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A Semilinear Elliptic Equation with Dirac Measure as Right-Hand Side

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Abstract. We investigate solutions to the problem

$$\left.\begin{aligned} \Delta u &= \lambda e^u + m\delta && \text{in } \mathcal{D}'(\Omega) \\ u &= g && \text{a.e. on } \partial\Omega, \end{aligned}\right\}$$

where δ is the Dirac measure and λ, m are real parameters, $m > 0$. We discuss the existence and uniqueness of solutions in dependence of these parameters. For the homogeneous Dirichlet problem in a ball we give multiplicity results.

Keywords: *Gelfand equation with Dirac measure, distributional solutions, existence and multiplicity results, phase plane analysis*

AMS subject classification: 35 B 32, 35 D 05, 35 D 10, 35 J 65

1. Introduction

In this paper we investigate L^1-solutions of the Dirichlet problem

$$\left.\begin{aligned} \Delta u &= \lambda e^u + m\delta && \text{in } \mathcal{D}'(\Omega) \\ u &= g && \text{a.e. on } \partial\Omega. \end{aligned}\right\} \qquad (1)$$

Here δ is the Dirac measure, and Ω is a bounded C^2-domain of $\mathbb{R}^n$, containing the support of the Dirac measure. The parameters λ and m are real, m is restricted to be positive. We note that for dimension $n = 1$ problem (1) can be solved completely for all real λ and m. In the case of positive λ there is a unique solution for every pair (λ, m). For negative λ there are critical bounds for the parameters, such that a solution either must or cannot exist.

We will investigate problem (1) for dimensions $n > 1$ in dependence of the parameters. For every positive pair (λ, m) we show the existence and uniqueness of the solution. In the case of negative λ and positive m we find bounds for these parameters, for which we state existence respectively non-existence of the solution. If we specify the problem for homogeneous Dirichlet data, we find multiple solutions for certain parameters.

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Nonlinear elliptic equations with measures have been investigated by different authors (see, e.g., L. Boccardo and T. Gallouet [3], L. Boccardo and F. Murat [4], T. Kilpelinen and X. Xu [12]. These results are statements on existence and uniqueness of the solution. Assumptions on the measure determine the choice of the solution space. In the cited papers the measure is assumed to be a Radon measure with supplementary properties such as boundedness (see, e.g., [3]) or bounded variation and absolute continuity with respect to the p-capacity (see T. Kilpelinen and X. Xu [12]). The investigations in [3, 4, 12] are restricted to Sobolev spaces, e.g. $W^{1,1}_{loc}(\Omega)$. The techniques are based on the approximation of the measure by means of functions in Sobolev spaces and the use of estimates in suitable L^p-norms. In our case (Dirac measure) it is impossible to obtain such estimates.

F. Rothe [15] proposes to use the homogeneous Dirichlet problem

$$\left.\begin{array}{ll} \Delta u = |x|^\alpha u|u|^\beta + m\delta & \text{in } D'(\Omega) \\ u = 0 & \text{on } \partial\Omega \end{array}\right\}$$

to find examples for the sharpness of regularity results for $m = 0$. He gives an example of a distributional solution, but no solution class is specified.

In the present paper we find solutions in larger spaces than Sobolev spaces. Outside of a small neighbourhood of the boundary no assumptions on the first derivative are needed. Our technique is based on separating the singular part of the equation. The reduced differential equation will be regular and can be treated in Sobolev spaces.

First we must formulate our problem correctly, i.e. we must specify the boundary conditions of problem (1) and make assumptions, how to understand the non-linearity e^u in weak sense. We use the following notations:

$\mathcal{D}(\Omega)$	space of test functions on Ω, i.e. $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$
$\mathcal{D}'(\Omega)$	space of the corresponding Schwartz distributions on Ω
$B(x,r)$	ball with center $x \in \mathbb{R}^n$ and radius $r > 0$
B	unit sphere in $\mathbb{R}^n$
$\Gamma(\cdot)$	Gamma function
$E(\cdot)$	elementary solution of the Laplace operator. It is given as

$$E(x) = \begin{cases} \frac{1}{2\pi} \ln|x| & \text{in } \mathbb{R}^2 \\ \frac{\Gamma(\frac{n}{2})}{(n-2)2\pi^{\frac{n}{2}}} |x|^{2-n} & \text{in } \mathbb{R}^n,\ n > 2. \end{cases}$$

$K(\Omega)$ Kato class, i.e. the class of measurable functions $g : \Omega \to \mathbb{R}$ with

$$\lim_{\alpha \downarrow 0} \left(\sup_{x\in\Omega} \int_{|x-y|\le\alpha} E(x-y)|g(y)|\,dy \right) = 0.$$

Especially, one has $L^p(\Omega) \subset K(\Omega)$ for $p > \frac{n}{2}$ (see B. Simon [17]). Further, we define the strip

$$S(\varepsilon) = \left(\bigcup_{x\in\partial\Omega} B(x,\varepsilon) \right) \cap \Omega$$

for $\varepsilon > 0$.

Definition 1. Our solution classes are

$$R^{1,p}(\Omega) = \left\{ u \in L^1(\Omega) \middle| \, e^u \in L^1_{loc}(\Omega), \text{ and } u|_{S(\varepsilon)} \in W^{1,p}(S(\varepsilon)) \text{ for some } \varepsilon \in (0,1) \right\}$$

and

$$\tilde{R}^{1,p}(\Omega) = \left\{ u \in L^1(\Omega) \middle| \, e^u \in K(\Omega), \text{ and } u|_{S(\varepsilon)} \in W^{1,p}(S(\varepsilon)) \text{ for some } \varepsilon \in (0,1) \right\}.$$

Notice that these classes contain also functions with much stronger singularities than the elementary solution E. In both classes $R^{1,p}(\Omega)$ and $\tilde{R}^{1,p}(\Omega)$ problem (1) is formulated correctly, if we suppose boundary values $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$. The first equation of (1) is equivalent to

$$\int_\Omega u(x)\,\Delta\varphi(x)\,dx = -\lambda \int_\Omega e^{u(x)}\,\varphi(x)\,dx + m\varphi(0) \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Obviously, we have $\tilde{R}^{1,p}(\Omega) \subset R^{1,p}(\Omega)$. For positive λ and m we can work in the class $R^{1,p}(\Omega)$.

In Section 1 we will prove the following

Theorem 1. *Consider problem* (1) *for positive* λ, m *and boundary values* $g \in W^{2-\frac{1}{p},p}(\partial\Omega)$, $p > n$. *Then there exists a unique solution in* $R^{1,p}(\Omega)$.

In Section 2 we consider the case $\lambda < 0$ and prove the following Theorem 2. Here we state the existence of bounds for λ such that there exists at least one solution respectively there is no solution. In the latter case we will work – for technical reasons – in the class $\tilde{R}^{1,p}(\Omega)$.

Theorem 2. *Consider the problem*

$$\left.\begin{aligned} \Delta u &= -\lambda e^u + m\delta \qquad && \text{in } \mathcal{D}'(\Omega) \\ u &= g && (x \in \partial\Omega) \end{aligned}\right\} \tag{2}$$

for parameters $m, \lambda > 0$ *and* $p > n$. *Then:*

(a) *There is* $\delta_0 = \delta_0(n,\Omega) > 0$ *such that problem* (2) *possesses a solution* $u \in R^{1,p}(\Omega)$ *for every* $0 < \lambda \le \delta_0$.

(b) *Define* δ_1 *as the first (positive) eigenvalue of the problem*

$$\left.\begin{aligned} -\Delta h &= \lambda e^{mE+f} h \qquad && (x \in \Omega) \\ h &= 0 && (x \in \partial\Omega) \end{aligned}\right\} \tag{3}$$

where the function f *is defined as the solution of the problem*

$$\left.\begin{aligned} \Delta f &= 0 \qquad && (x \in \Omega) \\ f &= g - mE && (x \in \partial\Omega). \end{aligned}\right\} \tag{4}$$

Then there is no solution $u \in \tilde{R}^{1,p}(\Omega)$ *of problem* (2) *for* $\lambda > \delta_1$.

The main goal of our paper is to establish multiplicity results for the homogeneous Dirichlet problem (Theorems 3 - 5). For its formulation we first need some concepts, which will be introduced later. The formulation of these results will be given in Section 3. The main part of the proof for Theorems 4 and 5 is done by phase plane analysis similar as in I.M. Gelfand [7].

1. Existence and uniqueness for positive parameters

We consider the case $\lambda, m \geq 0$ and proceed with the proof of Theorem 1 stated above.

Proof of Theorem 1. The ansatz

$$u = v + mE \tag{5}$$

yields the following problem for the function v:

$$\left.\begin{aligned} \Delta v &= \lambda e^{mE} e^{v} && \text{in } \Omega \\ v &= g - mE && \text{on } \partial\Omega. \end{aligned}\right\} \tag{6}$$

It is possible to construct weak sub- and supersolutions for the latter problem. Using a result due to S. I. Pokhozhaev [13] we can find a solution $v \in W^{2,p}(\Omega)$ of problem (6). Then $u = v + mE$ is a solution of problem (1).

Now we will show the existence of a constant $C = C(g) > 0$ such that for every solution $u \in R^{1,p}(\Omega)$ we have the estimate

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) < C. \tag{7}$$

For this purpose we use the regularization $\vartheta_\varepsilon : \mathbb{R}^n \to \mathbb{R}^+$ $(\varepsilon > 0)$ given by

$$\vartheta_\varepsilon(x) = \begin{cases} k_\varepsilon \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}\right) & \text{for } |x| < \varepsilon \\ 0 & \text{for } |x| \geq \varepsilon \end{cases} \quad \text{with } k_\varepsilon^{-1} = \varepsilon^n \int_{|y|\leq 1} exp\left(\frac{1}{|y|^2 - 1}\right) dy.$$

By definition of $R^{1,p}(\Omega)$ there exists a constant C such that for every $u \in R^{1,p}(\Omega), p > n$ we can find a strip $S(\gamma)$ with

$$\sup_{x\in S(\gamma)} u(x) < C. \tag{8}$$

We take a C^2-domain $\Omega_1 \subset\subset \Omega$ with $\Omega \setminus S(\gamma) \subset \Omega_1$. By u_1 we denote the restriction of u to Ω_1. Next we will show that u_1 is bounded from above. We take $\varepsilon < \operatorname{dist}(\partial\Omega, \partial\Omega_1)$ and consider the convolution

$$(u_1 * \vartheta_\varepsilon)(x) = \int_{B(0,\varepsilon)} u(x+z)\vartheta_\varepsilon(z)\,dz \qquad (x \in \Omega_1).$$

It is in $C^\infty(\overline{\Omega}_1)$. Owing to $\Delta(u_1 * \vartheta_\varepsilon) \geq 0$ in $\mathcal{D}'(\Omega_1)$ and the classical maximum principle it follows

$$\sup_{\overline{\Omega}_1}(u_1 * \vartheta_\varepsilon) \leq \sup_{\partial\Omega_1}(u_1 * \vartheta_\varepsilon). \tag{9}$$

The estimate

$$\sup_{x\in\partial\Omega_1} (u_1 * \vartheta_\varepsilon)(x) \leq C \qquad (\varepsilon < \operatorname{dist}(\partial\Omega_1, \partial S(\gamma)))$$

implies together with (9)

$$\sup_{\overline{\Omega}_1}(u_1 * \vartheta_\varepsilon) \leq C \qquad \big(\varepsilon < \operatorname{dist}(\partial\Omega_1, \partial S(\gamma))\big). \tag{10}$$

Suppose now $\operatorname{ess\,sup}_{\Omega_1} u_1 > C$, i.e. there exist $\alpha > 0$ and a set of positive measure $A \subset \Omega_1$ such that $u_1(x) > C + \alpha$ for all $x \in A$. By (10) it holds

$$u_1(x) - (u_1 * \vartheta_\varepsilon)(x) > C + \alpha - C = \alpha \qquad (x \in A). \tag{11}$$

Consequently, $\|u_1 - u_1 * \vartheta_\varepsilon\|_{L^1(\Omega_1)} \geq \alpha \operatorname{meas}(A)$. On the other hand, $\lim_{\varepsilon \to 0} \|u_1 - u_1 * \vartheta_\varepsilon\|_{L^1(\Omega_1)} = 0$. This is a contradiction. Consequently, $\operatorname{ess\,sup}_{\Omega_1} u_1 \leq C$. Owing to the independence of C on Ω_1 we obtain (7). As a consequence for every solution $u \in R^{1,p}(\Omega)$ one has $\Delta(u - mE) = \lambda e^u$ in $\mathcal{D}'(\Omega)$ where the right-hand side of that equation is in $L^\infty(\Omega)$. Then $w := u - mE \in W^{2,p}(\Omega)$.

Suppose now the existence of two solutions $u_1, u_2 \in R^{1,p}(\Omega)$. Then we find for $w_1 := u_1 - mE$ and $w_2 := u_2 - mE$

$$\left.\begin{aligned} \Delta(w_1 - w_2) &= \lambda e^{mE}(e^{w_1} - e^{w_2}) && \text{in } \mathcal{D}'(\Omega) \\ w_1 - w_2 &= 0 && \text{on } \partial\Omega. \end{aligned}\right\}$$

The application of the mean value theorem and the Aleksandrov maximum principle (see Gilbarg and Trudinger [9]) yields now $w_1 = w_2$ ∎

Remarks. 1. Instead of the assumption $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we make use of the stronger assumption $g \in W^{2-\frac{1}{p},p}(\partial\Omega)$. This is needed for the application of Pokhozhaev's supersolution technique [13].

2. We have $\sup_\Omega u \leq \sup_{\partial\Omega} u$ for the solution of Theorem 1 (maximum principle).

3. Supposing for the domain only $\Omega \in C^{1,1}$ and boundary values $g \in W^{\frac{1}{2},2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ we can show the existence of a solution $u \in R^{1,2}(\Omega)$ of problem (1) for every $\lambda, m > 0$. Here we use a supersolution technique due to J. Deuel and P. Hess [6].

4. The uniqueness statement of Theorem 1 can be generalized in the following way. In problem (1) we take a non-negative function $f = f(x, u)$ instead of e^u. In the definition of $R^{1,p}(\Omega)$ we replace the demand $e^u \in L^1_{loc}(\Omega)$ by $f(\cdot, u(\cdot)) \in L^1_{loc}(\Omega)$.

5. Owing to the representation (5) of our solution u of problem (1) one can show the continuous dependence of u on both parameters λ and m in certain L^q-norms, $q > 1$. For further details cf. R. Spielmann [16].

6. A similar version of Theorem 1 can be proved, if we take instead of δ a Leray form δ_Γ on a relatively compact C^1-submanifold $\Gamma \subset \mathbb{R}^n$, $\overline{\Gamma} \subset \Omega$, defined by $\langle \delta_\Gamma, \varphi \rangle := \int_\Gamma \varphi \, d\Gamma$ for $\varphi \in \mathcal{D}(\Omega)$.

2. Existence and non-existence for negative λ

Now we investigate problem (1) for negative parameters λ. We will reformulate it to obtain problem (2) with $\lambda > 0$. The aim of this section is to prove Theorem 2.

In the function class $\tilde{R}^{1,p}(\Omega)$ the following statement holds.

Lemma 1. *Suppose that $u \in \tilde{R}^{1,p}(\Omega)$ is a solution of problem (2) with boundary values $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$, $p > n$. Then $u \sim v + mE$, whereby $v \in W^{2,p}_{loc}(\Omega) \cap W^{1,p}(\Omega)$ is* *a solution of the auxiliary problem*

$$\left.\begin{aligned} -\Delta v &= \lambda e^{mE} e^{v} && \text{in } \Omega \\ v &= g - mE && \text{on } \partial\Omega. \end{aligned}\right\} \tag{12}$$

Proof. The equation in problem (2) can be written as $\Delta(u - mE) = -\lambda e^{u}$ in $\mathcal{D}'(\Omega)$. By definition of $\tilde{R}^{1,p}(\Omega)$ we have $e^{u} \in K(\Omega)$. Applying [17: Proposition A.2.4] we obtain $v := u - mE \in C(\overline{\Omega})$. Consequently, $e^{u} \in L^{\infty}(\Omega)$. This implies the assertion ∎

The next lemma is the "weak version" of a well-known classical result (cf. J. Bebernes and D. Eberly [2: Lemma 2.4]).

Lemma 2. *Suppose a bounded domain $\Omega \in C^2$, and positive functions $f_0 \in L^p(\Omega)$ $(p > n)$ and $f_1 \in L^{\infty}(\Omega)$ such that*

$$f(x,u) \geq f_0(x) + f_1(x)u \qquad ((x,u) \in \overline{\Omega} \times [0,\infty)).$$

Furthermore, denote by λ_0 the minimal (positive) eigenvalue of the problem

$$\left.\begin{aligned} -\Delta u &= \lambda f_1(x) u && (x \in \Omega) \\ u &= 0 && (x \in \partial\Omega). \end{aligned}\right\} \tag{13}$$

Then for $\lambda \geq \lambda_0$ the problem

$$\left.\begin{aligned} -\Delta u &= \lambda f(x,u) && (x \in \Omega) \\ u &= g && (x \in \partial\Omega) \end{aligned}\right\} \tag{14}$$

has no non-negative solutions $u \in W^{1,p}(\Omega)$.

Proof. Assume that the non-negative function $\overline{u} \in W^{1,p}(\Omega)$ is a solution of problem (14) for some $\lambda \geq \lambda_0$. Then it is a supersolution of the problem

$$\left.\begin{aligned} -\Delta u &= \lambda(f_0(x) + f_1(x)u) && (x \in \Omega) \\ u &= 0 && (x \in \partial\Omega). \end{aligned}\right\}$$

Regarding $\underline{u}(x) := 0$ as a subsolution we apply the supersolution technique of J. Deuel and P. Hess [6] to find a non-negative solution $u \in W^{1,2}(\Omega)$ of the last equation. Improving regularity by iteration we obtain $u \in W^{2,p}(\Omega)$. Owing to

$$-\Delta u(x) > \lambda f_1(x) u(x) \geq 0, \qquad u(x) \not\equiv 0 \ \ (x \in \Omega), \qquad u(x) = 0 \ \ (x \in \partial\Omega)$$

there follows $u(x) > 0 \ \ (x \in \Omega)$.

Let $w \in W^{1,2}(\Omega)$ be the non-negative eigenfunction of problem (13) corresponding to the eigenvalue λ_0. By a standard iteration technique one can show that $w \in W^{2,\frac{n}{2}}(\Omega)$. Now we can apply Green's second formula and obtain

$$\begin{aligned} 0 &= \int_{\partial\Omega} \left(u \frac{\partial w}{\partial \nu} - w \frac{\partial u}{\partial \nu} \right) dS \\ &= \int_{\Omega} (u \Delta w - w \Delta u)\, dx \\ &= \int_{\Omega} \Big(w(x)\big[\lambda f_0(x) + \lambda f_1(x) u(x)\big] - u(x)\big[\lambda f_1(x) w(x)\big] \Big) dx. \end{aligned}$$

It follows that

$$(\lambda_0 - \lambda) \int_\Omega f_1(x)u(x)w(x)\,dx = \lambda \int_\Omega w(x)f_0(x)\,dx > 0$$

and $\lambda < \lambda_0$. This is a contradiction to our assumption $\lambda \geq \lambda_0$ ∎

Now we prove the above mentioned Theorem 2 about the existence and non-existence of solutions of problem (2).

Proof of Theorem 2. Part (a): The proof of this statement is similar to the existence proof in Theorem 1.

Part (b): Regarding Lemma 1 it suffices to show that the auxiliary problem (12) has no solution $v \in W^{1,p}(\Omega)$ for every $\lambda > \delta_1$. Assume $v \in W^{1,p}(\Omega)$ is a solution of problem (12) for the parameter $\lambda \geq \delta_1$. By the ansatz $v = h + f$ we obtain for h the problem

$$\left.\begin{aligned} -\Delta h &= \lambda e^{mE+f} e^h && (x \in \Omega) \\ h &= 0 && (x \in \partial\Omega). \end{aligned}\right\} \tag{15}$$

Obviously, h is positive. Applying Lemma 2 to the last equation and taking into account $e^{mE+f}e^h \geq e^{mE+f} + e^{mE+f}h \quad (h \geq 0)$ we obtain the assertion ∎

3. Bifurcation of radially symmetric solutions

We investigate now problem (2) in the unit sphere B for homogeneous Dirichlet boundary conditions, i.e. we consider the problem

$$\left.\begin{aligned} \Delta u &= -\lambda e^u + m\delta && \text{in } \mathcal{D}'(B) \\ u &= 0 && (x \in \partial B) \end{aligned}\right\} \tag{16}$$

for parameters $m, \lambda > 0$. In the case $m = 0$ problem (16) is called the *Gelfand problem*. I. M. Gelfand [7], and D. Joseph and T. Lundgren [10] found for every $\lambda > 0$ the number of its solutions $u \in C^2(B) \cap C(\overline{B})$. For a survey see also J. Bebernes and D. Eberly [2].

Taking into account the Hölder continuity of e^{mE} we can conclude from Lemma 1 the representation $u \sim v + mE$ with a function $v \in C^{2,\alpha}(B)$, $\alpha \in (0,1)$ for every solution $u \in \tilde{R}^{1,p}(B)$ of problem (16). Remark that the symmetry principle [8] is not applicable to v.

Definition 2. A solution $u \in \tilde{R}^{1,p}(B)$ of problem (16) is called *radially symmetric*, if the corresponding function $v \in C^{2,\alpha}(\overline{B})$ according to Lemma 1 is radially symmetric. We denote this by $u(r) \sim v(r) + mE(r) \quad (0 < r \leq 1)$.

From problem (12) it follows that $v = v(r)$ solves the ordinary differential equation

$$\frac{d^2v}{dr} + \frac{n-1}{r}\frac{dv}{dr} + \lambda e^{mE(r)} e^{v(r)} = 0 \qquad (0 < r < 1) \tag{17}$$

with boundary conditions

$$\left.\frac{dv}{dr}\right|_{r=0} = 0 \qquad \text{and} \qquad v(1) = C(m), \tag{18}$$

whereby

$$C(m) = \begin{cases} 0 & \text{for } n = 2 \\ \frac{m\Gamma\left(\frac{n}{2}\right)}{(n-2)2\pi^{\frac{n}{2}}} & \text{for } n > 2. \end{cases}$$

For $n = 2$ this equation can be transformed by means of

$$r = e^{-t}, \qquad \tilde{v}(t) = v(r), \qquad z(t) = \tilde{v}(t) - \left(2 + \frac{m}{2\pi}\right)$$

to an exact solvable differential equation for $z = z(t)$. Solving it we get

Theorem 3. *The distribution problem* (16) *possesses for* $B \subset \mathbb{R}^2$

- *two radially symmetric solutions for every* $\lambda \in (0, \frac{1}{2}(2 + \frac{m}{2\pi})^2)$
- *one radially symmetric solution for* $\lambda = \frac{1}{2}(2 + \frac{m}{2\pi})^2$
- *no radially symmetric solutions for* $\lambda > \frac{1}{2}(2 + \frac{m}{2\pi})^2$.

Definition 3. We say that in λ occurs an *m-bifurcation*, if problem (16) and the Gelfand problem have different numbers of solutions for the value λ.

According to Theorem 3 and I. M. Gelfand [7] an m-bifurcation occurs for $\lambda \in \left[2, \frac{1}{2}(2 + \frac{m}{2\pi})^2\right]$.

Now we consider radially symmetric solutions of problem (16) for dimensions $n > 2$. We must carry out a bifurcation analysis of problem (17) - (18), but this is much more comprehensive than for the case $n = 2$. Therefore we will give the proofs for the following theorems in Section 4.

Theorem 4. *We consider problem* (16) *in* $B \subset \mathbb{R}^n$, $2 < n < 10$.

(a) *For every* $m > 0$ *there exists* $\lambda_1 = \lambda_1(m) > \lambda_{FK}$ *such that problem* (16) *possesses for every* $\lambda \in (0, \lambda_1]$ *at least one radially symmetric solution* $u \in \tilde{R}^{1,p}(B)$. *Here we denote by* λ_{FK} *the so-called Frank-Kamenetzki parameter (cf. J. Bebernes, D. Eberly* [2] *and Definition* 6 *in the following section).*

(b) *For every* $k \in \mathbb{N}$ *there exist* $m_k > 0$ *and values* $0 < \lambda_1(k) < \lambda_2(k) < \lambda_3(k)$ *such that problem* (16) *possesses for* $\lambda \in (\lambda_1(k), \lambda_2(k))$, $m = m_k$ *at least* k *radially symmetric solutions* $u \in \tilde{R}^{1,p}(B)$ *and for* $\lambda > \lambda_3(k)$, $m = m_k$ *no solutions* $u \in \tilde{R}^{1,p}(B)$.

Theorem 5. *Consider problem* (16) *in* $B \subset \mathbb{R}^n$, $n \geq 10$.

(a) *For every* $m > 0$ *there exists* $\lambda_1 = \lambda_1(m) > 2(n-2)$ *such that problem* (16) *possesses for every* $\lambda \in (0, \lambda_1]$ *at least one radially symmetric solution* $u \in \tilde{R}^{1,p}(B)$.

(b) *There exist* $\tilde{m} > 0$ *and values* $0 < \lambda_1 < \lambda_2 < \lambda_3$ *such that problem* (16) *possesses for* $\lambda \in (\lambda_1, \lambda_2)$, $m = \tilde{m}$ *at least two radially symmetric solutions* $u \in \tilde{R}^{1,p}(B)$ *and for* $\lambda > \lambda_3$, $m = \tilde{m}$ *no solutions* $u \in \tilde{R}^{1,p}(B)$.

A comparison with the bifurcation results for the Gelfand problem obtained by D. Joseph and T. Lundgren [10] yields m-bifurcations for every $n > 2$.

4. Proof of Theorems 4 and 5

For $n > 2$ problem (17) - (18) has the form

$$\frac{d^2v}{dr^2} + \frac{n-1}{r}\frac{dv}{dr} + \lambda \exp(-C(m)r^{2-n})e^{v(r)} = 0 \qquad (0 < r < 1) \tag{19}$$

with $\frac{dv}{dr}\big|_{r=0} = 0, v(1) = C(m)$ and the constant $C(m) = \frac{m\Gamma(\frac{n}{2})}{(n-2)2\pi^{\frac{n}{2}}}$.

The proofs of Theorems 4 and 5 reduce to the bifurcation analysis of equation (19). For the latter we use a phase plane analysis, i.e. we will transform this equation into a dynamical system. The technique of phase plane analysis has been developed by I. M. Gelfand [7], further we refer to D. Joseph and T. Lundgren [10], J. Bebernes and D. Eberly [2]. We use the dynamical system

$$\frac{dx}{dt} = x(y-2) \tag{20}$$

$$\frac{dy}{dt} = (n-2)y - x \tag{21}$$

which occured also in I. M. Gelfand [7]. First we give the following sketch of the proof for the bifurcation analysis of equation (19):

Step 1: We classify the trajectories of system (20) - (21) in the right half of the phase plane and specify their asymptotic behaviour by means of an *asymptotic coefficient*.

Step 2: We transform the solutions v of equation (19) into trajectories $(x(t), y(t))$ of system (20) - (21). Moreover, the parameters λ and m will be transformed into conditions for the initial value and the asymptotic coefficient of the obtained trajectory. Then we define a mapping, the so-called *asymptotic function*, between the initial values and the asymptotic coefficients. We prove some properties of the asymptotic function, important for the characterization of its level sets.

Step 3: Let $N(C)$ denote a level set of the asymptotic function. We investigate its shape for $2 < n < 10$.

Step 4: We investigate the shape of the level sets $N(C)$ for $n \geq 10$.

Step 5: By means of the shape of $N(C)$ we deduce bifurcation results of equation (19).

We proceed with

Step 1: Investigation of the associated dynamical system. We classify the trajectories of our dynamical system and give a concept to describe its asymptotical behaviour.

Lemma 3. *The system* $(20)-(21)$ *has critical points* $(0,0)$ *and* $(2(n-2), 2)$. *There exists a unique heteroclinic orbit* $T_H = (x_H(t), y_H(t))$ $(t \in \mathbb{R})$ *joining these points. In the right half of the phase plane* $\mathbb{R}_+ \times \mathbb{R}$ *we find furthermore*

- *trajectories on the ordinate* $T_{ord} = \left(0, y(0)e^{(n-2)t}\right)$ $(t \in \mathbb{R})$

• *trajectories* $T = (x(t), y(t))$ $(t \in \mathbb{R})$ *with the properties* $x(t) > 0$ $(t \in \mathbb{R})$ *as well as* $\lim_{t \to +\infty} x(t) = 0$, $\lim_{t \to +\infty} y(t) = -\infty$ *and* $\lim_{t \to -\infty}(x(t), y(t)) = (2(n-2), 2)$.

Proof. The uniqueness of the heteroclinic orbit has been shown by I. M. Gelfand [7] for $n = 3$, and by D. Joseph and T. Lundgren [10] for $n > 3$. The rest of the proof is standard ∎

Definition 4. Let be $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$ and consider the trajectory $T = (x(t), y(t))$ $(t \in \mathbb{R})$ of system (20) - (21) with $(x(0), y(0)) = (x, y)$. Assume, for some constant γ one has

$$\lim_{t \to +\infty} e^t \left(y(t) + \gamma e^{(n-2)t}\right) = 0.$$

Then γ will be called the *asymptotic coefficient* in the point (x, y).

If there exists some asymptotic coefficient in the point (x, y), it is determined uniquely. Furthermore, it is easy to see that neither in points of the heteroclinic orbit nor in $(2(n-2), 2)$ a positive asymptotic coefficient can exist. Later we will show the existence of asymptotic coefficients for all other points in the right half of the phase plane $\mathbb{R}_+ \times \mathbb{R}$.

Now we come to

Step 2: Transformation theorem and asymptotic coefficients. The following theorem points out the connection between our problem (16) and the asymptotical behaviour of the trajectories of the dynamical system (20) - (21).

Theorem 6. *The solutions of the auxiliary problem* (19) *can be characterized in the following way.*

1. *Let* v *be a solution of equation* (19). *Then we find a solution* $(x(t), y(t))$ *of system* (20) – (21) *with* $x(0) = \lambda$ *whereby* $\gamma = (n-2)C$ *is the asymptotic coefficient in* $(x(0), y(0))$.

2. *Let be given a solution* $(x(t), y(t))$ *of system* (20) – (21) *with* $x(0) = \lambda$ *whereby* $\gamma = (n-2)C$ *is the asymptotic coefficient in* $(x(0), y(0))$. *Then the function*

$$v(r) = \ln \frac{x(t(r))}{2(n-2)r^2} + Cr^{2-n} + \alpha \qquad \textit{with} \;\; t(r) = -\ln r \;\; (0 < r < 1)$$

and the constant $\alpha = \ln \frac{2(n-2)}{\lambda}$ *is a solution of equation* (19).

Proof. Let $v = v(r)$ be a solution of equation (19). By means of the transformation

$$\left.\begin{aligned} r &= e^{-t} \\ \tilde{v}(t) &= v(r) \end{aligned}\right\} \qquad (0 < t < +\infty) \tag{22}$$

and the constant $\alpha := \ln \frac{2(n-2)}{\lambda}$ we obtain from equation (19)

$$\frac{d^2 \tilde{v}}{dt^2} - (n-2)\frac{d\tilde{v}}{dt} + 2(n-2)\exp\left(\tilde{v} - C\exp((n-2)t) - 2t - \alpha\right) = 0$$

for $0 < t < +\infty$ with $\tilde{v}(0) = C$, $\frac{d\tilde{v}}{dt}(+\infty) = 0$ and $\tilde{v}(+\infty) < +\infty$. With the help of the transformation

$$z(t) = \tilde{v}(t) - 2t - C\exp((n-2)t) - \alpha \tag{23}$$

we get now

$$\frac{d^2 z}{dt^2} - (n-2)\frac{dz}{dt} - 2(n-2) + 2(n-2)e^z = 0 \qquad (0 < t < +\infty)$$

with boundary values $z(0) = -\alpha$, $\frac{dz}{dt}(+\infty) = -\infty$ and $z(+\infty) = -\infty$. By means of the transformation

$$x(t) = 2(n-2)\exp(z(t)) \tag{24}$$

$$y(t) = \frac{dz}{dt} + 2 \tag{25}$$

we obtain system (20) - (21) with the boundary conditions

$$\left.\begin{aligned} x(0) &= \lambda \\ y(0) &= -v'(1) - C(n-2) \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x(+\infty) &= 0 \\ y(+\infty) &= -\infty. \end{aligned}\right\}$$

For $\beta := -v'(1)$ it follows $y(0) = \beta$. From formula (23) we derive

$$\frac{dv}{dr} = \frac{dz}{dt}\frac{dt}{dr} + 2\frac{dt}{dr} + C(2-n)r^{1-n} = -\frac{1}{r}\big(y(t) - C(2-n)r^{2-n}\big).$$

Then we find from $v'(0) = 0$ the condition $\lim_{t\to+\infty} e^t\big(y(t) + \gamma\exp((n-2)t)\big) = 0$ with $\gamma = (n-2)C$, and the first assertion of Theorem 6 is proved. The second assertion can be shown by reversing the proof given above ∎

In the right half of the phase plane we define the set

$$M = \mathbb{R}_+ \times \mathbb{R} \setminus \Big(\big\{(2(n-2), 2)\big\} \cup \big\{(x_H(t), y_H(t)) \big| \, t \in \mathbb{R}\big\}\Big).$$

Theorem 7. *In every point $(x, y) \in M$ there exists an asymptotic coefficient.*

Proof. For all points on the ordinate our assertion follows from Lemma 3. We consider now $(x, y) \in M$ with $x > 0$ and $y < 0$. For the trajectory with initial dates $(x(0), y(0)) = (x, y)$ one has $\lim_{t\to+\infty} y(t) = -\infty$ and $x(t) > 0$ for all $t > 0$. With the expression

$$\left.\begin{aligned} x(t) &= \xi(t)\exp(-2t) \\ y(t) &= \eta(t)\exp((n-2)t) \end{aligned}\right\} \tag{26}$$

we obtain from (20) - (21)

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \xi(t)\eta(t)\exp((n-2)t) \\ \frac{d\eta}{dt} &= -\xi(t)\exp(-nt). \end{aligned}\right\} \tag{27}$$

The functions $\xi = \xi(t)$ and $\eta = \eta(t)$ are monotone decreasing for $t > 0$ and ξ possesses a continuous inverse function $t = t(\xi)$. Furthermore, there exists $\lim_{t\to\infty} \xi(t) = C_\xi \geq 0$. From (27) it follows

$$\begin{aligned}\frac{\eta'}{\xi'} &= -\frac{\exp((-2n+2)t)}{\eta}\\ \frac{(\frac{1}{2}\eta^2)'}{\xi'} &= -\exp((-2n+2)t).\end{aligned}$$

Defining

$$\begin{aligned}w(t) &= \frac{1}{2}\eta^2(t) \qquad &(t \geq 0)\\ \tilde{w}(\xi) &= w(t(\xi)) \qquad &(\xi \in (C_\xi, \xi(0)])\end{aligned}$$

we get

$$\frac{d\tilde{w}}{d\xi} = -\exp\left((-2n+2)t(\xi)\right) \quad (C_\xi < \xi \leq \xi(0))$$

$$\tilde{w}(\xi(0)) - \tilde{w}(C_\xi) = -\int_{C_\xi}^{\xi(0)} \exp\left((-2n+2)t(\xi)\right) d\xi$$

as the integral and consequently the limit $\lim_{t\to\infty} w(t) = \tilde{w}(C_\xi)$ exists because of the boundedness of the integrand. Accordingly, there exists $\lim_{t\to\infty} \eta(t) = -\sqrt{2\tilde{w}(C_\xi)}$.

We want to show that the asymptotic coefficient in $(x(0), y(0))$ is equal to $\gamma := \sqrt{2\tilde{w}(C_\xi)}$. We have

$$e^t\left(y(t) + \gamma \exp((n-2)t)\right) = \frac{\eta(t) + \gamma}{\exp((1-n)t)}.$$

For $t \to \infty$ the last fraction is an expression of the form $\frac{0}{0}$. Applying L'Hospital's rule we obtain

$$\begin{aligned}\lim_{t\to\infty} e^t\left(y(t) + \gamma\exp((n-2)t)\right) &= \lim_{t\to\infty} \frac{\eta(t)+\gamma}{\exp((1-n)t)}\\ &= \lim_{t\to\infty} \frac{\eta'(t)}{(1-n)\exp((1-n)t)}\\ &= -\lim_{t\to\infty} \frac{\xi(t)\exp(-nt)}{(1-n)\exp((1-n)t)}\\ &= 0.\end{aligned}$$

Consequently, we find in (x, y) the asymptotic coefficient $\gamma = \sqrt{2\tilde{w}(C_\xi)}$. Now we show that for every point $(x, y) \in M$ an asymptotic coefficient exists. Without loss of generality let be $x > 0$ and $y \geq 0$. We consider the trajectory $(x(t), y(t))$ with initial dates $x(0) = x$ and $y(0) = y$. Then there exists $\tau > 0$ such that $x(\tau) > 0$ and $y(\tau) < 0$. Define $\eta := t - \tau$. Accordingly, $(\tilde{x}(\tau), \tilde{y}(\tau)) := (x(t), y(t))$ are also solutions of system

(20) - (21). According to the previous considerations the point $(\tilde{x}(0), \tilde{y}(0))$ has a positive asymptotic coefficient γ. Therefore

$$\lim_{\eta \to +\infty} e^{\eta}\big(\tilde{y}(\eta) + \gamma \exp((n-2)\eta)\big) = 0$$

$$e^{-\tau} \lim_{t \to +\infty} e^{t}\big(y(t) + \gamma \exp((2-n)\tau)\exp((n-2)t)\big) = 0.$$

Then $(x(0), y(0))$ has the asymptotic coefficient $\gamma \exp((2-n)\tau)$ ∎

Hence, there exists a function $\Phi : M \to \mathbb{R}$ associating every point of M to its asymptotic coefficient. We call it the *asymptotic function*. From the last proof it follows

Corollary 1. *Let be given a trajectory $T = (x(t), y(t))$ $(t \in \mathbb{R})$ with initial dates in the set M. Then for two arbitrary points $(x(t_0), y(t_0))$ and $(x(t_1), y(t_1))$ it follows*

$$\Phi\big(x(t_1), y(t_1)\big) = \exp\big((2-n)(t_0 - t_1)\big)\, \Phi\big(x(t_0), y(t_0)\big).$$

We will show the continuity of the asymptotic function on the subset

$$M_\Phi := \mathbb{R}_+ \times \mathbb{R} \setminus \Big(\big\{(2(n-2), 2)\big\} \cup \big\{(x_H(t), y_H(t)) \,|\, t \in \mathbb{R}\big\} \cup \big\{(0, y) \,|\, y \geq 0\big\}\Big)$$

which is open in $\mathbb{R}_+ \times \mathbb{R}$. First we prove a representation formula.

Lemma 4. *Let $(x, y) \in M$ be the starting point of the trajectory $T = (x(t), y(t))$, that means we have $(x(0), y(0)) = (x, y)$. Then*

$$\Phi(x, y) = -y + \int_0^{+\infty} x(\eta) \exp((2-n)\eta)\, d\eta.$$

Proof. The assertion follows from Lemma 3 for all points on the ordinate. Consider now initial values $(x, y) \in M$ with $x > 0$. From (21) we obtain

$$y(t) = \exp((n-2)t) \left(y(0) - \int_0^t x(\eta) \exp((2-n)\eta)\, d\eta \right).$$

We consider the expression

$$A(t) = e^{t} \left[y(t) + \exp((n-2)t) \left(-y(0) + \int_0^{+\infty} x(\eta) \exp((2-n)\eta)\, d\eta \right) \right]$$

$$= -\frac{\int_{+\infty}^{t} x(\eta) \exp((2-n)\eta)\, d\eta}{\exp((1-n)t)}.$$

The last version is for $t \to +\infty$ of the form $\frac{0}{0}$. Applying L'Hospital's rule, we find

$$\lim_{t \to +\infty} e^{t} \left[y(t) + \left(-y(0) + \int_0^{+\infty} x(\eta) \exp((2-n)\eta)\, d\eta \right) \exp((n-2)t) \right]$$

$$= - \lim_{t\to+\infty} \frac{x(t)e^t}{1-n}.$$

For arbitrary real η also $(\tilde{x}(t), \tilde{y}(t)) := (x(t+\eta), y(t+\eta))$ is a solution of system (20) - (21), and it yields

$$- \lim_{t\to+\infty} \frac{x(t)e^t}{1-n} = \frac{e^\eta}{n-1} \lim_{t\to+\infty} x(t+\eta)e^t = \frac{e^\eta}{n-1} \lim_{t\to+\infty} \tilde{x}(t)e^t.$$

There exists $\eta > 0$ such that $y(\eta) < 0$ and consequently $\tilde{y}(0) < 0$. By means of expression (26) we find functions $(\tilde{u}(t), \tilde{v}(t))$, which satisfy (27). Then $\lim_{t\to+\infty} \tilde{u}(t) < \infty$ and consequently $\lim_{t\to+\infty} \tilde{x}(t)e^t = 0$. Hence

$$\lim_{t\to+\infty} e^t \left[y(t) + \left(-y(0) + \int_0^{+\infty} x(\eta) \exp((2-n)\eta)\, d\eta \right) \exp((n-2)t) \right] = 0$$

and the statement is proved ∎

Theorem 8. *The asymptotic function Φ is continuously differentiable on the set M_Φ.*

Proof. We consider the trajectory $T = \big(x(t,p,q), y(t,p,q)\big)$ $(t \in \mathbb{R})$ of system (20) - (21) with the initial dates $x(0,p,q) = p$ and $y(0,p,q) = q$. Now we want to investigate the function

$$\Phi(p,q) = -q + \int_0^{+\infty} x(t,p,q)\, e^{(2-n)t} dt \qquad \big((p,q) \in \mathbb{R}_+ \times \mathbb{R}\big).$$

First we choose $(p_0, q_0) \in M_\Phi$ with $p_0 > 0$. For every $\varepsilon > 0$ we find t_ε such that $x(t, p_0, q_0) < \varepsilon$ for all $t \geq t_\varepsilon$. The mapping $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ associates every point of the phase plane to its image after the time t_ε. For the sake of the continuous dependence of $\big(x(t,p,q), y(t,p,q)\big)$ on (p,q) there exists a ball

$$\tilde{B} = B\big(\big(x(t_\varepsilon, p_0, q_0), y(t_\varepsilon, p_0, q_0)\big), \delta_1\big) \subset M_\Phi \cap ([0,\varepsilon) \times \mathbb{R}_-),$$

which will be mapped by means of Ψ^{-1} on the open set $\Psi^{-1}\tilde{B} \subset M_\Phi$. Then yields $x(t_\varepsilon, p, q) < \varepsilon$ for every trajectory $\big(x(t,p,q), y(t,p,q)\big)$ with initial dates $(p,q) \in \Psi^{-1}\tilde{B}$. For $(p,q) \in \Psi^{-1}\tilde{B}$ it follows

$$\int_{t_\varepsilon}^{+\infty} x(t,p,q)\, e^{(2-n)t} dt < \varepsilon \int_0^{+\infty} e^{(2-n)t} dt$$

and the uniform convergence of $\int_0^{+\infty} x(t,p,q)\, e^{(2-n)t}\, dt$ is established. By means of the continuous dependence of $x(t,p,q)$ on (p,q) and the limit $\lim_{t\to+\infty} x(t,p,q) = 0$ we receive the continuity of $x(t,p,q)\, e^{(2-n)t}$ in the set $\overline{\mathbb{R}_+ \times \Psi^{-1}\tilde{B}}$. Accordingly, the integral

$\int_0^{+\infty} x(t,p,q)\, e^{(2-n)t} dt$ is continuous in the set $\overline{\Psi^{-1}\tilde{B}}$. In this way we have established the continuity of the function $\Phi = \Phi(p,q)$ on $M_\Phi \setminus \{(0,q) \,|\, q \in \mathbb{R}\}$. Analogously, we show the continuity of Φ in points $(0, q_0)$ with $q_0 < 0$.

Now we demonstrate for $(p,q) \in M_\Phi$ the uniform convergence of the integrals

$$I_1(p,q) = \int_0^{+\infty} \frac{\partial x(t,p,q)}{\partial p} e^{(2-n)t} dt \quad \text{and} \quad I_2(p,q) = \int_0^{+\infty} \frac{\partial x(t,p,q)}{\partial q} e^{(2-n)t} dt.$$

We choose $(p_0, q_0) \in M_\Phi$ with $p_0 > 0$. For arbitrary $K > 0$ and $\varepsilon > 0$ we find as above t_ε and $\delta > 0$ such that for all $(p,q) \in B((p_0,q_0),\delta)$

$$\left.\begin{aligned} x(t,p,q) &< \varepsilon \\ y(t,p,q) &< -K \end{aligned}\right\} \quad (t \geq t_\varepsilon). \tag{28}$$

We determine the function $\frac{\partial x(t,p,q)}{\partial p}$ from the variational equation (cf. V. I. Arnold [1])

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial x(t,p,q)}{\partial p} \\ \frac{\partial y(t,p,q)}{\partial p} \end{bmatrix} = \begin{bmatrix} y(t,p,q) - 2 & x(t,p,q) \\ -1 & n-2 \end{bmatrix} \begin{bmatrix} \frac{\partial x(t,p,q)}{\partial p} \\ \frac{\partial y(t,p,q)}{\partial p} \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial x(0,p,q)}{\partial p} \\ \frac{\partial y(0,p,q)}{\partial p} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let be

$$A(t,p,q) := \begin{bmatrix} y(t,p,q) - 2 & x(t,p,q) \\ -1 & n-2 \end{bmatrix}.$$

By $\langle \cdot, \cdot \rangle$ we denote the inner product of $\mathbb{R}^2$ induced by the Euclidean norm. With the notations

$$u(t,p,q) = \left(\frac{\partial x(t,p,q)}{\partial p}, \frac{\partial y(t,p,q)}{\partial p} \right)^{\mathsf{T}} \quad \text{and} \quad h(t,p,q) = \langle u(t,p,q), u(t,p,q) \rangle$$

it follows for all $(t,p,q) \in [t_\varepsilon, +\infty) \times B((p_0,q_0),\delta)$

$$\begin{aligned} \frac{d}{dt} h(t,p,q) &= 2\langle A(t,p,q)\, u(t,p,q), u(t,p,q) \rangle \\ &\leq 2\left(n - 2 + \left| \frac{x(t,p,q) - 1}{2} \right| \right) h(t,p,q). \end{aligned}$$

Let be $\tilde{C} := 2\left(n - 2 + \frac{\varepsilon+1}{2}\right)$. Then we obtain

$$\frac{d}{dt} h(t,p,q) \leq \tilde{C}\, h(t,p,q) \qquad \big((t,p,q) \in [t_\varepsilon, +\infty) \times B((p_0,q_0),\delta)\big).$$

Integrating this inequality we find a constant $C_1 > 0$ such that

$$\left| \frac{\partial y(t,p,q)}{\partial p} \right| \leq C_1 \exp(\tilde{C}\, t) \qquad \big((t,p,q) \in [t_\varepsilon, +\infty) \times B((p_0,q_0),\delta)\big). \tag{29}$$

With regard to the estimate $\frac{dy}{dt} \le (n-2)y$ $(t \in \mathbb{R})$ we find a constant $C_2 > 0$ such that

$$y(t,p,q) \le -C_2 \exp((n-2)t) \qquad ((t,p,q) \in [t_\varepsilon, +\infty) \times B((p_0,q_0),\delta)). \tag{30}$$

Estimating (20) and using (30) we obtain

$$\frac{dx}{dt} \le -C_2 \exp((n-2)t)x \qquad ((t,p,q) \in [t_\varepsilon, +\infty) \times B((p_0,q_0),\delta)).$$

Integrating the latter, we find a constant $C_3 > 0$ such that

$$x(t,p,q) \le C_3 \exp\left(-\frac{C_2}{n-2} \exp((n-2)t)\right) \quad ((t,p,q) \in [t_\varepsilon, +\infty) \times B((p_0,q_0),\delta)). \tag{31}$$

Regarding to the estimates (29) and (31) we find for $b(t,p,q) := x(t,p,q)\,\frac{\partial y(t,p,q)}{\partial p}$

$$\lim_{t\to+\infty} b(t,p,q) = 0 \quad \text{uniformly with respect to } (p,q) \in B((p_0,q_0),\delta). \tag{32}$$

Define $a(t,p,q) := y(t,p,q) - 2$ and $\phi(t,p,q) := \exp(\int_{t_\varepsilon}^t a(s)\,ds)$. From the variational equation we derive for $t \ge t_\varepsilon$

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial x(t,p,q)}{\partial p} &= a(t,p,q)\,\frac{\partial x(t,p,q)}{\partial p} + b(t,p,q) \\ \frac{\partial x(t,p,q)}{\partial p} &= \phi(t,p,q) + \int_{t_\varepsilon}^t \phi(t,p,q)\,\phi^{-1}(\tau,p,q)\,b(\tau,p,q)\,d\tau \end{aligned}$$

whereby $\frac{\partial}{\partial t}\phi(t,p,q) = a(t,p,q)\,\phi(t,p,q)$ $(t \ge t_\varepsilon)$. Using (28) we get for $\tau \in [t_\varepsilon, t]$

$$\begin{aligned} \phi(t,p,q)\,\phi^{-1}(\tau,p,q) &< \exp\big(-(K+2)(t-\tau)\big) \\ \phi(t,p,q) &< \exp\big(-(K+2)(t-t_\varepsilon)\big). \end{aligned}$$

Hence it follows for all $(p,q) \in B((p_0,q_0),\delta)$ and $t \ge t_\varepsilon$

$$\left|\frac{\partial x(t,p,q)}{\partial p}\right| < \exp\big(-(K+2)(t-t_\varepsilon)\big) + \int_{t_\varepsilon}^{t} \exp\big(-(K+2)(t-\tau)\big)\,b(\tau,p,q)\,d\tau.$$

On combining this inequality and (32), we find

$$\lim_{t\to+\infty} \frac{\partial x(t,p,q)}{\partial p} = 0 \quad \text{uniformly with respect to } (p,q) \in B((p_0,q_0),\delta). \tag{33}$$

Hence the integral $I_1(p,q)$ converges uniformly in $B((p_0,q_0),\delta)$. Analogously we verify the uniform convergence of $I_2(p,q)$ in $B((p_0,q_0),\delta)$.

Define $D := B\big((x(t_\varepsilon,p_0,q_0), y(t_\varepsilon,p_0,q_0)),\delta\big)$. By means of the continuous differentiability of $\big(x(t,p,q), y(t,p,q)\big)$ on (p,q) and (33) the continuity of the function $\frac{\partial x(t,p,q)}{\partial p}e^{(2-n)t}$ in $\overline{\mathbb{R}}_+ \times \overline{D}$ follows. Analogously we verify the continuity of $\frac{\partial x(t,p,q)}{\partial q}e^{(2-n)t}$ in $\overline{\mathbb{R}}_+ \times \overline{D}$. Hence $\int_0^{+\infty} x(t,p,q)\,e^{(2-n)t}\,dt$ is continuously differentiable in $\overline{D}$. In this way we verified the continuous differentiability of Φ in the set $M_\Phi \setminus \{(0,q) | q \in \mathbb{R}\}$. Analogously we show the continuous differentiability of Φ in all points $(0,q_0)$ with $q_0 < 0$ (instead of $\tilde{B}$ and $B((p_0,q_0),\delta)$ we choose fitting hemispheres) ∎

Corollary 1 implies for $t_1 = t$ and $t_0 = 0$ the equation

$$\frac{d\Phi(x(t), y(t))}{dt} = (n-2)\,\Phi(x(t), y(t)).$$

On the other hand we find

$$\begin{aligned}\frac{d\Phi(x(t), y(t))}{dt} &= \frac{\partial\Phi(x(t), y(t))}{\partial x}\,\frac{dx(t)}{dt} + \frac{\partial\Phi(x(t), y(t))}{\partial y}\,\frac{dy(t)}{dt} \\ &= \frac{\partial\Phi}{\partial x}\,x(y-2) + \frac{\partial\Phi}{\partial y}\,\big((n-2)y - x\big).\end{aligned}$$

In this way we obtain

Theorem 9. *In the set M_Φ one has*

$$\frac{\partial\Phi}{\partial x}\,x(y-2) + \frac{\partial\Phi}{\partial y}\,\big((n-2)y - x\big) = (n-2)\Phi.$$

Definition 5. Ψ_t $(t \in \mathbb{R})$ denotes the *flow* of the system (20) - (21). For every $\alpha > 0$, $N(\alpha) = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R} \mid \Phi(x, y) = \alpha\}$ denotes the *level set* of the asymptotic function Φ in the right half of the phase plane.

Corollary 1 implies now

Theorem 10. *For every $\alpha \in \mathbb{R}$ one has $\Psi_t(N(\alpha)) = N\big(\exp((n-2)t)\,\alpha\big)$ $(t \in \mathbb{R})$. (The flow transforms the level sets into another.)*

Now we proceed with

Step 3: Investigation of the level sets for $2 < n < 10$. Let be $2 < n < 10$. We choose a parametrization $(x_H(t), y_H(t))$ of the heteroclinic orbit. Then there exists $t_0 = \max\{t \in \mathbb{R} \mid y_H(t) = 2\}$.

Definition 6. The value $\lambda_{FK} = x_H(t_0)$ is called the *Frank-Kamenetski parameter* of the Gelfand problem for $2 < n < 10$ (cf. J. Bebernes and D. Eberly [2] and Figure 1).

The ray $\mathcal{K} = \{(x, 2) \mid x > \lambda_{FK}\}$ divides the set M_Φ into two subsets M_0 and M_1. The subset M_0 consists of pairs (x, y) with $y < 2$ (the shaded area in Figure 1). So we obtain the decomposition $M_\Phi = M_0 \cup M_1$.

Now we want to investigate the subset $N(\alpha) \cap \overline{M}_0$. The proofs of the following two lemmata are trivial and omitted.

Lemma 5. *Consider $T = (x(t), y(t))$ with initial dates $(x(t_0), y(t_0)) \in M_0 \backslash \{(0, y) \mid y \in \mathbb{R}\}$. Then:*

1. *The functions $x = x(t)$ and $y = y(t)$ are monotone decreasing for $t > t_0$.*

2. *For every $\eta > t_0$ the curve $K = \{(x(t), y(t)) \mid t_0 \le t \le \eta\}$ can be represented in the phase plane as the graph of a monotone increasing function $y = f(x)$ $(x \in [x(\eta), x(t_0)])$.*

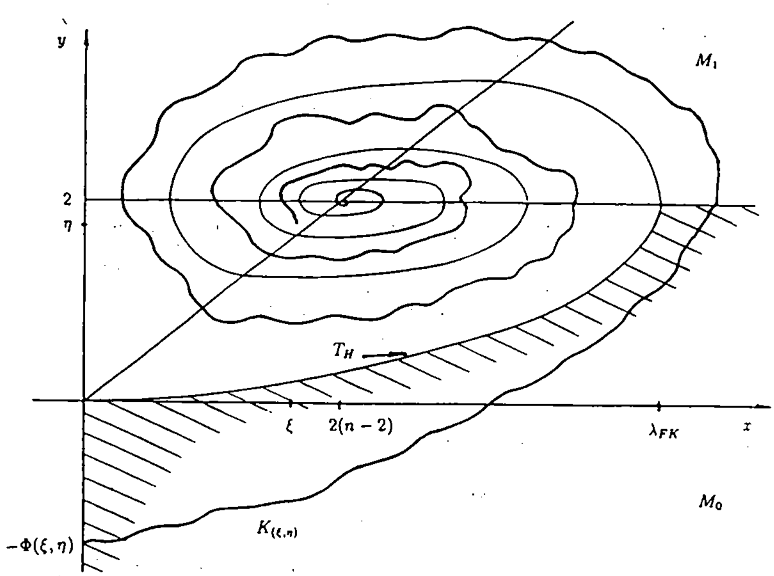

Figure 1

Lemma 6. *For all* $(x, y) \in M_0 \setminus \{(0,0)\}$ *one has* $x > (n-2)y$.

Lemma 7. *Suppose* $\alpha > 0$. *Then there exist an* $a_0 = a_0(\alpha) > \lambda_{FK}$ *and a monotone increasing continuous function* $\mathcal{F}_0$ *such that the set* $N(\alpha) \cap \overline{M}_0$ *can be represented as the graph of* $y = \mathcal{F}_0(x)$ $(x \in [0, a_0])$, *i.e.*

$$N(\alpha) \cap \overline{M}_0 = \{(x,y) |\, y = \mathcal{F}_0(x),\ x \in [0, a_0]\}.$$

Furthermore, $\mathcal{F}_0(0) = -\alpha$ *and* $\mathcal{F}_0(a_0) = 2$. *The curve* $N(\alpha) \cap \overline{M}_0$ *is rectifiable.*

To prove this statement we need the following three lemmata.

Lemma 8. *For every fixed* $x_0 \geq 0$ *the function* $\Phi(x_0, \cdot)$ *is strongly monotone decreasing on* M_0, *i.e. for points* $(x_0, y_0), (x_0, y_1) \in M_0$ *one has*

$$y_0 < y_1 \quad \Longrightarrow \quad \Phi(x_0, y_1) < \Phi(x_0, y_0).$$

Proof. In the case of $x_0 = 0$ our assumption follows from Lemma 3. Now we consider two points (x_0, y_0) and (x_0, y_1) in the set $M_0 \setminus \{(0,y) |\, y < 0\}$, whereby $y_0 < y_1$. Let be

$$T_0 = (x_0(t), y_0(t)) \qquad \text{and} \qquad T_1 = (x_1(t), y_1(t)) \qquad (t \in \mathbb{R})$$

trajectories with initial dates $(x_0(0), y_0(0)) = (x_0, y_0)$ and $(x_1(0), y_1(0)) = (x_0, y_1)$. For every $\tilde{x} \in (0, x_0)$ the trajectory T_0 intersects the straight line $x = \tilde{x}$ in exactly one point of M_0. The trajectory T_0 attains this point at the time $t_0 > 0$. Consequently, we have $x_0(t_0) = \tilde{x}$. Analogously we show the existence of an intersection point $(x_1(t_1), y_1(t_1)) \in M_0$ of T_1 and $x = \tilde{x}$, which will be reached by T_0 at the time $t_1 > 0$.

By Lemma 5 there are functions $f_0 = f_0(x)$ and $f_1 = f_1(x)$ such that

$$\{(x_0(t), y_0(t)) |\, 0 \leq t \leq t_0\} = \{(x,y) |\, x \in I_0 = [x_0(t_0), x_0(0)],\ y = f_0(x)\}$$
$$\{(x_1(t), y_1(t)) |\, 0 \leq t \leq t_1\} = \{(x,y) |\, x \in I_1 = [x_1(t_1), x_1(0)],\ y = f_1(x)\}$$

where $I_0 = I_1 = [\tilde{x}, x_0]$. The graphs of f_0 and f_1 have no common points in the phase plane. From $f_0(x_0) = y_0 < y_1 = f_1(x_0)$ now for all $x \in [\tilde{x}, x_0]$ the inequality $f_0(x) < f_1(x)$ follows. Especially, one has $y_0(t_0) = f_0(\tilde{x}) < f_1(\tilde{x}) = y_1(t_1)$. We choose a sufficiently small $\tilde{x}$. Then we get by help of Theorem 8 and Lemma 3

$$0 < \Phi(x_1(t_1), y_1(t_1)) < \Phi(x_0(t_0), y_0(t_0)). \tag{34}$$

The point (x_0, y_1) is an inner point of M_0. By means of the monotony of f_1 we find $f_1(x) < 2$ for all $x \in [\tilde{x}, x_0]$. Remembering $f_0(x) < f_1(x)$ $(x \in [\tilde{x}, x_0])$ we get now

$$\frac{1}{2 - f_0(x)} < \frac{1}{2 - f_1(x)} \qquad (\tilde{x} \le x \le x_0). \tag{35}$$

From (20)

$$t_0 = \int_{\tilde{x}}^{x_0} \frac{1}{2 - f_0(x)} \frac{dx}{x} \qquad \text{and} \qquad t_1 = \int_{\tilde{x}}^{x_0} \frac{1}{2 - f_1(x)} \frac{dx}{x} \tag{36}$$

follow. From these equations and (35) $t_0 < t_1$ follows. Therefore $\exp((2-n)t_1) < \exp((2-n)t_0)$. By means of (34) we find

$$\exp\left((2-n)t_1\right) \Phi\big(x_1(t_1), y_1(t_1)\big) < \exp\left((2-n)t_0\right) \Phi\big(x_0(t_0), y_0(t_0)\big).$$

Our assertion results now from Corollary 1 ∎

Lemma 9. *For every fixed y_0 the function $\Phi(\cdot, y_0)$ is in M_0 strongly monotone increasing, i.e.*

$$x_0 < x_1 \quad \Longrightarrow \quad \Phi(x_0, y_0) < \Phi(x_1, y_0).$$

Proof. The assertion follows for $x_0 = 0$ from Lemmata 3 and 4. Consider now points $(x_0, y_0) \in M_0$ and (x_1, y_0) with $0 < x_0 < x_1$. Then we have $(x_1, y_0) \in M_0$. We introduce the trajectory $(x_0(t), y_0(t))$ with initial dates $(x_0(0), y_0(0)) = (x_0, y_0)$ and the trajectory $(x_1(t), y_1(t))$ with initial dates $(x_1(0), y_1(0)) = (x_1, y_0)$. For a given $\tilde{y} < y_0$ there exist times $t_0 > 0$ and $t_1 > 0$ such that $y_0(t_0) = y_1(t_1) = \tilde{y}$. The curves $K_0 = \{(x_0(t), y_0(t)) | 0 \le t \le t_0\}$ and $K_1 = \{(x_1(t), y_1(t)) | 0 \le t \le t_1\}$ are completely in M_0. Now we want to compare t_0 and t_1. By Lemma 5 there are monotone increasing functions $f_0 = f_0(y)$ and $f_1 = f_1(y)$ such that

$$K_0 = \Big\{(x,y) \Big| \, x = f_0(y), \; y \in I_0 = [y_0(t_0), y_0(0)]\Big\}$$
$$K_1 = \Big\{(x,y) \Big| \, x = f_1(y), \; y \in I_1 = [y_1(t_1), y_1(0)]\Big\}$$

where $I_0 = I_1 = [\tilde{y}, y_0]$. Analogously to the derivation of (35) we find

$$\frac{1}{f_1(y) - (n-2)y} < \frac{1}{f_0(y) - (n-2)y} \qquad (y \in [\tilde{y}, y_0]). \tag{37}$$

From (21)

$$t_0 = \int_{\tilde{y}}^{y_0} \frac{dy}{f_0(y) - (n-2)y} \qquad \text{and} \qquad t_1 = \int_{\tilde{y}}^{y_0} \frac{dy}{f_1(y) - (n-2)y}$$

follows. By (37) we get

$$t_1 < t_0. \tag{38}$$

Bearing in mind the monotony of $y_0(t)$ and $y_1(t)$ for $t > 0$, we get

$$y_1(t) < y_0(t) \qquad (t > 0). \tag{39}$$

By the method of variation of constants it follows from (21) that

$$y_0(t) = \exp((n-2)t)\left(y_0(0) - \int_0^t x_0(\eta)\exp((2-n)\eta)\,d\eta\right)$$

$$y_1(t) = \exp((n-2)t)\left(y_1(0) - \int_0^t x_1(\eta)\exp((2-n)\eta)\,d\eta\right).$$

Therefore

$$\exp((2-n)t)(y_0(t) - y_1(t)) = \int_0^t (x_1(\eta) - x_0(\eta))\exp((2-n)\eta)\,d\eta. \tag{40}$$

By Theorem 7 there exists $\int_0^\infty (x_1(\eta) - x_0(\eta))\exp((2-n)\eta)\,d\eta$. By (39) for $t > 0$ the inequality $\exp((2-n)t)(y_0(t) - y_1(t)) > 0$ follows. Therefore

$$\int_0^{+\infty} (x_1(\eta) - x_0(\eta))\exp((2-n)\eta)\,d\eta = \lim_{t\to+\infty} \exp((2-n)t)(y_0(t) - y_1(t)) \geq 0$$

and

$$\int_0^{+\infty} x_1(\eta)\exp((2-n)\eta)\,d\eta \geq \int_0^{+\infty} x_0(\eta)\exp((2-n)\eta)\,d\eta.$$

By Lemma 4 there follows

$$\Phi(x_1(0), y_1(0)) \geq \Phi(x_0(0), y_0(0)). \tag{41}$$

For $(C_1, C_2) \in M_0$ we define the ray $S(C_1, C_2) = \{(x,y) \mid x \geq C_1,\ y = C_2\}$. By (41) the asymptotic function Φ is monotone non-decreasing on every such ray. We want to show that it is monotone increasing. For this purpose we assume the existence of two points (x_0, y_0) and (x_1, y_0) in M_0 with $x_0 < x_1$ such that

$$\Phi(x_0, y_0) = \Phi(x_1, y_0). \tag{42}$$

We denote the trajectories with the initial dates (x_0, y_0) and (x_1, y_0) by T_0 and T_1. Consequently we have $\Phi(x_0(0), y_0(0)) = \Phi(x_1(0), y_1(0))$. Now we fix $\tilde{y} < y_0$. By Lemma 5 there exists $t_0 > 0$ and $t_1 > 0$ such that $y_0(t_0) = y_1(t_1) = \tilde{y}$. Repeating the considerations for the proof of inequality (38) we find $t_1 < t_0$. Therefore $\exp((n-2)t_1) < \exp((n-2)t_0)$. By (42) and Corollary 1 we get

$$\exp((n-2)t_1)\,\Phi(x_1(0), y_1(0)) < \exp((n-2)t_0)\,\Phi(x_0(0), y_0(0))$$

$$\Phi(x_0(t_0), y_0(t_0)) > \Phi(x_1(t_1), y_1(t_1)).$$

This is a contradiction to (42) ∎

Now we can give the announced proof of Lemma 7.

Proof of Lemma 7. Owing to $(0,-\alpha) \in N(\alpha) \cap M_0$ we define $\mathcal{F}_0(0) := -\alpha$. By Theorem 8 and Lemmata 8 and 9 we can construct a monotone increasing continuous function $y = \mathcal{F}_0(x)$, whose graph equals the set $N(\alpha) \cap \overline{M}_0$. Now let be δ_1 the first eigenvalue of problem (3), posed in $\Omega = B$ (cf. Theorem 2). As follows from the proof of Theorem 2/(b), there is no solution of equation (19) for parameters $\lambda > \delta_1$.

We take now some $\lambda > \delta_1$ and assume $\lambda \in \operatorname{dom} \mathcal{F}_0$. Then we must find $y = \mathcal{F}_0(\lambda)$ such that $(\lambda, y) \in N(\alpha) \cap M_0$. We investigate the trajectory $(x(t), y(t))$ with $(x(0), y(0)) = (\lambda, y)$. By Theorem 6 we find a solution of equation (19) for the parameter $\lambda > \delta_1$. This is a contradiction. Consequently, the domain of the function $y = \mathcal{F}_0(x)$ is an interval $[0, a_0]$. There are the following two possibilities for $\mathcal{F}(x)$ as $x \to a_0$:

1. $\mathcal{F}_0(x)$ converges for $x \to a_0$ to a point of the heteroclinic orbit, such that $x_H(t) = a_0$ and $\lim_{x \to a_0} \mathcal{F}_0(x) = y_H(t)$ hold for some $t \in \mathbb{R}$.
2. $\lim_{x \to a_0} \mathcal{F}_0(x) = 2$.

We assume the first hypothesis. Owing to the continuity of $\mathcal{F}_0$ we find $\varepsilon > 0$ such that

$$y < -\frac{\alpha}{2} \qquad ((x,y) \in N(\alpha) \cap M_0 \cap B((0,-\alpha),\varepsilon)).$$

We choose $(x_0, y_0) \in N(\alpha) \cap M_0 \cap B((0,-\alpha),\varepsilon) \setminus \{(0,-\alpha)\}$ and consider the trajectory $(x(t), y(t))$ with $(x(0), y(0)) = (x_0, y_0)$. Then we have $\Phi(x(t), y(t)) = \alpha \exp((n-2)t)$ $(t \in \mathbb{R})$. Consequently $\Phi(x(t), y(t)) < \alpha$ $(t < 0)$, and the graph of $\mathcal{F}_0(x)$ cannot intersect the curve $\{(x(t), y(t)) | \, t < 0\}$ for $x > x_0$. For a sufficiently small δ we consider the environment $B((x_0, y_0), \delta) \cap M_0$ of (x_0, y_0) in M_0. Due to Lemmata 3 - 5 the curve

$$N(\alpha) \cap M_0 \cap B((x_0, y_0), \delta) \setminus \{(x, \mathcal{F}_0(x)) | \, x \leq x_0\}$$

is located in the phase plane below the curve

$$\{(x(t), y(t)) | \, t < 0\} \cap B((x_0, y_0), \delta).$$

Owing to the continuity of the curves $N(\alpha) \cap M_0$ and $\{(x(t), y(t)) | \, t < 0\} \cap M_0$ we conclude that $N(\alpha) \cap M_0$ is located below $\{(x(t), y(t)) | \, t < 0\} \cap M_0$. Considering $(x(t), y(t))$ and the heteroclinic orbit in the phase plane we see that our first hypothesis is false. So we have $\lim_{x \to a_0} \mathcal{F}_0(x) = 2$ for some $a_0 > 2(n-2)$. The rectifiability of the curve $N(\alpha) \cap M_0$ is a consequence of the monotony and continuity of $\mathcal{F}_0$ ∎

Corollary 2. *On the ray $\mathcal{K} = \{(x, 2) | \, x > \lambda_{FK}\}$ we find the following properties of the asymptotic function:*

- *The restriction $\Phi(x, 2)$ is strongly monotone increasing for $x > \lambda_{FK}$.*
- *One has $\lim_{x \to \lambda_{FK}} \Phi(x, 2) = 0$ and $\lim_{x \to \infty} \Phi(x, 2) = \infty$.*

Theorem 11. *For every $\alpha > 0$ we can represent the set $N(\alpha)$ as a non-intersecting rectifiable C^1-curve K_α. Denoting its arc length parameter by s, we find $b \in (0, \infty]$ with*

$$N(\alpha) = K_\alpha(s) = \{(x_\alpha(s), y_\alpha(s)) | \, 0 < s < b\} \qquad \text{and} \qquad \lim_{s \to 0} K_\alpha(s) = (0, -\alpha).$$

Furthermore, there exists $a \in (0,b)$ such that

$$N(\alpha) \cap M_0 = \{(x_\alpha(s), y_\alpha(s)) | 0 < s \le a\}$$
$$N(\alpha) \cap M_1 = \{(x_\alpha(s), y_\alpha(s)) | a \le s < b\}.$$

The point $(x_\alpha(a), y_\alpha(a)) \in N(\alpha)$ is the only intersection point of $N(\alpha)$ with the ray

$$\mathcal{K} = \{(x,2) | x > \lambda_{FK}\}.$$

Proof. By Lemma 7 we can represent the set $N(\alpha) \cap M_0$ as a non-intersecting rectifiable C^1-curve. Let be a its arc length. Then we have $N(\alpha) \cap M_0 = \{(x_\alpha(s), y_\alpha(s)) | 0 < s \le a\}$. By Corollary 2 the point $(x_\alpha(a), y_\alpha(a)) \in N(\alpha)$ is the only intersection point of $N(\alpha)$ and the ray $\mathcal{K} = \{(x,2) | x > \lambda_{FK}\}$.

We want to show that $N(\alpha) \cap M_1$ is a connected curve. Suppose that the points $(x_0, y_0) \in M_1$ and $(x_1, y_1) \in M_1$ belong to the same level set $N(\alpha)$. Consequently, there exists $t > 0$ such that

$$\Psi_t(x_0, y_0) \in N\big(\alpha \exp(n-2)t\big) \cap M_0 \quad \text{and} \quad \Psi_t(x_1, y_1) \in N\big(\alpha \exp(n-2)t\big) \cap M_0.$$

With regard to Lemma 7 there exists in M_0 a unique curve K beginning in $\Psi_t(x_0, y_0)$ and ending in $\Psi_t(x_1, y_1)$, on which one has $\Phi(K) = \alpha \exp(n-2)t$. We consider the curve $\Psi_{-t}K$. It is contained in $N(\alpha)$. Therefore it has at most one intersection point with the ray $\mathcal{K}$ (cf. Corollary 2). The initial point (x_0, y_0) and the end point (x_1, y_1) of $\Psi_{-t}K$ are in M_1. Therefore $\Psi_{-t}K$ is completely contained in M_1 and we have $\Phi\Psi_{-t}K = \alpha$. Consequently the set $N(\alpha) \cap M_1$ contains a uniquely determined curve

$$\tilde{K}\big((x_0, y_0), (x_1, y_1)\big) := \Psi_{-t}K \tag{43}$$

connecting our points (x_0, y_0) and (x_1, y_1). Hence $N(\alpha) \cap M_1$ is connected.

Regarding Theorem 9 and the implicit function theorem we find for every $(x_0, y_0) \in N(\alpha) \cap M_1$ some $\varepsilon > 0$ such that

$$N(\alpha) \cap M_1 \cap \big((x_0 - \varepsilon, x_0 + \varepsilon) \times (y_0 - \varepsilon, y_0 + \varepsilon)\big)$$

can be represented in one of the following ways as graph of a continuous function:

$$y = y(x) \qquad \big(x \in (x_0 - \varepsilon, x_0 + \varepsilon)\big) \tag{44}$$
$$x = x(y) \qquad \big(y \in (y_0 - \varepsilon, y_0 + \varepsilon)\big). \tag{45}$$

Therefore $N(\alpha) \cap M_1$ is a non-intersecting, connected C^1-curve. The rectifiability of $N(\alpha) \cap M_1$ is a consequence of the rectifiability of $N(\alpha) \cap M_0$ (cf. Lemma 7) and Theorem 10 ∎

Now we proceed with

Step 4: Investigation of the level sets for $n \ge 10$. For $n \ge 10$ we consider in the phase plane of system (20) - (21) the connected curve

$$\mathcal{K} := T_H \cup \{(x,2) | x \ge 2(n-2)\}.$$

It decomposes the set M_Φ into two subsets M_0 and M_1. By M_0 we denote the lower and by M_1 the upper of them. (M_0 is shaded in Figure 2.)

Lemma 10. *Let be $n \geq 10$. For every $\alpha > 0$ we can represent the set $N(\alpha) \cap M_0$ as graph of a monotone increasing continuous function $y = \mathcal{F}_0(x)$ $(x \in [0, a_0])$, i.e. there exists $a_0 = a_0(\alpha) > 2(n-2)$ such that*

$$N(\alpha) \cap M_0 = \{(x,y) | \, y = \mathcal{F}_0(x), \, x \in [0, a_0]\}.$$

Furthermore, $\mathcal{F}_0(0) = -\alpha$ und $\mathcal{F}_0(a_0) = 2$. On the ray $\{(x,2) | \, x > 2(n-2)\}$ we find the following properties of the asymptotic function:

- *The restriction $\Phi(x,2)$ is strongly monotone increasing for $x > 2(n-2)$.*
- *One has $\lim_{x \to 2(n-2)} \Phi(x,2) = 0$ and $\lim_{x \to \infty} \Phi(x,2) = \infty$.*

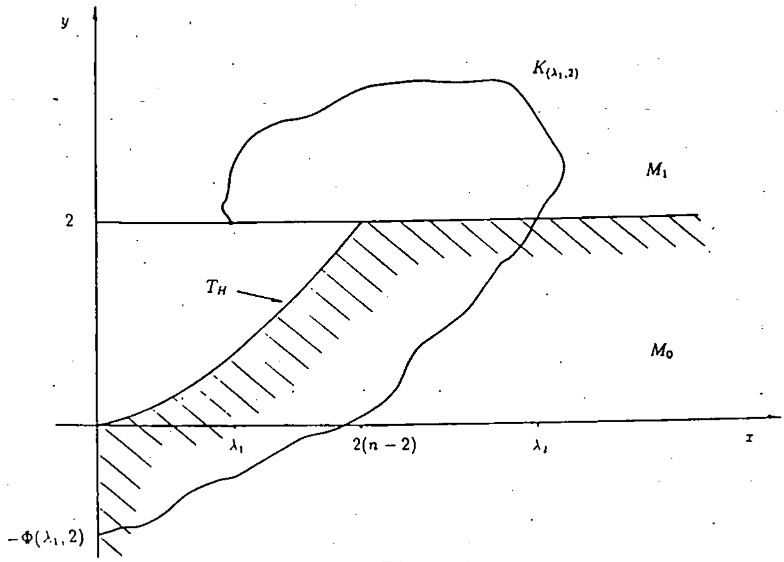

Figure 2

Theorem 12. *We consider the level sets $N(\alpha)$ of Φ for $n \geq 10$. For every $\alpha > 0$ we can represent $N(\alpha)$ as a non-intersecting, rectifiable C^1-curve K_α. Denoting its parameter of arc length by s, we find $b \in (0, \infty]$ with*

$$N(\alpha) = K_\alpha(s) = \{(x_\alpha(s), y_\alpha(s)) | \, 0 < s < b\} \quad \text{and} \quad \lim_{s \to 0} K_\alpha(s) = (0, -\alpha).$$

Furthermore, there exists $a \in (0, b)$ such that

$$N(\alpha) \cap M_0 = \{(x_\alpha(s), y_\alpha(s)) | \, 0 < s \leq a\}$$
$$N(\alpha) \cap M_1 = \{(x_\alpha(s), y_\alpha(s)) | \, a \leq s < b\}.$$

The point $(x_\alpha(a), y_\alpha(a)) \in N(\alpha)$ is the only intersection point of $N(\alpha)$ with the ray

$$\mathcal{K} = \{(x,2) | \, x > 2(n-2)\}.$$

The proofs of Lemma 10 and Theorem 12 are analogous to those of Lemma 7 and Theorem 11 and hence omitted.

Now we come to

Step 5: Proof of the final results. By means of the shape of the level sets $N(\alpha)$ we deduce bifurcation results of equation (19) and prove Theorems 4 and 5.

Proof of Theorem 4. Part (a): We define $\lambda_1(m) = a_0(m\Gamma(\frac{n}{2})/2\pi^{\frac{n}{2}})$, whereby $a_0(\cdot)$ is as in Lemma 7. By Lemma 7 it follows that the straight line $g(\lambda) := \{(\lambda, y) | y \in \mathbb{R}\}$ possesses for every $\lambda \in (0, \lambda_1]$ at least one intersection point with the level set $N(m\Gamma(\frac{n}{2})/2\pi^{\frac{n}{2}})$. By Theorem 6 equation (19) possesses for parameters $\lambda \in (0, \lambda_1]$ and m at least one solution, i.e. we obtain assertion (a).

Part (b): In the phase plane of system (20) - (21) we consider a point $(\xi, \eta) \in M_1$. By Theorem 11 there exists a unique curve $K_{(\xi,\eta)} \subset N(\Phi(\xi, \eta))$ with initial point (ξ, η) and end point $(0, -\Phi(\xi, \eta))$. This curve cannot intersect the heteroclinic orbit, because every level set $N(\alpha)$ $(\alpha > 0)$ is completely in M_Φ. We choose $k \in \mathbb{N}$. Now we take a point $(\xi, \eta) \in M_1$ sufficiently close to $(2(n-2), 2)$. Owing to the *shape of the heteroclinic orbit* of system (20) - (21) for $2 < n < 10$ (cf. D. Joseph and T. Lundgren [10]) and the *disjointness of* T_H *and* $K_{(\xi,\eta)}$ we find values $0 < \lambda_1 < 2(n-2) < \lambda_2$ such that the straight line $g(\lambda) = \{(\lambda, y) | y \in \mathbb{R}\}$ has for every $\lambda \in (\lambda_1, \lambda_2)$ at least k intersection points with the curve $K_{(\xi,\eta)}$ (cf. Figure 1). By Theorem 6 we find that equation (19) possesses for parameters $\lambda \in (\lambda_1, \lambda_2)$ and $m = (2\pi)^{\frac{n}{2}}\Phi(\xi, \eta)/\Gamma(\frac{n}{2})$ at least k solutions. Hence we obtain assertion (b) for $\lambda \in (\lambda_1, \lambda_2)$ and $m = (2\pi)^{\frac{n}{2}}\Phi(\xi, \eta)/\Gamma(\frac{n}{2})$ ∎

Proof of Theorem 5. Part (a): We proceed as in the proof of Theorem 4/(a). Instead of Lemma 7 we use Lemma 10 and instead of λ_{FK} we take the value $2(n-2)$.

Part (b): The most significant difference to the proof of Theorem 4/(b) is the shape of the heteroclinic orbit of system (20) - (21) for $n \geq 10$ (cf. D. Joseph and T. Lundgren [10] or Figure 2). We consider a point $(\xi, \eta) \in M_1$ in the phase plane of system (20) - (21) for $n \geq 10$. Regarding Theorem 12 there exists a unique curve $K_{(\xi,\eta)} \subset N(\Phi(\xi, \eta))$ with initial point (ξ, η) and end point $(0, -\Phi(\xi, \eta))$. This curve cannot intersect the heteroclinic orbit. Now we consider a point $(\xi, \eta) \in M_1$ with $\xi \in (0, 2(n-2))$, $\eta = 2$ and define $\lambda_1 := \xi$. According to Lemma 10 we denote by λ_2 the uniquely determined value, for which $\lambda_2 > 2(n-2)$ and $\Phi(\lambda_2, 2) = \Phi(\xi, \eta)$ holds. Then the straight line $g(\lambda) := \{(\lambda, y) | y \in \mathbb{R}\}$ possesses for every $\lambda \in (\lambda_1, \lambda_2)$ at least two intersection points with the curve $K_{(\lambda_1,2)} \subset N(\Phi(\lambda_1, 2))$ (cf. Figure 2). Owing to Theorem 6 equation (19) has for parameters $\lambda \in (\lambda_1, \lambda_2)$ and $m = (2\pi)^{\frac{n}{2}}\Phi(\lambda_1, 2)/\Gamma(\frac{n}{2})$ at least two solutions. Hence we obtain assertion (b) for $\lambda \in (\lambda_1, \lambda_2)$ and $m = (2\pi)^{\frac{n}{2}}\Phi(\lambda_1, 2)/\Gamma(\frac{n}{2})$ ∎

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