{aligned} \tag{2.1.21}
$$

with $B_{\pm} = \nabla^2 f(u_{\pm})$ being the Hessian of f at $u_{\pm}$. Similarly, the "ε^0" term of the boundary condition in (2.1.18) is

$$
(\sigma + \partial_\tau \phi)[u] = [f(u)]
$$

which implies

$$
\partial_\tau \phi = 0 \tag{2.1.22}
$$

i.e. ϕ is independent of $\tau = \frac{t}{\epsilon}$ by using (2.1.8) and $[u] \neq 0$, the "ε^1" term of the boundary condition in (2.1.18) is

$$
(\partial_t \phi + \partial_\tau \varphi)[u] + [(\sigma I - A)U] = 0. \tag{2.1.23}
$$

Defining the operator

$$
P_{\pm}(\partial_\tau, \partial_\theta) = \partial_\tau \pm (A_{\pm} - \sigma I)\partial_\theta
$$

and denoting its symbol by $p_\pm(\lambda, \alpha)$, it is easy to see that $\mathrm{char} P_\pm = \{c(\mp(\lambda_i^\pm - \sigma), 1) | c \in \mathbb{R}, \ i = 1, \ldots, m\}$.

Let the operator $\mathbb{E}_\pm$ be the extension in $C^0(\Omega^+ : \mathbb{R}^2)$ of the following action on the space of trigonometric polynomials:

$$\mathbb{E}_\pm u(t,x) e^{i(\lambda\tau + \alpha\theta)} = \begin{cases} \Pi_\pm(\lambda,\alpha) u(t,x) e^{i(\lambda\tau + \alpha\theta)} & \text{if } (\lambda,\alpha) \in \mathrm{char}\, P_\pm \\ 0 & \text{otherwise.} \end{cases} \quad (2.1.24)$$

Here, for $(\lambda,\alpha) \in \mathrm{char} P_\pm$, $\Pi_\pm(\lambda,\alpha)$ is the projector in $\mathbb{C}^m$ on the kernal of $p_\pm(\lambda,\alpha)$ with respect to the decomposition

$$\mathbb{C}^m = \mathrm{sp}(r_1^\pm) \oplus \cdots \oplus \mathrm{sp}(r_m^\pm),$$

with $r_i^\pm = r_i(u_\pm)$ given by (2.1.3) and $\mathrm{sp}(r_i^\pm)$ being the span of $r_i^\pm$. Then, as J. L. Joly et al. in [8, 9], on $C^1(\Omega^+ : \mathbb{R}^2)$, we have the following:

(1) $\mathbb{E}_\pm U = U$ is equivalent to $P_\pm(\partial_\tau, \partial_\theta) U = 0$.

(2) For any $V \in C^1(\Omega^+ : \mathbb{R}^2)$, $\mathbb{E}_\pm P_\pm(\partial_\tau, \partial_\theta) V = 0$.

Acting the operator $\mathbb{E}_\pm$ on (2.1.20) - (2.1.21) and using (2.1.22), it follows that the leading terms of $(u_\pm^\epsilon, \phi^\epsilon)$ satisfy

$$\left. \begin{array}{r} \mathbb{E}_\pm U_\pm = U_\pm \\[4pt] \mathbb{E}_\pm \left(\partial_t U_\pm \pm (A_\pm - \sigma I) \partial_x U_\pm \pm B_\pm(\partial_\theta U_\pm, U_\pm) \mp \chi \partial_\theta U_\pm \right) = 0 \\[4pt] \chi[u] + [(\sigma I - A)U] = 0 \quad (x = \theta = 0) \\[4pt] U_\pm |_{t=\tau=0} = U_{\pm,0}(x,\theta) \end{array} \right\} \quad (2.1.25)$$

where $\chi(t,\tau) = d_t \phi + \partial_\tau \varphi$.

Let us analyse the problem (2.1.25) in detail. Define

$$\mathbb{E}_0 u(t,x;\tau,\theta) = \bar{u}(t,x) = \lim_{\rho \to \infty} \frac{1}{(2\rho)^2} \int_{-\rho}^{+\rho} \int_{-\rho}^{+\rho} u(t,x;\tau,\theta) \, d\tau \, d\theta$$

to be the mean value operator of $u \in C^0(\Omega^+ : \mathbb{R}^2)$ in (τ,θ). For any fixed $k \in \{1,\ldots,m\}$, let $\mathbb{E}_k^\pm$ be the extension, similar to $\mathbb{E}^\pm$, of the operator

$$\mathbb{E}_k^\pm u(t,x) e^{i(\lambda\tau + \alpha\theta)} = \begin{cases} \Pi_k^\pm u(t,x) e^{i(\lambda\tau + \alpha\theta)} & \text{when } (\lambda,\alpha) = c(\mp(\lambda_k^\pm - \sigma), 1) \\ 0 & \text{otherwise} \end{cases}$$

where $c \in \mathbb{R} \setminus \{0\}$ and $\Pi_k^\pm$ is the projector in $\mathbb{C}^m$ on the kernal of $p_\pm(\mp(\lambda_k^\pm - \sigma), 1)$. Obviously, we have

$$\mathbb{E}^\pm = \sum_{k=1}^m \mathbb{E}_k^\pm + \mathbb{E}_0 \quad (2.1.26)$$

and, for any $(\lambda, \alpha) \doteq c\big(\mp (\lambda_k^\pm - \sigma), 1\big)$ with $c \in \mathbb{R}\backslash\{0\}$,

$$\Pi_k^\pm u(t,x)e^{i(\lambda\tau+\alpha\theta)} = \big(l_k^\pm u(t,x)\big)e^{i(\lambda\tau+\alpha\theta)}r_k^\pm$$

by using (2.1.4).

Let $U_\pm$ be the functions determined from (2.1.25), $U_{\pm,k} = \mathbb{E}_k^\pm U_\pm$ and $\overline{U}_\pm = \mathbb{E}_0 U_\pm$. Then from (2.1.26) we have

$$U_\pm = \mathbb{E}^\pm U_\pm = \sum_{k=1}^m U_{\pm,k} + \overline{U}_\pm.$$

The fact $U_{\pm,k} = \mathbb{E}_k^\pm U_\pm$ shows that $U_{\pm,k}(t,x;\tau,\theta)$ can be viewed as a function of $\big(t,x,\theta\mp(\lambda_k^\pm-\sigma)\tau\big)$, from which it follows that there is a scalar function $\sigma_k^\pm(t,x,\theta)$ almost periodic in $\theta \in \mathbb{R}$ such that

$$U_{\pm,k} = \Pi_k^\pm U_{\pm,k} = (l_k^\pm U_{\pm,k})r_k^\pm = \sigma_k^\pm\big(t,x,\theta\mp(\lambda_k^\pm-\sigma)\tau\big)r_k^\pm. \tag{2.1.27}$$

This implies

$$\partial_\theta U_\pm = \sum_{k=1}^m \sigma_k^{\pm'}\big(t,x;\theta\mp(\lambda_k^\pm-\sigma)\tau\big)r_k^\pm \tag{2.1.28}$$

with $\sigma_k^{\pm'}$ denoting the derivative of $\sigma_k^\pm(t,x;\theta)$ in $\theta \in \mathbb{R}$.

On the other hand, it is obvious that

$$\mathbb{E}_k^\pm = \Pi_k^\pm M_k^\pm = M_k^\pm \Pi_k^\pm$$

where

$$M_k^\pm u(t,x;\tau,\theta) = \lim_{\rho\to\infty} \frac{1}{2\rho}\int_{-\rho}^{+\rho} u\Big(t,x;\tau+s,\theta\pm(\lambda_k^\pm-\sigma)s\Big)ds$$

is the mean value operator in the direction $(\tau,\theta) = \big(1,\pm(\lambda_k^\pm-\sigma)\big)$ for any $k \in \{1,\ldots,m\}$. Therefore, by using (2.1.28) we have

$$\mathbb{E}^\pm\big((d_t\phi + \partial_\tau\varphi)\partial_\theta U_\pm\big)$$

$$= \sum_{k=1}^m \mathbb{E}_k^\pm\big((d_t\phi + \partial_\tau\varphi)\partial_\theta U_\pm\big)$$

$$= \sum_{k=1}^m M_k^\pm \Pi_k^\pm \left((d_t\phi + \partial_\tau\varphi)\sum_{i=1}^m \sigma_i^{\pm'}\big(t,x;\theta\mp(\lambda_i^\pm-\sigma)\tau\big)r_i^\pm\right)$$

$$= \sum_{k=1}^m M_k^\pm\Big((d_t\phi + \partial_\tau\varphi)\sigma_k^{\pm'}\big(t,x;\theta\mp(\lambda_k^\pm-\sigma)\tau\big)\Big)r_k^\pm \tag{2.1.29}$$

$$= \sum_{k=1}^m d_t\phi\,\sigma_k^{\pm'}\big(t,x;\theta\mp(\lambda_k^\pm-\sigma)\tau\big)r_k^\pm$$

$$= \mathbb{E}^\pm(d_t\phi\,\partial_\theta U_\pm)$$

Acting the operator $\mathbb{E}_0$ on (2.1.25) and using (2.1.29), we obtain that $(\overline{U}_\pm, \phi)$ satisfy

$$
\left.
\begin{aligned}
\partial_t \overline{U}_\pm \pm (A_\pm - \sigma I)\partial_x \overline{U}_\pm = 0 \quad & (t, x > 0) \\
d_t \phi[u] + [(\sigma I - A)\bar{U}] = 0 \quad & (x = 0) \\
\phi(0) = 0 \quad & \\
\overline{U}_\pm(0, x) = \overline{U}_{\pm,0}(x) \quad &
\end{aligned}
\right\}
\tag{2.1.30}
$$

with

$$
\overline{U}_{\pm,0}(x) = \lim_{\rho \to 0} \frac{1}{2\rho} \int_{-\rho}^{+\rho} U_{\pm,0}(x, \theta)\, d\theta.
$$

As A.Majda and R.Rosales in [15], we can easily reformulate the problem (2.1.25) as an integro-differential system for $\sigma_k^\pm$ introduced in (2.1.27).

2.2 Compatibility conditions and main results. Let us study compatibility conditions for the problems (2.1.18), (2.1.25) and (2.1.30). Since (2.1.30) is deduced from (2.1.25) with $\chi(t, \tau) = d_t \phi + \partial_\tau \varphi$, it is obvious that the compatibility conditions of (2.1.30) immediately follows from those of (2.1.25).

Since the boundary condition in (2.1.18) must be valid at $\{x = t = 0\}$, the zero-th order compatibility condition for the problem (2.1.18) is

$$
(\sigma + \varepsilon d_t \phi^\varepsilon(0))(\varepsilon[u_0^\varepsilon] + [u]) - [f(u + \varepsilon u_0^\varepsilon)] = 0
\tag{2.2.1}
$$

with $[u_0^\varepsilon] = u_{+,0}^\varepsilon(0) - u_{-,0}^\varepsilon(0)$. It is well-known (see, e.g., P. D. Lax [11]) that the shock speed $\sigma + \varepsilon d_t \phi^\varepsilon(0)$ and the state ahead of the shock, $u_+ + \varepsilon u_{+,0}^\varepsilon(0)$, define a unique state behind the shock, $u_- + \varepsilon u_{-,0}^\varepsilon(0)$, so that the zero-th order condition (2.2.1) is satisfied. Moreover, when $(d_t \phi^\varepsilon(0), u_{+,0}^\varepsilon(0))$ are bounded in $\varepsilon \in (0, \varepsilon_0]$, we also have the boundedness of $u_{-,0}^\varepsilon(0)$ in $\varepsilon \in (0, \varepsilon_0]$.

Differentiating the boundary condition in (2.1.18) with respect to t, and evaluating the result at $x = t = 0$, we obtain

$$
d_t^2 \phi^\varepsilon(0)(\varepsilon[u_0^\varepsilon] + [u]) + (\sigma + \varepsilon d_t \phi^\varepsilon(0))\left[\partial_t u^\varepsilon|_{t=0}\right] - \left[A(u + \varepsilon u_0^\varepsilon)\partial_t u^\varepsilon|_{t=0}\right] = 0.
\tag{2.2.2}
$$

On the other hand, from the equation and initial data in (2.1.18) we have

$$
\partial_t u_\pm^\varepsilon(0, 0) = \mp\left(A\big(u_\pm + \varepsilon u_{\pm,0}^\varepsilon(0)\big) - (\sigma + \varepsilon d_t \phi^\varepsilon(0))I\right)d_x u_{\pm,0}^\varepsilon(0).
\tag{2.2.3}
$$

Substituting (2.2.3) into (2.2.2), it follows the first order compatibility condition for the problem (2.1.18):

$$
\begin{aligned}
d_t^2 \phi^\varepsilon(0)(\varepsilon[u_0^\varepsilon] + [u]) &+ \left((\sigma + \varepsilon d_t \phi^\varepsilon(0))I - A\big(u_+ + \varepsilon u_{+,0}^\varepsilon(0)\big)\right)^2 d_x u_{+,0}^\varepsilon(0) \\
&+ \left((\sigma + \varepsilon d_t \phi^\varepsilon(0))I - A\big(u_- + \varepsilon u_{-,0}^\varepsilon(0)\big)\right)^2 d_x u_{-,0}^\varepsilon(0) = 0.
\end{aligned}
\tag{2.2.4}
$$

In order to deduce compatibility conditions for the problem (2.1.25), let us first diagonalize this problem. Set

$$T_\pm = (r_1^\pm, \ldots, r_m^\pm) \qquad (2.2.5)$$

with $r_i^\pm = r_i(u_\pm)$ determined from (2.1.3) by setting $u = u_\pm$. From (2.1.4), it is obvious that $T_\pm^{-1} = (l_1^\pm, \ldots, l_m^\pm)^T$ with $l_i^\pm = l_i(u_\pm)$. Define

$$\widetilde{U}_\pm = T_\pm^{-1} U_\pm.$$

Then by the same computation as from (2.1.20) and (2.1.21) - (2.1.25), we deduce that $(\widetilde{U}_\pm, \chi)$ satisfy the problem

$$\left.\begin{array}{r}
\widetilde{\mathbb{E}}_\pm \widetilde{U}_\pm = \widetilde{U}_\pm \iff (\partial_\tau \pm (\Lambda_\pm - \sigma I)\partial_\theta)\widetilde{U}_\pm = 0 \\[2mm]
\partial_t \widetilde{U}_\pm \pm (\Lambda_\pm - \sigma I)\partial_x \widetilde{U}_\pm \pm \widetilde{\mathbb{E}}_\pm (\widetilde{B}_\pm(\partial_\theta \widetilde{U}_\pm, \widetilde{U}_\pm) - \chi \partial_\theta \widetilde{U}_\pm) = 0 \\[2mm]
\chi[u] + (\sigma I - A_+)T_+ \widetilde{U}_+ - (\sigma I - A_-)T_- \widetilde{U}_- = 0 \quad \text{on} \quad x = \theta = 0 \\[2mm]
\widetilde{U}_\pm|_{t=\tau=0} = \widetilde{U}_{\pm,0}(x,\theta) := T_\pm^{-1} U_{\pm,0}(x,\theta)
\end{array}\right\} \quad (2.2.6)$$

where $\widetilde{\mathbb{E}}_\pm$ is defined similar to $\mathbb{E}_\pm$ in (2.1.24) with $\Pi_\pm(\lambda, \alpha)$ replaced by $\widetilde{\Pi}_\pm(\lambda, \alpha)$, the projector in $\mathbb{C}^m$ on the kernel of $\tilde{p}_\pm(\lambda, \alpha)$, the symbol of $\widetilde{P}_\pm(\partial_\tau, \partial_\theta) = \partial_\tau + (\Lambda_\pm - \sigma I)\partial_\theta$ with $\Lambda_\pm = \text{diag}[\lambda_1^\pm, \ldots, \lambda_m^\pm]$, and the bilinear form $\widetilde{B}_\pm(\tilde{u}, \tilde{v})$ equal to $T_\pm^{-1} B_\pm(T_\pm \tilde{u}, T_\pm \tilde{v})$.

Let $\widetilde{U}_\pm = (\widetilde{U}_\pm^1, \ldots, \widetilde{U}_\pm^m)^T$. From the first and last lines of (2.2.6) we get

$$\widetilde{U}_\pm^i(0, x; \tau, \theta) = \widetilde{U}_{\pm,0}^i(x, \theta \mp (\lambda_i^\pm - \sigma)\tau).$$

For simplicity, we denote

$$\begin{aligned}
\widetilde{U}_\pm(0, x; \tau, \theta) &= \widetilde{U}_{\pm,0}(x, \theta \mp (\Lambda_\pm - \sigma I)\tau) \\
&= \left(\widetilde{U}_{\pm,0}^1(x, \theta \mp (\lambda_1^\pm - \sigma)\tau), \ldots, \widetilde{U}_{\pm,0}^m(x, \theta \mp (\lambda_m^\pm - \sigma)\tau)\right)^T.
\end{aligned} \qquad (2.2.7)$$

Evaluating the boundary condition in (2.2.6) at $x = t = \theta = 0$, we obtain the zero-th order compatibility condition for the problem (2.2.6)

$$\begin{aligned}
\chi(0, \tau)[u] + (\sigma I - A_+)T_+ \widetilde{U}_{+,0}(0, (\sigma I - \Lambda_+)\tau) \\
- (\sigma I - A_-)T_- \widetilde{U}_{-,0}(0, (\Lambda_- - \sigma I)\tau) = 0
\end{aligned} \qquad (2.2.8)$$

which easily implies the zero-th order compatibility condition for the problem (2.1.30)

$$d_t \phi(0)[u] + (\sigma I - A_+)\overline{U}_{+,0}(0) - (\sigma - A_-)\overline{U}_{-,0}(0) = 0 \qquad (2.2.9)$$

by taking the mean value of (2.2.8) with respect to $\tau \in \mathbb{R}$.

Differentiating the boundary condition in (2.2.6) with respect to t, we have

$$\partial_t \chi[u] + (\sigma I - A_+)T_+ \partial_t \widetilde{U}_+ - (\sigma I - A_-)T_- \partial_t \widetilde{U}_- = 0 \quad \text{on} \quad x = \theta = 0. \qquad (2.2.10)$$

From the second line of (2.2.6), it follows

$$\partial_t \widetilde{U}_\pm|_{t=0} = \mp(\Lambda_\pm - \sigma I)\partial_x \widetilde{U}_{\pm,0} \mp \widetilde{\mathbb{E}}_\pm \left(\widetilde{B}_\pm(\partial_\theta \widetilde{U}_{\pm,0}, \widetilde{U}_{\pm,0}) - \chi(0,\tau)\partial_\theta \widetilde{U}_{\pm,0} \right) \quad (2.2.11)$$

with $\widetilde{U}_{\pm,0}(x,\tau,\theta) = \widetilde{U}_{\pm,0}(x, \theta \mp (\Lambda_\pm - \sigma I)\tau)$ given by (2.2.7). Substituting (2.2.11) into (2.2.10) at $t = x = \theta = 0$, it immediately follows the first order compatibility condition for the problem (2.2.6)

$$\partial_t \chi(0,\tau)[u]$$

$$+ (\sigma I - A_+)\dot{T}_+(\sigma I - \Lambda_+)\partial_x \widetilde{U}_{+,0}\big(0, (\sigma I - \Lambda_+)\tau\big)$$

$$+ (\sigma I - A_-)T_-(\sigma I - \Lambda_-)\partial_x \widetilde{U}_{-,0}\big(0, (\Lambda_- - \sigma I)\tau\big)$$

$$- \Big\{ (\sigma I - A_+)T_+\widetilde{\mathbb{E}}_+ \left((\widetilde{B}_+(\partial_\theta \widetilde{U}_{+,0}, \widetilde{U}_{+,0}) - \chi(0,\tau)\partial_\theta \widetilde{U}_{+,0}) \right) \qquad (2.2.12)$$

$$+ (\sigma I - A_-)T_-\widetilde{\mathbb{E}}_- \left((\widetilde{B}_-(\partial_\theta \widetilde{U}_{-,0}, \widetilde{U}_{-,0}) - \chi(0,\tau)\partial_\theta \widetilde{U}_{-,0}) \right) \Big\}\Big|_{x=\theta=0} = 0$$

with $\chi(0,\tau)$ determined from (2.2.8).

Similarly, we can deduce higher order compatibility conditions for (2.1.18) and (2.2.6). As A. Majda in [13: Proposition 2.2], we have the following

Proposition 2.1.

(1) *Suppose that* $u^\varepsilon_{\pm,0}(0) = a^\varepsilon_{\pm,0}$ *and* $d_t\phi^\varepsilon(0) = \sigma^\varepsilon$ *satisfy the zero-th order condition* (2.2.1) *with* $\{a^\varepsilon_{\pm,0}, \sigma^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *being bounded, let* P^ε_+ *and* P^ε_- *be the projectors in* $\mathbb{C}^m$ *on the spaces*

$$\mathrm{sp}\Big\{ r_{j+1}(u_+ + \varepsilon a^\varepsilon_{+,0}), \ldots, r_m(u_+ + \varepsilon a^\varepsilon_{+,0}) \Big\}$$

and

$$\mathrm{sp}\Big\{ r_1(u_- + \varepsilon a^\varepsilon_{-,0}), \ldots, r_{j-1}(u_- + \varepsilon a^\varepsilon_{-,0}) \Big\},$$

with respect to the decompositions

$$\mathbb{C}^m = \bigoplus_{k=1}^m \mathrm{sp}\big(r_k(u_+ + \varepsilon a^\varepsilon_{+,0})\big) \qquad and \qquad \mathbb{C}^m = \bigoplus_{k=1}^m \mathrm{sp}\big(r_k(u_- + \varepsilon a^\varepsilon_{-,0})\big),$$

respectively, with $sp(\vec{a})$ *being the span of* $\vec{a}$. *Assume constants* $g^\varepsilon_\pm \in \mathbb{C}^m$ *satisfying* $P^\varepsilon_\pm g^\varepsilon_\pm = 0$ *and* $\{\varepsilon g^\varepsilon_\pm\}_{\varepsilon \in (0,\varepsilon_0]}$ *being bounded. Then, there are* $(u^\varepsilon_{\pm,0}(x), \phi^\varepsilon(t))$ *bounded in* $C^1_\varepsilon(\omega^+) \times \widetilde{C}^2_\varepsilon[0, T_0]$ *such that*

$$u^\varepsilon_{\pm,0}(0) = a^\varepsilon_{\pm,0}, \quad \phi^\varepsilon(0) = 0, \quad d_t\phi^\varepsilon(0) = \sigma^\varepsilon, \quad (I - P^\varepsilon_\pm)d_x u^\varepsilon_{\pm,0}(0) = g^\varepsilon_\pm,$$

and the first order compatibility condition (2.2.4) *is satisfied.*

(2) *Let* P_+ *and* P_- *be the projectors in* $\mathbb{C}^m$ *on the spaces* $sp\{r^+_{j+1}, \ldots, r^+_m\}$ *and* $sp\{r^-_1, \ldots, r^-_{j-1}\}$ *with respect to the decompositions*

$$\mathbb{C}^m = \bigoplus_{k=1}^m \mathrm{sp}(r^+_k) \qquad and \qquad \mathbb{C}^m = \bigoplus_{k=1}^m \mathrm{sp}(r^-_k),$$

respectively. For any given $V_\pm(\theta) \in C_p^1(\mathbb{R})$ *and* $W_\pm(\theta) \in C_p^0(\mathbb{R})$ *satisfying* $P_\pm V_\pm = 0$ *and* $P_\pm W_\pm = 0$, *respectively, there are*

$$\big(U_{\pm,0}(x,\theta), \chi(t,\tau)\big) \in C^1(\omega^+ : \mathbb{R}) \times C^1([0,T_0] : \mathbb{R})$$

such that

$$(I - P_\pm)U_{\pm,0}(0,\theta) = V_\pm(\theta) \qquad and \qquad (I - P_\pm)\partial_x U_{\pm,0}(0,\theta) = W_\pm(\theta),$$

and the compatibility conditions for the problem (2.1.25) *up to order one are satisfied.*

Proof. Assertion (1): Let us diagonalize the condition (2.2.4). Set

$$T_\pm^\epsilon = \Big(r_1(u_\pm + \epsilon a_{\pm,0}^\epsilon), \ldots, r_m(u_\pm + \epsilon a_{\pm,0}^\epsilon)\Big).$$

Obviously, from the normalization (2.1.4), we have

$$(T_\pm^\epsilon)^{-1} = \Big(l_1(u_\pm + \epsilon a_{\pm,0}^\epsilon), \ldots, l_m(u_\pm + \epsilon a_{\pm,0}^\epsilon)\Big)^T.$$

By taking the transformation

$$v_{\pm,0}^\epsilon(x) = (T_\pm^\epsilon)^{-1} u_{\pm,0}^\epsilon(x)$$

in (2.2.4), it follows that (2.2.4) is equivalent to that $\big(d_x v_{\pm,0}^\epsilon(0), d_t^2 \phi^\epsilon(0)\big)$ satisfy the condition

$$
\begin{aligned}
&d_t^2 \phi^\epsilon(0)([u] + \epsilon[a_0^\epsilon]) \\
&+ \sum_{k=1}^m \big(\sigma + \epsilon\sigma^\epsilon - \lambda_k(u_+ + \epsilon a_{+,0}^\epsilon)\big)^2 d_x v_{+,0}^{\epsilon,k}(0) r_k(u_+ + \epsilon a_{+,0}^\epsilon) \\
&+ \sum_{k=1}^m \big(\sigma + \epsilon\sigma^\epsilon - \lambda_k(u_- + \epsilon a_{-,0}^\epsilon)\big)^2 d_x v_{-,0}^{\epsilon,k}(0) r_k(u_- + \epsilon a_{-,0}^\epsilon) = 0.
\end{aligned}
\tag{2.2.13}
$$

The hypotheses $(I - P_\pm^\epsilon)d_x u_{\pm,0}^\epsilon(0) = g_\pm^\epsilon$ give rise to

$$
\begin{aligned}
g_+^\epsilon &= \sum_{k=1}^j d_x v_{+,0}^{\epsilon,k}(0) r_k(u_+ + \epsilon a_{+,0}^\epsilon) \\
g_-^\epsilon &= \sum_{k=j}^m d_x v_{-,0}^{\epsilon,k}(0) r_k(u_- + \epsilon a_{-,0}^\epsilon).
\end{aligned}
\tag{2.2.14}
$$

By applying the stability condition (2.1.10) in (2.2.13) we know that, for any given $g_\pm^\epsilon$, (2.2.13) defines uniquely $d_t^2 \phi^\epsilon(0)$ and

$$V^\epsilon = \Big\{ d_x v_{-,0}^{\epsilon,1}(0), \ldots, d_x v_{-,0}^{\epsilon,j-1}(0), d_x v_{+,0}^{\epsilon,j+1}(0), \ldots, d_x v_{+,0}^{\epsilon,m}(0) \Big\}.$$

Moreover, when $\{\varepsilon g_\pm^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ is bounded, we have that $\{\varepsilon d_t^2 \phi^\varepsilon(0), \varepsilon V^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ is also bounded.

It remains to construct $(u_{\pm,0}^\varepsilon(x), \phi^\varepsilon(t))$ bounded in $C_\varepsilon^1(\omega^+) \times \widetilde{C}_\varepsilon^2[0, T_0]$, such that $u_{\pm,0}^\varepsilon(0) = a_{\pm,0}^\varepsilon$, $\phi^\varepsilon(0) = 0$, $d_t \phi^\varepsilon(0) = \sigma^\varepsilon$, and $(d_x u_{\pm,0}^\varepsilon(0), d_t^2 \phi^\varepsilon(0))$ are determined as above. For simplicity, let us only discuss the case $\phi^\varepsilon(t)$. For instance, by setting

$$\phi^\varepsilon(t) = \frac{1}{2} \int_0^t \left(\sigma^\varepsilon (e^{i\frac{s}{\varepsilon}} + e^{-i\frac{s}{\varepsilon}}) - i\varepsilon d_t^2 \phi^\varepsilon(0)(e^{i\frac{s}{\varepsilon}} - e^{-i\frac{s}{\varepsilon}}) \right) ds$$

it immediately follows $\phi^\varepsilon \in \widetilde{C}_\varepsilon^2[0, T_0]$ satisfying all demands for any $T_0 > 0$.

Assertion (2): As in (2.2.6), set $\widetilde{U}_\pm = T_\pm^{-1} U_\pm$ with $T_\pm$ given in (2.2.5). The hypotheses

$$(I - P_\pm)U_{\pm,0}(0, \theta) = V_\pm(\theta) \qquad \text{and} \qquad (I - P_\pm)\partial_x U_{\pm,0}(0, \theta) = W_\pm(\theta)$$

imply

$$\left.\begin{aligned} V_+(\theta) &= \sum_{k=1}^{j} \widetilde{U}_{+,0}^k(0,\theta) r_k^+ \\ V_-(\theta) &= \sum_{k=j}^{m} \widetilde{U}_{-,0}^k(0,\theta) r_k^- \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} W_+(\theta) &= \sum_{k=1}^{j} \partial_x \widetilde{U}_{+,0}^k(0,\theta) r_k^+ \\ W_-(\theta) &= \sum_{k=j}^{m} \partial_x \widetilde{U}_{-,0}^k(0,\theta) r_k^- . \end{aligned}\right\} \quad (2.2.15)$$

By substituting (2.2.15) into (2.2.8) and (2.2.12), and using the stability condition (2.1.10), we immediately get the functions

$$\left(U_{\pm,0}(0,\theta), \chi(0,\theta)\right) \in C_p^1(\mathbb{R}) \qquad \text{and} \qquad \left(\partial_x U_{\pm,0}(0,\theta), \partial_t \chi(0,\theta)\right) \in C_p^0(\mathbb{R}).$$

Similar to the proof of Assertion (1), it suffices to construct $\chi(t, \tau) \in C^1([0, T_0] : \mathbb{R})$ with $\chi(0, \tau)$ and $\partial_t \chi(0, \tau)$ being the functions determined from (2.2.8) and (2.2.12). Let $\chi(0, \tau) = a(\tau)$ and $\partial_t \chi(0, \tau) = b(\tau)$. It is easy to verify that the function

$$\chi(t, \tau) = \frac{a(\tau + t) + a(\tau - t)}{2} + \frac{1}{2} \int_{\tau - t}^{\tau + t} b(s) \, ds \in C^1([0, T_0] : \mathbb{R}) \qquad (2.2.16)$$

is the one we look for ∎

The main assumption of this paper is the following one:

(MA) Given the initial data $u_{\pm,0}^\varepsilon(x) \in C^1(\omega^\pm)$ satisfying the compatibility conditions (2.2.1) and (2.2.4) for the problem (2.1.18) for any $\varepsilon \in (0, \varepsilon_0]$, there are $U_{\pm,0}(x, \theta) \in C^1(\omega^\pm : \mathbb{R})$ satisfying the compatibility conditions (2.2.8) and (2.2.12), such that we have the asymptotic property (2.1.12).

Remark 2.1. When $u_{\pm,0}^\epsilon \in C^1(\omega^\pm)$ and $U_{\pm,0} \in C^1(\omega^\pm, \mathbb{R})$ satisfy the asymptotics (2.1.12), $u_{\pm,0}^\epsilon(0) = \partial_x u_{\pm,0}^\epsilon(0) = 0$, and $U_{\pm,0}(0,\theta) = \partial_x U_{\pm,0}(0,\theta) = 0$, the assumption (MA) is obviously valid.

Now, we can state the main result of this paper as follows.

Theorem 2.1. *Under the above assumption* (MA), *we have the following.*

(1) *There are constants* $T, \varepsilon_0 > 0$ *such that the problem* (2.1.18) *has unique solutions* $u_\pm^\epsilon$ *and* ϕ^ϵ *bounded in* $C_\epsilon^1(\Omega_T^+)$ *and* $\widetilde{C}_\epsilon^2[0,T]$, *respectively, for any* $\varepsilon \in (0, \varepsilon_0]$, *where* $\Omega_T^+ = \Omega^+ \cap \{t \leq T\}$.

(2) *There are unique solutions* $U_\pm \in C^1(\Omega_T^+ : \mathbb{R}^2)$, $\chi \in C^1([0,T] : \mathbb{R})$ *and* $\phi \in C^2[0,T]$ *to the problems* (2.1.25) *and* (2.1.30).

(3) *For the above solutions* $(u_\pm^\epsilon, \phi^\epsilon)$, *we have the asymptotic properties*

$$\left\| u_\pm^\epsilon(t,x) - U_\pm\left(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\right) \right\|_{\epsilon,1,\Omega_T^+} = o(1) \tag{2.2.17}$$

and

$$\left.\begin{aligned} \left\| d_t \phi^\epsilon(t) - \chi\left(t, \tfrac{t}{\epsilon}\right) \right\|_{\epsilon,1,[0,T]} = o(1) \\ \left\| \phi^\epsilon(t) - \phi(t) \right\|_{L^\infty[0,T]} = o(1) \end{aligned}\right\} \tag{2.2.18}$$

when $\varepsilon \to 0$.

Remark 2.2. From the results in (2.2.18), we can easily obtain the asymptotic property of the shock front $\{x = \sigma t + \varepsilon\phi^\epsilon(t)\}$.

3. Existence of oscillatory shock waves

This section is devoted to the proof of Theorem 2.1/(1), which gives the existence and uniqueness of the exact solutions $(u_\pm^\epsilon, \phi^\epsilon)$ to the problem (2.1.18).

3.1 Construction of approximate solutions. Introduce the notations

$$L_\pm^\epsilon(u_\pm^\epsilon, \phi^\epsilon) = \partial_t \pm \Big(A(u_\pm + \varepsilon u_\pm^\epsilon) - (\sigma + \varepsilon d_t \phi^\epsilon)I\Big)\partial_x \tag{3.1.1}$$

and

$$G^\epsilon(u_+^\epsilon, u_-^\epsilon, d_t\phi^\epsilon) = \frac{1}{\epsilon}\Big((\sigma + \varepsilon d_t\phi^\epsilon)(\varepsilon[u^\epsilon] + [u]) - [f(u + \varepsilon u^\epsilon)]\Big). \tag{3.1.2}$$

The function $f(u_\pm + \varepsilon u_\pm^\epsilon)$ admits Taylor's expansion at the constant state $u_\pm$ as

$$f(u_\pm + \varepsilon u_\pm^\epsilon) = f(u_\pm) + A_\pm(\varepsilon u_\pm^\epsilon) + \varepsilon^2 \int_0^1 (1-\eta)\nabla^2 f(u_\pm + \eta\varepsilon u_\pm^\epsilon)(u_\pm^\epsilon, u_\pm^\epsilon)\, d\eta.$$

This implies

$$\begin{aligned} G^\epsilon(u_+^\epsilon, u_-^\epsilon, d_t\phi^\epsilon) = d_t\phi^\epsilon(\varepsilon[u^\epsilon] + [u]) + [(\sigma I - A)u^\epsilon] \\ - \varepsilon \int_0^1 (1-\eta)[\nabla^2 f(u + \eta\varepsilon u^\epsilon)(u^\epsilon, u^\epsilon)]\, d\eta \end{aligned} \tag{3.1.3}$$

by using the Rankine-Hugoniot condition (2.1.8), where $[\cdot]$ denotes the jump of the related function on $\{x = 0\}$. Obviously, the problem (2.1.18) can be written as

$$\left.\begin{array}{rl} L_\pm^\epsilon(u_\pm^\epsilon, \phi^\epsilon)u_\pm^\epsilon = 0 & (t, x > 0) \\[4pt] G^\epsilon(u_+^\epsilon, u_-^\epsilon, d_t\phi^\epsilon) = 0 & (x = 0) \\[4pt] \phi^\epsilon(0) = 0 & \\[4pt] u_\pm^\epsilon(0, x) = u_{\pm,0}^\epsilon(x). & \end{array}\right\} \qquad (3.1.4)$$

Under the assumption (MA) in Subsection 2.2, we try to construct approximate solutions $(u_\pm^{\epsilon,0}, \phi^{\epsilon,0})$ of this problem, which are bounded in $C_\epsilon^1(\Omega^+) \times \widetilde{C}_\epsilon^2[0, T_0]$ $((T_0, 0) \in \Omega)$ and satisfy

$$\left.\begin{array}{rl} L_\pm^\epsilon(u_\pm^{\epsilon,0}, \phi^{\epsilon,0})u_\pm^{\epsilon,0}|_{t=0} = 0 & \\[4pt] d_t^k G^\epsilon(u_+^{\epsilon,0}, u_-^{\epsilon,0}, d_t\phi^{\epsilon,0})|_{t=0} = 0 & (k = 0, 1) \\[4pt] \phi^{\epsilon,0}(0) = 0 & \\[4pt] u_\pm^{\epsilon,0}(0, x) = u_{\pm,0}^\epsilon(x). & \end{array}\right\} \qquad (3.1.5)$$

Set $m_\pm^{k,\epsilon}(x) = \partial_t^k u_\pm^{\epsilon,0}(0, x)$ for $k \in \{0, 1\}$. The initial condition in (3.1.4) implies that $m_\pm^{0,\epsilon}(x) = u_{\pm,0}^\epsilon(x)$ are bounded in $C_\epsilon^1(\omega^+)$. From the compatibility condition (2.2.1), we immediately obtain that the sequence $\{d_t\phi^{\epsilon,0}(0)\}_{\epsilon\in(0,\epsilon_0]}$ is bounded when $\epsilon_0 > 0$ is small enough, such that, for any $\epsilon \in (0, \epsilon_0]$,

$$\left|\epsilon\big(u_{+,0}^\epsilon(0) - u_{-,0}^\epsilon(0)\big)\right| \le \tfrac{1}{2}|u_+ - u_-|. \qquad (3.1.6)$$

From the equation in (3.1.4), we deduce $m_\pm^{1,\epsilon}(x) \in C^0(\omega^+)$ and $\epsilon m_\pm^{1,\epsilon}(x)$ are bounded in $C_\epsilon^0(\omega^+)$ by using the boundedness of $\epsilon\partial_x u_{\pm,0}^\epsilon(x)$ in $C_\epsilon^0(\omega^+)$.

Let us construct the approximate solutions $u_\pm^{\epsilon,0}$ of (3.1.4) by the following lemma.

Lemma 3.1. *Given functions $m_\pm^{0,\epsilon} \in C_\epsilon^1(\omega^+)$, $\epsilon m_\pm^{1,\epsilon} \in C_\epsilon^0(\omega^+)$ and the sequence $\{d_t\phi^{\epsilon,0}(0)\}_{\epsilon\in(0,\epsilon_0]}$ bounded as above, there are functions $u_\pm^{\epsilon,0}$ bounded in $C_\epsilon^1(\Omega^+)$ such that*

$$u_\pm^{\epsilon,0}(0, x) = m_\pm^{0,\epsilon}(x) \qquad and \qquad \partial_t u_\pm^{\epsilon,0}(0, x) = m_\pm^{1,\epsilon}(x) \qquad (3.1.7)$$

$$\partial_t u_\pm^{\epsilon,0} \pm \Big(A(u_\pm + \epsilon u_\pm^{\epsilon,0}) - (\sigma + \epsilon d_t\phi^{\epsilon,0}(0))I\Big)\partial_x u_\pm^{\epsilon,0} = 0 \quad on\ t = 0 \qquad (3.1.8)$$

and

$$\partial_t u_\pm^{\epsilon,0} \pm (A_\pm - \sigma I)\partial_x u_\pm^{\epsilon,0} \quad bounded\ in\ C_\epsilon^0(\Omega^+). \qquad (3.1.9)$$

Proof. The conclusion (3.1.8) is a simple consequence from (3.1.7) and the choice of $m_\pm^{1,\epsilon}$. It is sufficient to construct bounded $u_\pm^{\epsilon,0}$ in $C_\epsilon^1(\Omega^+)$ satisfying (3.1.7) and (3.1.9). Denote the extensions of $m_\pm^{0,\epsilon}$ in ω still by $m_\pm^{0,\epsilon}$, with $m_\pm^{0,\epsilon} \in C_\epsilon^1(\omega)$. Set

$$m_\pm^{1,\epsilon}(x) = \mp\Big(A\big(u_\pm + \epsilon m_\pm^{0,\epsilon}(x)\big) - (\sigma + \epsilon d_t\phi^\epsilon(0))I\Big)d_x m_\pm^{0,\epsilon}(x). \qquad (3.1.10)$$

Obviously, we have that $\varepsilon m_{\pm}^{1,\varepsilon}(x)$ are bounded sequences in $C_\varepsilon^0(\omega)$ and that $m_{\pm}^{1,\varepsilon}(x)$ are extensions in ω of $m_{\pm}^{1,\varepsilon}(x) = \partial_t u_{\pm}^{\varepsilon,0}(0,x)$ obtained from (3.1.5).

Let us solve the linear Cauchy problem of $u_{\pm}^{\varepsilon,0}$ in Ω

$$\left.\begin{aligned} \partial_t u_{\pm}^{\varepsilon,0} \pm (A_\pm - \sigma I)\partial_x u_{\pm}^{\varepsilon,0} &= m_{\pm}^{1,\varepsilon} \pm (A_\pm - \sigma I)d_x m_{\pm}^{0,\varepsilon} \\ u_{\pm}^{\varepsilon,0}(0,x) &= m_{\pm}^{0,\varepsilon}(x) \end{aligned}\right\} \qquad (3.1.11)$$

from which the conclusions (3.1.7) and (3.1.9) are obviously valid by using the fact that

$$m_{\pm}^{1,\varepsilon} \pm (A_\pm - \sigma I)d_x m_{\pm}^{0,\varepsilon} = \pm\Big(A_\pm - A(u_\pm + \varepsilon m_{\pm}^{0,\varepsilon}) + \varepsilon d_t \phi^{\varepsilon,0}(0)I\Big)d_x m_{\pm}^{0,\varepsilon} \qquad (3.1.12)$$

are bounded in $C_\varepsilon^0(\Omega)$. Hence, it suffices to verify that the solutions $u_{\pm}^{\varepsilon,0}$ of the problem (3.1.11) are bounded in $C_\varepsilon^1(\Omega^+)$. As J. L. Joly et al. in [8], it is obvious that $u_{\pm}^{\varepsilon,0}$ are bounded in $C_\varepsilon^0(\Omega^+)$ by using (3.1.12).

To check that $\varepsilon \nabla u_{\pm}^{\varepsilon,0}$ are bounded in $C_\varepsilon^0(\Omega^+)$, let us diagonalize the problem (3.1.11). Set $T_\pm = (r_1^\pm, \ldots, r_m^\pm)$ as in (2.2.5) and define

$$v_{\pm}^{\varepsilon,0} = T_\pm^{-1} u_{\pm}^{\varepsilon,0}.$$

Then $v_{\pm}^{\varepsilon,0}$ satisfy

$$\left.\begin{aligned} \partial_t v_{\pm}^{\varepsilon,0} \pm (\Lambda_\pm - \sigma I)\partial_x v_{\pm}^{\varepsilon,0} &= \widetilde{m}_{\pm}^{1,\varepsilon} \pm (\Lambda_\pm - \sigma I)d_x \widetilde{m}_{\pm}^{0,\varepsilon} \\ v_{\pm}^{\varepsilon,0}(0,x) &= \widetilde{m}_{\pm}^{0,\varepsilon}(x) \end{aligned}\right\} \qquad (3.1.13)$$

where $\widetilde{m}_{\pm}^{1,\varepsilon} = T_\pm^{-1} m_{\pm}^{1,\varepsilon}$ and $\widetilde{m}_{\pm}^{0,\varepsilon} = T_\pm^{-1} m_{\pm}^{0,\varepsilon}$. Obviously, the same assumptions as those of $m_{\pm}^{i,\varepsilon}(x)$ are still valid for $\widetilde{m}_{\pm}^{i,\varepsilon}(x)$, with $i \in \{0,1\}$. It is easy to see that the solutions $v_{\pm}^{\varepsilon,0}$ of the problem (3.1.13) can be expressed as

$$v_{\pm,i}^{\varepsilon,0}(t,x) = \widetilde{m}_{\pm,i}^{0,\varepsilon}(x) \pm \frac{1}{\lambda_i^\pm - \sigma} \int_{x\pm(\sigma-\lambda_i^\pm)t}^{x} \widetilde{m}_{\pm,i}^{1,\varepsilon}(\tau)\, d\tau \qquad (3.1.14)$$

for any $i \in \{1, \ldots, m\}$. From here, we immediately obtain that $\varepsilon \nabla v_{\pm}^{\varepsilon,0}$ are bounded in $C_\varepsilon^0(\Omega^+)$ which is equivalent to the boundedness of $\varepsilon \nabla u_{\pm}^{\varepsilon,0}$ in $C_\varepsilon^0(\Omega^+)$ ∎

Set $a_\varepsilon = d_t \phi^\varepsilon(0)$ and $b_\varepsilon = d_t^2 \phi^\varepsilon(0)$. From the compatibility conditions (2.2.1) and (2.2.4), we know that the sequences $\{a_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ and $\{\varepsilon b_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ are bounded when (3.1.6) holds. Define the approximate solution $\phi^{\varepsilon,0} \in \widetilde{C}_\varepsilon^2[0,T_0]$ of (3.1.4) by the following lemma.

Lemma 3.2. *Let $u_{\pm}^{\varepsilon,0} \in C_\varepsilon^1(\Omega^+)$ be the approximate solutions of (3.1.4) given as above. Then there is a $\phi^{\varepsilon,0}$ bounded in $\widetilde{C}_\varepsilon^2[0,T_0]$ for $\varepsilon \in (0,\varepsilon_0]$, with $\varepsilon_0 > 0$ small enough, such that*

$$\left.\begin{aligned} d_t^k G^\varepsilon\big(u_+^{\varepsilon,0}, u_-^{\varepsilon,0}, d_t\phi^{\varepsilon,0}\big)\big|_{t=0} &= 0 \quad (k = 0,1) \\ \phi^{\varepsilon,0}(0) = 0, \quad d_t\phi^{\varepsilon,0}(0) = a_\varepsilon, \quad d_t^2\phi^{\varepsilon,0}(0) &= b_\varepsilon. \end{aligned}\right\} \qquad (3.1.15)$$

Proof. The compatibility condition (2.2.1) can be written as

$$
\begin{aligned}
d_t\phi^\varepsilon(0)\big([u] + \varepsilon[u^{\varepsilon,0}(0)]\big) + (\sigma I - A_+)u_+^{\varepsilon,0}(0,0) \\
- (\sigma I - A_-)u_-^{\varepsilon,0}(0,0) = \varepsilon g^\varepsilon(0)
\end{aligned}
\tag{3.1.16}
$$

where $[u^{\varepsilon,0}(t)] = u_+^{\varepsilon,0}(t,0) - u_-^{\varepsilon,0}(t,0)$, and

$$
\begin{aligned}
g^\varepsilon(t) = \frac{1}{\varepsilon^2}\Big\{& f(u_+ + \varepsilon u_+^{\varepsilon,0}) - f(u_+) - \varepsilon A_+ u_+^{\varepsilon,0} \\
& - f(u_- + \varepsilon u_-^{\varepsilon,0}) + f(u_-) + \varepsilon A_- u_-^{\varepsilon,0}\Big\}(t,0)
\end{aligned}
\tag{3.1.17}
$$

is bounded in $C_\varepsilon^1[0,T_0]$, by using the Rankine-Hugoniot condition (2.1.8), the boundedness of $u_\pm^{\varepsilon,0}$ in $C_\varepsilon^1(\Omega^+)$, and Taylor's expansion of $f(u_\pm + \varepsilon u_\pm^{\varepsilon,0})$. As in the proof of Lemma 3.1, performing the transformations $v_\pm^{\varepsilon,0} = T_\pm^{-1}u_\pm^{\varepsilon,0}$ in (3.1.16), it follows

$$
\begin{aligned}
d_t\phi^\varepsilon(0)\big([u] + \varepsilon[u^{\varepsilon,0}(0)]\big) + (\sigma I - A_+)T_+ v_+^{\varepsilon,0}(0,0) \\
- (\sigma I - A_-)T_- v_-^{\varepsilon,0}(0,0) = \varepsilon g^\varepsilon(0).
\end{aligned}
\tag{3.1.18}
$$

Obviously, from the stability condition (2.1.10), we have that the matrix

$$
M^\varepsilon(t) = \left([u] + \varepsilon[u^{\varepsilon,0}(t)], \big\{(\sigma - \lambda_i^+)r_i^+\big\}_{i=j+1}^m, \big\{(\lambda_i^- - \sigma)r_i^-\big\}_{i=1}^{j-1}\right)
\tag{3.1.19}
$$

is invertible when $\varepsilon \in (0,\varepsilon_0]$, with ε_0 small enough. Hence, (3.1.18) gives rise to

$$
d_t\phi^\varepsilon(0) = \vec{e}_1 \cdot (M^\varepsilon(0))^{-1}\mathcal{G}^\varepsilon(0)
\tag{3.1.20}
$$

where $\vec{e}_1 = (1,0,\ldots,0)$ and

$$
\mathcal{G}^\varepsilon(t) = \varepsilon g^\varepsilon(t) - \sum_{i=1}^j (\sigma - \lambda_i^+)\, r_i^+ v_{+,i}^{\varepsilon,0}(t,0) - \sum_{i=j}^m (\lambda_i^- - \sigma)r_i^- \, v_{-,i}^{\varepsilon,0}(t,0).
\tag{3.1.21}
$$

Let us solve the initial value problem

$$
\left.\begin{aligned}
d_t\phi^{\varepsilon,0}(t) &= \vec{e}_1 \cdot (M^\varepsilon(t))^{-1}\mathcal{G}^\varepsilon(t) \\
\phi^{\varepsilon,0}(0) &= 0.
\end{aligned}\right\}
\tag{3.1.22}
$$

The existence of a solution $\phi^{\varepsilon,0} \in C^2[0,T_0]$ is clear for this problem. By using the boundedness of $g^\varepsilon(t)$ and $v_\pm^{\varepsilon,0}(t,0)$ in $C_\varepsilon^1[0,T_0]$, we obtain that $\phi^{\varepsilon,0}(t)$ is bounded in $\tilde{C}_\varepsilon^2[0,T_0]$. On the other hand, from the above discussion, we know that the solution $\phi^{\varepsilon,0}$ of the problem (3.1.22) satisfies $d_t\phi^{\varepsilon,0}(0) = a_\varepsilon$ and $G^\varepsilon\big(u_+^{\varepsilon,0}, u_-^{\varepsilon,0}, d_t\phi^{\varepsilon,0}\big)\big|_{t=0} = 0$. If we can verify that $\phi^{\varepsilon,0}$ also satisfies

$$
d_t^2\phi^{\varepsilon,0}(0) = b_\varepsilon,
\tag{3.1.23}
$$

then from the definition of b_ε, and the first order compatibility condition in (2.1.18), we immediately have

$$d_t G^\varepsilon\big(u_+^{\varepsilon,0}, u_-^{\varepsilon,0}, d_t \phi^{\varepsilon,0}\big)\big|_{t=0} = 0.$$

Let us prove the assertion (3.1.23) as follows. Differentiating the equation in (3.1.22), it follows

$$d_t^2 \phi^{\varepsilon,0}(0) = \vec{e}_1 \cdot (M^\varepsilon(0))^{-1} d_t \mathcal{G}^\varepsilon(0) + \vec{e}_1 \cdot d_t (M^\varepsilon(0))^{-1} \mathcal{G}^\varepsilon(0). \qquad (3.1.24)$$

From the definitions of $M^\varepsilon(t)$ and $\mathcal{G}^\varepsilon(t)$ in (3.1.19) and (3.1.21), respectively, we have

$$d_t M^\varepsilon(0) = \big(\varepsilon[\partial_t u^{\varepsilon,0}(0)], 0, \ldots, 0\big) \quad \text{and} \quad (M^\varepsilon(0))^{-1} \mathcal{G}^\varepsilon(0) = (a_\varepsilon, *)^T \qquad (3.1.25)$$

by using $d_t \phi^{\varepsilon,0}(0) = a_\varepsilon$, where $*$ is an $1 \times (m-1)$ vector the explicit expression of which we do not need. By substituting the formula

$$d_t(M^\varepsilon(0))^{-1} = -(M^\varepsilon(0))^{-1}(d_t M^\varepsilon(0))(M^\varepsilon(0))^{-1}$$

into (3.1.24), and using (3.1.25), we obtain

$$d_t^2 \phi^{\varepsilon,0}(0) = \vec{e}_1 \cdot (M^\varepsilon(0))^{-1}\big(d_t \mathcal{G}^\varepsilon(0) - \varepsilon a_\varepsilon[\partial_t u^{\varepsilon,0}(0)]\big). \qquad (3.1.26)$$

On the other hand, from (2.2.2) we have

$$\begin{aligned}
b_\varepsilon\big([u] + \varepsilon[u^{\varepsilon,0}(0)]\big) &+ \big[(\sigma I - A(u))\partial_t u^{\varepsilon,0}(0)\big] \\
&= \Big[\big((A(u + \varepsilon u^{\varepsilon,0}(0)) - A(u) - \varepsilon a_\varepsilon\big)\partial_t u^{\varepsilon,0}(0)\Big] \\
&= \varepsilon d_t g^\varepsilon(0) - \varepsilon a_\varepsilon[\partial_t u^{\varepsilon,0}(0)]
\end{aligned} \qquad (3.1.27)$$

with $g^\varepsilon(t)$ defined in (3.1.17). In a way similar to (3.1.20), from (3.1.27) we deduce

$$b_\varepsilon = \vec{e}_1 \cdot (M^\varepsilon(0))^{-1}\big(d_t \mathcal{G}^\varepsilon(0) - \varepsilon a_\varepsilon[\partial_t u^{\varepsilon,0}(0)]\big) \qquad (3.1.28)$$

where $\mathcal{G}^\varepsilon(t)$ is defined in (3.1.21). Comparing (3.1.26) with (3.1.28), it concludes the assertion (3.1.23) ∎

Summing up, we have the following

Proposition 3.1. *Under the assumption* (MA) *in Subsection 2.2, there are approximate solutions* $(u_\pm^{\varepsilon,0}, \phi^{\varepsilon,0})$ *to the problem* (3.1.4), *such that* $(u_\pm^{\varepsilon,0}, \phi^{\varepsilon,0})$ *are bounded in* $C_\varepsilon^1(\Omega^+) \times \widetilde{C}_\varepsilon^2[0, T_0]$, *and satisfy*

$$\left.\begin{aligned}
L_\pm^\varepsilon(u_\pm^{\varepsilon,0}, \phi^{\varepsilon,0})u_\pm^{\varepsilon,0}|_{t=0} &= 0 \\
d_t^k G^\varepsilon\big(u_+^{\varepsilon,0}, u_-^{\varepsilon,0}, d_t \phi^{\varepsilon,0}\big)\big|_{t=0} &= 0 \quad (k = 0, 1) \\
\phi^{\varepsilon,0}(0) &= 0 \\
u_\pm^{\varepsilon,0}(0, x) &= u_{\pm,0}^\varepsilon(x)
\end{aligned}\right\} \qquad (3.1.29)$$

3.2 Iteration scheme and proof of Theorem 2.1/(1). From the definition (3.1.2) of G^ϵ, it is easy to see that the Fréchet derivative of G^ϵ with respect to its arguments at $(v_+, v_-, d_t\varphi)$ is

$$G^{\epsilon\prime}_{(u^\epsilon_+, u^\epsilon_-, d_t\phi^\epsilon)}(v_+, v_-, d_t\varphi)$$
$$= ([u] + \epsilon[u^\epsilon])d_t\varphi + (\sigma + \epsilon d_t\phi^\epsilon)[v] - [A(u + \epsilon u^\epsilon)v]. \tag{3.2.1}$$

As A. Majda in [13], given approximate solutions $(u^{\epsilon,0}_\pm, \phi^{\epsilon,0})$ by Proposition 3.1, we solve the problem (3.1.4) by the iteration scheme

$$\left.\begin{aligned}
L^\epsilon_\pm(u^{\epsilon,\nu}_\pm, \phi^{\epsilon,\nu})u^{\epsilon,\nu+1}_\pm &= 0 \\
G^{\epsilon\prime}_{(u^{\epsilon,\nu}_\pm, d_t\phi^{\epsilon,\nu})}(u^{\epsilon,\nu+1}_\pm, d_t\phi^{\epsilon,\nu+1}) &= -G^\epsilon(u^{\epsilon,\nu}_\pm, d_t\phi^{\epsilon,\nu}) \\
&\quad + G^{\epsilon\prime}_{(u^{\epsilon,\nu}_\pm, d_t\phi^{\epsilon,\nu})}(u^{\epsilon,\nu}_\pm, d_t\phi^{\epsilon,\nu}) \\
\phi^{\epsilon,\nu+1}(0) &= 0 \\
u^{\epsilon,\nu+1}_\pm(0,x) &= u^\epsilon_{\pm,0}(x),
\end{aligned}\right\} \tag{3.2.2}$$

i.e. we use the usual Picard iteration for the equation, and the Newton iteration for the boundary condition. The zero-th and first order compatibility conditions in (3.2.2) for any $\nu \geq 0$ were verified by A.Majda in [13: Section 3] even for case of several space variables by using (3.1.29).

To study the problem (3.2.2), let us first consider the linear problem

$$\left.\begin{aligned}
L^\epsilon_\pm(u^\epsilon_\pm, \phi^\epsilon)v^\epsilon_\pm &= f^\epsilon_\pm \\
G^{\epsilon\prime}_{(u^\epsilon_+, u^\epsilon_-, d_t\phi^\epsilon)}(v^\epsilon_+, v^\epsilon_-, d_t\varphi^\epsilon) &= g^\epsilon(t) \\
\varphi^\epsilon(0) &= 0 \\
v^\epsilon_\pm(0,x) &= u^\epsilon_{\pm,0}(x)
\end{aligned}\right\} \tag{3.2.3}$$

where $\{u^\epsilon_\pm\}$ and $\{\phi^\epsilon\}$ are bounded in $C^1_\epsilon(\Omega^+_{T_0})$ and $\widetilde{C}^2_\epsilon[0, T_0]$, respectively, $f^\epsilon_\pm \in C^1(\Omega^+_{T_0})$ and $g^\epsilon \in C^1[0, T_0]$ satisfy the compatibility conditions of (3.2.3) up to order one.

To alleviate the burden of notations, in the remainder of this section, setting $\omega^+_s = \Omega^+ \cap \{t = s\}$ we use $\|u(t)\|$ and $\|u(t)\|_1 = \|u(t)\| + \|\nabla u(t)\|$ to denote the $L^\infty(\omega^+_t)$ and $W^{1,\infty}(\omega^+_t)$ norms, respectively, of $u(t, \cdot)$. Analogously, we use $\|u\|_t$ and $\|u\|_{1,t}$ to denote the $L^\infty(\Omega^+_t)$ and $W^{1,\infty}(\Omega^+_t)$ norms, respectively, of $u(\cdot)$. For any $\phi \in L^\infty[0, T]$, the norm $\|\phi\|_{L^\infty[0,t]}$ is also denoted by $\|\phi\|_t$, for any $t \in (0, T)$.

For the problem (3.2.3), we have the following results, the proof of which will be given in the next subsection.

Proposition 3.2.

(1) *Suppose that the families $f^\epsilon_\pm \in C^0(\Omega^+_{T_0})$ and $g^\epsilon \in C^0[0, T_0]$ are bounded and satisfy the zero-th order compatibility condition of the problem (3.2.3). Then there exist*

unique weak solutions $(v_\pm^\varepsilon, \varphi^\varepsilon) \in C^0(\Omega_{T_0}^+) \times C^1[0, T_0]$ to the problem (3.2.3). Moreover, there is a constant $C > 0$ such that

$$|d_t \varphi^\varepsilon(t)| + \|v_\pm^\varepsilon(t)\| \leq C e^{CMt} \left(\|g^\varepsilon\|_t + \|u_{\pm,0}^\varepsilon\| + \int_0^t \|f_\pm^\varepsilon(s)\| \, ds \right) \tag{3.2.4}$$

for any $t \in (0, T_0]$, where $M \geq 1 + \varepsilon \big(\|\nabla u_+^\varepsilon\|_{T_0} + \|\nabla u_-^\varepsilon\|_{T_0} + \|d_t^2 \phi^\varepsilon\|_{T_0} \big)$.

(2) *If $(f_\pm^\varepsilon, g^\varepsilon)$ have the additional regularity $(f_\pm^\varepsilon, g^\varepsilon) \in C^1$ and satisfy the first order compatibility condition of the problem (3.2.3), then the unique solutions $(v_\pm^\varepsilon, \varphi^\varepsilon)$ of (3.2.3) obtained above belong to $C^1(\Omega_{T_0}^+) \times C^2[0, T_0]$. Moreover, we have the estimate*

$$|\varepsilon d_t^2 \varphi^\varepsilon(t)| + \|\varepsilon \nabla_{(t,x)} v_\pm^\varepsilon(t)\|$$

$$\leq C \exp(CMt e^{CMt}) \bigg(\|\varepsilon d_t g^\varepsilon\|_t + \|\varepsilon d_x u_{\pm,0}^\varepsilon\| + \|\varepsilon f_\pm^\varepsilon(0)\|$$

$$+ \varepsilon M \big(\|g^\varepsilon\|_t + \|u_{\pm,0}^\varepsilon\| \big) + \int_0^t \big(\varepsilon M \|f_\pm^\varepsilon(s)\| + \|\varepsilon \partial_t f_\pm^\varepsilon(s)\| \big) ds \bigg) \tag{3.2.5}$$

for any $t \in (0, T_0]$.

For any ν, given $(u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})$ bounded in $C_\varepsilon^1(\Omega_T^+) \times \widetilde{C}_\varepsilon^2[0, T]$, let us study the iteration scheme (3.2.2). Fix

$$M \geq 1 + \|u_+^{\varepsilon,\nu}\|_{\varepsilon,1,\Omega_T^+} + \|u_-^{\varepsilon,\nu}\|_{\varepsilon,1,\Omega_T^+} + \|d_t \phi^{\varepsilon,\nu}\|_{\varepsilon,1,[0,T]}$$

and

$$\eta \geq \varepsilon \left(\|u_+^{\varepsilon,\nu}\|_T + \|u_-^{\varepsilon,\nu}\|_T + \|d_t \phi^{\varepsilon,\nu}\|_T \right)$$

with $\eta \leq \frac{|u_+ - u_-|}{2}$. Then we have the following result.

Lemma 3.3. *There are a constant $C(\eta)$ depending on η, and two increasing functions $C_1(\cdot)$ and $C_2(\cdot)$ such that, for any $t \in (0, T]$,*

$$|d_t \phi^{\varepsilon,\nu+1}(t)| + \|u_\pm^{\varepsilon,\nu+1}(t)\| \leq C e^{CMt} \Big(\|u_{\pm,0}^\varepsilon\| + \varepsilon C_1 \big(\|u_\pm^{\varepsilon,\nu}\|_t + \|d_t \phi^{\varepsilon,\nu}\|_t \big) \Big) \tag{3.2.6}$$

and

$$|\varepsilon d_t^2 \phi^{\varepsilon,\nu+1}(t)| + \|\varepsilon \nabla u_\pm^{\varepsilon,\nu+1}(t)\|$$

$$\leq C \exp(CMt e^{CMt}) \Big(\|\varepsilon d_x u_{\pm,0}^\varepsilon\| + \varepsilon M \|u_{\pm,0}^\varepsilon\|$$

$$+ \varepsilon C_2 \big(\|u_\pm^{\varepsilon,\nu}\|_{\varepsilon,1,\Omega_t^+} + \|d_t \phi^{\varepsilon,\nu}\|_{\varepsilon,1,[0,t]} \big) \Big). \tag{3.2.7}$$

Proof. By applying the estimate (3.2.4) in the problem (3.2.2), we obtain

$$|d_t \phi^{\varepsilon,\nu+1}(t)| + \|u_\pm^{\varepsilon,\nu+1}(t)\| \leq C e^{CMt} \big(\|u_{\pm,0}^\varepsilon\| + \|g^{\varepsilon,\nu}\|_t \big) \tag{3.2.8}$$

where

$$g^{\varepsilon,\nu} = -G^{\varepsilon}(u_{\pm}^{\varepsilon,\nu}, d_t\phi^{\varepsilon,\nu}) + G_{(u_{\pm}^{\varepsilon,\nu}, d_t\phi^{\varepsilon,\nu})}^{\varepsilon\prime}(u_{\pm}^{\varepsilon,\nu}, d_t\phi^{\varepsilon,\nu})$$

$$= \varepsilon d_t\phi^{\varepsilon,\nu}[u^{\varepsilon,\nu}] + \int_0^1 \Big[\big(A(u + \eta\varepsilon u^{\varepsilon,\nu}) - A(u + \varepsilon u^{\varepsilon,\nu})\big)u^{\varepsilon,\nu} \Big] d\eta. \tag{3.2.9}$$

Obviously, we have an increasing function $C_1(\cdot)$ such that

$$\|g^{\varepsilon,\nu}\|_t \leq \varepsilon C_1 \big(\|u_{\pm}^{\varepsilon,\nu}\|_t + \|d_t\phi^{\varepsilon,\nu}\|_t \big). \tag{3.2.10}$$

Substituting this estimate into (3.2.8), the estimate (3.2.6) follows immediately. Similarly, by employing the estimate (3.2.5) for the problem (3.2.2), and using (3.2.9) and (3.2.10), we get the conclusion (3.2.7) ∎

The next result is devoted to the iteration scheme (3.2.2), from which we immediately obtain the conclusion of Theorem 2.1/(1).

Theorem 3.1. *There are constants $T > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the iteration scheme (3.2.2) defines a sequence $(u_{\pm}^{\varepsilon,\nu}, \phi^{\varepsilon,\nu}) \in C_{\varepsilon}^1(\Omega_T^+) \times \widetilde{C}_{\varepsilon}^2(0,T)$ satisfying the following conditions.*

(1) *There is a constant η, $0 < \eta < \frac{|u_+ - u_-|}{2}$, such that, for any ν and $\varepsilon \in (0, \varepsilon_0]$, one has*

$$\|u_{\pm}^{\varepsilon,\nu}\|_{\varepsilon,1,\Omega_T^+} + \|d_t\phi^{\varepsilon,\nu}\|_{\varepsilon,1,[0,T]} \leq K \tag{3.2.11}$$

and

$$\varepsilon \big(\|u_{\pm}^{\varepsilon,\nu}\|_T + \|d_t\phi^{\varepsilon,\nu}\|_T \big) \leq \eta. \tag{3.2.12}$$

(2) *For each fixed $\varepsilon \in (0, \varepsilon_0]$, the sequence $(u_{\pm}^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})$ converges in $C^1(\Omega_T^+) \times C^2[0,T]$ to solutions $(u_{\pm}^{\varepsilon}, \phi^{\varepsilon}) \in C^1(\Omega_T^+) \times C^2[0,T]$ of the problem (3.1.4).*

(3) *Moreover, $u_{\pm}^{\varepsilon,\nu} \to u_{\pm}^{\varepsilon}$ in $C^0(\Omega_T^+)$, and $\phi^{\varepsilon,\nu} \to \phi^{\varepsilon}$ in $C^1[0,T]$ as $\nu \to \infty$, uniformly in $\varepsilon \in (0, \varepsilon_0]$.*

Proof. Assertion (1): Choose

$$K_0 > C(\eta)\Big(\|u_{+,0}^{\varepsilon}\| + \|u_{-,0}^{\varepsilon}\| + 1 \Big) + \|u_{\pm}^{\varepsilon,0}\|_{T_0} + \|d_t\phi^{\varepsilon,0}\|_{T_0}$$

and

$$K_1 > C(\eta)\Big(\|u_{+,0}^{\varepsilon}\|_{\varepsilon,1,\omega^+} + \|u_{-,0}^{\varepsilon}\|_{\varepsilon,1,\omega^+} + 1 \Big) + \|u_{\pm}^{\varepsilon,0}\|_{\varepsilon,1,\Omega_{T_0}^+} + \|d_t\phi^{\varepsilon,0}\|_{\varepsilon,1,[0,T_0]}$$

with $C(\eta) > 0$ being the constant in (3.2.6) and (3.2.7). Then, there are constants $T > 0$ and $\varepsilon_0 > 0$ such that, for any $\nu \geq 0$,

$$\left. \begin{array}{c} \|u_{\pm}^{\varepsilon,\nu}\|_T + \|d_t\phi^{\varepsilon,\nu}\|_T \leq K_0 \\[2mm] \|\varepsilon\nabla u_{\pm}^{\varepsilon,\nu}\|_T + \|\varepsilon d_t^2\phi^{\varepsilon,\nu}\|_T \leq K_1 \\[2mm] \varepsilon\big(\|u_{\pm}^{\varepsilon,\nu}\|_T + \|d_t\phi^{\varepsilon,\nu}\|_T \big) \leq \eta \end{array} \right\} \tag{3.2.13}$$

when $\varepsilon \in (0, \varepsilon_0]$. Indeed, obviously, (3.2.13) is valid for $\nu = 0$ when $\varepsilon_0 > 0$ is small enough. Suppose (3.2.13) holds for some $\nu \geq 0$ and let us consider the case $\nu + 1$. In (3.2.13), the first line immediately implies the last line by taking $\varepsilon_0 > 0$ small enough. From the definition of (K_0, K_1), we know that there are constants $T > 0$ and $\varepsilon_0 > 0$, small enough, such that

$$
\left.
\begin{array}{r}
Ce^{CKT}\Big(\|u_{\pm,0}^\varepsilon\| + \varepsilon_0 C_1(K_0)\Big) \leq K_0 \\[2mm]
C\exp(CKTe^{CKT})\Big(\|u_{\pm,0}^\varepsilon\|_{\varepsilon,1,\omega+} + \varepsilon_0 K\|u_{\pm,0}^\varepsilon\| + \varepsilon_0 C_2(K)\Big) \leq K_1
\end{array}
\right\}
\tag{3.2.14}
$$

where $K = K_0 + K_1$, $C_1(\cdot)$ and $C_2(\cdot)$ are two increasing functions given in Lemma 3.3. By taking constants $T > 0$ and $\varepsilon_0 > 0$ from (3.2.14), with $T \in (0, T_0]$, and using Lemma 3.3, we obtain that (3.2.13) is valid for $\nu + 1$. Hence, $(u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})$ are bounded in $C_\varepsilon^1(\Omega_T^+) \times \widetilde{C}_\varepsilon^2(0, T]$ and satisfy (3.2.11) and (3.2.12).

Assertion (2): From (3.2.2), we know that $(u_\pm^{\varepsilon,\nu+1} - u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu+1} - \phi^{\varepsilon,\nu})$ satisfy the problem

$$
\left.
\begin{array}{r}
L_\pm^\varepsilon(u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})(u_\pm^{\varepsilon,\nu+1} - u_\pm^{\varepsilon,\nu}) = F_\pm^{\varepsilon,\nu} \\[2mm]
G_{(u_+^{\varepsilon,\nu}, u_-^{\varepsilon,\nu}, d_t\phi^{\varepsilon,\nu})}^{\varepsilon l}\Big(u_+^{\varepsilon,\nu+1} - u_+^{\varepsilon,\nu}, u_-^{\varepsilon,\nu+1} - u_-^{\varepsilon,\nu}, d_t(\phi^{\varepsilon,\nu+1} - \phi^{\varepsilon,\nu})\Big) = \ddot{\mathcal{G}}^{\varepsilon,\nu} \\[2mm]
(u_\pm^{\varepsilon,\nu+1} - u_\pm^{\varepsilon,\nu})(0, x) = 0 \\[2mm]
(\phi^{\varepsilon,\nu+1} - \phi^{\varepsilon,\nu})(0) = 0
\end{array}
\right\}
\tag{3.2.15}
$$

where

$$
F_\pm^{\varepsilon,\nu} = L_\pm^\varepsilon(u_\pm^{\varepsilon,\nu-1}, \phi^{\varepsilon,\nu-1})u_\pm^{\varepsilon,\nu} - L_\pm^\varepsilon(u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})u_\pm^{\varepsilon,\nu}
\tag{3.2.16}
$$

and

$$
\begin{aligned}
\mathcal{G}^{\varepsilon,\nu} = {} & \varepsilon d_t(\phi^{\varepsilon,\nu} - \phi^{\varepsilon,\nu-1})[u^{\varepsilon,\nu-1} - u^{\varepsilon,\nu}] \\[2mm]
& + \Big[\Big(A(u + \varepsilon u^{\varepsilon,\nu}) - A(u + \varepsilon u^{\varepsilon,\nu-1})\Big)u^{\varepsilon,\nu}\Big] \\[2mm]
& + \int_0^1 \Big[\Big(A(u + \eta\varepsilon u^{\varepsilon,\nu}) - A(u + \varepsilon u^{\varepsilon,\nu})\Big)u^{\varepsilon,\nu} \\[2mm]
& \qquad - \Big(A(u + \eta\varepsilon u^{\varepsilon,\nu-1}) - A(u + \varepsilon u^{\varepsilon,\nu-1})\Big)u^{\varepsilon,\nu-1}\Big] d\eta.
\end{aligned}
\tag{3.2.17}
$$

For $F_\pm^{\varepsilon,\nu}$ and $\mathcal{G}_\pm^{\varepsilon,\nu}$, clearly, we have

$$
\left.
\begin{array}{r}
\|F_\pm^{\varepsilon,\nu}(s)\| \leq C\Big(|d_t(\phi^{\varepsilon,\nu} - \phi^{\varepsilon,\nu-1})(s)| + \|(u_\pm^{\varepsilon,\nu} - u_\pm^{\varepsilon,\nu-1})(s)\|\Big) \\[2mm]
\|\mathcal{G}_\pm^{\varepsilon,\nu}\|_t \leq C\varepsilon\Big(\|d_t(\phi^{\varepsilon,\nu} - \phi^{\varepsilon,\nu-1})\|_t + \|u_\pm^{\varepsilon,\nu} - u_\pm^{\varepsilon,\nu-1}\|_t\Big)
\end{array}
\right\}
\tag{3.2.18}
$$

by using the boundedness of $\{u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu}\}$ in $C_\varepsilon^1(\Omega_T^+) \times \widetilde{C}_\varepsilon^2[0, T]$. Employing (3.2.4) for the problem (3.2.15), using (3.2.18), and setting

$$
a^\nu(t) = \left\|u_+^{\varepsilon,\nu+1} - u_+^{\varepsilon,\nu}\right\|_t + \left\|u_-^{\varepsilon,\nu+1} - u_-^{\varepsilon,\nu}\right\|_t + \left\|d_t(\phi^{\varepsilon,\nu+1} - \phi^{\varepsilon,\nu})\right\|_t,
$$

we obtain

$$a^\nu(t) \le C \left(\varepsilon a^{\nu-1}(t) + \int_0^t a^{\nu-1}(s)\,ds \right) \tag{3.2.19}$$

for any $t \in [0,T]$. From here, we immediately obtain that there are constants $T > 0$ and $\varepsilon_0 > 0$, small enough, such that $(u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})$ converge in $C^0(\Omega_T^+) \times C^1(0,T]$ uniformly in $\varepsilon \in (0,\varepsilon_0]$, and their limits $(u_\pm^\varepsilon, \phi^\varepsilon)$ belong to $C^0(\Omega_T^+) \times C^1[0,T]$.

Assertion (3): On the other hand, as P. Hartman and A. Wintner in [6], and J. L. Joly et al. in [8], it can be shown that, for ε fixed, $\nabla u_\pm^{\varepsilon,\nu}$ and $d_t^2 \phi^{\varepsilon,\nu}$ are not only uniformly bounded as in (3.2.11), but also equicontinuous, which implies that the convergences $u_\pm^{\varepsilon,\nu} \to u_\pm^\varepsilon$ and $\phi^{\varepsilon,\nu} \to \phi^\varepsilon$ also hold in C^1 and C^2, respectively. Hence, we conclude that $(u_\pm^\varepsilon, \phi^\varepsilon) \in C^1(\Omega_T^+) \times C^2[0,T]$ are solutions to the problem (3.1.4) ∎

3.3 Study of linear problems. This subsection is to study the linear problem (3.2.3). Most of this part extends the investigation of P. Hartman and A. Wintner in [6], and of J. L. Joly, G. Métivier and J. Rauch in [8: Subsection 6.2] for Cauchy problems of semilinear and quasilinear systems in one space variable to the case of boundary value problems.

At first, let us diagonalize the problem (3.2.3). As in the proof of Lemma 3.1, set

$$T_\pm^\varepsilon = \left(r_1(u_\pm + \varepsilon u_\pm^\varepsilon), \dots, r_m(u_\pm + \varepsilon u_\pm^\varepsilon) \right) \tag{3.3.1}$$

and

$$(T_\pm^\varepsilon)^{-1} = \left(l_1(u_\pm + \varepsilon u_\pm^\varepsilon), \dots, l_m(u_\pm + \varepsilon u_\pm^\varepsilon) \right)^T, \tag{3.3.2}$$

with $\{r_i, l_i\}_{i=1}^m$ given in (2.1.3). By the normalization (2.1.4), it is obvious that $(T_\pm^\varepsilon)^{-1}$ is the inverse matrix of $T_\pm^\varepsilon$. Set

$$v_\pm^\varepsilon = T_\pm^\varepsilon \tilde{v}_\pm^\varepsilon \tag{3.3.3}$$

and

$$\widetilde{L}_\pm^\varepsilon(u_\pm^\varepsilon, \phi^\varepsilon) = \partial_t \pm \left(\Lambda(u_\pm + \varepsilon u_\pm^\varepsilon) - (\sigma + \varepsilon d_t \phi^\varepsilon)I \right)\partial_x, \tag{3.3.4}$$

with

$$\Lambda(u) = \mathrm{diag}\left[\lambda_1(u), \dots, \lambda_m(u)\right] \tag{3.3.5}$$

being a diagonal matrix with eigenvalues as its entries. By making use of the fact

$$(\partial T_\pm^\varepsilon) \cdot (T_\pm^\varepsilon)^{-1} = -T_\pm^\varepsilon \cdot \partial (T_\pm^\varepsilon)^{-1},$$

it is easy to see that the problem (3.2.3) is equivalent to

$$\left. \begin{array}{c} \widetilde{L}_\pm^\varepsilon(u_\pm^\varepsilon, \phi^\varepsilon)\tilde{v}_\pm^\varepsilon = (T_\pm^\varepsilon)^{-1} f_\pm^\varepsilon + \left(\widetilde{L}_\pm^\varepsilon(u_\pm^\varepsilon, \phi^\varepsilon)(T_\pm^\varepsilon)^{-1} \right) T_\pm^\varepsilon \tilde{v}_\pm^\varepsilon \\[4pt] \widetilde{G}_{(u_+^\varepsilon, u_-^\varepsilon, d_t\phi^\varepsilon)}^{\varepsilon\prime}(\tilde{v}_+^\varepsilon, \tilde{v}_-^\varepsilon, d_t\varphi^\varepsilon) = g^\varepsilon(t) \\[4pt] \varphi^\varepsilon(0) = 0 \\[4pt] \tilde{v}_\pm^\varepsilon(0,x) = \tilde{v}_{\pm,0}^\varepsilon(x) = (T_\pm^\varepsilon)^{-1} u_{\pm,0}^\varepsilon(x) \end{array} \right\} \tag{3.3.6}$$

with

$$
\widetilde{G}^{\varepsilon\prime}_{(u^\varepsilon_+,u^\varepsilon_-,d_t\phi^\varepsilon)}(\tilde{v}^\varepsilon_+,\tilde{v}^\varepsilon_-,d_t\varphi^\varepsilon)
$$

$$
= \sum_{i=1}^m \Big(\sigma + \varepsilon d_t\phi^\varepsilon - \lambda_i(u_+ + \varepsilon u^\varepsilon_+) \Big) r_i(u_+ + \varepsilon u^\varepsilon_+)\tilde{v}^\varepsilon_{+,i} \tag{3.3.7}
$$

$$
+ \sum_{i=1}^m \Big(\lambda_i(u_- + \varepsilon u^\varepsilon_-) - \sigma - \varepsilon d_t\phi^\varepsilon \Big) r_i(u_- + \varepsilon u^\varepsilon_-)\tilde{v}^\varepsilon_{-,i} + ([u] + \varepsilon[u^\varepsilon])d_t\varphi^\varepsilon.
$$

To study the problem (3.3.6), let us first consider the diagonal problem

$$
\left.\begin{array}{r}
\widetilde{L}^\varepsilon_\pm(u^\varepsilon_\pm,\phi^\varepsilon)v^\varepsilon_\pm = f^\varepsilon_\pm \\[2mm]
\widetilde{G}^{\varepsilon\prime}_{(u^\varepsilon_+,u^\varepsilon_-,d_t\phi^\varepsilon)}(v^\varepsilon_+,v^\varepsilon_-,d_t\varphi^\varepsilon) = g^\varepsilon(t) \\[2mm]
\varphi^\varepsilon(0) = 0 \\[2mm]
v^\varepsilon_\pm(0,x) = v^\varepsilon_{\pm,0}(x)
\end{array}\right\} \tag{3.3.8}
$$

where $f^\varepsilon_\pm \in C^1(\Omega^+_{T_0})$ and $g^\varepsilon \in C^1[0,T_0]$ satisfy the compatibility conditions of (3.3.8) up to order one. We decompose $v^\varepsilon_\pm$ into

$$
v^\varepsilon_{+,I} = (v^\varepsilon_{+,1},\ldots,v^\varepsilon_{+,j})^T \quad \text{and} \quad v^\varepsilon_{+,II} = (v^\varepsilon_{+,j+1},\ldots,v^\varepsilon_{+,m})^T \tag{3.3.9}
$$

$$
v^\varepsilon_{-,I} = (v^\varepsilon_{-,j},\ldots,v^\varepsilon_{-,m})^T \quad \text{and} \quad v^\varepsilon_{-,II} = (v^\varepsilon_{-,1},\ldots,v^\varepsilon_{-,j-1})^T. \tag{3.3.10}
$$

The same decompositions of $(f^\varepsilon_\pm, v^\varepsilon_{\pm,0})$ as above are also denoted by $f^\varepsilon_{\pm,I}$, $f^\varepsilon_{\pm,II}$ and $v^{\varepsilon,I}_{\pm,0}$, $v^{\varepsilon,II}_{\pm,0}$. From the Lax entropy condition (2.1.7), we know that given $u^\varepsilon_\pm \in C^1_\varepsilon(\Omega^+_{T_0})$ and $\phi^\varepsilon \in \widetilde{C}^2_\varepsilon[0,T_0]$ as in (3.2.3), when $\eta > 0$ is small enough such that

$$
\|\varepsilon u^\varepsilon_\pm\|_{L^\infty(\Omega^+_{T_0})} < \eta \quad \text{and} \quad \|\varepsilon d_t\phi^\varepsilon\|_{L^\infty[0,T_0]} \leq \eta \tag{3.3.11}
$$

holds for $\varepsilon \in (0,\varepsilon_0]$, then

$$
\lambda_i(u_+ + \varepsilon u^\varepsilon_+) - \sigma - \varepsilon d_t\phi^\varepsilon \begin{cases} < 0 & \text{when } i \in \{1,\ldots,j\} \\ > 0 & \text{when } i \in \{j+1,\ldots,m\} \end{cases}
$$

$$
\tag{3.3.12}
$$

$$
\lambda_i(u_- + \varepsilon u^\varepsilon_-) - \sigma - \varepsilon d_t\phi^\varepsilon \begin{cases} < 0, & \text{when } i \in \{1,\ldots,j-1\} \\ > 0 & \text{when } i \in \{j,\ldots,m\} \end{cases}
$$

which implies that (3.3.8) is an initial value problem for the components $V^\varepsilon_{\pm,I}$, and a mixed one for the components $V^\varepsilon_{\pm II}$. Therefore, by using the method of J. L. Joly et al. [8: Lemmas 6.2.1 and 6.2.2], we immediately obtain the following

Lemma 3.4.

(1) *For any bounded* $u_\pm^\epsilon \in C_\epsilon^1(\Omega_{T_0}^+)$ *and* $\phi^\epsilon \in \widetilde{C}_\epsilon^2[0, T_0]$ *satisfying* (3.3.11), $f_{\pm,I}^\epsilon \in C^0(\Omega_{T_0}^+)$ *and* $v_{\pm,0}^\epsilon \in C^0(\omega^+)$, *there are unique weak solutions* $v_{\pm,I}^\epsilon \in C^0(\Omega_{T_0}^+)$ *to the I-part of the problem* (3.3.8). *Moreover, for any* $t \in (0, T_0]$,

$$\|v_{\pm,I}^\epsilon(t)\| \leq \|v_{\pm,0}^{\epsilon,I}\| + \int_0^t \|f_{\pm,I}^\epsilon(s)\| ds. \tag{3.3.13}$$

(2) *There is a constant* $C > 0$ *such that if* $M \geq 1 + \epsilon\|\nabla u_\pm^\epsilon\|_{T_0}$, *then*

$$\omega(\delta, t; v_{\pm,I}^\epsilon) \leq C e^{CMt} \omega(\delta; v_{\pm,0}^{\epsilon,I}) + \delta\|f_{\pm,I}^\epsilon\|_t + \int_0^t C e^{CM(t-s)} \omega(\delta, s; f_{\pm,I}^\epsilon) ds \tag{3.3.14}$$

where

$$\omega(\delta, t; u) = \sup |u(s, x) - u(s', x')| \tag{3.3.15}$$

denotes the modulus of continuity of u *with supremum taken over* (s, x) *and* (s', x') *in* Ω_t^+ *such that* $|(s, x) - (s', x')| \leq \delta$.

For the *II*-part of the problem (3.3.8)

$$\left. \begin{array}{r} \partial_t v_{\pm,II}^\epsilon + \Theta_{\pm,II}(\epsilon u_\pm^\epsilon, \epsilon d_t \phi^\epsilon)\partial_x v_{\pm,II}^\epsilon = f_{\pm,II}^\epsilon \\[2mm] M^\epsilon \cdot (d_t \varphi^\epsilon, v_{+,II}^\epsilon, v_{-,II}^\epsilon)^T = g^\epsilon(t) + B^\epsilon(v_{+,I}^\epsilon, v_{-,I}^\epsilon) \\[2mm] \varphi^\epsilon(0) = 0 \\[2mm] v_{\pm,II}^\epsilon(0, x) = v_{\pm,0}^{\epsilon,II}(x) \end{array} \right\} \tag{3.3.16}$$

where $v_{\pm,I}^\epsilon \in C^0(\Omega^+)$ are given by Lemma 3.4,

$$\Theta_{\pm,II}(\epsilon u_\pm^\epsilon, \epsilon d_t \phi^\epsilon) = \pm\left(\Lambda_{\pm,II}(u_\pm + \epsilon u_\pm^\epsilon) - (\sigma + \epsilon d_t \phi^\epsilon)I\right)$$

are diagonal matrices with positive entries,

$$\Lambda_{+,II}(u_+ + \epsilon u_+^\epsilon) = \text{diag}\left[\lambda_{j+1}(u_+ + \epsilon u_+^\epsilon), \ldots, \lambda_m(u_+ + \epsilon u_+^\epsilon)\right]$$

$$\Lambda_{-,II}(u_- + \epsilon u_-^\epsilon) = \text{diag}\left[\lambda_1(u_- + \epsilon u_-^\epsilon), \ldots, \lambda_{j-1}(u_- + \epsilon u_-^\epsilon)\right],$$

$$B^\epsilon(v_{+,I}^\epsilon, v_{-,I}^\epsilon) = \sum_{i=1}^{j}\left(\lambda_i(u_+ + \epsilon u_+^\epsilon) - \sigma - \epsilon d_t \phi^\epsilon\right)r_i(u_+ + \epsilon u_+^\epsilon)v_{+,i}^\epsilon$$

$$+ \sum_{i=j}^{m}\left(\sigma + \epsilon d_t \phi^\epsilon - \lambda_i(u_- + \epsilon u_-^\epsilon)\right)r_i(u_- + \epsilon u_-^\epsilon)v_{-,i}^\epsilon$$

and the matrix

$$M^\epsilon = \left([u] + \epsilon[u^\epsilon], \left((\sigma + \epsilon d_t \phi^\epsilon - \lambda_i(u_+ + \epsilon u_+^\epsilon))r_i(u_+ + \epsilon u_+^\epsilon) \right)_{i=j+1,\ldots,m}, \right.$$

$$\left. \left((\lambda_i(u_- + \epsilon u_-^\epsilon) - \sigma - \epsilon d_t \phi^\epsilon)r_i(u_- + \epsilon u_-^\epsilon) \right)_{i=1,\ldots,j-1} \right)^T$$

is invertible from the stability condition (2.1.10) when (3.3.11) holds with $\eta > 0$ small enough.

Without loss of generality, let us investigate the component $v_{-,1}^\epsilon$ in the problem (3.3.16). Obviously, we know that $v_{-,1}^\epsilon$ satisfies

$$\left. \begin{aligned} \partial_t v_{-,1}^\epsilon - \left(\lambda_1(u_- + \epsilon u_-^\epsilon) - \sigma - \epsilon d_t \phi^\epsilon \right) \partial_x v_{-,1}^\epsilon &= f_{-,1}^\epsilon \\ v_{-,1}^\epsilon(t,0) &= a_{-,1}^\epsilon(t) \\ v_{-,1}^\epsilon(0,x) &= v_{-,0}^{\epsilon,1}(x) \end{aligned} \right\} \quad (3.3.17)$$

where $a_{-,1}^\epsilon(t)$ is the $(m-j+2)$-th component of the vector $(M^\epsilon)^{-1} \left(g^\epsilon(t) + B^\epsilon(v_{+,I}^\epsilon, v_{-,I}^\epsilon) \right)$, and the compatibility conditions of (3.3.17) up to order one are valid.

For the problem (3.3.17), similar to J. L. Joly et al. in [8], by integrating along characteristic curves, we obtain the following

Lemma 3.5.

(1) *For any* $v_{-,0}^{\epsilon,1} \in C^0(\omega^+)$ *and* $f_{-,1}^\epsilon \in C^0(\Omega_{T_0}^+)$, *there is a unique weak solution* $v_{-,1}^\epsilon \in C^0(\Omega_{T_0}^+)$ *to the problem* (3.3.17). *Moreover, for any* $t \in (0, T_0]$,

$$\|v_{-,1}^\epsilon(t)\| \leq \|a_{-,1}^\epsilon\|_t + \|v_{-,0}^{\epsilon,1}\| + \int_0^t \|f_{-,1}^\epsilon(s)\| \, ds. \quad (3.3.18)$$

(2) *If* $M \geq 1 + \epsilon \|\nabla u_-^\epsilon\|_{T_0} + \epsilon \|d_t^2 \phi^\epsilon\|_{T_0}$, *then*

$$\omega(\delta, t; v_{-,1}^\epsilon) \leq Ce^{CMt} \left(\omega(\delta, t; a_{-,1}^\epsilon) + \omega(\delta, v_{-,0}^{\epsilon,1}) \right.$$

$$\left. + \delta \|f_{-,1}^\epsilon\|_t + \int_0^t \omega(\delta, s; f_{-,1}^\epsilon) \, ds \right). \quad (3.3.19)$$

Proof. Let $s \to (s, \gamma_\epsilon(s;t,x))$ be the characteristic curve of (3.3.17) through (t,x) with $\gamma_\epsilon(s;t,x)$ being the solution of the problem

$$\left. \begin{aligned} d_s \gamma_\epsilon(s;t,x) &= \sigma + \epsilon d_t \phi^\epsilon(s) - \lambda_1 \left(u_- + \epsilon u_-^\epsilon \left(s, \gamma_\epsilon(s;t,x)\right) \right) \\ \gamma_\epsilon(t;t,x) &= x. \end{aligned} \right\} \quad (3.3.20)$$

Let $s_\epsilon(t, x)$ be the root of

$$\gamma_\epsilon(s_\epsilon(t, x); t, x) = 0 \tag{3.3.21}$$

and

$$\widetilde{\Omega}^+_{T_0} = \Big\{ (s, t, x) \Big| \, \max(0, s_\epsilon(t, x)) \leq s \leq x \, \text{ for } \, (t, x) \in \Omega^+_{T_0} \Big\}.$$

From the theory of ordinary differential equations, we have $\gamma_\epsilon(s; t, x) \in C^1(\widetilde{\Omega}^+_{T_0})$. For $(t, x) \in \Omega^+_{T_0}$, we have two cases:

Case (1): $s_\epsilon(t, x) < 0$. In this case, (3.3.17) is a Cauchy problem for $v^\epsilon_{-,1}(t, x)$, and for its solution we have the explicit formula

$$v^\epsilon_{-,1}(t, x) = v^{\epsilon,1}_{-,0}(\gamma_\epsilon(0; t, x)) + \int_0^t f^\epsilon_{-,1}(s, \gamma_\epsilon(s; t, x)) ds. \tag{3.3.22}$$

Case (2): $s_\epsilon(t, x) \geq 0$. The solution of problem (3.3.17) can be expressed as

$$v^\epsilon_{-,1}(t, x) = a^\epsilon_{-,1}(s_\epsilon(t, x)) + \int_{s_\epsilon(t,x)}^t f^\epsilon_{-,1}(s, \gamma_\epsilon(s; t, x)) ds. \tag{3.3.23}$$

From (3.3.22) and (3.3.23), we immediately deduce the estimate (3.3.18).

Next, we consider the estimate (3.3.19). For any $\delta > 0$, $t \in (0, T_0]$ and $(t_i, x_i) \in \Omega^+_t$ $(i = 1, 2)$ with $|(t_1, x_1) - (t_2, x_2)| < \delta$, we divide the estimate of $v^\epsilon_{-,1}(t_1, x_1) - v^\epsilon_{-,1}(t_2, x_2)$ into three cases:

Case (α): $s_\epsilon(t_i, x_i) < 0$ $(i = 1, 2)$. As above, (3.3.17) is a Cauchy problem for $v^\epsilon_{-,1}(t_i, x_i)$. By using a result of J. L. Joly et al. in [8: Lemma 6.2.2], we obtain

$$\big| v^\epsilon_{-,1}(t_1, x_1) - v^\epsilon_{-,1}(t_2, x_2) \big|$$

$$\leq C e^{CMt} \omega(\delta, v^{\epsilon,1}_{-,0}) + \delta \| f^\epsilon_{-,1} \|_t + \int_0^t C e^{CM(t-s)} \omega(\delta, s; f^\epsilon_{-,1}) ds. \tag{3.3.24}$$

Case (β): $s_\epsilon(t_i, x_i) \geq 0$ $(i = 1, 2)$. From (3.3.23), we have

$$v^\epsilon_{-,1}(t_1, x_1) - v^\epsilon_{-,1}(t_2, x_2)$$

$$= a^\epsilon_{-,1}(s_\epsilon(t_1, x_1)) - a^\epsilon_{-,1}(s_\epsilon(t_2, x_2))$$

$$+ \int_{s_\epsilon(t_1,x_1)}^{t_1} f^\epsilon_{-,1}(s, \gamma_\epsilon(s; t_1, x_1)) ds - \int_{s_\epsilon(t_2,x_2)}^{t_2} f^\epsilon_{-,1}(s, \gamma_\epsilon(s; t_2, x_2)) ds. \tag{3.3.25}$$

The definition (3.3.21) of $s_\epsilon(t, x)$ implies

$$\partial s_\epsilon(t, x) = -\Big(d_s \gamma_\epsilon(s_\epsilon(t, x), t, x) \Big)^{-1} (\partial \gamma_\epsilon)(s_\epsilon(t, x); t, x)$$

$$= \Big(\lambda_1 \big(u_- + \epsilon u^\epsilon_-(s_\epsilon, \gamma_\epsilon(s_\epsilon)) \big) - \sigma - \epsilon d_t \phi^\epsilon(s_\epsilon) \Big)^{-1} (\partial \gamma_\epsilon)(s_\epsilon(t, x); t, x) \tag{3.3.26}$$

for $\partial = \partial_t$ or $\partial = \partial_x$. Applying the estimate of γ_ϵ (see [8: Formula (6.2.8)])

$$\left|\partial_{(t,x)}\gamma_\epsilon(s;t,x)\right| \le C e^{CM(t-s)} \tag{3.3.27}$$

in (3.3.26), it follows

$$\left|\partial_{(t,x)}s_\epsilon(t,x)\right| \le C e^{CMt} \qquad \text{when } s_\epsilon(t,x) \ge 0 \tag{3.3.28}$$

with another constant $C > 0$. Employing (3.3.28) for (3.3.25), it is easy to obtain

$$\left|v_{-,1}^\epsilon(t_1,x_1) - v_{-,1}^\epsilon(t_2,x_2)\right|$$
$$\le C e^{CMt}\left(\omega(\delta,t;a_{-,1}^\epsilon) + \delta\|f_{-,1}^\epsilon\|_t + \int_0^t \omega(\delta,s;f_{-,1}^\epsilon)\,ds\right). \tag{3.3.29}$$

Case (γ): $s_\epsilon(t_1,x_1) \ge 0$ and $s_\epsilon(t_2,x_2) < 0$. From (3.3.22) and (3.3.23), we obtain

$$v_{-,1}^\epsilon(t_1,x_1) - v_{-,1}^\epsilon(t_2,x_2)$$
$$= a_{-,1}^\epsilon\big(s_\epsilon(t_1,x_1)\big) - v_{-,0}^{\epsilon,1}\big(\gamma_\epsilon(0;t_2,x_2)\big)$$
$$+ \int_{s_\epsilon(t_1,x_1)}^{t_1} f_{-,1}^\epsilon\big(s,\gamma_\epsilon(s;t_1,x_1)\big)ds - \int_0^{t_2} f_{-,1}^\epsilon\big(s,\gamma_\epsilon(s;t_2,x_2)\big)ds. \tag{3.3.30}$$

Obviously, when $(t_i,x_i) \in \Omega_t^+$ and $|(t_1,x_1) - (t_2,x_2)| < \delta$, using (3.3.27) and (3.3.28) we have

$$0 \le s_\epsilon(t_1,x_1) \le s_\epsilon(t_1,x_1) - s_\epsilon(t_2,x_2) \le C e^{CMt}\delta, \tag{3.3.31}$$

and

$$0 \le \gamma_\epsilon(0;t_2,x_2) \le \gamma_\epsilon(0;t_2,x_2) - \gamma_\epsilon(0;t_1,x_1) \le C e^{CMt}\delta. \tag{3.3.32}$$

Applying (3.3.31) and (3.3.32) in (3.3.30), and using the compatibility condition $a_{-,1}^\epsilon(0) = v_{-,0}^{\epsilon,1}(0)$ of (3.3.13), it follows

$$\left|v_{-,1}^\epsilon(t_1,x_1) - v_{-,1}^\epsilon(t_2,x_2)\right|$$
$$\le C e^{CMt}\left(\omega(\delta,t;a_{-,1}^\epsilon) + \omega(\delta,v_{-,0}^{\epsilon,1}) + \delta\|f_{-,1}^\epsilon\|_t + \int_0^t \omega(\delta,s;f_{-,1}^\epsilon)\,ds\right). \tag{3.3.33}$$

Summing the above three cases up, it concludes the result (3.3.19) ∎

From Lemmas 3.4 with 3.5 together it follows

Lemma 3.6.

(1) *For any $f_\pm^\epsilon \in C^0(\Omega_{T_0}^+)$ and $v_{\pm,0}^\epsilon \in C^0(\omega^+)$ satisfying the zero-th order compatibility condition of (3.3.8), there are unique weak solutions $(v_\pm^\epsilon, \varphi^\epsilon) \in C^0(\Omega_{T_0}^+) \times C^1[0,T_0]$ to the problem (3.3.8). Moreover, there is a constant $C > 0$ such that, for any $t \in (0,T_0]$,*

$$|d_t\varphi^\epsilon(t)| + \|v_\pm^\epsilon(t)\| \le C\left(\|g^\epsilon\|_t + \|v_{\pm,0}^\epsilon\| + \int_0^t \|f_\pm^\epsilon(s)\|\,ds\right). \tag{3.3.34}$$

(2) *If $M \geq 1 + \varepsilon\big(\|\nabla u^\varepsilon_+\|_{T_0} + \|\nabla u^\varepsilon_-\|_{T_0} + \|d^2_t \phi^\varepsilon\|_{T_0}\big)$, then*

$$\omega(\delta, t; d_t\varphi^\varepsilon) + \omega(\delta, t; v^\varepsilon_\pm)$$

$$\leq C e^{CMt}\left\{ \omega(\delta, t; g^\varepsilon) + \omega(\delta, v^\varepsilon_{\pm,0}) + \delta\|f^\varepsilon_\pm\|_t + \int_0^t \omega(\delta, s; f^\varepsilon_\pm)\,ds \right\}. \qquad (3.3.35)$$

It is clear that if $f^\varepsilon_\pm$ and g^ε are in C^1 and satisfy the compatibility conditions of the problem (3.3.8) up to order to one, then the solutions $(v^\varepsilon_\pm, \varphi^\varepsilon)$ of that problem belong to $C^1(\Omega^+_{T_0}) \times C^2[0, T_0]$. But, the important remark in P. Hartman and A. Wintner [6], and J. L. Joly et al. [8: Lemma 6.2.3] for Cauchy problems can be generalized to the case of initial-boundary value problems as follows:

Lemma 3.7. *With the same conditions as above, if $g^\varepsilon(t) \in C^1[0, T_0]$ and $f^\varepsilon_\pm \in C^0(\Omega^+_{T_0})$ have the form*

$$f^\varepsilon_{\pm,i} = \rho^\varepsilon_{\pm,i}\Big(\partial_t\sigma^\varepsilon_{\pm,i} \pm \big(\lambda_i(u_\pm + \varepsilon u^\varepsilon_\pm) - \sigma - \varepsilon d_t\phi^\varepsilon\big)\partial_x\sigma^\varepsilon_{\pm,i}\Big), \qquad (3.3.36)$$

with $(\rho^\varepsilon_{\pm,i}, \sigma^\varepsilon_{\pm,i}) \in C^1$ for any $i \in \{1, \ldots, m\}$, then the solutions $(v^\varepsilon_\pm, \varphi^\varepsilon)$ of the problem (3.3.8) belong to $C^1(\Omega^+_{T_0}) \times C^2[0, T_0]$.

This result can be proved in a way similar to [8: Lemma 6.2.3]. Moreover, as J. L. Joly et al. shown, estimates of $(\nabla v^\varepsilon_\pm, d^2_t\varphi)$ in L^∞ and of the modulus of continuity of $(\nabla v^\varepsilon_\pm, d^2_t\varphi)$ in $C^0(\Omega^+_{T_0})$ can be obtained as in P. Hartman and A.Wintner [6].

Proof of Proposition 3.2. Assertion (1): Let us turn to the study of the problem (3.2.3). As in the discussion at the beginning of this subsection, it is sufficient to consider the diagonal problem (3.3.6), which is solved by the iteration scheme

$$\left.\begin{aligned}
\widetilde{L}^\varepsilon_\pm(u^\varepsilon_\pm, \phi^\varepsilon)\tilde{v}^{\varepsilon,\nu+1}_\pm &= (T^\varepsilon_\pm)^{-1}f^\varepsilon_\pm \\
&\quad + \big(\widetilde{L}^\varepsilon_\pm(u^\varepsilon_\pm, \phi^\varepsilon)(T^\varepsilon_\pm)^{-1}\big)T^\varepsilon_\pm\tilde{v}^{\varepsilon,\nu}_\pm \\
\widetilde{G}^{\varepsilon\prime}_{(u^\varepsilon_+, u^\varepsilon_-, d_t\phi^\varepsilon)}\big(\tilde{v}^{\varepsilon,\nu+1}_+, \tilde{v}^{\varepsilon,\nu+1}_-, d_t\varphi^{\varepsilon,\nu+1}\big) &= g^\varepsilon(t) \\
\varphi^{\varepsilon,\nu+1}(0) &= 0 \\
\tilde{v}^{\varepsilon,\nu+1}_\pm(0, x) &= \tilde{v}^\varepsilon_{\pm,0}(x)
\end{aligned}\right\} \qquad (3.3.37)$$

with the first approximate solution $(\varphi^{\varepsilon,0}, \tilde{v}^{\varepsilon,0}_\pm) \in \widetilde{C}^2_\varepsilon[0, T] \times C^1_\varepsilon(\Omega^+_T)$ constructed in a way similar to Proposition 3.1. Under the assumption of Proposition 3.2/(1), by employing the estimate (3.3.34) for the problem (3.3.37), there is a constant $C > 0$ such that, for any $\nu \geq 0$ and $t \in (0, T_0]$,

$$|d_t\varphi^{\varepsilon,\nu+1}(t)| + \|\tilde{v}^{\varepsilon,\nu+1}_\pm(t)\|$$

$$\leq C\left(\|g^\varepsilon\|_t + \|\tilde{v}^\varepsilon_{\pm,0}\| + \int_0^t \big(\|f^\varepsilon_\pm(s)\| + M\|\tilde{v}^{\varepsilon,\nu}_\pm(s)\|\big)\,ds\right) \qquad (3.3.38)$$

with $M \geq 1 + \varepsilon(\|\nabla u_+^\varepsilon\|_{T_0} + \|\nabla u_-^\varepsilon\|_{T_0} + \|d_t^2 \phi^\varepsilon\|_{T_0})$. By induction on ν for (3.3.38), it follows that $\varphi^{\varepsilon,\nu} \in C^1[0, T_0]$ and $\tilde{v}_\pm^{\varepsilon,\nu} \in C^0(\Omega_{T_0}^+)$ are bounded with the estimate

$$|d_t \varphi^{\varepsilon,\nu}(t)| + \|\tilde{v}_\pm^{\varepsilon,\nu}(t)\| \leq Ce^{CMt}\left(\|g^\varepsilon\|_t + \|\tilde{v}_{\pm,0}^\varepsilon\| + \int_0^t \|f_\pm^\varepsilon(s)\| ds\right) \qquad (3.3.39)$$

for any $\nu \geq 0$ and any $t \in (0, T_0]$. Moreover, by employing (3.3.34) for the problem of $(\tilde{v}_\pm^{\varepsilon,\nu+1} - \tilde{v}_\pm^{\varepsilon,\nu}, \varphi^{\varepsilon,\nu+1} - \varphi^{\varepsilon,\nu})$

$$\left.\begin{array}{l}
\tilde{L}_\pm^\varepsilon(u_\pm^\varepsilon, \phi^\varepsilon)(\tilde{v}_\pm^{\varepsilon,\nu+1} - \tilde{v}_\pm^{\varepsilon,\nu}) = \left(\tilde{L}_\pm^\varepsilon(u_\pm^\varepsilon, \phi^\varepsilon)(T_\pm^\varepsilon)^{-1}\right)T_\pm^\varepsilon(\tilde{v}_\pm^{\varepsilon,\nu} - \tilde{v}_\pm^{\varepsilon,\nu-1}) \\[2mm]
\tilde{G}_{(u_+^\varepsilon, u_-^\varepsilon, d_t \phi^\varepsilon)}^{\varepsilon\prime}\left(\tilde{v}_+^{\varepsilon,\nu+1} - \tilde{v}_+^{\varepsilon,\nu}, \tilde{v}_-^{\varepsilon,\nu+1} - \tilde{v}_-^{\varepsilon,\nu}, d_t(\varphi^{\varepsilon,\nu+1} - \varphi^{\varepsilon,\nu})\right) = 0 \\[2mm]
\varphi^{\varepsilon,\nu+1} - \varphi^{\varepsilon,\nu} = \tilde{v}_\pm^{\varepsilon,\nu+1} - \tilde{v}_\pm^{\varepsilon,\nu} = 0 \quad \text{on } t = 0
\end{array}\right\} \qquad (3.3.40)$$

we obtain

$$\left|d_t(\varphi^{\varepsilon,\nu+1} - \varphi^{\varepsilon,\nu})(t)\right| + \left\|(\tilde{v}_\pm^{\varepsilon,\nu+1} - \tilde{v}_\pm^{\varepsilon,\nu})(t)\right\|$$
$$\leq CM \int_0^t \left(\left\|(\tilde{v}_+^{\varepsilon,\nu} - \tilde{v}_+^{\varepsilon,\nu-1})(s)\right\| + \left\|(\tilde{v}_-^{\varepsilon,\nu} - \tilde{v}_-^{\varepsilon,\nu-1})(s)\right\|\right)ds$$

which implies

$$\left|d_t(\varphi^{\varepsilon,\nu+1} - \varphi^{\varepsilon,\nu})(t)\right| + \left\|(\tilde{v}_\pm^{\varepsilon,\nu+1} - \tilde{v}_\pm^{\varepsilon,\nu})(t)\right\| \leq \frac{(CMt)^\nu}{\nu!}\left(\|\tilde{v}_+^{\varepsilon,0}\|_{T_0} + \|\tilde{v}_-^{\varepsilon,0}\|_{T_0}\right) \quad (3.3.41)$$

by induction on ν. From (3.3.39) and (3.3.41), and the uniqueness of weak solutions $(\tilde{v}_\pm^\varepsilon, d_t \varphi^\varepsilon)$ in L^∞ for the problem (3.3.6), which is a simple consequence from the estimate (3.3.34) in (3.3.6) with $(f_\pm^\varepsilon, g^\varepsilon, u_{\pm,0}^\varepsilon) = 0$, we immediately deduce that the sequences $\tilde{v}_\pm^{\varepsilon,\nu}$ and $\varphi^{\varepsilon,\nu}$ converge in $C^0(\Omega_{T_0}^+)$ and $C^1[0, T_0]$ uniformly in $\varepsilon \in (0, \varepsilon_0]$, the limits $\tilde{v}_\pm^\varepsilon \in C^0(\Omega_{T_0}^+)$ and $\varphi^\varepsilon \in C^1[0, T_0]$ are the unique weak solutions of the problem (3.3.6), and they satisfy the estimate

$$|d_t \varphi^\varepsilon(t)| + \|\tilde{v}_\pm^\varepsilon(t)\| \leq Ce^{CMt}\left(\|g^\varepsilon\|_t + \|\tilde{v}_{\pm,0}^\varepsilon\| + \int_0^t \|f_\pm^\varepsilon(s)\| ds\right)$$

for any $t \in (0, T_0]$.

Assertion (2): Suppose the assumptions of Proposition 3.1/(2) are valid. The right-hand side of the iteration equation (3.3.37) is of the form (3.3.36). By applying Lemma 3.7 in (3.3.37) we conclude that $(\tilde{v}_\pm^{\varepsilon,\nu}, \varphi^{\varepsilon,\nu}) \in C^1(\Omega_{T_0}^+) \times C^2[0, T_0]$ are bounded sequences for fixed ε. As J. L. Joly et al. in [8], a lemma resembling Lemma 3.6/(2) applied to $\nabla \tilde{v}_\pm^{\varepsilon,\nu}$ and $d_t^2 \varphi^{\varepsilon,\nu}$ shows that, for ε fixed, the families $\{\nabla \tilde{v}_\pm^{\varepsilon,\nu}, d_t^2 \varphi^{\varepsilon,\nu}\}_{\nu \in \mathbb{N}}$ are equicontinuous. Therefore $\tilde{v}_\pm^{\varepsilon,\nu} \to \tilde{v}_\pm^\varepsilon$ in $C^1(\Omega_{T_0}^+)$ and $\varphi^{\varepsilon,\nu} \to \varphi^\varepsilon$ in $C^2[0, T_0]$. Going back to

the problem (3.2.3), it follows that, for each fixed $\varepsilon \in (0, \varepsilon_0]$, the solutions $(v_\pm^\varepsilon, \varphi^\varepsilon)$ of the problem (3.2.3) belong to $C^1(\Omega_{T_0}^+) \times C^2[0, T_0]$.

To estimate $\nabla v_\pm^\varepsilon$ and $d_t^2 \varphi^\varepsilon$, by setting $z_\pm^\varepsilon = \varepsilon \partial_t v_\pm^\varepsilon$ and $\Phi^\varepsilon = \varepsilon d_t \varphi^\varepsilon$ and differentiating (3.2.3) with respect to t, we obtain that $(z_\pm^\varepsilon, \Phi^\varepsilon)$ satisfy the problem

$$
\left.
\begin{aligned}
L_\pm^\varepsilon(u_\pm^\varepsilon, \phi^\varepsilon) z_\pm^\varepsilon &= Q_\pm^\varepsilon \\
G_{(u_+^\varepsilon, u_-^\varepsilon, d_t \phi^\varepsilon)}^{\varepsilon \prime}(z_+^\varepsilon, z_-^\varepsilon, d_t \Phi^\varepsilon) &= \mathcal{G}^\varepsilon(t) \\
z_\pm^\varepsilon(0, x) &= z_{\pm,0}^\varepsilon(x)
\end{aligned}
\right\}
\tag{3.3.42}
$$

where

$$
z_{\pm,0}^\varepsilon(x) = \varepsilon f_\pm^\varepsilon(0, x) \mp \varepsilon \Big(A(u_\pm + \varepsilon u_\pm^\varepsilon) - (\sigma + \varepsilon d_t \phi^\varepsilon) I \Big)(0, x) d_x u_{\pm,0}^\varepsilon(x)
\tag{3.3.43}
$$

and

$$
\begin{aligned}
Q_\pm^\varepsilon &= \varepsilon \partial_t f_\pm^\varepsilon - \nabla A_\pm(u_\pm + \varepsilon u_\pm^\varepsilon)\Big(\varepsilon \partial_t u_\pm^\varepsilon, A_\pm^\varepsilon \cdot (\varepsilon f_\pm^\varepsilon - z_\pm^\varepsilon)\Big) \\
&\quad + \varepsilon d_t^2 \phi^\varepsilon A_\pm^\varepsilon \cdot (\varepsilon f_\pm^\varepsilon - z_\pm^\varepsilon)
\end{aligned}
\tag{3.3.44}
$$

with

$$
A_\pm^\varepsilon = \Big(A(u_\pm + \varepsilon u_\pm^\varepsilon) - (\sigma + \varepsilon d_t \phi^\varepsilon) I \Big)^{-1}
$$

and

$$
\mathcal{G}^\varepsilon(t) = \varepsilon d_t g^\varepsilon - [\varepsilon \partial_t u^\varepsilon] \Phi^\varepsilon - \varepsilon d_t^2 \phi^\varepsilon [\varepsilon v^\varepsilon] + [\nabla A(u + \varepsilon u^\varepsilon)(\varepsilon \partial_t u^\varepsilon, \varepsilon v^\varepsilon)].
\tag{3.3.45}
$$

Applying the estimate (3.2.4) in the problem (3.3.42), it follows

$$
|d_t \Phi^\varepsilon(t)| + \|z_\pm^\varepsilon(t)\| \leq C e^{CMt} \left(\|\mathcal{G}^\varepsilon\|_t + \|z_{\pm,0}^\varepsilon\| + \int_0^t \|Q_\pm^\varepsilon(s)\| \, ds \right).
\tag{3.3.46}
$$

From (3.3.44), we have

$$
\|Q_\pm^\varepsilon(s)\| \leq C \Big(\|\varepsilon \partial_t f_\pm^\varepsilon(s)\| + \varepsilon M \|f_\pm^\varepsilon(s)\| + M \|z_\pm^\varepsilon(s)\| \Big).
\tag{3.3.47}
$$

Obviously, (3.3.45) gives rise to

$$
\|\mathcal{G}^\varepsilon\|_t \leq \|\varepsilon d_t g^\varepsilon\|_t + M \Big(\|\Phi^\varepsilon\|_t + C(\|\varepsilon v_+^\varepsilon\|_t + \|\varepsilon v_-^\varepsilon\|_t) \Big)
\tag{3.3.48}
$$

which implies

$$
\|\mathcal{G}^\varepsilon\|_t \leq \|\varepsilon d_t g^\varepsilon\|_t + \varepsilon C M e^{CMt} \left(\|g^\varepsilon\|_t + \|u_{\pm,0}^\varepsilon\| + \int_0^t \|f_\pm^\varepsilon(s)\| \, ds \right)
\tag{3.3.49}
$$

by using (3.2.4). Substituting (3.3.47) and (3.3.49) into (3.3.46), and using Gronwall's inequality, we obtain

$$
\begin{aligned}
|d_t \Phi^\varepsilon(t)| + \|z_\pm^\varepsilon(t)\| \leq {}& C \exp(CMt e^{CMt}) \bigg(\|\varepsilon d_t g^\varepsilon\|_t + \|\varepsilon f_\pm^\varepsilon(0)\| + \|\varepsilon d_x u_{\pm,0}^\varepsilon\| \\
&+ \varepsilon M (\|g^\varepsilon\|_t + \|u_{\pm,0}^\varepsilon\|) + \int_0^t \Big(\|\varepsilon \partial_t f_\pm^\varepsilon(s)\| + \varepsilon M \|f_\pm^\varepsilon(s)\| \Big) ds \bigg).
\end{aligned}
$$

The estimate of $\partial_x v_\pm^\varepsilon$ can be easily obtained from the equation in (3.2.3) ∎

4. Existence of profiles

4.1 Construction of approximate solutions. Under the assumption of the compatibility conditions of the problems (2.1.25) and (2.1.30) up to order one being valid for the initial data $U_{\pm,0} \in C^1(\omega^+ : \mathbb{R})$, we immediately get the existence of $\phi \in C^2[0, T_0]$ from the problem (2.1.30).

With the functions $m_{\pm}^{0,\epsilon} \in C_\epsilon^1(\omega)$ determined in the proof of Lemma 3.1, we still denote $U_{\pm,0}$ to be the proper extension of $U_{\pm,0}$ in ω such that $U_{\pm,0} \in C^1(\omega : \mathbb{R})$ satisfy the asymptotic property

$$m_{\pm}^{0,\epsilon}(x) - U_{\pm,0}\left(x, \tfrac{x}{\epsilon}\right) = o(1) \tag{4.1.1}$$

in $C_\epsilon^1(\omega)$ when $\epsilon \to 0$. Let $U_{\pm}^0 \in C^1(\Omega^+ : \mathbb{R}^2)$ be unique solutions to the linear problem

$$\left.\begin{aligned} \mathbb{E}_{\pm} U_{\pm}^0 &= U_{\pm}^0 \\ (\partial_t \pm (A_{\pm} - \sigma I)\partial_x)U_{\pm}^0 &= 0 \\ U_{\pm}^0|_{t=\tau=0} &= U_{\pm,0}(x,\theta). \end{aligned}\right\} \tag{4.1.2}$$

Then we have the following

Proposition 4.1. *Suppose that $u_{\pm}^{\epsilon,0} \in C_\epsilon^1(\Omega^+)$ are the approximate solutions constructed in Lemma 3.1. Then the asymptotic property*

$$u_{\pm}^{\epsilon,0}(t,x) - U_{\pm}^0\left(t,x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\right) = o(1) \tag{4.1.3}$$

in $C_\epsilon^1(\Omega^+)$ is valid, when $\epsilon \to 0$.

Proof. In this proof, we will always use $o(1)$ to denote any infinity small quantity when $\epsilon \to 0$. Obviously, to prove the assertion (4.1.3) is equivalent to prove

$$v_{\pm}^{\epsilon,0}(t,x) - V_{\pm}^0\left(t,x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\right) = o(1) \tag{4.1.4}$$

in $C_\epsilon^1(\Omega^+)$ where $v_{\pm}^{\epsilon,0} = T_{\pm}^{-1} u_{\pm}^{\epsilon,0}$ are solutions to the problem (3.1.13) and $V_{\pm}^0(t,x;\tau,\theta)$ satisfy

$$\left.\begin{aligned} \widetilde{\mathbb{E}}_{\pm} V_{\pm}^0 &= V_{\pm}^0 \\ (\partial_t \pm (\Lambda_{\pm} - \sigma I)\partial_x)V_{\pm}^0 &= 0 \\ V_{\pm}^0|_{t=\tau=0} &= \widetilde{U}_{\pm,0}(x,\theta) = T_{\pm}^{-1} U_{\pm,0}(x,\theta) \end{aligned}\right\} \tag{4.1.5}$$

with the mean value operator $\widetilde{\mathbb{E}}_{\pm}$ being defined in (2.2.6) and $T_{\pm} = (r_1^{\pm}, \dots, r_m^{\pm})^T$. From the zero-th order compatibility condition (2.2.1) of (2.1.18), we have

$$d_t\phi^{\epsilon,0}(0)[u] - (A_+ - \sigma I)m_+^{0,\epsilon}(0) + (A_- - \sigma I)m_-^{0,\epsilon}(0) = o(1)$$

which is equivalent to

$$d_t\phi^{\epsilon,0}(0)[u] - (A_+ - \sigma I)T_+\widetilde{m}_+^{0,\epsilon}(0) + (A_- - \sigma I)T_-\widetilde{m}_-^{0,\epsilon}(0) = o(1) \tag{4.1.6}$$

by setting $\widetilde{m}_{\pm}^{0,\epsilon} = T_{\pm}^{-1} m_{\pm}^{0,\epsilon}$. By applying the stability condition (2.1.10) in (4.1.6), and using (4.1.1), it gives rise to

$$
\begin{aligned}
d_t \phi^{\epsilon,0}(0) &= A\Big(\widetilde{m}_{-,j}^{0,\epsilon}, \ldots, \widetilde{m}_{-,m}^{0,\epsilon}, \widetilde{m}_{+,1}^{0,\epsilon}, \ldots, \widetilde{m}_{+,j}^{0,\epsilon}\Big)(0) + o(1) \\
&= A\Big(\widetilde{U}_{-,0}^{j}, \ldots, \widetilde{U}_{-,0}^{m}, \widetilde{U}_{+,0}^{1}, \ldots, \widetilde{U}_{+,0}^{j}\Big)(0,0) + o(1)
\end{aligned}
\tag{4.1.7}
$$

where $A(\cdot)$ is a linear function with constant coefficients. From (3.1.12), we have

$$
m_{\pm}^{1,\epsilon} \pm (A_{\pm} - \sigma I) d_x m_{\pm}^{0,\epsilon} = \mp B_{\pm}(m_{\pm}^{0,\epsilon}, \epsilon d_x m_{\pm}^{0,\epsilon}) \pm \epsilon d_t \phi^{\epsilon,0}(0) d_x m_{\pm}^{0,\epsilon} + o(1)
$$

which is equivalent to

$$
\begin{aligned}
\widetilde{m}_{\pm}^{1,\epsilon} &\pm (\Lambda_{\pm} - \sigma I) d_x \widetilde{m}_{\pm}^{0,\epsilon} \\
&= \mp \widetilde{B}_{\pm}(\widetilde{m}_{\pm}^{0,\epsilon}, \epsilon d_x \widetilde{m}_{\pm}^{0,\epsilon}) \pm \epsilon d_t \phi^{\epsilon,0}(0) d_x \widetilde{m}_{\pm}^{0,\epsilon} + o(1) \\
&= \mp \Big[\widetilde{B}_{\pm}(\widetilde{U}_{\pm,0}, \partial_\theta \widetilde{U}_{\pm,0}) - A(\widetilde{U}_{-,0}^{j}, \ldots, \widetilde{U}_{+,0}^{j})(0) \partial_\theta \widetilde{U}_{\pm,0} \Big]\Big|_{\theta = \frac{x}{\epsilon}} + o(1)
\end{aligned}
\tag{4.1.8}
$$

by using (4.1.1) and (4.1.7), where the bilinear form $\widetilde{B}_{\pm}(\cdot, \cdot)$ is defined in (2.2.6). Employing the theory of classical linear geometric optics (see, e.g., J. L. Joly et al. in [8: Formulas (6.1.10) - (6.1.16)]) for the problem (3.1.13), we obtain

$$
v_{\pm}^{\epsilon,0}(t,x) - V_{\pm}^{0}\big(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big) = o(1)
\tag{4.1.9}
$$

in $L^{\infty}(\Omega^+)$ by using (4.1.8) and the obvious fact

$$
\widetilde{\mathbb{E}}^{\pm}\Big(\mp \widetilde{B}_{\pm}(\widetilde{U}_{\pm,0}, \partial_\theta \widetilde{U}_{\pm,0}) \pm A(\widetilde{U}_{-,0}^{j}, \ldots, \widetilde{U}_{+,0}^{j})(0) \partial_\theta \widetilde{U}_{\pm,0} \Big) = 0
$$

where $V_{\pm}^{0} \in \mathcal{C}^{1}(\Omega : \mathbb{R}^{2})$ is the unique solution to the problem (4.1.5).

It remains to verify that

$$
\epsilon \nabla_{(t,x)} \Big(v_{\pm}^{\epsilon,0}(t,x) - V_{\pm}^{0}\big(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big) \Big) = o(1)
\tag{4.1.10}
$$

in $L^{\infty}(\Omega^+)$. From the equality (3.1.10)

$$
m_{\pm}^{1,\epsilon}(x) = \mp \Big(A\big(u_{\pm} + \epsilon m_{\pm}^{0,\epsilon}(x)\big) - \big(\sigma + \epsilon d_t \phi^{\epsilon}(0)\big) I \Big) d_x m_{\pm}^{0,\epsilon}(x)
$$

we have

$$
\epsilon \Big(m_{\pm}^{1,\epsilon}(x) \pm (A_{\pm} - \sigma I) \partial_x \big(U_{\pm,0}\big(x, \tfrac{x}{\epsilon}\big)\big) \Big) = o(1)
$$

in $L^{\infty}(\omega)$ which implies

$$
\epsilon \widetilde{m}_{\pm}^{1,\epsilon}(x) = \mp \epsilon (\Lambda_{\pm} - \sigma I) \partial_x \big(\widetilde{U}_{\pm,0}\big(x, \tfrac{x}{\epsilon}\big)\big) + o(1)
\tag{4.1.11}
$$

in $L^\infty(\omega)$. Applying this in the expression (3.1.14)

$$v_{\pm,i}^{\varepsilon,0}(t,x) = \widetilde{m}_{\pm,i}^{0,\varepsilon}(x) \pm \frac{1}{\lambda_i^\pm - \sigma} \int\limits_{x\pm(\sigma-\lambda_i^\pm)t}^{x} \widetilde{m}_{\pm,i}^{1,\varepsilon}(\tau)\,d\tau$$

we can easily obtain

$$\varepsilon\partial_t v_{\pm,i}^{\varepsilon,0}(t,x) = \mp(\lambda_i^\pm - \sigma)(\varepsilon\partial_x)\left(\widetilde{U}_{\pm,0}^i\left(x\pm(\sigma-\lambda_i^\pm)t, \frac{x\pm(\sigma-\lambda_i^\pm)t}{\varepsilon}\right)\right) + o(1)$$

and

$$\varepsilon\partial_x v_{\pm,i}^{\varepsilon,0}(t,x) = (\varepsilon\partial_x)\left(\widetilde{U}_{\pm,0}^i\left(x\pm(\sigma-\lambda_i^\pm)t, \frac{x\pm(\sigma-\lambda_i^\pm)t}{\varepsilon}\right)\right) + o(1)$$

in $L^\infty(\Omega^+)$, which is equivalent to the assertion (4.1.10) ∎

With $V_\pm^0 \in \mathcal{C}^1(\Omega^+ : \mathbb{R}^2)$ given in (4.1.5), from the zero-th order compatibility condition (2.2.8) of the problem (2.2.6), we have

$$\chi(0,\tau) = \vec{e}_1 M^{-1}\left(\sum_{i=1}^{j}(\lambda_i^+ - \sigma)r_i^+ V_{+,i}(0,0;(\sigma-\lambda_i^+)\tau)\right.$$
$$\left. + \sum_{i=j}^{m}(\sigma-\lambda_i^-)r_i^- V_{-,i}(0,0;(\lambda_i^- - \sigma)\tau))\right) \qquad (4.1.12)$$

where

$$M = \left([u], \{(\sigma-\lambda_i^+)r_i^+\}_{i=j+1}^{m}, \{(\lambda_i^- - \sigma)r_i^-\}_{i=1}^{j-1}\right) \qquad (4.1.13)$$

is an invertible matrix from stability condition (2.1.10), and the solution $V_{\pm,k}^0(t,x;\tau,\theta)$ of the problem (4.1.5) is regarded as a function of $(t,x;\theta\mp(\lambda_k^\pm - \sigma)\tau)$, which is written as $V_{\pm,k}^0(t,x;\theta\mp(\lambda_k^\pm - \sigma)\tau)$ for each $k \in \{1,\ldots,m\}$.

According to (4.1.12), let $\chi^0(t,\tau) \in \mathcal{C}^1([0,T]:\mathbb{R})$ $((T,0)\in\Omega)$ be the function

$$\chi^0(t,\tau) = \vec{e}_1 M^{-1}\left(\sum_{i=1}^{j}(\lambda_i^+ - \sigma)r_i^+ V_{+,i}(t,0;(\sigma-\lambda_i^+)\tau)\right.$$
$$\left. + \sum_{i=j}^{m}(\sigma-\lambda_i^-)r_i^- V_{-,i}(t,0;(\lambda_i^- - \sigma)\tau)\right). \qquad (4.1.14)$$

Then, as a simply consequence of (4.1.3), we have the following result.

Corollary 4.1. *Suppose that* $(u_\pm^{\varepsilon,0}, \phi^{\varepsilon,0}) \in C_\varepsilon^1(\Omega^+) \times \widetilde{C}_\varepsilon^2[0,T]$ *are the approximate solutions given in Proposition 3.1, and* $(U_\pm^0, \chi^0) \in \mathcal{C}^1(\Omega^+ : \mathbb{R}^2) \times \mathcal{C}^1([0,T]:\mathbb{R})$ *are constructed at above. Then*

$$\left\|d_t\phi^{\varepsilon,0}(t) - \chi^0\left(t,\tfrac{t}{\varepsilon}\right)\right\|_{\varepsilon,1,[0,T]} = o(1) \qquad (4.1.15)$$

when $\varepsilon \to 0$.

Proof. From (3.1.22), we have

$$d_t \phi^{\varepsilon,0}(t) = \vec{e}_1 (M^\varepsilon(t))^{-1}$$
$$\times \left(\varepsilon g^\varepsilon + \sum_{i=1}^{j} (\lambda_i^+ - \sigma) r_i^+ v_{+,i}^{\varepsilon,0} + \sum_{i=j}^{m} (\sigma - \lambda_i^-) r_i^- v_{-,i}^{\varepsilon,0} \right)(t,0) \qquad (4.1.16)$$

where g^ε is bounded in $C_\varepsilon^1[0, T_0]$ and

$$M^\varepsilon(t) = \left([u] + \varepsilon[u^{\varepsilon,0}(t)], \{ (\sigma - \lambda_i^+) r_i^+ \}_{i=j+1}^m, \{ (\lambda_i^- - \sigma) r_i^- \}_{i=1}^{j-1} \right). \qquad (4.1.17)$$

By comparing the equalities (4.1.16) with (4.1.14), and using the simple consequence of (4.1.4)

$$v_{\pm,i}^{\varepsilon,0}(t,0) - V_{\pm,i}^0 \left(t, 0; \frac{\pm(\sigma - \lambda_i^\pm)t}{\varepsilon} \right) = o(1)$$

in $C_\varepsilon^1[0, T]$ we immediately obtain the assertion (4.1.15) ∎

4.2 Proof of Theorem 2.1/(2). In the remainder of this paper, without loss of generality, we suppose that $A_\pm$ is the diagonal matrix

$$A_\pm = \Lambda_\pm = \mathrm{diag}\left[\lambda_1^\pm, \ldots, \lambda_m^\pm \right]. \qquad (4.2.1)$$

For this diagonal case, the stability condition (2.1.10) implies that the matrix

$$\left(e_1, \ldots, e_{j-1}, [u], e_{j+1}, \ldots, e_m \right)^T \qquad (4.2.2)$$

is invertible where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ with the i-th slot element being one is the standard basis.

With $U_\pm^0 \in C^1(\Omega^+ : \mathbb{R}^2)$ and $\chi^0 \in C^1([0,T] : \mathbb{R})$ given by (4.1.2) and (4.1.14), respectively, we solve the nonlinear problem (2.1.25) by the iteration scheme

$$\left. \begin{array}{c} \mathbb{E}_\pm U_\pm^{\nu+1} = U_\pm^{\nu+1} \\[2mm] \partial_t U_\pm^{\nu+1} \pm (\Lambda_\pm - \sigma I) \partial_x U_\pm^{\nu+1} \\[2mm] \mp \mathbb{E}_\pm \left(\chi^\nu \partial_\theta U_\pm^{\nu+1} - B_\pm (\partial_\theta U_\pm^{\nu+1}, U_\pm^\nu) \right) = 0 \\[2mm] \chi^{\nu+1}[u] + (\sigma I - \Lambda_+) U_+^{\nu+1} - (\sigma I - \Lambda_-) U_-^{\nu+1} = 0 \quad \text{on } x = \theta = 0 \\[2mm] U_\pm^{\nu+1}|_{t=\tau=0} = U_{\pm,0}(x,\theta). \end{array} \right\} \qquad (4.2.3)$$

It is easy to verify that the compatibility conditions here up to order one are valid for each $\nu \geq 0$.

For any fixed $k \in \{1, \ldots, m\}$, let us define the mean value operator $\mathbb{E}_\pm^k$ in the same way as $\mathbb{E}_\pm$ by replacing $P_\pm^k(\partial_\tau, \partial_\theta) = \partial_\tau \pm (\lambda_k^\pm - \sigma) \partial_\theta$ for the operator $P_\pm(\partial_\tau, \partial_\theta)$ in (2.1.24), i.e. for any $u(t, x; \tau, \theta) \in C^0(\Omega^+ : \mathbb{R}^2)$,

$$(\mathbb{E}_\pm^k u)(t, x; \tau, \theta) = \lim_{\rho \to \infty} \frac{1}{2\rho} \int_{-\rho}^{+\rho} u\left(t, x; \tau + s, \theta \pm (\lambda_k^\pm - \sigma)s \right) ds. \qquad (4.2.4)$$

As J. L. Joly et al. in [8: Proposition 6.3.1], at first we have the following

Lemma 4.1. *For any* $(u_\pm, v_\pm) \in (\mathcal{C}^1(\Omega^+ : \mathbb{R}^2))^2$ *satisfying*

$$\mathbb{E}_\pm u_\pm = u_\pm \qquad and \qquad \mathbb{E}_\pm v_\pm = v_\pm, \tag{4.2.5}$$

if we denote by $B_\pm(u,v)$ *the bilinear form*

$$B_\pm(u,v) = \left(B_\pm^1(u,v), \ldots, B_\pm^m(u,v)\right)^T$$

with $B_\pm^k(u,v) = \sum_{i,l=1}^m b_{\pm,k}^{il} u^i v^l$, *then*

$$\mathbb{E}_\pm^k B_\pm^k(\partial_\theta u_\pm, v_\pm) = \gamma_\pm^k(v_\pm)\partial_\theta u_\pm^k + \Xi_\pm^k(u_\pm, \partial_\theta v_\pm) \tag{4.2.6}$$

where

$$\gamma_\pm^k(v_\pm) = \mathbb{E}_\pm^k\left(\sum_{l=1}^m b_{\pm,k}^{kl} v_\pm^l\right) \tag{4.2.7}$$

and

$$\Xi_\pm^k(u_\pm, v_\pm) = \mathbb{E}_\pm^k\left(\sum_{i \neq k, l \neq k} \gamma_{\pm,k}^{il} u_\pm^i v_\pm^l\right) \tag{4.2.8}$$

with

$$\gamma_{\pm,k}^{il} = -b_{\pm,k}^{il} \frac{\lambda_k^\pm - \lambda_l^\pm}{\lambda_k^\pm - \lambda_i^\pm}. \tag{4.2.9}$$

Proof. From the definitions, we have

$$\mathbb{E}_\pm^k B_\pm^k(\partial_\theta u_\pm, v_\pm) = \sum_{i,l=1}^m b_{\pm,k}^{il} \mathbb{E}_\pm^k(\partial_\theta u_\pm^i v_\pm^l). \tag{4.2.10}$$

We split the right-hand side herein into three parts:

Case (1): $i = k$. Since $\mathbb{E}_\pm u_\pm = u_\pm$, it is obvious that $u_\pm^i$ can be regarded as a function of $(t, x; \theta \mp (\lambda_i^\pm - \sigma)\tau)$, hence

$$\mathbb{E}_\pm^k(\partial_\theta u_\pm^k v_\pm^l = \mathbb{E}_\pm^k(v_\pm^l)\partial_\theta u_\pm^k. \tag{4.2.11}$$

Case (2): $i \neq k$, $l = k$. By using $\mathbb{E}_\pm^k v_\pm^k = v_\pm^k$ we obtain

$$\mathbb{E}_\pm^k(\partial_\theta u_\pm^i v_\pm^k) = \mathbb{E}_\pm^k(\partial_\theta u_\pm^i)v_\pm^k = 0. \tag{4.2.12}$$

Case (3): $i \neq k$ and $l \neq k$. From the definition (4.2.4) of $\mathbb{E}_\pm^k$, we have

$$\mathbb{E}_\pm^k(\partial_\theta u_\pm^i v_\pm^l) = -\frac{\lambda_k^\pm - \lambda_l^\pm}{\lambda_k^\pm - \lambda_i^\pm} \mathbb{E}_\pm^k(u_\pm^i \partial_\theta v_\pm^l). \tag{4.2.13}$$

Substituting these three cases into (4.2.10), the conclusion (4.2.6) follows ∎

Employing Lemma 4.1, we know that the problem (4.2.3) is equivalent to the following one:

$$\left.\begin{aligned}
\mathbb{E}^k_\pm U^{\nu+1}_{\pm,k} &= U^{\nu+1}_{\pm,k} \\
X^\pm_k U^{\nu+1}_{\pm,k} \pm \left(\gamma^k_\pm(U^\nu_\pm) - \mathbb{E}_0 \chi^\nu\right)\partial_\theta U^{\nu+1}_{\pm,k} \pm \Xi^k_\pm(U^{\nu+1}_\pm, \partial_\theta U^\nu_\pm) &= 0 \\
\chi^{\nu+1}[u] + (\sigma I - \Lambda_+)U^{\nu+1}_+ - (\sigma I - \Lambda_-)U^{\nu+1}_- &= 0 \text{ on } x = \theta = 0 \\
U^{\nu+1}_\pm|_{t=r=0} &= U_{\pm,0}(x,\theta)
\end{aligned}\right\} \quad (4.2.14)$$

where $X^\pm_k = \partial_t \pm (\lambda^\pm_k - \sigma)\partial_x$ are scalar operators for each $k \in \{1,\dots,m\}$ and

$$(\mathbb{E}_0\chi)(t) = \lim_{\rho\to\infty} \frac{1}{2\rho}\int_{-\rho}^{+\rho} \chi(t,\tau)\,d\tau.$$

To study this problem, let us first consider the linear problem

$$\left.\begin{aligned}
\mathbb{E}^k_\pm U_{\pm,k} &= U_{\pm,k} \quad (1 \le k \le m) \\
X^\pm_k U_{\pm,k} \pm \left(\gamma^k_\pm(V_\pm) - \mathbb{E}_0 K\right)\partial_\theta U_{\pm,k} \pm \Xi^k_\pm(U_\pm, \partial_\theta V_\pm) &= \mathbb{E}^k_\pm f_{\pm,k} \\
\chi[u] + (\sigma I - \Lambda_+)U_+ - (\sigma I - \Lambda_-)U_- &= 0 \quad \text{on } x = \theta = 0 \\
U_\pm|_{t=r=0} &= U_{\pm,0}(x,\theta)
\end{aligned}\right\} \quad (4.2.15)$$

where, for any fixed $T_0 > 0$, $K \in \mathcal{C}^1([0,T_0] : \mathbb{R})$, $V_\pm \in \mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$, $f_\pm \in \mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$ and $U_{\pm,0} \in \mathcal{C}^1(\omega^+ : \mathbb{R})$ satisfying the compatibility conditions of (4.2.15) up to order one.

In the remainder of this section, we will use $\|U(t)\|$ and $\|U(t)\|_1 = \|U(t)\| + \|\nabla U(t)\|$ to denote the $L^\infty(\omega^+_t \times \mathbb{R}^2)$ and $W^{1,\infty}(\omega^+_t \times \mathbb{R}^2)$ norms, respectively, of $U(t,\cdot)$. Analogously, we will use $\|U\|_t$ and $\|U\|_{1,t}$ to denote the $L^\infty(\Omega^+_t \times \mathbb{R}^2)$ and $W^{1,\infty}(\Omega^+_t \times \mathbb{R}^2)$ norms, respectively, of $U(\cdot)$. Further, C and M will denote constants depending only upon $(\|V_\pm\|_{L^\infty(\Omega^+_{T_0} \times \mathbb{R}^2)}, \|K\|_{L^\infty([0,T_0] \times \mathbb{R})})$ and $(\|\nabla V_\pm\|_{L^\infty(\Omega^+_{T_0} \times \mathbb{R}^2)}, \|\nabla K\|_{L^\infty([0,T_0] \times \mathbb{R})})$, respectively, and C_1 denote a constant independent of any function appearing in the problem (4.2.15). Each notation of χ is similar to that of U.

For the linear problem (4.2.15), we have the following results, the proof of which will be given in the next subsection.

Proposition 4.2.

(1) *With the above assumptions, there exist unique solutions $U_\pm \in \mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$ and $\chi \in \mathcal{C}^1([0,T_0] : \mathbb{R})$ to the problem (4.2.15). Moreover, for any $t \in [0,T_0]$,*

$$\|\chi(t)\| + \|U_\pm(t)\| \le C_1 e^{C_1 M t}\left(\|U_{\pm,0}\| + \int_0^t \|f_\pm(s)\|\,ds\right) \qquad (4.2.16)$$

and

$$\|\chi(t)\|_1 + \|U_\pm(t)\|_1$$
$$\leq C \exp(CMte^{CMt})$$
$$\times \left(\|U_{\pm,0}\|_1 + \|V_\pm(0)\|_1 \|U_{\pm,0}\| + \|f_\pm(0)\| + \int_0^t \|f_\pm(s)\|_1 ds \right). \tag{4.2.17}$$

(2) *There is a $T > 0$ depending upon $\|V_\pm\|_{W^{1,\infty}(\Omega_{T_0}^+ \times \mathbb{R}^2)}$ and $\|K\|_{W^{1,\infty}([0,T_0] \times \mathbb{R})}$ such that, for any $t \in (0,T]$, the estimate for the modulus of continuity of $(U_\pm, \chi)$*

$$\omega(\delta, t; \chi) + \omega(\delta, t; U_\pm)$$
$$\leq Ce^{CMt} \left(\omega(\delta, U_{\pm,0}) + \delta \|f_\pm\|_t + \delta M \left(\|U_{\pm,0}\| + \int_0^t \|f_\pm(s)\| \, ds \right) \right) \tag{4.2.18}$$
$$+ Ce^{CMt} \int_0^t \left(\omega(\delta, s; f_\pm) + (\|U_{\pm,0}\| + s\|f_\pm\|_s) \omega(\delta, s; \partial_\theta V_\pm) \right) ds$$

holds where $\omega(\delta, t; u)$ is defined in a way similar to (3.3.15) :

$$\omega(\delta, t; u) = \sup |u(s, x; \tau, \theta) - u(s', x'; \tau', \theta')| \tag{4.2.19}$$

with the supremum being taken over $(s, x; \tau, \theta)$ and $(s', x'; \tau', \theta')$ in $\Omega_t^+ \times \mathbb{R}^2$ such that $|(s, x; \tau, \theta) - (s', x'; \tau', \theta')| \leq \delta$.

As J. L. Joly et al. in [8], by using (4.2.18), we also can establish a similar estimate on the modulus of continuity of $(\nabla U_\pm, \nabla \chi)$ for the problem (4.2.15).

Theorem 4.1. *For the iteration scheme (4.2.3), there is a constant $T > 0$ such that the solution sequences $\{U_\pm^\nu, \chi^\nu\}$ are convergent in $C^1(\Omega_T^+ : \mathbb{R}^2) \times C^1([0,T] : \mathbb{R})$, and their limits $(U_\pm, \chi) \in C^1(\Omega_T^+ : \mathbb{R}^2) \times C^1([0,T] : \mathbb{R})$ are unique solutions to the problem (2.1.25).*

Proof. From the above discussion, we know that it is sufficient to discuss the iteration scheme (4.2.14). Applying the estimates (4.2.16) and (4.2.17) in the problem (4.2.14), we obtain that, for any $\nu \geq 0$ and $t \in (0, T_0]$,

$$\|\chi^{\nu+1}(t)\| + \|U_\pm^{\nu+1}(t)\| \leq C_1 e^{C_1 t M_t^\nu} \|U_{\pm,0}\| \tag{4.2.20}$$

and

$$\|\chi^{\nu+1}(t)\|_1 + \|U_\pm^{\nu+1}(t)\|_1 \leq C_t^\nu \exp\left(t C_t^\nu M_t^\nu e^{t C_t^\nu M_t^\nu}\right) \|U_{\pm,0}\|_1 (1 + \|U_{\pm,0}\|) \tag{4.2.21}$$

where $C_1 > 0$ is a constant, and

$$C_t^\nu = C\left(\|U_\pm^\nu\|_t, \|\chi^\nu\|_t\right) \qquad \text{and} \qquad M_t^\nu = M\left(\|U_\pm^\nu\|_{1,t}, \|\chi^\nu\|_{1,t}\right)$$

are two positive increasing functions with respect to their arguments. Let us choose two constants $K \geq \|U_{\pm,0}\|$ and $K_1 \geq \|U_{\pm,0}\|_1$ large enough, and a constant $T_1 \in (0, T_0]$ small enough, such that

$$\left. \begin{array}{r} C_1 e^{C_1 M(K_1)T_1} \|U_{\pm,0}\| \leq K \\ C(K) \exp\left(T_1 C(K) M(K_1) e^{T_1 C(K) M(K_1)}\right) \|U_{\pm,0}\|_1 \left(1 + \|U_{\pm,0}\|\right) \leq K_1. \end{array} \right\} \quad (4.2.22)$$

From (4.2.20) and (4.2.21), by induction on ν, it is easy to verify that $\{U_{\pm}^{\nu}, \chi^{\nu}\}$ are bounded in $C^1(\Omega_{T_1}^+ : \mathbb{R}^2) \times C^1([0, T] : \mathbb{R})$. Set $w_{\pm}^{\nu} = U_{\pm}^{\nu+1} - U_{\pm}^{\nu}$ and $\tilde{\chi}^{\nu} = \chi^{\nu+1} - \chi^{\nu}$. From (4.2.14) we know that $(w_{\pm}^{\nu}, \tilde{\chi}^{\nu})$ satisfies

$$\left. \begin{array}{r} \mathbb{E}_{\pm}^k w_{\pm,k}^{\nu} = w_{\pm,k}^{\nu} \quad (1 \leq k \leq m) \\ X_k^{\pm} w_{\pm,k}^{\nu} \pm \left(\gamma_{\pm}^k(U_{\pm}^{\nu}) - \mathbb{E}_0 \chi^{\nu}\right)\partial_{\theta} w_{\pm,k}^{\nu} \pm \Xi_{\pm}^k(w_{\pm}^{\nu}, \partial_{\theta} U_{\pm}^{\nu}) = G_{\pm,k}^{\nu} \\ \tilde{\chi}^{\nu}[u] + (\sigma I - \Lambda_+)w_+^{\nu} - (\sigma I - \Lambda_-)w_-^{\nu} = 0 \quad \text{on } x = \theta = 0 \\ w_{\pm}^{\nu}|_{t=\tau=0} = 0 \end{array} \right\} \quad (4.2.23)$$

where

$$G_{\pm,k}^{\nu} = \mp\left(\mathbb{E}_{\pm}^k B_{\pm}^k(\partial_{\theta} U_{\pm}^{\nu}, w_{\pm}^{\nu-1}) - \mathbb{E}_0 \tilde{\chi}^{\nu-1} \partial_{\theta} U_{\pm,k}^{\nu}\right)$$

which implies, for any $t \in (0, T_1]$,

$$\|G_{\pm}^{\nu}(t)\| \leq K_2 \left(\|w_{\pm}^{\nu-1}(t)\| + \|\tilde{\chi}^{\nu-1}(t)\|\right) \quad (4.2.24)$$

with K_2 depending only upon the uniform bound of $\{\|U_{\pm}^{\nu}\|_{1,T_1}\}$. Applying the estimate (4.2.16) in the problem (4.2.23), and using (4.2.24), the convergence of $\{U_{\pm}^{\nu}, \chi^{\nu}\}$ in $L^{\infty}(\Omega_{T_1}^+ \times \mathbb{R}^2) \times L^{\infty}([0, T_1] \times \mathbb{R})$ follows immediately, and the limits are in $C^0(\Omega_{T_1}^+ : \mathbb{R}^2) \times C^0([0, T_1] : \mathbb{R})$. As J. L. Joly et al. in [8], we can prove the existence of a constant $T_2 \in (0, T_1]$ such that the derivatives of $U_{\pm}^{\nu}$ and χ^{ν} are equicontinuous on $\Omega_{T_2}^+ \times \mathbb{R}^2$ and $[0, T_2] \times \mathbb{R}$, respectively. Therefore, the convergence of $U_{\pm}^{\nu}$ and χ^{ν} holds in C^1 on any compact subset of $\Omega_{T_2}^+ \times \mathbb{R}^2$ and $[0, T_2] \times \mathbb{R}$, respectively. From [8: Proposition 4.1.2], we conclude that the limits $(U_{\pm}, \chi)$ of $\{U_{\pm}^{\nu}, \chi^{\nu}\}$ are in $C^1(\Omega_{T_2}^+ : \mathbb{R}^2) \times C^1([0, T_2] : \mathbb{R})$. Finally, by using the same argument as in [8], we obtain that $(U_{\pm}, \chi) \in C^1(\Omega_{T_2}^+ : \mathbb{R}^2) \times C^1([0, T_2] : \mathbb{R})$ are unique solutions to the problem (2.1.25) ∎

4.3 Study of linear problems. Before studying the linear problem (4.2.15), at first, let us consider the diagonal systems

$$\left. \begin{array}{r} \mathbb{E}_{\pm}^k U_{\pm,k} = U_{\pm,k} \quad (1 \leq k \leq m) \\ X_k^{\pm} U_{\pm,k} \pm \left(\gamma_{\pm}^k(V_{\pm}) - \mathbb{E}_0 K\right)\partial_{\theta} U_{\pm,k} = \mathbb{E}_{\pm}^k f_{\pm,k} \\ \chi[u] + (\sigma I - \Lambda_+)U_+ - (\sigma I - \Lambda_-)U_- = 0 \quad \text{on } x = \theta = 0 \\ U_{\pm}|_{t=\tau=0} = U_{\pm,0}(x, \theta) \end{array} \right\} \quad (4.3.1)$$

where the notations are the same as in (4.2.15), $K \in \mathcal{C}^1([0, T_0] : \mathbb{R})$, $V_\pm \in \mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$, $f_\pm \in \mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$ and $U_{\pm,0} \in \mathcal{C}^1(\omega^+ : \mathbb{R})$ satisfying the compatibility conditions of (4.3.1) up to order one. As in (3.3.9) and (3.3.10), decompose $U_\pm$ into

$$U_{+,I} = (U_{+,1}, \ldots, U_{+,j})^T \quad \text{and} \quad U_{+,II} = (U_{+,j+1}, \ldots, U_{+,m})^T \quad (4.3.2)$$

$$U_{-,I} = (U_{-,j}, \ldots, U_{-,m})^T \quad \text{and} \quad U_{-,II} = (U_{-,1}, \ldots, U_{-,j-1})^T. \quad (4.3.3)$$

The same decompositions of $f_\pm$ and $U_{\pm,0}$ as above are also denoted by $f_{\pm,I}$, $f_{\pm,II}$ and $U^I_{\pm,0}$, $U^{II}_{\pm,0}$, respectively.

From the Lax entropy condition (2.1.7), we know that (4.3.1) is an initial value problem for the components $U_{\pm,I}$ and a mixed problem for $U_{\pm,II}$. Applying the result of J. L. Joly et al. [8: Lemmas 6.3.2 - 6.3.4] in the I-part of the system (4.3.1), we obtain the following

Lemma 4.2.

(1) *Suppose* $K \in \mathcal{C}^1([0, T_0] : \mathbb{R})$, $V_\pm \in \mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$ *and* $f_{\pm,I} \in \mathcal{C}^0(\Omega^+_{T_0} : \mathbb{R}^2)$. *Then there exist unique weak solutions* $U_{\pm,I} \in \mathcal{C}^0(\Omega^+_{T_0} : \mathbb{R}^2)$ *to the I-part of the problem* (4.3.1). *Moreover, we have*

$$\|U_{\pm,I}(t)\| \leq \|U^I_{\pm,0}\| + \int_0^t \|f_{\pm,I}(s)\| \, ds. \quad (4.3.4)$$

(2) *For the modulus of continuity of* $U_{\pm,I}$, *we have*

$$\omega(\delta, t; U_{\pm,I}) \leq Ce^{CMt}\omega(\delta, U^I_{\pm,0}) + \delta\|f_{\pm,I}\|_t + \int_0^t Ce^{CM(t-s)}\omega(\delta, s; f_{\pm,I}) \, ds. \quad (4.3.5)$$

(3) *If we have the additional regularity* $f_{\pm,I} \in \mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$, *then the weak solutions* $U_{\pm,I}$ *obtained in Part* (1) *belong to* $\mathcal{C}^1(\Omega^+_{T_0} : \mathbb{R}^2)$ *and satisfy*

$$\|u_{\pm,I}(t)\|_1 \leq Ce^{CMt}\|U^I_{\pm,0}\|_1 + \|f_{\pm,I}(0)\| + \int_0^t Ce^{CM(t-s)}\|f_{\pm,I}(s)\|_1 \, ds. \quad (4.3.6)$$

Let us study the II-part of the problem (4.3.1). From the stability condition (4.2.4), we know that the boundary condition in (4.3.1) can be reformulated as

$$M \cdot \left(\chi(t, \tau), U_{+,II}, U_{-,II}\right)^T = B(U_{+,I}, U_{-,I}) \quad (4.3.7)$$

where the matrix

$$M = \left([u], (\sigma - \lambda^+_{j+1})e_{j+1}, \ldots, (\sigma - \lambda^+_m)e_m, (\lambda^-_1 - \sigma)e_1, \ldots, (\lambda^-_{j-1} - \sigma)e_{j-1}\right)^T$$

is invertible and

$$B(U_{+,I}, U_{-,I}) = \sum_{i=1}^{j} (\lambda_i^+ - \sigma) U_{+,i} e_i + \sum_{i=j}^{m} (\sigma - \lambda_i^-) U_{-,i} e_i.$$

Hence, from (4.3.1) we know that $U_{\pm,II}$ and χ satisfy the problem

$$\left.\begin{array}{c} \mathbb{E}_\pm^k U_{\pm,k} = U_{\pm,k} \\[2mm] X_k^\pm U_{\pm,k} \pm (\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K) \partial_\theta U_{\pm,k} = \mathbb{E}_\pm^k f_{\pm,k} \\[2mm] U_{\pm,k}|_{z=\theta=0} = a_{\pm,k}(U_{+,I}, U_{-,I})(t,\tau) \\[2mm] \chi(t,\tau) = a(U_{+,I}, U_{-,I})(t,\tau) \\[2mm] U_{\pm,II}|_{t=\tau=0} = U_{\pm,0}^{II}(x,\theta) \end{array}\right\} \qquad (4.3.8)$$

where $a(\cdot)$ and $a_{\pm,k}(\cdot)$ are linear in their arguments, $k \in \{j+1, \ldots, m\}$ for "+" and $k \in \{1, \ldots, j-1\}$ for "−". For this problem, similar to J. L. Joly et al. in [8], by integrating along characteristic curves we obtain the following

Lemma 4.3.

(1) *For any given* $K \in \mathcal{C}^1([0,T_0] : \mathbb{R})$, $V_\pm \in \mathcal{C}^1(\Omega_{T_0}^+ : \mathbb{R}^2)$ *and* $f_\pm \in \mathcal{C}^0(\Omega_{T_0}^+ : \mathbb{R}^2)$, *there are unique weak solutions* $U_{\pm,II} \in \mathcal{C}^0(\Omega_{T_0}^+ : \mathbb{R}^2)$ *and* $\chi \in \mathcal{C}^0([0,T_0] : \mathbb{R})$ *to the problem* (4.3.8). *Moreover*,

$$\|\chi(t)\| + \|U_{\pm,II}(t)\| \le C_1 \|U_{\pm,I}\|_t + \|U_{\pm,0}^{II}\| + \int_0^t \|f_{\pm,II}(s)\| \, ds. \qquad (4.3.9)$$

(2) *For the modulus of continuity of* $(U_{\pm,II}, \chi)$, *we have*

$$\omega(\delta, t; \chi) + \omega(\delta, t; U_{\pm,II}) \le C e^{CMt} \Big(\omega(\delta, t; U_{\pm,I}) + \omega(\delta; U_{\pm,0}^{II}) \Big)$$
$$+ C_1 \delta \|f_{\pm,II}\|_t + C \int_0^t e^{CM(t-s)} \omega(\delta, s; f_{\pm,II}) \, ds. \qquad (4.3.10)$$

(3) *If we have the additional regularity* $f_\pm \in \mathcal{C}^1(\Omega_{T_0}^+ : \mathbb{R}^2)$, *then the solutions* $(U_{\pm,II}, \chi)$ *obtained in part* (1) *belong to* $\mathcal{C}^1(\Omega_{T_0}^+ : \mathbb{R}^2) \times \mathcal{C}^1([0,T_0] : \mathbb{R})$ *and satisfy*

$$\|\chi(t)\|_1 + \|U_{\pm,II}(t)\|_1 \le C e^{CMt} \big(\|U_{\pm,I}\|_{1,t} + \|U_{\pm,0}^{II}\|_1 \big)$$
$$+ C_1 \|f_{\pm,II}(0)\| + \int_0^t C e^{CM(t-s)} \|f_{\pm,II}(s)\|_1 \, ds. \qquad (4.3.11)$$

Proof. All results of $\chi(t,\tau)$ are clear. It is sufficient to discuss $U_{\pm,II}$. From $\mathbb{E}_\pm^k U_{\pm,k} = U_{\pm,k}$, we know that $U_{\pm,k}$ can be regarded as a function of $\big(t, x; \theta \mp (\lambda_k^\pm - \sigma)\tau\big)$,

which can be written as $U_{\pm,k}\big(t,x;\theta\mp(\lambda_k^\pm-\sigma)\tau\big)$. Similarly, the functions $\mathbb{E}_\pm^k f_{\pm,k}$ depend only upon $\big(t,x;\theta\mp(\lambda_k^\pm-\sigma)\tau\big)$, which can be written as $\mathbb{E}_\pm^k f_{\pm,k} = F_{\pm,k}\big(t,x;\theta\mp(\lambda_k^\pm-\sigma)\tau\big)$. Therefore, from (4.3.8) we know that $U_{\pm,k}(t,x;\theta)$ satisfy

$$
\left.
\begin{aligned}
X_k^\pm U_{\pm,k} \pm \big(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\big)\partial_\theta U_{\pm,k} &= F_{\pm,k} \\
U_{\pm,k}|_{x=0} &= b_{\pm,k}(t,\theta) \\
U_{\pm,k}|_{t=0} &= U_{\pm,0}^k(x,\theta)
\end{aligned}
\right\}
\tag{4.3.12}
$$

where $b_{\pm,k}(t,\theta) = a_{\pm,k}\big(t,\frac{\theta}{\pm(\sigma-\lambda_k^\pm)}\big)$ are almost periodic in $\theta \in \mathbb{R}$. The characteristic curves of $X_k^\pm \pm \big(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\big)\partial_\theta$ are

$$
s \longrightarrow \Big(s, x + \mu_{\pm,k}^1(s;t,x,\theta); \theta + \mu_{\pm,k}^2(s;t,x,\theta)\Big)
$$

where $\mu_{\pm,k}^1$ and $\mu_{\pm,k}^2$ are solutions to the problem

$$
\left.
\begin{aligned}
d_s\mu_{\pm,k}^1(s;t,x,\theta) &= \pm(\lambda_k^\pm - \sigma) \\
d_s\mu_{\pm,k}^2(s;t,x,\theta) &= \pm\big(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\big)\big(s, x + \mu_{\pm,k}^1(s); \theta + \mu_{\pm,k}^2(s)\big) \\
\mu_{\pm,k}^1(t;t,x,\theta) &= \mu_{\pm,k}^2(t;t,x,\theta) = 0
\end{aligned}
\right\}
\tag{4.3.13}
$$

which immediately implies

$$
\mu_{\pm,k}^1(s;t,x,\theta) = \pm(\lambda_k^\pm - \sigma)(s - t).
\tag{4.3.14}
$$

Set

$$
\widetilde{\Omega}_{T_0}^+ = \Big\{(s;t,x)\,\Big|\,0 \le s \le t, (t,x) \in \Omega_{T_0}^+\Big\}.
$$

Since $\mu_{\pm,k}^1$ are independent of θ, $\gamma_\pm^k$ are linear in $V_\pm$, and $V_\pm$ are almost periodic in θ, as J. L. Joly et al. in [8: Lemma 6.3.2], we can obtain that the problem (4.3.13) admits unique solutions $\mu_{\pm,k}^2 \in \mathcal{C}^1(\widetilde{\Omega}_{T_0}^+ : \mathbb{R})$, and they satisfy

$$
\big|\nabla_{(s;t,x,\theta)}\mu_{\pm,k}^2\big| \le Ce^{CM|t-s|}.
\tag{4.3.15}
$$

The solutions $U_{\pm,k}$ of the problem (4.3.12) are given as follows:

(i) When $\eta = t \mp \frac{x}{\lambda_k^\pm - \sigma} \ge 0$, then

$$
U_{\pm,k}(t,x;\theta) = b_{\pm,k}\big(\eta, \theta + \mu_{\pm,k}^2(\eta)\big)
$$
$$
+ \int_\eta^t F_{\pm,k}\big(s, x + \mu_{\pm,k}^1(s); \theta + \mu_{\pm,k}^2(s)\big)\,ds.
\tag{4.3.16}
$$

(ii) When $t \mp \frac{x}{\lambda_k^{\mp} - \sigma} < 0$, then

$$U_{\pm,k}(t,x;\theta) = U_{\pm,0}^k \big(x + \mu_{\pm,k}^1(0), \theta + \mu_{\pm,k}^2(0) \big)$$

$$+ \int_0^t F_{\pm,k} \Big(s, x + \mu_{\pm,k}^1(s); \theta + \mu_{\pm,k}^2(s) \Big) ds. \qquad (4.3.17)$$

Here $\mu_{\pm,k}^i(s) = \mu_{\pm,k}^i(s;t,x,\theta)$. From (4.3.16) and (4.3.17), we immediately conclude that $U_{\pm,II}$ belong to $C^0(\Omega_{T_0}^+; \mathbb{R}^2)$ and satisfy the estimate (4.3.9) by noting that $a_{\pm,k}$ are linear in (4.3.8). By applying the estimate (4.3.15) in (4.3.16) and (4.3.17), and using the compatibility conditions of (4.3.12), it gives rise to estimates (4.3.10) and (4.3.11) ∎

Taking together Lemmas 4.2 with 4.3, it follows

Lemma 4.4.

(1) *For any given* $K \in C^1([0,T_0] : \mathbb{R})$, $V_\pm \in C^1(\Omega_{T_0}^+ : \mathbb{R}^2)$, $f_\pm \in C^0(\Omega_{T_0}^+ : \mathbb{R}^2)$ *and* $U_{\pm,0} \in C^0(\omega^+ : \mathbb{R})$ *satisfying the zero-th order compatibility condition of* (4.3.1), *there are unique weak solutions* $U_\pm \in C^0(\Omega_{T_0}^+ : \mathbb{R}^2)$ *and* $\chi \in C^0([0,T_0] : \mathbb{R})$ *to the problem* (4.3.1). *Moreover,*

$$\|\chi(t)\| + \|U_\pm(t)\| \leq C_1 \left(\|U_{\pm,0}\| + \int_0^t \|f_\pm(s)\| \, ds \right). \qquad (4.3.18)$$

(2) *For the modulus of continuity of* $(U_\pm, \chi)$, *we have*

$$\omega(\delta, t; \chi) + \omega(\delta, t; U_\pm) \leq C e^{CMt} \left(\omega(\delta, U_{\pm,0}) + \delta \|f_\pm\|_t + \int_0^t \omega(\delta, s; f_\pm) \, ds \right). \qquad (4.3.19)$$

(3) *If we have the additional regularity* $f_\pm \in C^1(\Omega_{T_0}^+ : \mathbb{R}^2)$ *and* $U_{\pm,0} \in C^1(\omega^+ : \mathbb{R})$ *satisfying the first order compatibility condition of* (4.3.1), *then the solutions* $(U_\pm, \chi)$ *of the problem* (4.3.1) *belong to* $C^1(\Omega_{T_0}^+ : \mathbb{R}^2) \times C^1([0,T_0] : \mathbb{R})$ *and satisfy*

$$\|\chi(t)\|_1 + \|U_\pm(t)\|_1 \leq C e^{CMt} \left(\|U_{\pm,0}\|_1 + \|f_\pm(0)\| + \int_0^t \|f_\pm(s)\|_1 \, ds \right). \qquad (4.3.20)$$

Now, let us study the linear problem (4.2.15).

Proof of Proposition 4.2. We solve the problem (4.2.15) by the iteration scheme

$$\left. \begin{aligned} \mathbb{E}_\pm^k U_{\pm,k}^{\nu+1} &= U_{\pm,k}^{\nu+1} \quad (1 \leq k \leq m) \\ X_k^\pm U_{\pm,k}^{\nu+1} \pm \big(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K \big) \partial_\theta U_{\pm,k}^{\nu+1} & \\ \pm \Xi_\pm^k(U_\pm^\nu, \partial_\theta V_\pm) &= \mathbb{E}_\pm^k f_{\pm,k} \\ \chi^{\nu+1}[u] + (\sigma I - \Lambda_+)U_+^{\nu+1} - (\sigma I - \Lambda_-)U_-^{\nu+1} &= 0 \quad \text{on } x = \theta = 0 \\ U_\pm^{\nu+1}|_{t=r=0} &= U_{\pm,0}(x,\theta) \end{aligned} \right\} \qquad (4.3.21)$$

with $U_\pm^0$ given by (4.1.2). It is sufficient to consider the part of $U_\pm^\nu$, because all properties of χ^ν and its limit $\chi \in C^1([0,T] : \mathbb{R})$ can be easily deduced from the boundary condition in (4.3.21). We divide our proof into the following three lemmas.

Lemma 4.5. *The sequences $\{U_\pm^\nu\}$ are convergent in $C^0(\Omega_{T_0}^+ : \mathbb{R}^2)$, and the limits $U_\pm$ are unique weak solutions of the problem (4.2.15). Moreover, we have*

$$\|U_\pm(t)\| \le C_1 e^{C_1 M t}\left(\|U_{\pm,0}\| + \int_0^t \|f_\pm(s)\|\,ds\right) \tag{4.3.22}$$

with $M = \max\left(\|V_+\|_{1,T_0}, \|V_-\|_{1,T_0}\right)$.

Proof. Obviously, we have

$$\|\Xi_\pm^k(U_\pm, \partial_\theta V_\pm)(t)\| \le C_1\|V_\pm(t)\|_1\|U_\pm(t)\|. \tag{4.3.23}$$

Applying (4.3.18) in (4.3.21), it follows

$$\|U_\pm^{\nu+1}(t)\| \le C_1\left(\|U_{\pm,0}\| + \int_0^t \left(\|f_\pm(s)\| + \|V_\pm(s)\|_1\,\|U_\pm^\nu(s)\|\right)ds\right)$$

which implies, by induction,

$$\|U_\pm^\nu(t)\| \le C_1 e^{C_1 M t}\left(\|U_{\pm,0}\| + \int_0^t \|f_\pm(s)\|\,ds\right) \tag{4.3.24}$$

for any $\nu \ge 0$, where $M = \max\left(\|V_+\|_{1,T_0}, \|V_-\|_{1,T_0}\right)$. Clearly, the last estimate gives rise to the boundedness of $\{U_\pm^\nu\}$ in $C^0(\Omega_{T_0}^+ : \mathbb{R}^2)$.

From (4.3.21), we know that $w_\pm^\nu = U_\pm^{\nu+1} - U_\pm^\nu$ satisfies the problem

$$\left.\begin{array}{c}\mathbb{E}_\pm^k w_{\pm,k}^\nu = w_{\pm,k}^\nu \\[4pt] X_k^\pm w_{\pm,k}^\nu \pm \left(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\right)\partial_\theta w_{\pm,k}^\nu \pm \Xi_\pm^k(w_\pm^{\nu-1}, \partial_\theta V_\pm) = 0 \\[4pt] \tilde{\chi}^\nu[u] + (\sigma I - \Lambda_+)w_+^\nu - (\sigma I - \Lambda_-)w_-^\nu = 0 \quad \text{on } x = \theta = 0 \\[4pt] w_\pm^\nu|_{t=\tau=0} = 0\end{array}\right\} \tag{4.3.25}$$

with $\tilde{\chi}^\nu(t,\tau) = (\chi^{\nu+1} - \chi^\nu)(t,\tau)$, which immediately implies the convergence of $\{U_\pm^\nu\}$ in $C^0(\Omega_{T_0}^+ : \mathbb{R}^2)$ by applying (4.3.18) in (4.3.25) and using (4.3.23). Obviously, the limits $U_\pm$ of $\{U_\pm^\nu\}$ in $C^0(\Omega_{T_0}^+ : \mathbb{R}^2)$ are unique weak solutions to the problem (4.3.15), and they satisfy the estimate (4.3.22) by using (4.3.24) ∎

Lemma 4.6. *There is a constant $T > 0$ depending upon $\|V_\pm\|_{W^{1,\infty}}$ and $\|K\|_{W^{1,\infty}}$ such that, for any $t \in (0, T]$, we have the estimate*

$$\omega(\delta, t; U_\pm)$$

$$\le Ce^{CMt}\left(\omega(\delta, U_{\pm,0}) + \delta\|f_\pm\|_t + \delta M\|U_{\pm,0}\| + \delta M \int_0^t \|f_\pm(s)\|\,ds\right) \tag{4.3.26}$$

$$+ Ce^{CMt}\int_0^t \left(\omega(\delta, s; \partial_\theta V_\pm)(\|U_{\pm,0}\| + s\|f_\pm\|_s) + \omega(\delta, s; f_\pm)\right)ds$$

on the modulus of continuity of the solutions $U_\pm$ to the problem (4.2.15).

Proof. At first, from the definition we have the fact

$$\omega\big(\delta, s; \Xi_\pm^k(U_\pm, \partial_\theta V_\pm)\big) \leq C_1 \Big(\omega(\delta, s; \partial_\theta V_\pm)\|U_\pm\|_s + \omega(\delta, s; U_\pm)\|\partial_\theta V_\pm\|_s\Big). \qquad (4.3.27)$$

For the problem (4.2.15), by removing the term $\Xi_\pm^k(U_\pm, \partial_\theta V_\pm)$ to the right-hand side and using (4.3.19), we have

$$\omega(\delta, t; U_\pm)$$

$$\leq Ce^{CMt}\Big(\omega(\delta, U_{\pm,0}) + \delta\|f_\pm\|_t + \delta\|\partial_\theta V_\pm\|_t\|U_\pm\|_t\Big)$$

$$+ Ce^{CMt}\int_0^t \Big(\omega(\delta, s; f_\pm) + \omega(\delta, s; \partial_\theta V_\pm)\|U_\pm\|_s + \omega(\delta, s; U_\pm)\|\partial_\theta V_\pm\|_s\Big)ds$$

$$\leq Ce^{CMt}\bigg(\omega(\delta, U_{\pm,0}) + \delta\|f_\pm\|_t + \delta M\|U_{\pm,0}\| + \delta M \int_0^t \|f_\pm(s)\|\, ds\bigg)$$

$$+ Ce^{CMt}\int_0^t \Big(\omega(\delta, s; f_\pm) + (\|U_{\pm,0}\| + s\|f_\pm\|_s)\omega(\delta, s; \partial_\theta V_\pm)$$

$$+ M\omega(\delta, s; U_\pm)\Big)ds.$$

This implies that when $T > 0$ satisfies

$$CMTe^{CMT} \leq \frac{1}{2}, \qquad (4.3.28)$$

then we have the conclusion (4.3.26) for any $t \in (0, T]$ ∎

Lemma 4.7. *The solutions $U_\pm$ of the problem* (4.2.15) *belong to $C^1(\omega_{T_0}^+ : \mathbb{R}^2)$ and satisfy*

$$\|U_\pm(t)\|_1 \leq C \exp(CMte^{CMt})$$

$$\times \bigg(\|U_{\pm,0}\|_1 + \|V_\pm(0)\|_1 \|U_{\pm,0}\| + \|f_\pm(0)\| + \int_0^t \|f_\pm(s)\|_1\, ds\bigg). \qquad (4.3.29)$$

Proof. From Lemma 4.5, we know that $U_\pm$ are the limits of $\{U_\pm^\nu\}$ defined by (4.3.21). As J. L. Joly et al. in [8], from the definition (4.2.8) of $\Xi_\pm^k$ we have

$$\big\|\Xi_\pm^k(U_\pm, \partial_\theta V_\pm)(t)\big\|_1 \leq C_1\|V_\pm(t)\|_1 \|U_\pm(t)\|_1 \qquad (4.3.30)$$

which implies $U_\pm^\nu \in C^1(\Omega_{T_0}^+ : \mathbb{R}^2)$ for each $\nu \geq 0$. Applying (4.3.20) in (4.3.21), it follows

$$\|U_\pm^{\nu+1}(t)\|_1 \leq Ce^{CMt}\bigg(\|U_{\pm,0}\|_1 + \|f_\pm(0)\| + \|V_\pm(0)\|_1 \|U_{\pm,0}\|$$

$$+ \int_0^t \Big(\|f_\pm(s)\|_1 + M\|U_\pm^\nu(s)\|_1\Big)ds\bigg). \qquad (4.3.31)$$

From here, by induction on ν we obtain that $\{U_\pm^\nu\}$ are bounded in $\mathcal{C}^1(\Omega_{T_0}^+ : \mathbb{R}^2)$, and for all $\nu \geq 0$, the estimates

$$\|U_\pm^\nu(t)\|_1 \leq C \exp(CMte^{CMt})$$

$$\times \left(\|U_{\pm,0}\|_1 + \|V_\pm(0)\|_1 \|U_{\pm,0}\| + \|f_\pm(0)\| + \int_0^t \|f_\pm(s)\|_1 \, ds \right) \qquad (4.3.32)$$

are valid for all $t \in (0, T_0]$. Applying (4.3.20) in (4.3.25), we can easily deduce the convergence of $\{U_\pm^\nu\}$ in $\mathcal{C}^1(\Omega_{T_0}^+ : \mathbb{R}^2)$, with limits $U_\pm \in \mathcal{C}^1(\Omega_{T_0}^+ : \mathbb{R}^2)$ being the solutions to the problem (4.2.15). Obviously, from (4.3.32) we know that $U_\pm$ satisfy the estimate (4.3.29) ∎

5. Asymptotic properties

This section is devoted to the study of the asymptotic property of exact solutions $(u_\pm^\varepsilon, \phi^\varepsilon)$ to the problem (2.1.18), which gives the proof of Theorem 2.1/(3).

Let $T > 0$ be the smaller one between those obtained in Theorems 3.1 and 4.1. In this section, we will always use $o(1)$ to denote any infinite small quantity when $\varepsilon \to 0$. At first, we claim that the asymptotic property (2.2.18) of $\phi^\varepsilon(t)$ can be easily deduced from the property (2.2.17) of $u_\pm^\varepsilon$, which is stated as the following result.

Proposition 5.1. *Suppose that* $(u_\pm^\varepsilon, \phi^\varepsilon) \in C_\varepsilon^1(\Omega_T^+) \times \widetilde{C}_\varepsilon^2[0, T]$, $(U_\pm, \chi) \in \mathcal{C}^1(\Omega_T^+ : \mathbb{R}^2) \times \mathcal{C}^1([0, T] : \mathbb{R})$ *and* $\phi \in C^2[0, T]$ *are the unique solutions of the problems* (2.1.18), (2.1.25) *and* (2.1.30), *respectively, and* $u_\pm^\varepsilon$ *satisfy the asymptotic property*

$$\left\| u_\pm^\varepsilon(t, x) - U_\pm\left(t, x; \tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon}\right) \right\|_{\varepsilon, 1, \Omega_T^+} = o(1) \qquad \text{when } \varepsilon \to 0. \qquad (5.1)$$

Then

$$\left\| d_t \phi^\varepsilon(t) - \chi\left(t, \tfrac{t}{\varepsilon}\right) \right\|_{\varepsilon, 1, [0, T]} = o(1)$$

$$\left\| \phi^\varepsilon(t) - \phi(t) \right\|_{L^\infty[0, T]} = o(1) \qquad (5.2)$$

when $\varepsilon \to 0$.

Proof. The proof of the first result in (5.2) is the same as in Corollary 4.1, and the second result in (5.2) is similarly obtained in [18: Theorem 5.2]. So we omit their proofs here ∎

Now, let us establish the asymptotics (5.1). From (3.2.2), we know that the exact solutions $(u_\pm^\varepsilon, \phi^\varepsilon)$ of (2.1.18) are the limits of $(u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})$ in $C_\varepsilon^1(\Omega_T^+) \times \widetilde{C}_\varepsilon^2[0, T]$ with $(u_\pm^{\varepsilon,\nu+1}, \phi^{\varepsilon,\nu+1})$ satisfying

$$\left. \begin{aligned} & \partial_t u_\pm^{\varepsilon,\nu+1} \pm \left(A(u_\pm + \varepsilon u_\pm^{\varepsilon,\nu}) - (\sigma + \varepsilon d_t \phi^{\varepsilon,\nu})I \right) \partial_x u_\pm^{\varepsilon,\nu+1} = 0 \\ & d_t \phi^{\varepsilon,\nu+1}[u] + (\sigma I - \Lambda_+)u_+^{\varepsilon,\nu+1} - (\sigma I - \Lambda_-)u_-^{\varepsilon,\nu+1} = \varepsilon g^{\varepsilon,\nu} \\ & \phi^{\varepsilon,\nu+1}(0) = 0 \\ & u_\pm^{\varepsilon,\nu+1}(0, x) = u_{\pm,0}^\varepsilon(x) \end{aligned} \right\} \qquad (5.3)$$

where $(u_\pm^{\varepsilon,0}, \phi^{\varepsilon,0})$ are the approximate solutions constructed in Proposition 3.1, and

$$
\begin{aligned}
g^{\varepsilon,\nu} = {} & d_t\phi^{\varepsilon,\nu}[u^{\varepsilon,\nu} - u^{\varepsilon,\nu+1}] - d_t\phi^{\varepsilon,\nu+1}[u^{\varepsilon,\nu}] \\
& + \int_0^1 \left[\nabla^2 f(u + \eta\varepsilon u^{\varepsilon,\nu})(u^{\varepsilon,\nu}, u^{\varepsilon,\nu+1} - \eta u^{\varepsilon,\nu})\right]d\eta
\end{aligned}
\tag{5.4}
$$

is bounded in $C_\varepsilon^1[0,T]$ for all $\nu \geq 0$. Moreover, the convergence of $(u_\pm^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})$ in $C^0(\Omega_T^+) \times C^1[0,T]$ is uniform for all $\varepsilon \in (0,\varepsilon_0]$.

On the other hand, from (4.2.3) we know that the leading profiles $(U_\pm, \chi)$ are the limits of $(U_\pm^\nu, \chi^\nu)$ in $C^1(\Omega_T^+ : \mathbb{R}^2) \times C^1([0,T] : \mathbb{R})$, where $(U_\pm^{\nu+1}, \chi^{\nu+1})$ satisfy the problem

$$
\left.
\begin{aligned}
&\mathbb{E}_\pm U_\pm^{\nu+1} = U_\pm^{\nu+1} \\
&\partial_t U_\pm^{\nu+1} \pm (\Lambda_\pm - \sigma I)\partial_x U_\pm^{\nu+1} \\
&\mp \mathbb{E}_\pm\left(\chi^\nu \partial_\theta U_\pm^{\nu+1} - B_\pm(\partial_\theta U_\pm^{\nu+1}, U_\pm^\nu)\right) = 0 \\
&\chi^{\nu+1}[u] + (\sigma I - \Lambda_+)U_+^{\nu+1} - (\sigma I - \Lambda_-)U_-^{\nu+1} = 0 \quad \text{on } x = \theta = 0 \\
&U_\pm^{\nu+1}|_{t=\tau=0} = U_{\pm,0}(x,\theta)
\end{aligned}
\right\}
\tag{5.5}
$$

with $(U_\pm^0, \chi^0) \in C^1(\Omega_T^+ : \mathbb{R}^2) \times C^1([0,T] : \mathbb{R})$ being constructed in Subsection 4.1. The proof of (5.1) is to make use of the ideas in J. L. Joly et al. [8: Subsections 6.4 - 6.8], and to extend their results to the case of boundary value problems. That is to say that the proof of

$$
\left\|u_\pm^\varepsilon(t,x) - U_\pm\left(t,x; \tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon}\right)\right\|_{L^\infty(\Omega_T^+)} = o(1)
\tag{5.6}
$$

is to use the argument of simultaneous Picard iteration, which means that, for any $\nu \geq 0$, we wish to prove

$$
\left\|u_\pm^{\varepsilon,\nu}(t,x) - U_\pm^\nu\left(t,x; \tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon}\right)\right\|_{L^\infty(\Omega_T^+)} = o(1)
\tag{5.7}
$$

which is valid for $\nu = 0$ by Proposition 4.1. By taking advantage of the uniform convergence of $\{u_\pm^{\varepsilon,\nu}\}$ in $C^0(\Omega_T^+)$, we can conclude (5.6) from the assertion (5.7). The asymptotics of derivatives of $u_\pm^\varepsilon$ is studied from the nonlinear problem (2.1.18).

The existence of $u_\pm^\varepsilon$ and $U_\pm$ to the following each problem can be obtained in the same way as in Sections 3 and 4, so we only give our whole attention to the asymptotics of $u_\pm^\varepsilon$.

As the first step, let us consider the linear diagonal problem

$$
\left.
\begin{aligned}
&\partial_t u_\pm^\varepsilon \pm \left(\Lambda(u_\pm + \varepsilon v_\pm^\varepsilon) - (\sigma + \varepsilon d_t\Phi^\varepsilon)I\right)\partial_x u_\pm^\varepsilon = f_\pm^\varepsilon \\
&d_t\phi^\varepsilon[u] + (\sigma I - \Lambda_+)u_+^\varepsilon - (\sigma I - \Lambda_-)u_-^\varepsilon = g^\varepsilon \quad \text{on } x = 0 \\
&\phi^\varepsilon(0) = 0 \\
&u_\pm^\varepsilon(0,x) = u_{\pm,0}^\varepsilon(x)
\end{aligned}
\right\}
\tag{5.8}
$$

where
$$\Lambda(u) = \text{diag}\big[\lambda_1(u), \ldots, \lambda_m(u)\big] \tag{5.9}$$

and $\Lambda_\pm = \Lambda(u_\pm)$. For the problem (5.8), suppose that $v_\pm^\epsilon$, $f_\pm^\epsilon$, Φ^ϵ, g^ϵ and $u_{\pm,0}^\epsilon$ are bounded in $C_\epsilon^1(\Omega_T^+)$, $C_\epsilon^0(\Omega_T^+)$, $\tilde{C}_\epsilon^1[0,T]$, $C_\epsilon^0[0,T]$ and $C_\epsilon^0(\omega^+)$, respectively. There are

$$V_\pm \in C^1(\Omega_T^+ : \mathbb{R}^2), \quad F_\pm \in C^0(\Omega_T^+ : \mathbb{R}^{2m}), \quad K \in C^0([0,T] : \mathbb{R}), \quad U_{\pm,0} \in C^0(\omega^+ : \mathbb{R})$$

such that
$$\mathbb{E}_\pm V_\pm = V_\pm \tag{5.10}$$

and, in $L^\infty(\Omega_T^+)$,

$$
\left.
\begin{aligned}
v_\pm^\epsilon(t,x) - V_\pm\big(t,x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big) &= o(1) \\
f_\pm^\epsilon(t,x) - F_\pm\big(t,x; \vec{\varphi}_+\big(\tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big), \vec{\varphi}_-\big(\tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big)\big) &= o(1) \\
d_t\Phi^\epsilon(t) - K\big(t, \tfrac{t}{\epsilon}\big) &= o(1) \\
u_{\pm,0}^\epsilon(x) - U_{\pm,0}\big(x, \tfrac{x}{\epsilon}\big) &= o(1) \\
g^\epsilon(t) &= o(1)
\end{aligned}
\right\} \tag{5.11}
$$

when $\epsilon \to 0$, where
$$\vec{\varphi}_\pm(\tau,\theta) = \big(\varphi_1^\pm(\tau,\theta), \ldots, \varphi_m^\pm(\tau,\theta)\big)$$

with $\varphi_i^\pm(\tau,\theta) = \theta \mp (\lambda_i^\pm - \sigma)\tau$ for all $i \in \{1, \ldots, m\}$. Furthermore, we suppose that the zero-th order compatibility conditions for the problem (5.8) as well as the problem

$$
\left.
\begin{aligned}
\mathbb{E}_\pm U_\pm &= U_\pm \\
X_k^\pm U_{\pm,k} \pm \big(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\big)\partial_\theta U_{\pm,k} &= \mathbb{E}_\pm^k F_{\pm,k} \\
\chi[u] + (\sigma I - \Lambda_+)U_+ - (\sigma I - \Lambda_-)U_- &= 0 \quad \text{on } x = \theta = 0 \\
U_\pm|_{t=\tau=0} &= U_{\pm,0}(x,\theta)
\end{aligned}
\right\} \tag{5.12}
$$

are valid, where $X_k^\pm = \partial_t \pm (\lambda_k^\pm - \sigma)\partial_x$ and

$$\gamma_\pm^k(V_\pm) = \mathbb{E}_\pm^k\left(\sum_{p=1}^m \frac{\partial \lambda_k}{\partial v_p}(u_\pm)V_{\pm,p}\right). \tag{5.13}$$

Then, we have the following result.

Proposition 5.2. *Under the above assumption,*

$$
\left.
\begin{aligned}
u_\pm^\epsilon(t,x) - U_\pm\big(t,x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big) &= o(1) \\
d_t\phi^\epsilon(t) - \chi\big(t, \tfrac{t}{\epsilon}\big) &= o(1)
\end{aligned}
\right\} \tag{5.14}
$$

in $L^\infty(\Omega_T^+)$ when $\epsilon \to 0$.

Proof. As in Proposition 5.1, it suffices to consider the first line of (5.14). The proof is to use the method of integration along characteristic curves. At first, let us investigate characteristic curves for (5.8) and (5.12).

For the problem (5.8), fix any $(t, x) \in \Omega_T^+$, let $\xi_k^\pm(s) = x \pm (\lambda_k^\pm - \sigma)(s - t)$, and let $s \to \left(s, \xi_k^\pm(s) + \varepsilon y_{\varepsilon,k}^\pm(s; t, x)\right)$ be the characteristic curves of $\partial_t \pm \left(\lambda_k(u_\pm + \varepsilon v_\pm^\varepsilon) - (\sigma + \varepsilon d_t \Phi^\varepsilon)\right) \partial_x$ through (t, x), which means that $y_{\varepsilon,k}^\pm(s; t, x)$ satisfy

$$
\left.
\begin{aligned}
d_s y_{\varepsilon,k}^\pm(s; t, x) &= \pm \varepsilon^{-1} \left(\lambda_k \left(u_\pm + \varepsilon v_\pm^\varepsilon \left(s, \xi_k^\pm(s) + \varepsilon y_{\varepsilon,k}^\pm(s)\right)\right)\right. \\
&\quad \left. - \lambda_k^\pm - \varepsilon d_t \Phi^\varepsilon(s)\right) \\
y_{\varepsilon,k}^\pm(t; t, x) &= 0.
\end{aligned}
\right\}
\tag{5.15}
$$

For the problem (5.12), fix any $(t, x, \theta) \in \Omega_T^+ \times \mathbb{R}$, and let $s \to \left(s, \xi_k^\pm(s), \theta + Y_k^\pm(s; t, x, \theta)\right)$ be the characteristic curve of $\partial_t \pm (\lambda_k^\pm - \sigma) \partial_x \pm \left(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\right) \partial_\theta$ through (t, x, θ), where $Y_k^\pm(s)$ satisfies the problem

$$
\left.
\begin{aligned}
d_s Y_k^\pm(s; t, x, \theta) &= \pm \left(\gamma_\pm^k(V_\pm)\left(s, \xi_k^\pm(s), \theta + Y_k^\pm(s)\right) - \mathbb{E}_0 K(s)\right) \\
Y_k^\pm(t; t, x, \theta) &= 0.
\end{aligned}
\right\}
\tag{5.16}
$$

For any $k \in \{1, \ldots, m\}$, denote by $\widetilde{\Omega}_T^+$ the set

$$
\widetilde{\Omega}_T^+ = \begin{cases} \{(s, t, x) \mid 0 \le s \le t \quad ((t, x) \in \Omega_T^+)\} & \text{for } k \in I \\ \{(s, t, x) \mid \max\left(0, t \pm \frac{x}{\sigma - \lambda_k^\mp}\right) \le s \le t \quad ((t, x) \in \Omega_T^+)\} & \text{for } k \in II \end{cases}
$$

where

$$
I = \begin{cases} \{1, \ldots, j\} & \text{for the case " + "} \\ \{j, \ldots, m\} & \text{for the case " - "} \end{cases} \qquad \text{and} \qquad II = \{1, \ldots, m\} \setminus I
$$

as given in the decompositions (3.3.9) and (3.3.10).

For the problems (5.15) and (5.16), we quote a result from J. L. Joly et al. [8: Proposition 6.4.2] as follows.

Lemma 5.1. $y_{\varepsilon,k}^\pm$ are bounded in $C_\varepsilon^1(\widetilde{\Omega}_T^+)$, $Y_k^\pm \in C^1(\widetilde{\Omega}_T^+ : \mathbb{R})$, and

$$
y_{\varepsilon,k}^\pm(s; t, x) - Y_k^\pm\left(s; t, x, \frac{x \mp (\lambda_k^\pm - \sigma)t}{\varepsilon}\right) = o(1)
\tag{5.17}
$$

in $L^\infty(\widetilde{\Omega}_T^+)$ when $\varepsilon \to 0$.

Now, we turn to prove the first line of (5.14). For any $k \in I$, both (5.8) and (5.12) are the initial value problems for $u_{\pm,k}^\varepsilon$ and $U_{\pm,k}$, and by using J. L. Joly et al. [8: Proposition 6.4.1] we immediately obtain the result (5.14).

Let us discuss the case $k \in II$. As before, it is sufficient to investigate the component $u^\varepsilon_{-,1}(t,x)$. For any $(t,x) \in \Omega^+_T$, let $s(t,x) = \frac{x}{\lambda^-_1 - \sigma} + t$, and $s_\varepsilon = s_\varepsilon(t,x)$ be the root of the algebraic equation

$$x - (\lambda^-_1 - \sigma)(s - t) + \varepsilon y^-_{\varepsilon,1}(s;t,x) = 0.$$

That is to say that $(s_\varepsilon(t,x),0)$ and $(s(t,x),0)$ are the intersection points of the characteristic curves of

$$\partial_t - (\lambda_1(u_- + \varepsilon v^\varepsilon_-) - (\sigma + \varepsilon d_t \Phi^\varepsilon))\partial_x \qquad \text{and} \qquad X^-_1 = \partial_t - (\lambda^-_1 - \sigma)\partial_x,$$

respectively, at the boundary $\{x = 0\}$. Denote

$$\Omega^+_{T,1} = \{(t,x) \in \Omega^+_T | s(t,x) \le 0\}$$
$$\Omega^+_{T,1_\varepsilon} = \{(t,x) \in \Omega^+_T | s_\varepsilon(t,x) \le 0\}$$
$$\Omega^+_{T,2} = \Omega^+_T \setminus \Omega^+_{T,1}$$
$$\Omega^+_{T,2_\varepsilon} = \Omega^+_T \setminus \Omega^+_{T,1_\varepsilon}.$$

Obviously, we know that, for any (t,x) in $\Omega^+_{T,1}$ and $\Omega^+_{T,1_\varepsilon}$, (5.12) and (5.8) are Cauchy problems for $U_{-,1}$ and $u^\varepsilon_{-,1}$, respectively; and for any (t,x) in $\Omega^+_{T,2}$ and $\Omega^+_{T,2_\varepsilon}$, (5.12) and (5.8) are boundary value problems, respectively.

When $(t,x) \in \Omega^+_{T,1} \cap \Omega^+_{T,1_\varepsilon}$, by using the result of [8: Proposition 6.4.1] again, we obtain the asymptotic expansion (5.14) for $u^\varepsilon_{-,1}(t,x)$. For any $(t,x) \in \Omega^+_{T,2} \cap \Omega^+_{T,2_\varepsilon}$, from (5.8) and (5.12) we have

$$u^\varepsilon_{-,1}(t,x) = u^\varepsilon_{-,1}(s_\varepsilon,0) + \int_{s_\varepsilon}^t f^\varepsilon_{-,1}\Big(\tau, \xi^-_1(\tau) + \varepsilon y^-_{\varepsilon,1}(\tau)\Big) d\tau \tag{5.18}$$

and

$$U_{-,1}(t,x,\theta) = U_{-,1}\big(s,0,\theta + Y^-_1(s;t,x,\theta)\big)$$
$$+ \int_s^t \mathbb{E}^1_- F_{-,1}\Big(\tau, \xi^-_1(\tau); \theta + Y^-_1(\tau)\Big) d\tau \tag{5.19}$$

where $s_\varepsilon = s_\varepsilon(t,x)$ and $s = s(t,x)$. Let

$$s_\varepsilon(t,x) - s(t,x) = \varepsilon \eta_\varepsilon(t,x). \tag{5.20}$$

Then from the definition of $s_\varepsilon(t,x)$, we have

$$\eta_\varepsilon(t,x) = \frac{1}{\lambda^-_1 - \sigma} y^-_{\varepsilon,1}\big(s(t,x) + \varepsilon \eta_\varepsilon(t,x); t,x\big) = \frac{1}{\lambda^-_1 - \sigma} y^-_{\varepsilon,1}\big(s(t,x); t,x\big) + o(1)$$

which implies

$$\eta_\varepsilon(t,x) = \frac{1}{\lambda^-_1 - \sigma} Y^-_1\left(s(t,x); t,x, \frac{x + (\lambda^-_1 - \sigma)t}{\varepsilon}\right) + o(1) \tag{5.21}$$

by using Lemma 5.1. By employing this and the asymptotics of the components $u^\epsilon_{\pm,I}$ for the boundary conditions in (5.8) and (5.12), we obtain

$$
\begin{aligned}
u^\epsilon_{-,1}(s_\epsilon,0) &= U_{-,1}\left(s_\epsilon,0;\frac{(\lambda^-_1 - \sigma)s_\epsilon}{\epsilon}\right) + o(1) \\
&= U_{-,1}\left(s,0;\theta + Y^-_1(s;t,x,\theta)\right)\Big|_{\theta=\frac{x+(\lambda^-_1-\sigma)t}{\epsilon}} + o(1).
\end{aligned}
\tag{5.22}
$$

Let us consider the integral terms on the right-hand sides of (5.18) and (5.19). Without loss of generality, we assume $s_\epsilon(t,x) \le s(t,x)$. By using the hypothesis (5.11) and $|s_\epsilon(t,x) - s(t,x)| = o(1)$, we obtain

$$
\begin{aligned}
\int^t_{s_\epsilon} &f^\epsilon_{-,1}\left(\tau,\xi^-_1(\tau) + \epsilon y^-_{\epsilon,1}(\tau)\right)d\tau \\
&= \int^t_s F_{-,1}\left(\tau,\xi^-_1(\tau);\bar{\varphi}_\pm\left(\frac{\tau}{\epsilon},\frac{\xi^-_1(\tau)}{\epsilon}\right) + Y^-_1(\tau)\right)d\tau + o(1)
\end{aligned}
\tag{5.23}
$$

where $Y^-_1(\tau) = Y^-_1(\tau;t,x,\frac{x+(\lambda^-_1-\sigma)t}{\epsilon})$ and $\xi^-_1(\tau) = x - (\lambda^-_1 - \sigma)(\tau - t)$. Applying the result of the non-stationary phase in J. L. Joly et al. [8: Theorem 4.4.2] to the right-hand side of (5.23), it follows

$$
\begin{aligned}
\int^t_{s_\epsilon} &f^\epsilon_{-,1}\left(\tau,\xi^-_1(\tau) + \epsilon y^-_{\epsilon,1}(\tau)\right)d\tau \\
&= \int^t_s \mathbb{E}^1_- F_{-,1}\left(\tau,\xi^-_1(\tau);\theta + Y^-_1(\tau;t,x,\theta)\right)d\tau\Big|_{\theta=\frac{x+(\lambda^-_1-\sigma)t}{\epsilon}} + o(1).
\end{aligned}
\tag{5.24}
$$

Substituting (5.22) and (5.24) into (5.18) and (5.19), it immediately follows that, for any $(t,x) \in \Omega^+_{T,2} \cap \Omega^+_{T,2_\epsilon}$,

$$
u^\epsilon_{-,1}(t,x) - U_{-,1}\left(t,x;\frac{x+(\lambda^-_1-\sigma)t}{\epsilon}\right) = o(1).
\tag{5.25}
$$

For any $(t,x) \in \Omega^+_{T,1} \cap \Omega^+_{T,2_\epsilon}$, the solutions of the problems (5.8) and (5.12) can be expressed as

$$
u^\epsilon_{-,1}(t,x) = u^\epsilon_{-,1}(s_\epsilon,0) + \int^t_{s_\epsilon} f^\epsilon_{-,1}\left(\tau,\xi^-_1(\tau) + \epsilon y^-_{\epsilon,1}(\tau)\right)d\tau
\tag{5.26}
$$

and

$$
\begin{aligned}
U_{-,1}(t,x,\theta) &= U^1_{-,0}\left(x + (\lambda^-_1 - \sigma)t,\theta + Y^-_1(0;t,x,\theta)\right) \\
&+ \int^t_0 \dot{\mathbb{E}}^1_- F_{-,1}\left(\tau,\xi^-_1(\tau);\theta + Y^-_1(\tau)\right)d\tau.
\end{aligned}
\tag{5.27}
$$

Since $s_\epsilon(t,x) - s(t,x) = \epsilon\eta_\epsilon(t,x)$ with $s_\epsilon(t,x) > 0$ and $s(t,x) \le 0$, we have $0 < s_\epsilon(t,x) \le \epsilon\eta_\epsilon(t,x)$, which implies $|s_\epsilon(t,x)| = o(1)$. Hence, by the same argument as in (5.24), we obtain

$$
\int_{s_\epsilon}^{t} f_{-,1}^{\epsilon}\Big(\tau, \xi_1^-(\tau) + \epsilon y_{\epsilon,1}^-(\tau)\Big)d\tau
$$

$$
= \int_{s_\epsilon}^{t} \mathbb{E}_-^1 F_{-,1}\Big(\tau, \xi_1^-(\tau); \theta + Y_1^-(\tau; t,x,\theta)\Big)d\tau\Big|_{\theta = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon}} + o(1) \qquad (5.28)
$$

$$
= \int_0^{t} \mathbb{E}_-^1 F_{-,1}\Big(\tau, \xi_1^-(\tau); \theta + Y_1^-(\tau; t,x,\theta)\Big)d\tau\Big|_{\theta = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon}} + o(1).
$$

From $0 \le -s(t,x) \le \epsilon\eta_\epsilon(t,x)$, we have $|x + (\lambda_1^- - \sigma)t| = |(\lambda_1^- - \sigma)s(t,x)| = o(1)$ which implies

$$
U_{-,0}^1\Big(x + (\lambda_1^- - \sigma)t, \theta + Y_1^-(0; t,x,\theta)\Big)\Big|_{\theta = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon}}
$$
$$
= U_{-,1}\Big(s_\epsilon, 0; \theta + Y_1^-(0; t,x,\theta)\Big)\Big|_{\theta = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon}} + o(1) \qquad (5.29)
$$

by using the zero-th order compatibility condition of (5.12).

On the other hand, from the definition of $s_\epsilon(t,x)$, we have

$$
\frac{(\lambda_1^- - \sigma)s_\epsilon}{\epsilon} = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon} + y_{\epsilon,1}^-(s_\epsilon; t,x)
$$
$$
= (\theta + Y_1^-(0; t,x,\theta))\Big|_{\theta = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon}} + o(1)
$$

which gives rise to

$$
U_{-,1}\Big(s_\epsilon, 0; \theta + Y_1^-(0; t,x,\theta)\Big)\Big|_{\theta = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon}} = U_{-,1}\Big(s_\epsilon, 0; \frac{(\lambda_1^- - \sigma)s_\epsilon}{\epsilon}\Big) + o(1)
$$
$$
= u_{-,1}^\epsilon\big(s_\epsilon(t,x), 0\big) + o(1) \qquad (5.30)
$$

by using the boundary conditions in (5.8) and (5.12).

Combining (5.29) with (5.30), it follows

$$
u_{-,1}^\epsilon\big(s_\epsilon(t,x), 0\big)
$$
$$
= U_{-,0}^1\Big(x + (\lambda_1^- - \sigma)t, \theta + Y_1^-(0; t,x,\theta)\Big)\Big|_{\theta = \frac{x + (\lambda_1^- - \sigma)t}{\epsilon}} + o(1). \qquad (5.31)
$$

By substituting (5.28) and (5.31) into (5.26) and (5.27), we obtain the same result as in (5.25).

For the case of $(t,x) \in \Omega_{T,2}^+ \cap \Omega_{T,1_\epsilon}^+$, we can get the conclusion (5.25) in the same way as above. In summary, for any $(t,x) \in \Omega_T^+$, we obtain the asymptotic expansion (5.14) for the component $u_{-,1}^\epsilon(t,x)$ ∎

Let us consider the semilinear problem with linear diagonal principal part

$$\left.\begin{aligned}
\partial_t u_\pm^\epsilon \pm \Big(\Lambda(u_\pm + \varepsilon v_\pm^\epsilon) - (\sigma + \varepsilon d_t \Phi^\epsilon)I\Big)\partial_x u_\pm^\epsilon & \\
+ m_\pm(\varepsilon v_\pm^\epsilon, w_\pm^\epsilon)u_\pm^\epsilon + Q_\pm(\varepsilon v_\pm^\epsilon, u_\pm^\epsilon) &= f_\pm^\epsilon \\
d_t\phi^\epsilon[u] + (\sigma I - \Lambda_+)u_+^\epsilon - (\sigma I - \Lambda_-)u_-^\epsilon &= g^\epsilon \quad \text{on } x = 0 \\
\phi^\epsilon(0) &= 0 \\
u_\pm^\epsilon(0, x) &= u_{\pm,0}^\epsilon(x)
\end{aligned}\right\} \qquad (5.32)$$

where every notation and assumption are the same as in (5.8),

$$m_\pm(v, w) = \sum_l m_{\pm,l}(v)w_l \qquad \text{and} \qquad Q_\pm^k(v, u) = \sum_{i,p} Q_{\pm,k}^{ip}(v)u_i u_p, \qquad (5.33)$$

$m_\pm(v, w)$ are linear in w and $Q_\pm(v, u)$ are quadratic forms in u, with $m_{\pm,l}$ and $(Q_{\pm,k}^{ip})_{ip}$ being $(m \times m)$-matrices. Suppose that $w_\pm^\epsilon$ are bounded in $C_\epsilon^0(\Omega_T^+)$ and satisfy the asymptotic property

$$w_\pm^\epsilon(t, x) - W_\pm\Big(t, x; \vec{\varphi}_+\big(\tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big), \vec{\varphi}_-\big(\tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big)\Big) = o(1) \qquad (5.34)$$

in $L^\infty(\Omega_T^+)$ where $W_\pm(t, x; \theta) \in C^0(\Omega_T^+ : \mathbb{R}^{2m})$. For the problem (5.32), we have the following

Proposition 5.3. *Suppose that the zero-th order compatibility conditions for the problem (5.32) and the following problem (5.36) are valid. Then there is $T_1 \in (0, T]$ such that, in $L^\infty(\Omega_{T_1}^+)$,*

$$\left.\begin{aligned}
u_\pm^\epsilon(t, x) - U_\pm\big(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big) &= o(1) \\
d_t\phi^\epsilon(t) - \chi\big(t, \tfrac{t}{\epsilon}\big) &= o(1)
\end{aligned}\right\} \qquad (5.35)$$

where $(U_\pm, \chi) \in C^0(\Omega_{T_1}^+ : \mathbb{R}^2) \times C^0([0, T_1] : \mathbb{R})$ are the unique solutions to the problem

$$\left.\begin{aligned}
\mathbb{E}_\pm U_\pm &= U_\pm \\
X_k^\pm U_{\pm,k} \pm \big(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\big)\partial_\theta U_{\pm,k} & \\
+ \mathbb{E}_\pm^k\Big(\sum_{l,p} \overline{m}_{\pm,l}^{kp} W_{\pm,l} U_{\pm,p} + \sum_{i,p} \overline{Q}_{\pm,k}^{ip} U_{\pm,i} U_{\pm,p}\Big) &= \mathbb{E}_\pm^k F_{\pm,k} \\
\chi[u] + (\sigma I - \Lambda_+)U_+ - (\sigma I - \Lambda_-)U_- &= 0 \quad \text{on } x = \theta = 0 \\
U_\pm|_{t=\tau=0} &= U_{\pm,0}(x, \theta)
\end{aligned}\right\} \qquad (5.36)$$

with every notation being the same as in (5.12), $\overline{m}_{\pm,l} = m_{\pm,l}(0)$ and $\overline{Q}_{\pm,k}^{ip} = Q_{\pm,k}^{ip}(0)$.

This result can be easily obtained by using Proposition 5.2, and the idea of the proof can be found in J. L. Joly et al. [8: Proposition 6.5.1]. Of course, this propositon can be generalized to the case of $m_\pm$ and $Q_\pm$ depending upon (t, x) also.

Consider the linear problem with non-diagonal principal part

$$
\left.
\begin{aligned}
\partial_t u_\pm^\epsilon \pm \Big(A(u_\pm + \epsilon v_\pm^\epsilon) - (\sigma + \epsilon d_t \Phi^\epsilon) I \Big) \partial_x u_\pm^\epsilon & \\
+ m_\pm (\epsilon v_\pm^\epsilon, w_\pm^\epsilon) u_\pm^\epsilon &= f_\pm^\epsilon \\
d_t \phi^\epsilon[u] + (\sigma I - \Lambda_+) u_+^\epsilon - (\sigma I - \Lambda_-) u_-^\epsilon &= g^\epsilon \quad \text{on } x = 0 \\
\phi^\epsilon(0) &= 0 \\
u_\pm^\epsilon(0, x) &= u_{\pm,0}^\epsilon(x)
\end{aligned}
\right\}
\tag{5.37}
$$

where all hypotheses are the same as in Propositions 5.2 and 5.3. Moreover, let

$$
v_\pm^\epsilon(t, x) - V_\pm \big(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon} \big) = o(1) \quad \text{in } C_\epsilon^1(\Omega_T^+) \tag{5.38}
$$

with $V_\pm(t, x; \tau, \theta) \in C^1(\Omega_T^+ : \mathbb{R}^2)$ being the same as in (5.10): $\mathbb{E}_\pm V_\pm = V_\pm$. As before, $A_\pm$ are assumed to be the diagonal matrices $\Lambda_\pm$.

Proposition 5.4. *Under the assumption of the zero-th order compatibility conditions for the problem* (5.37) *and the following problem* (5.40) *being valid, we have*

$$
\left.
\begin{aligned}
u_\pm^\epsilon(t, x) - U_\pm \big(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon} \big) &= o(1) \\
d_t \phi^\epsilon(t) - \chi \big(t, \tfrac{t}{\epsilon} \big) &= o(1)
\end{aligned}
\right\}
\tag{5.39}
$$

in $L^\infty(\Omega_T^+)$, where $(U_\pm, \chi) \in C^0(\Omega_T^+ : \mathbb{R}^2) \times C^0([0, T] : \mathbb{R})$ are the unique weak solutions to the problem

$$
\left.
\begin{aligned}
\mathbb{E}_\pm U_\pm &= U_\pm \\
X_k^\pm U_{\pm,k} \pm \mathbb{E}_\pm^k \Big(B_\pm^k (\partial_\theta U_\pm, V_\pm) - K \partial_\theta U_{\pm,k} & \\
\pm \sum_{l,p} \overline{m}_{\pm,l}^{kp} W_{\pm,l} U_{\pm,p} \Big) &= \mathbb{E}_\pm^k F_{\pm,k} \\
\chi[u] + (\sigma I - \Lambda_+) U_+ - (\sigma I - \Lambda_-) U_- &= 0 \ on \ x = \theta = 0 \\
U_\pm |_{t = \tau = 0} &= U_{\pm,0}(x, \theta).
\end{aligned}
\right\}
\tag{5.40}
$$

Proof. Suppose that $T_\pm(v)$ are the diagonalizers of $A(u_\pm + v)$,

$$
T_\pm^{-1}(v) A(u_\pm + v) T_\pm(v) = \Lambda(u_\pm + v). \tag{5.41}
$$

Obviously, $T_\pm(v)$ can be easily computed from the right eigenvectors $\{r_k(u)\}_{k=1}^m$ of $A(u)$, and $T_\pm(0) = I$. By performing the transformation

$$
\hat{u}_\pm^\epsilon = T_\pm^{-1}(\epsilon v_\pm^\epsilon) u_\pm^\epsilon \tag{5.42}
$$

in the problem (5.37), we know that $\hat{u}_\pm^\epsilon$ satisfies

$$\left.\begin{array}{r}\partial_t \hat{u}_\pm^\epsilon \pm \Big(\Lambda(u_\pm + \varepsilon v_\pm^\epsilon) - (\sigma + \varepsilon d_t \Phi^\epsilon)I\Big)\partial_x \hat{u}_\pm^\epsilon + \hat{m}_\pm \hat{u}_\pm^\epsilon = \hat{f}_\pm^\epsilon \\ d_t \phi^\epsilon[u] + (\sigma I - \Lambda_+)\hat{u}_+^\epsilon - (\sigma I - \Lambda_-)\hat{u}_-^\epsilon = o(1) \quad \text{on } x = 0 \\ \phi^\epsilon(0) = 0 \\ \hat{u}_\pm^\epsilon(0, x) = \hat{u}_{\pm,0}^\epsilon(x) := T_\pm^{-1}(\varepsilon v_\pm^\epsilon)u_{\pm,0}^\epsilon(x)\end{array}\right\} \quad (5.43)$$

where $\hat{f}_\pm^\epsilon = T_\pm^{-1}(\varepsilon v_\pm^\epsilon)f_\pm^\epsilon$ and $\widehat{m}_\pm^\epsilon = T_\pm^{-1}(\varepsilon v_\pm^\epsilon)m_\pm T_\pm(\varepsilon v_\pm^\epsilon) + \hat{n}_\pm^\epsilon$, with

$$\begin{aligned}\hat{n}_\pm^\epsilon &= T_\pm^{-1}(\varepsilon v_\pm^\epsilon)\partial_t T_\pm(\varepsilon v_\pm^\epsilon) \\ &\pm \Big(\Lambda(u_\pm + \varepsilon v_\pm^\epsilon) - (\sigma + \varepsilon d_t \Phi^\epsilon)I\Big)T_\pm^{-1}(\varepsilon v_\pm^\epsilon)\partial_x T_\pm(\varepsilon v_\pm^\epsilon).\end{aligned} \quad (5.44)$$

For any $\partial = \partial_t$ or $\partial = \partial_x$, we have

$$\partial T_\pm(\varepsilon v_\pm^\epsilon) = \sum_{p=1}^m \frac{\partial T_\pm}{\partial v_p}(0)(\varepsilon\partial)v_{\pm,p}^\epsilon + o(1). \quad (5.45)$$

Substituting (5.45) into (5.44), and using the assumption (5.38), it follows

$$\hat{n}_\pm^\epsilon(t, x) = N_\pm\big(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big) + o(1) \quad (5.46)$$

in $L^\infty(\Omega_T^+)$ where $N_\pm = (N_\pm^{kl})_{m \times m}$ with

$$N_\pm^{kl} = \pm \sum_{p=1}^m (\lambda_k^\pm - \lambda_p^\pm)\frac{\partial T_\pm^{kl}}{\partial v_p}(0)\partial_\theta V_{\pm,p}\left(t, x; \frac{x \mp (\lambda_p^\pm - \sigma)t}{\epsilon}\right) + o(1). \quad (5.47)$$

Here, we still regard $V_{\pm,p}(t, x; \tau, \theta)$ as functions of $\big(t, x; \theta \mp (\lambda_p^\pm - \sigma)\tau\big)$.

Employing Proposition 5.3 for the problem (5.43) with the case $Q_\pm = 0$, we have

$$\left.\begin{array}{r}\hat{u}_\pm^\epsilon(t, x) - U_\pm\big(t, x; \tfrac{t}{\epsilon}, \tfrac{x}{\epsilon}\big) = o(1) \\ d_t \phi^\epsilon(t) - \chi\big(t, \tfrac{t}{\epsilon}\big) = o(1)\end{array}\right\} \quad (5.48)$$

in $L^\infty(\Omega_T^+)$, where $(U_\pm, \chi) \in \mathcal{C}^0(\Omega_T^+ : \mathbb{R}^2) \times \mathcal{C}^0([0, T] : \mathbb{R})$ are the unique weak solutions to the problem

$$\left.\begin{array}{r}\mathbb{E}_\pm U_\pm = U_\pm \\ X_k^\pm U_{\pm,k} \pm \big(\gamma_\pm^k(V_\pm) - \mathbb{E}_0 K\big)\partial_\theta U_{\pm,k} \\ +\mathbb{E}_\pm^k\left(\sum_p \Big(N_\pm^{kp} + \sum_l \overline{m}_{\pm,l}^{kp} W_{\pm,l}\Big)U_{\pm,p}\right) = \mathbb{E}_\pm^k F_{\pm,k} \\ \chi[u] + (\sigma I - \Lambda_+)U_+ - (\sigma I - \Lambda_-)U_- = 0 \quad \text{on } x = \theta = 0 \\ U_\pm|_{t=\tau=0} = U_{\pm,0}(x, \theta).\end{array}\right\} \quad (5.49)$$

Obviously, we have

$$\hat{u}_\pm^\epsilon(t,x) - u_\pm^\epsilon(t,x) = o(1) \tag{5.50}$$

in $L^\infty(\Omega_T^+)$ by making use of $T_\pm(0) = I$.

By comparing the problems (5.40) with (5.49), and using Lemma 4.1, we know that it suffices to verify

$$\mathbb{E}_\pm^k \left(\sum_{p=1}^m N_\pm^{kp} U_{\pm,p} \right) = \pm \Xi_\pm^k(U_\pm, \partial_\theta V_\pm). \tag{5.51}$$

From (5.41), we have

$$\frac{\partial A}{\partial v_l}(u_\pm) = \frac{\partial T_\pm}{\partial v_l}(0)\Lambda_\pm - \Lambda_\pm \frac{\partial T_\pm}{\partial v_l}(0) + \frac{\partial \Lambda}{\partial v_l}(u_\pm)$$

which implies

$$\frac{\partial^2 f_k(u_\pm)}{\partial v_i \partial v_l} = \frac{\partial T_\pm^{ki}}{\partial v_l}(0)\lambda_i^\pm - \lambda_k^\pm \frac{\partial T_\pm^{ki}}{\partial v_l}(0) + \delta_{ki}\frac{\partial \lambda_k}{\partial v_l}(u_\pm)$$

by using the fact $A = \nabla f = \left(\frac{\partial f_k}{\partial v_i} \right)_{m \times m}$. Hence, the coefficients $b_{\pm,k}^{il}$ of the bilinear forms $B_\pm^k(\cdot, \cdot)$ given in Lemma 4.1 have the expressions

$$b_{\pm,k}^{il} = \frac{\partial T_\pm^{ki}}{\partial v_l}(0)(\lambda_i^\pm - \lambda_k^\pm) + \delta_{ki}\frac{\partial \lambda_k}{\partial v_l}(u_\pm)$$

which implies

$$\begin{aligned}
\Xi_\pm^k(U_\pm, \partial_\theta V_\pm) &= \mathbb{E}_\pm^k \left(\sum_{i \neq k, l \neq k} \frac{\lambda_k^\pm - \lambda_l^\pm}{\lambda_i^\pm - \lambda_k^\pm} b_{\pm,k}^{il} U_{\pm,i} \partial_\theta V_{\pm,l} \right) \\
&= \mathbb{E}_\pm^k \left(\sum_{i \neq k, l \neq k} (\lambda_k^\pm - \lambda_l^\pm) \frac{\partial T_\pm^{ki}}{\partial v_l}(0) U_{\pm,i} \partial_\theta V_{\pm,l} \right)
\end{aligned} \tag{5.52}$$

by using (4.2.8). By noting the basic fact $\mathbb{E}_\pm^k(\partial_\theta V_{\pm,p}) = 0$ for any $p \neq k$, and using the expression (5.47) of $N_\pm^{kl}$, we immediately obtain the assertion (5.51) ∎

Now, let us consider the asymptotics of derivatives of solutions to the linear version of the problem (5.3): Study the linear problem

$$\left. \begin{aligned}
\partial_t u_\pm^\epsilon \pm \left(A(u_\pm + \epsilon v_\pm^\epsilon) - (\sigma + \epsilon d_t \Phi^\epsilon)I \right)\partial_x u_\pm^\epsilon &= f_\pm^\epsilon \\
d_t \phi^\epsilon[u] + (\sigma I - \Lambda_+)u_+^\epsilon - (\sigma I - \Lambda_-)u_-^\epsilon &= g^\epsilon \quad \text{on } x = 0 \\
\phi^\epsilon(0) &= 0 \\
u_\pm^\epsilon(0, x) &= u_{\pm,0}^\epsilon(x)
\end{aligned} \right\} \tag{5.53}$$

where $v_\pm^\epsilon$ have the same property as in (5.38), $f_\pm^\epsilon$, Φ^ϵ, g^ϵ and $u_{\pm,0}^\epsilon$ are bounded in $C_\epsilon^1(\Omega_T^+)$, $\widetilde{C}_\epsilon^2[0,T]$, $C_\epsilon^1[0,T]$ and $C_\epsilon^1(\omega^+)$, respectively, and have the asymptotic properties

$$
\left.\begin{aligned}
f_\pm^\epsilon(t,x) - F_\pm\big(t,x;\vec\varphi_+\big(\tfrac{t}{\epsilon},\tfrac{x}{\epsilon}\big),\vec\varphi_-\big(\tfrac{t}{\epsilon},\tfrac{x}{\epsilon}\big)\big) &= o(1) \\
d_t\Phi^\epsilon(t) - K\big(t,\tfrac{t}{\epsilon}\big) &= o(1) \\
u_{\pm,0}^\epsilon(x) - U_{\pm,0}\big(x,\tfrac{x}{\epsilon}\big) &= o(1) \\
g^\epsilon(t) &= o(1)
\end{aligned}\right\}
\tag{5.54}
$$

in C_ϵ^1, where $F_\pm \in \mathcal{C}^1(\Omega_T^+ : \mathbb{R}^{2m})$, $K \in \mathcal{C}^1([0,T] : \mathbb{R})$ and $U_{\pm,0} \in \mathcal{C}^1(\omega^+ : \mathbb{R})$. Furthermore, we suppose that the compatibility conditions of (5.53) and the problem

$$
\left.\begin{aligned}
\mathbb{E}_\pm U_\pm &= U_\pm \\
X_k^\pm U_{\pm,k} \pm \mathbb{E}_\pm^k\big(B_\pm^k(\partial_\theta U_\pm, V_\pm) - K\partial_\theta U_{\pm,k}\big) &= \mathbb{E}_\pm^k F_{\pm,k} \\
\chi[u] + (\sigma I - \Lambda_+)U_+ - (\sigma I - \Lambda_-)U_- &= 0 \quad \text{on } x = \theta = 0 \\
U_\pm|_{t=r=0} &= U_{\pm,0}(x,\theta)
\end{aligned}\right\}
\tag{5.55}
$$

up to order one are valid.

Proposition 5.5. *Under the above assumptions for the problems (5.53) and (5.55), we have that* $(u_\pm^\epsilon, \phi^\epsilon) \in C_\epsilon^1(\Omega_T^+) \times \widetilde{C}_\epsilon^2[0,T]$ *are bounded,* $(U_\pm, \chi) \in \mathcal{C}^1(\Omega_T^+ : \mathbb{R}^2) \times \mathcal{C}^1([0,T] : \mathbb{R})$ *and*

$$
\left.\begin{aligned}
u_\pm^\epsilon(t,x) - U_\pm\big(t,x;\tfrac{t}{\epsilon},\tfrac{x}{\epsilon}\big) &= o(1) \\
d_t\phi^\epsilon(t) - \chi\big(t,\tfrac{t}{\epsilon}\big) &= o(1)
\end{aligned}\right\}
\tag{5.56}
$$

in $C_\epsilon^1(\Omega_T^+)$ *when* $\epsilon \to 0$.

Proof. At above, we have obtained the asymptotics (5.56) in $L^\infty(\Omega_T^+)$. If we can prove

$$
\epsilon\partial_t u_{\pm,k}^\epsilon(t,x) \pm (\lambda_k^\pm - \sigma)(\partial_\theta U_{\pm,k})\Big(t,x;\tfrac{x\mp(\lambda_k^\pm-\sigma)t}{\epsilon}\Big) = o(1)
\tag{5.57}
$$

in $L^\infty(\Omega_T^+)$ for any $k \in \{1,\dots,m\}$, then the asymptotics of $\epsilon\partial_x u_\pm^\epsilon$ can be easily deduced from the equations in (5.53) and (5.55). Hence, it is enough to consider the estimate (5.57).

Set $z_\pm^\epsilon = \epsilon\partial_t u_\pm^\epsilon$ and $A_\pm^\epsilon = \big(A(u_\pm + \epsilon v_\pm^\epsilon) - (\sigma + \epsilon d_t\Phi^\epsilon)I\big)^{-1}$. Then

$$
A_\pm^\epsilon(t,x) - (\Lambda_\pm - \sigma I)^{-1} = o(1)
\tag{5.58}
$$

in $C_\epsilon^1(\Omega_T^+)$ and $z_\pm^\epsilon$ satisfy

$$
\left.\begin{aligned}
\partial_t z_\pm^\epsilon \pm \Big(A(u_\pm + \epsilon v_\pm^\epsilon) - (\sigma + \epsilon d_t\Phi^\epsilon)I\Big)\partial_x z_\pm^\epsilon &= \mathcal{F}_\pm^\epsilon \\
\epsilon d_t^2\phi^\epsilon[u] + (\sigma I - \Lambda_+)z_+^\epsilon - (\sigma I - \Lambda_-)z_-^\epsilon &= \epsilon d_t g^\epsilon \quad \text{on } x = 0 \\
z_\pm^\epsilon(0,x) &= \pm(\sigma I - \Lambda_\pm)(\partial_\theta U_{\pm,0})\big(x,\tfrac{x}{\epsilon}\big) + o(1) \quad \text{in } L^\infty
\end{aligned}\right\}
\tag{5.59}
$$

from the problem (5.53), where

$$\mathcal{F}_\pm^\epsilon = \epsilon \partial_t f_\pm^\epsilon + B_\pm(\epsilon \partial_t v_\pm^\epsilon, A_\pm^\epsilon z_\pm^\epsilon) - A_\pm^\epsilon (\epsilon d_t^2 \Phi^\epsilon) z_\pm^\epsilon + o(1) \tag{5.60}$$

in $L^\infty(\Omega_T^+)$. We note that the problem (5.59) - (5.60) has the same form as (5.37), with $w_\pm^\epsilon = (\epsilon d_t^2 \Phi^\epsilon, \epsilon \partial_t v_\pm^\epsilon)$ and

$$m_\pm(\epsilon v_\pm^\epsilon, w_\pm^\epsilon) z_\pm^\epsilon = A_\pm^\epsilon(\epsilon d_t^2 \Phi^\epsilon) z_\pm^\epsilon - B_\pm(\epsilon \partial_t v_\pm^\epsilon, A_\pm^\epsilon z_\pm^\epsilon). \tag{5.61}$$

Set $m_\pm(v, w) = \sum_{l=0}^m m_{\pm,l}(v) w_l$, with $m_{\pm,l} = (m_{\pm,l}^{kp})_{k,p}$ being an $(m \times m)$-matrix. By simple computation, we deduce that

$$\overline{m}_{\pm,l}^{kp} = m_{\pm,l}^{kp}(0) = \begin{cases} (\lambda_p^\pm - \sigma)^{-1} \delta_{kp} & \text{if } l = 0 \\ (\sigma - \lambda_p^\pm)^{-1} b_{\pm,k}^{lp} & \text{if } l = 1, \ldots, m \end{cases} \tag{5.62}$$

where $b_{\pm,k}^{lp}$ is the coefficient of $B_\pm^k$ defined by $B_\pm^k(u, v) = \sum_{l,p} b_{\pm,k}^{lp} u_l v_p$. Applying Proposition 5.4 to the problem (5.59), we get

$$\left. \begin{array}{l} z_\pm^\epsilon(t, x) - Z_\pm\left(t, x; \frac{t}{\epsilon}, \frac{x}{\epsilon}\right) = o(1) \\[2mm] \epsilon d_t^2 \phi^\epsilon(t) - \xi\left(t, \frac{t}{\epsilon}\right) = o(1) \end{array} \right\} \tag{5.63}$$

in $L^\infty(\Omega_T^+)$, where $(Z_\pm, \xi) \in \mathcal{C}^0(\Omega_T^+ : \mathbb{R}^2) \times \mathcal{C}^0([0,T] : \mathbb{R})$ are the unique weak solutions to the problem

$$\left. \begin{array}{l} \mathbb{E}_\pm Z_\pm = Z_\pm \\[3mm] X_k^\pm Z_{\pm,k} \pm \mathbb{E}_\pm^k \left(B_\pm^k(\partial_\theta Z_\pm, V_\pm) - K \partial_\theta Z_{\pm,k} \right. \\[3mm] \qquad \left. + \sum_{l,p} \dfrac{\sigma - \lambda_l^\pm}{\sigma - \lambda_p^\pm} b_{\pm,k}^{lp} \partial_\theta V_{\pm,l} Z_{\pm,p} \right) = \mathbb{E}_\pm^k(\nabla_\Theta F_{\pm,k} \cdot \vec{a}) \\[3mm] \xi[u] + (\sigma I - \Lambda_+) Z_+ - (\sigma I - \Lambda_-) Z_- = 0 \quad \text{on } x = \theta = 0 \\[3mm] Z_\pm|_{t=\tau=0} = \pm(\sigma I - \Lambda_\pm)(\partial_\theta U_{\pm,0})(x, \theta) \end{array} \right\} \tag{5.64}$$

by using

$$\left. \begin{array}{l} \epsilon \partial_t f_\pm^\epsilon(t, x) - \nabla_\Theta F_\pm\left(t, x; \vec{\varphi}_+\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \vec{\varphi}_-\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)\right) \cdot \vec{a} = o(1) \\[3mm] \epsilon \partial_t v_{\pm,k}^\epsilon(t, x) \pm (\lambda_k^\pm - \sigma) \partial_\theta V_{\pm,k}\left(t, x; \frac{x \mp (\lambda_k^\pm - \sigma) t}{\epsilon}\right) = o(1) \end{array} \right\}$$

and

$$\mathbb{E}_\pm^k(\partial_\tau K Z_{\pm,k}) = 0 \qquad \text{for all} \quad k \in \{1, \ldots, m\},$$

with $\Theta \in \mathbb{R}^{2m}$ being the last $2m$ variables of $F_\pm$ and

$$\vec{a} = \left(\sigma - \lambda_1^+, \ldots, \sigma - \lambda_m^+, \lambda_1^- - \sigma, \ldots, \lambda_m^- - \sigma \right)^T.$$

To prove the assertion (5.57), it remains to verify that

$$Z_{\pm,k} = \widetilde{U}_{\pm,k}(t,x;\tau,\theta) := \pm(\sigma - \lambda_k^{\pm})(\partial_\theta U_{\pm,k})(t,x;\tau,\theta) \left.\right\}$$

$$\xi = \partial_\tau \chi(t,\tau) \hspace{5cm} \tag{5.65}$$

satisfy the problem (5.64), where $U_{\pm}$ and χ are the unique solutions of the problem (5.55). From that problem, it is easy to verify that the boundary and initial conditions in (5.64) are satisfied by $(\widetilde{U}_{\pm}, \partial_\tau \chi)$. On the other hand, from the definition of $B_{\pm}^k$, it can be checked that

$$(\sigma - \lambda_k^{\pm}) \partial_\theta \mathbb{E}_{\pm}^k B_{\pm}^k (\partial_\theta U_{\pm}, V_{\pm})$$

$$= \pm \mathbb{E}_{\pm}^k \left(B_{\pm}^k (\partial_\theta Z_{\pm}, V_{\pm}) + \sum_{l,p} \frac{\sigma - \lambda_l^{\pm}}{\sigma - \lambda_p^{\pm}} b_{\pm,k}^{lp} \partial_\theta V_{\pm,l} Z_{\pm,p} \right) \tag{5.66}$$

which implies that $\widetilde{U}_{\pm}$ satisfy the equation in (5.64) by acting the operator ∂_τ on the equation of (5.55). Thus, we obtain the assertion (5.65) ∎

By applying the above propositions in the problems (5.4) and (5.6), it immediately gives rise to the following result.

Theorem 5.1. *Suppose that* $(u_{\pm}^{\varepsilon,\nu}, \phi^{\varepsilon,\nu}) \in C_\varepsilon^1(\Omega_T^+) \times \widetilde{C}_\varepsilon^2[0,T]$ *and* $(U_{\pm}^{\nu}, \chi^{\nu}) \in C^1(\Omega_T^+ : \mathbb{R}^2) \times C^1([0,T] : \mathbb{R})$ *are the solution sequences of the problems* (5.4) *and* (5.6), *respectively, for each* $\nu \geq 0$. *Then*

$$u_{\pm}^{\varepsilon,\nu}(t,x) - U_{\pm}^{\nu}\left(t,x;\tfrac{t}{\varepsilon},\tfrac{x}{\varepsilon}\right) = o(1) \left.\right\}$$

$$d_t \phi^{\varepsilon,\nu}(t) - \chi^{\nu}\left(t,\tfrac{t}{\varepsilon}\right) = o(1) \hspace{3cm} \tag{5.67}$$

in $C_\varepsilon^1(\Omega_T^+)$ *when* $\varepsilon \to 0$.

By using Theorem 3.1/(3), the uniform convergence of $(u_{\pm}^{\varepsilon,\nu}, \phi^{\varepsilon,\nu})$ in $C^0(\Omega_T^+) \times C^1[0,T]$ with respect to $\varepsilon \in (0,\varepsilon_0]$, as a simple consequence from Theorem 5.1 we obtain the following

Corollary 5.1. *Suppose that* $(u_{\pm}^{\varepsilon}, \phi^{\varepsilon}) \in C_\varepsilon^1(\Omega_T^+) \times \widetilde{C}_\varepsilon^2[0,T]$ *and* $(U_{\pm}, \chi) \in C^1(\Omega_T^+ : \mathbb{R}^2) \times C^1([0,T] : \mathbb{R})$ *are the unique solutions of the problems* (2.1.18) *and* (2.1.25), *respectively. Then we have the asymptotic properties*

$$\left\| u_{\pm}^{\varepsilon}(t,x) - U_{\pm}\left(t,x;\tfrac{t}{\varepsilon},\tfrac{x}{\varepsilon}\right) \right\|_{L^\infty(\Omega_T^+)} = o(1) \left.\right\}$$

$$\left\| d_t \phi^{\varepsilon}(t) - \chi\left(t,\tfrac{t}{\varepsilon}\right) \right\|_{L^\infty[0,T]} = o(1) \hspace{2cm} \tag{5.68}$$

when $\varepsilon \to 0$.

To finish the proof of Theorem 2.1/(3), it remains to prove that

$$\varepsilon \nabla_{(t,x)}\left(u_{\pm}^{\varepsilon}(t,x) - U_{\pm}\left(t,x;\tfrac{t}{\varepsilon},\tfrac{x}{\varepsilon}\right)\right) = o(1) \left.\right\}$$

$$\varepsilon d_t\left(d_t \phi^{\varepsilon}(t) - \chi\left(t,\tfrac{t}{\varepsilon}\right)\right) = o(1) \hspace{2cm} \tag{5.69}$$

in $L^\infty(\Omega_T^+)$. As before, by using the equations and boundary conditions in (2.1.18) and (2.1.25), it is sufficient to verify the following

Theorem 5.2. *With the same notations as in Corollary 5.1, there is a constant* $T > 0$ *such that, for any* $k \in \{1, \ldots, m\}$,

$$\varepsilon \partial_t u^\varepsilon_{\pm,k}(t,x) \pm (\lambda^\pm_k - \sigma)(\partial_\theta U_{\pm,k})(t,x; \tfrac{z \mp (\lambda^\pm_k - \sigma)t}{\varepsilon}) = o(1) \tag{5.70}$$

in $L^\infty(\Omega^+_T)$ *when* $\varepsilon \to 0$.

Proof. By performing the transformation

$$z^\varepsilon_\pm(t,x) = \varepsilon T^{-1}_\pm(\varepsilon u^\varepsilon_\pm)\partial_t u^\varepsilon_\pm \tag{5.71}$$

in (2.1.18), with $T_\pm(v)$ being the diagonalizer of $A(u_\pm + v)$ as in (5.41), we know that $z^\varepsilon_\pm$ satisfy the problem

$$\left.\begin{array}{l}
\partial_t z^\varepsilon_\pm \pm \Big(\Lambda(u_\pm + \varepsilon u^\varepsilon_\pm) - (\sigma + \varepsilon d_t \phi^\varepsilon)I\Big)\partial_x z^\varepsilon_\pm + Q_\pm\big(\varepsilon u^\varepsilon_\pm, \varepsilon d^2_t \phi^\varepsilon, z^\varepsilon_\pm\big) = 0 \\[2mm]
\varepsilon d^2_t \phi^\varepsilon[u] + (\sigma I - \Lambda_+)z^\varepsilon_+ - (\sigma I - \Lambda_-)z^\varepsilon_- = o(1) \quad \text{in} \ \ L^\infty \\[2mm]
z^\varepsilon_\pm(0,x) = \pm(\sigma I - \Lambda_\pm)(\varepsilon d_x u^\varepsilon_{\pm,0})(x) + o(1) \quad \text{in} \ \ L^\infty
\end{array}\right\} \tag{5.72}$$

where

$$Q_\pm = \Big(T^{-1}_\pm(\varepsilon u^\varepsilon_\pm)\partial_t T_\pm(\varepsilon u^\varepsilon_\pm) \pm \Big(\Lambda(u_\pm + \varepsilon u^\varepsilon_\pm)$$

$$- (\sigma + \varepsilon d_t \phi^\varepsilon)I\Big)T^{-1}_\pm(\varepsilon u^\varepsilon_\pm)\partial_x T_\pm(\varepsilon u^\varepsilon_\pm)\Big)z^\varepsilon_\pm$$

$$- T^{-1}_\pm(\varepsilon u^\varepsilon_\pm)\Big(\partial_t A(u_\pm + \varepsilon u^\varepsilon_\pm) - \varepsilon d^2_t \phi^\varepsilon I\Big)A^\varepsilon_\pm T_\pm(\varepsilon u^\varepsilon_\pm)z^\varepsilon_\pm$$

with $A^\varepsilon_\pm = \big(A(u_\pm + \varepsilon u^\varepsilon_\pm) - (\sigma + \varepsilon d_t \phi^\varepsilon)I\big)^{-1}$. Obviously, we have

$$Q_\pm = \sum_{p=1}^m \frac{\partial T_\pm(0)}{\partial v_p} z^\varepsilon_{\pm,p} z^\varepsilon_\pm - (\Lambda_\pm - \sigma I)\sum_{p=1}^m \frac{\partial T_\pm(0)}{\partial v_p}(\lambda^\pm_p - \sigma)^{-1}z^\varepsilon_{\pm,p} z^\varepsilon_\pm$$

$$+ \varepsilon d^2_t \phi^\varepsilon(\Lambda_\pm - \sigma I)^{-1}z^\varepsilon_\pm - B_\pm\Big(z^\varepsilon_\pm, (\Lambda_\pm - \sigma I)^{-1}z^\varepsilon_\pm\Big) + o(1) \tag{5.73}$$

in $L^\infty(\Omega^+_T)$. The problem (5.72) has the same form as (5.32). By applying Proposition 5.3 in the problem (5.72), and using (5.73), we obtain

$$\left.\begin{array}{l}
z^\varepsilon_\pm(t,x) - Z_\pm\big(t,x; \tfrac{t}{\varepsilon}, \tfrac{x}{\varepsilon}\big) = o(1) \\[2mm]
\varepsilon d^2_t \phi^\varepsilon(t) - \xi\big(t, \tfrac{t}{\varepsilon}\big) = o(1)
\end{array}\right\} \tag{5.74}$$

in $L^\infty(\Omega^+_T)$, where $(Z_\pm, \xi) \in C^0(\Omega^+_T : \mathbb{R}^2) \times C^0([0,T] : \mathbb{R})$ are the unique weak solutions to the problem

$$\left.\begin{array}{l}
\mathbb{E}_\pm Z_\pm = Z_\pm \\[2mm]
X^\pm_k Z_{\pm,k} \pm \Big(\gamma^k_\pm(U_\pm) - \mathbb{E}_0\chi\Big)\partial_\theta Z_{\pm,k} \\[2mm]
+\mathbb{E}^k_\pm\left(\sum_l M^{kl}_\pm Z_{\pm,l} + \xi(\lambda^\pm_k - \sigma)^{-1}Z_{\pm,k}\right) \\[2mm]
\qquad -\mathbb{E}^k_\pm B^k_\pm\Big(Z_\pm, (\Lambda_\pm - \sigma I)^{-1}Z_\pm\Big) = 0 \\[2mm]
\xi[u] + (\sigma I - \Lambda_+)Z_+ - (\sigma I - \Lambda_-)Z_- = 0 \quad \text{on} \ x = \theta = 0 \\[2mm]
\qquad Z_\pm|_{t=\tau=0} = \pm(\sigma I - \Lambda_\pm)(\partial_\theta U_{\pm,0})(x,\theta)
\end{array}\right\} \tag{5.75}$$

with

$$M_{\pm}^{kl} = \sum_{p \neq k} \frac{\lambda_p^{\pm} - \lambda_k^{\pm}}{\lambda_p^{\pm} - \sigma} \frac{\partial T_{\pm}^{kl}(0)}{\partial v_p} Z_{\pm,p}. \tag{5.76}$$

To end the proof of (5.70), as in (5.65), it remains to verify that

$$\left. \begin{aligned} Z_{\pm,k} &= \widetilde{U}_{\pm,k}(t,x;\tau,\theta) = \pm(\sigma - \lambda_k^{\pm})(\partial_\theta U_{\pm,k})(t,x;\tau,\theta) \\ \xi &= \partial_\tau \chi(t,\tau) \end{aligned} \right\} \tag{5.77}$$

satisfy the problem (5.75), where $U_{\pm}$ and χ are the unique solutions of the problem (2.1.25). From here, it is easy to see that $\widetilde{U}_{\pm}$ and $\partial_\tau \chi$ satisfy the boundary and initial conditions in (5.75). As in the proof of Proposition 5.5, it is a direct computation to verify that $\widetilde{U}_{\pm}$ and $\partial_\tau \chi$ also satisfy the equation in (5.75) by acting the operator ∂_τ on the equation of (2.1.25) and using the expression of $B_{\pm}^k$. Thus, we have completed the proof of (5.70) ∎

Note in the proof. After the finish of this work, the author was informed that a similar problem had been investigated by A. Corli in [3].

References

[1] Cheverry, C.: *Justification de l'optique géométrique non linéaire pour un systeme de lois de conservation.* Preprint. Université de Rennes I, 1995, 50 pp.

[2] Chikhi, J.: *Sur la réflexion des oscillations pour un système à deux vitesses.* C.R. Acad. Sci. Paris 313 (1991), 675 – 678.

[3] Corli, A.: *Weakly nonlinear geometric optics for hyperbolic systems of conservation laws with shock waves.* Asympt. Anal. 10 (1995), 117 – 172.

[4] Corli, A. and M. Sablé-Tougeron: *Perturbation à variation bornée d'un choc de grande amplitude.* C.R. Acad. Sci. Paris 321 (1995), 537 – 540.

[5] DiPerna, R. and A. Majda: *The validity of nonlinear geometric optics for weak solutions of conservation laws.* Commun. Math. Phys. 98 (1985), 313 – 347.

[6] Hartman, P. and A. Wintner: *Hyperbolic partial differential equations.* Amer. J. Math. 74 (1952), 834 – 864.

[7] He, Y. and T. B. Moodie: *Two-wave interactions for hyperbolic signaling problems and the application to supersonic gas dynamics.* Stud. Appl. Math. 90 (1993), 249 – 277.

[8] Joly, J. L., Métivier, G. and J. Rauch: *Resonantly one dimensional nonlinear geometric optics.* J. Funct. Anal. 114 (1993), 106 – 231.

[9] Joly, J. L., Métivier, G. and J. Rauch: *Coherent and focusing multidimensional nonlinear geometric optics.* Ann. scient. Éc. Norm. Sup. (4ᵉ Serie) 28 (1995), 51 – 113.

[10] Katznelson, Y.: *Introduction to Harmonic Analysis.* New York: Dover 1976.

[11] Lax, P. D.: *Hyperbolic systems of conservation laws.* Part II. Comm. Pure Appl. Math. 10 (1957), 537 – 556.

[12] Majda, A.: *The stability of multi-dimensional shock fronts.* Mem. Amer. Math. Soc. 275 (1983), 95 pp.

[13] Majda, A.: *The existence of multi-dimensional shock fronts.* Mem. Amer. Math. Soc. 281 (1983), 93 pp.

[14] Majda, A. and M. Artola: *Nonlinear geometric optics for hyperbolic mixed problems.* In: Analyse Mathématique et Applications (eds. C. Agmon et al.). Paris: Guathier-Villars 1988, pp. 319 – 356.

[15] Majda, A. and R. R. Rosales: *Resonantly interacting weakly nonlinear hyperbolic waves.* Part I: *A single space variable.* Stud. Appl. Math. 71 (1984), 149 – 179.

[16] Sablé-Tougeron, M.: *Justification de l'optique géométrique non linéaire pour le probléme mixte: cas des concentrations.* Asympt. Anal. 12 (1996), 169 – 186.

[17] Schochet, S.: *Resonant nonlinear geometric optics for weak solutions of conservation laws.* J. Diff. Equ. 113 (1994), 473 – 504.

[18] Wang, Y. G.: *Nonlinear geometric optics for shock waves.* Part I: *Scalar case.* Z. Anal. Anw. 16 (1997), 607 – 619.

[19] Williams, M.: *Resonant reflection of multidimensional semilinear oscillations.* Commun. Part. Diff. Equ. 18 (1993), 1901 – 1959.

[20] Williams, M.: *Nonlinear geometric optics for hyperbolic boundary problems.* Preprint. Univ. North Carolina 1995, 60 pp.