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On a Theorem by W. von Wahl

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Abstract. Let A be the – not necessarily densely defined – generator of an analytic semigroup acting in some Banach space X. In the paper we prove a general theorem about the existence and uniqueness of solutions of

$$\left.\begin{aligned} u'(t) &= Au(t) + F(u(t)) \\ u(0) &= u_0. \end{aligned}\right\}$$

Our main assumption with respect to the non-linearity is that F is locally Lipschitz continuous with respect to certain intermediate spaces between $\mathcal{D}(A)$ and X. Our theorem extends results obtained by W. von Wahl [9] and A. Lunardi [2]. In the second part this theorem is applied to the Cahn-Hilliard equation with Dirichlet boundary conditions.

Keywords: *Abstract semilinear parabolic equations, differential operators with non-dense domain, intermediate spaces, Cahn-Hilliard equation*

AMS subject classification: 35 A 05, 35 K 30, 35 Q 72

1. Introduction

In this paper abstract semilinear parabolic equations are investigated. Our interest lies in results obtained by Von Wahl [9, 10] and Lunardi [2]. We consider the problem

$$\left.\begin{aligned} u'(t) &= Au(t) + F(u(t)) \quad (t > 0) \\ u(0) &= u_0 \end{aligned}\right\} \tag{1}$$

in some Banach space $(X, \|\cdot\|_X)$. The operator $A : \mathcal{D}(A) \subset X \to X$ is a generator of a bounded analytic semigroup on X, which is not necessarily strongly continuous at 0 (cf. [3]). Without loss of generality we assume that $0 \in \rho(A)$. The map $F : Y \to X$ is locally Lipschitz continuous, where Y is an intermediate space which belongs to the class J_θ between X and $\mathcal{D}(A)$ with $\theta \in (0,1)$ (precise definitions are given in Subsection 2.1). We note that the definition of the class J_θ is quite general. It includes interpolation spaces and domains of fractional powers.

Let A be densely defined in X and let X_α with $0 < \alpha < 1$ denote the space $\mathcal{D}((-A)^\alpha)$. A classical result, which goes back to Sobolevskiĭ and Tanabe [5, 7] states:

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If $u_0 \in X_\alpha$ and if $F : X_\alpha \to X$ is locally Lipschitz continuous, then there is a number $T(u_0) \in (0,\infty]$ and a unique element $u \in C\big([0,T(u_0)); X_\alpha\big)$, which is a classical solution of problem (1):

1. $u \in C\big((0,T(u_0)); \mathcal{D}(A)\big) \cap C^1\big((0,T(u_0)); X\big)$.
2. $u'(t) = Au(t) + F(u(t))$ for $0 < t < T(u_0)$ and $u(0) = u_0$.

Moreover, if $T(u_0) < \infty$, then $\limsup_{t\uparrow T(u_0)} \|u(t)\|_{X_\alpha} = \infty$.

In order to obtain solutions for more general initial values u_0 and in order to obtain a better test for global existence, Von Wahl introduced a second intermediate space in [7]. In that case $F : X_\alpha \to X$ is not merely locally Lipschitz continuous, but it must satisfy

$$\begin{aligned}\|F(u) - F(v)\|_X \\ &\leq g\big(\|u\|_{X_\beta} + \|v\|_{X_\beta}\big)\Big\{\|u-v\|_{X_\alpha} + \big(\|u\|_{X_\alpha} + \|v\|_{X_\alpha} + 1\big)\|u-v\|_{X_\beta}\Big\}\end{aligned}$$

with $0 < \beta < \alpha < 1$ and where $g : [0,\infty) \to [0,\infty)$ is a continuous map. In [9] Von Wahl proves: For every $u_0 \in X_\beta$ there is a number $T(u_0) \in (0,\infty]$ and a classical solution $u \in C\big([0,T(u_0)); X_\beta\big)$ of problem (1). This solution is unique in an appropriate sense. Moreover, if $T(u_0) < \infty$, then $\limsup_{t\uparrow T(u_0)} \|u(t)\|_{X_\beta} = \infty$. In [9] this theorem is applied to the Cahn-Hilliard equation.

On the other hand, in [2] Lunardi replaces X_α by the more general intermediate spaces of class J_α between X and $\mathcal{D}(A)$ and even allows A to be non-densely defined. However, this intermediate space, say Y, must satisfy

$$\textbf{(L)} \quad \begin{cases} 1. \ Y \hookrightarrow \mathcal{D}_A(\alpha,\infty) = \big\{x \in X : t \mapsto e^{tA}x \in C^{0,\alpha}([0,T]; X) \text{ for all } T > 0\big\} \\ 2. \ \text{The part of } A \text{ in Y is sectorial in } Y. \end{cases}$$

The second part of assumption (L) implies that A generates an analytic semigroup in Y. For more details on these particular conditions we refer to [2]. Lunardi assumes that $F : Y \to X$ satisfies

$$\begin{aligned}&\|F(u) - F(v)\|_X \\ &\quad \leq g\big(\|u\|_X + \|v\|_X\big) \\ &\qquad \times \Big\{\big(\|u\|_Y^{\zeta-1} + \|v\|_Y^{\zeta-1} + 1\big)\|u-v\|_Y + \big(\|u\|_Y^\zeta + \|v\|_Y^\zeta\big)\|u-v\|_X\Big\},\end{aligned}$$

where g is a continuous function and $\zeta \geq 1$. In [2] it is proved: If $\zeta\alpha < 1$ and if $u_0 \in X$, then there is a $T(u_0) > 0$ and a unique function $u \in C\big((0,T(u_0)); \mathcal{D}(A)\big) \cap C^1\big((0,T(u_0)); X\big)$ which satisfies:

1. $u'(t) = Au(t) + F(u(t))$ for $0 < t < T(u_0)$.
2. $\lim_{t\downarrow 0} \|A^{-1}u(t) - A^{-1}u_0\|_X = 0$.
3. $u \in BC_\alpha((0,T]; Y) \cap BC((0,T]; X)$ for all $T < T(u_0)$ (cf. Definition 2.3).
4. If $T(u_0) < \infty$, then $\limsup_{t\uparrow T(u_0)} \|u(t)\|_X = \infty$.

In fact Lunardi shows the existence of a mild solution (cf. [2: Definition 7.0.2]), which is equivalent to the above.

One sees that Von Wahl's theorem and Lunardi's theorem have much in common. Lunardi can relax the conditions on A and the intermediate space, if stricter assumptions on the Lipschitz condition of F are imposed. In that case more general initial values are allowed and, also due to condition (L), there is a better test for global existence.

Our aim is to analyze the interplay between the various conditions on the operator A, the map F and the intermediate space(s). Indeed, we will prove a general theorem, Theorem 3.2, which covers together with Proposition 3.12 all cases citated above. However, our conditions on A and the intermediate spaces are minimal: A is only supposed to be the generator of a bounded analytic semigroup in X, which is not necessarily densely defined. The intermediate spaces only satisfy Definition 2.2. In particular we do *not* suppose that condition (L) holds nor are the intermediate spaces supposed to be domains of fractional powers. All the above results as well as other more general results can be obtained by Theorem 3.2. In particular, if Y does not satisfy the condition (L), then Lunardi's result above still holds, if $\limsup_{t\uparrow T(u_0)} \|u(t)\|_X = \infty$ is replaced by $\limsup_{t\uparrow T(u_0)} \|u(t)\|_Y = \infty$. On the other hand we can also generalize Von Wahl's theorem. In that case A is not necessarily densely defined and X_α, X_β are replaced by general intermediate spaces. An other application is the case with $u_0 \in \mathcal{D}(A)$ and $F : Y \to X$ locally Lipschitz continuous, especially when A is not densely defined.

2. Preliminaries

2.1 Definitions. Let $(X, \|\cdot\|_X)$ be a Banach space and $A : \mathcal{D}(A) \subset X \to X$ a closed linear operator, which is not necessarily densely defined. We assume that A (or the complexification of A) satisfies the condition

$$(\mathbf{H_A}) \quad \begin{cases} 1.\ \rho(A) \supset \{z \in \mathbb{C} : \Re z > 0\} \cup \{0\} \\ 2.\ \exists\ M > 0 \text{ such that } \|z(z-A)^{-1}\| \le M \text{ for all } z \in \{z \in \mathbb{C} : \Re z > 0\}, \end{cases}$$

where $\rho(A)$ denotes the resolvent set of A.

We recall the following proposition of Sinestrari [4].

Proposition 2.1. *If A statisfies condition* ($\mathrm{H_A}$), *then there is a collection* $\{e^{tA}\}_{t\ge 0} \subset \mathcal{L}(X)$ *such that the following statements hold:*

1. $e^{0A} = I$ *and* $e^{(s+t)A} = e^{sA}e^{tA}$ *for all* $s, t \ge 0$.
2. *If* $t > 0$, *then* $e^{tA} : X \to \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$ *and* $A^n e^{tA} x = e^{tA} A^n x$ *for* $x \in \mathcal{D}(A^n)$ *and* $n \in \mathbb{N}$.
3. $t \mapsto e^{tA} \in C^\infty((0,\infty); \mathcal{L}(X))$ *and* $\frac{d^n}{dt^n} e^{tA} = A^n e^{tA}$ *for* $n \in \mathbb{N}$.
4. *There are* $M_1, M_2 > 0$ *such that* $\|e^{tA}\|_X \le M_1$ *and* $\|tAe^{tA}\|_X \le M_2$ *for all* $t > 0$.
5. $\lim_{t\downarrow 0} \|A^{-1}(e^{tA}x - x)\|_X = 0$ *for all* $x \in X$.
6. *For every* $x \in X$ *and* $t > 0$ *we have* $\int_0^t e^{sA} x\,ds \in \mathcal{D}(A)$ *and* $A\int_0^t e^{sA} x\,ds =$

$e^{tA}x - x$.

7. *If* $s \mapsto \|e^{sA}f(s)\|_X$ *and* $s \mapsto \|Ae^{sA}f(s)\|_X$ *with* $f \in C((0,t);X)$ *are integrable over* $(0,t)$, *then* $\int_0^t e^{sA}f(s)\,ds \in \mathcal{D}(A)$ *and* $A\int_0^t e^{sA}f(s)\,ds = \int_0^t Ae^{sA}f(s)\,ds$.

In view of this proposition we say that A generates a bounded analytic semigroup. Also note that e^{tA} is strongly continuous in X if and only if A is densely defined.

Definition 2.2. If $(Y, \|\cdot\|_Y)$ is a Banach space which satisfies

1. $\mathcal{D}(A) \hookrightarrow Y \hookrightarrow X$ (here we do not require $\overline{\mathcal{D}(A)} = Y$ nor $\overline{Y} = X$)
2. there are constants $C > 0$ and $\theta \in (0,1)$ such that $\|x\|_Y \le C\|x\|_{\mathcal{D}(A)}^{\theta}\|x\|_X^{1-\theta}$ for all $x \in \mathcal{D}(A)$,

then Y is said to be an *intermediate space* of class J_θ between X and $\mathcal{D}(A)$. This will be denoted by $Y \in J_\theta(X, \mathcal{D}(A))$ (cf. [2]).

Remark 2.3. If $\mathcal{D}_A(\theta,1) = \{x \in X : t \mapsto \|t^{-\theta}Ae^{tA}x\| \in L^1(0,1)\}$, then $\mathcal{D}_A(\theta,1) \hookrightarrow Y$ for all $Y \in J_\theta(X, \mathcal{D}(A))$. For more details we refer to [2: Section 2.2].

Definition 2.4. By $BC((0,T];Y)$ we denote the set of functions $u : (0,T] \to Y$ which are continuous and bounded (with respect to $\|\cdot\|_Y$). We say that $u \in BC_\theta((0,T];Y)$ for $\theta > 0$ if $t \mapsto t^\theta u(t)$ is an element of $BC((0,T];Y)$.

Definition 2.5. A map $F : Y \to X$ is said to be *locally Lipschitz continuous* if for every $R > 0$ there is a constant $M(R) > 0$ such that $\|F(u) - F(v)\|_X \le M(R)\,\|u - v\|_Y$ for all $u, v \in Y$ with $\|u\|_Y, \|v\|_Y \le R$.

2.2 The linear initial value problem. Let $(X, \|\cdot\|_X)$ be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator. For $u_0 \in X$ and $f \in C([0,T];X)$ we consider the linear initial value problem

$$\left.\begin{aligned} u'(t) &= Au(t) + f(t) \quad (0 \le t \le T) \\ u(0) &= u_0. \end{aligned}\right\} \tag{2}$$

Definition 2.6. A function $u \in C([0,T];X)$ is said to be an *integral solution* of problem (2) if

$$\int_0^t u(s)\,ds \in \mathcal{D}(A) \qquad \text{and} \qquad u(t) = u_0 + A\int_0^t u(s)\,ds + \int_0^t f(s)\,ds$$

for all $0 \le t \le T$.

If A is a Hille-Yosida operator in the sense of [3: Formula (1.1)] and if $u_0 \in \overline{\mathcal{D}(A)}$, then it is well known that there exists a unique integral solution. Moreover, we also note that an operator which satisfies condition ($\mathrm{H_A}$) is a Hille-Yosida operator (cf. [3]). The converse is not true in general. In view of Remark 2.3 above, [2: Proposition 4.2.1] and [4: Theorem 4.4] imply the following theorem.

Theorem 2.7. *Let A satisfy condition* $(\mathrm{H_A})$, *let* $Y \in J_\theta(X, \mathcal{D}(A))$ *with* $\theta \in (0,1)$ *and let* $u_0 \in \mathcal{D}(A)$. *Then the follwing statements are true:*

1. *If* $f \in C([0,T]; X)$, *then the unique integral solution to problem* (2) *belongs to* $C^{0,1-\theta}([0,T]; Y)$. *Moreover, there is a constant* $C > 0$ *independent of* f *such that* $\|u\|_{C^{0,1-\theta}([0,T];Y)} \leq C\|f\|_\infty$.

2. *If* $f \in C^{0,\alpha}([0,T]; X)$ *with* $\alpha \in (0,1)$, *then problem* (2) *has a unique integral solution* $u \in C([0,T]; X)$ *such that* $u(t) \in \mathcal{D}(A)$ *for all* $t \in [0,T]$, $u \in C^1((0,T]; X) \cap C((0,T]; \mathcal{D}(A))$, *and* $u'(t) = Au(t) + f(t)$ *for* $0 < t \leq T$.

Furthermore, by an application of [3: Proposition 12.4] and [2: Propositions 4.2.1 and 4.3.4] we find the following statement.

Theorem 2.8. *Let A satisfy condition* $(\mathrm{H_A})$, *let* $Y \in J_\theta(X, \mathcal{D}(A))$ *with* $\theta \in (0,1)$, *and let* $u_0 \in X$. *Furthermore, assume that* $f \in BC_\beta((0,T]; X)$ *with* $\beta \in (0,1)$. *If we define*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)\, ds \qquad (0 < t \leq T),$$

then $u \in BC_\theta((0,T]; Y)$ *and* $\lim_{t\downarrow 0} \|A^{-1}u(t) - A^{-1}u_0\|_X = 0$. *Moreover, u is the unique function in* $BC((0,T]; X)$ *such that*

$$\int_0^t u(s)\, ds \in \mathcal{D}(A) \qquad \text{and} \qquad u(t) = u_0 + A\int_0^t u(s)\, ds + \int_0^t f(s)\, ds$$

for all $0 < t \leq T$.

Furthermore, if – in addition – there exists for every $\varepsilon \in (0,T]$ *an* $\alpha \in (0,1)$ *such that* $f \in C^{0,\alpha}([\varepsilon,T]; X)$, *then* $u(t) \in \mathcal{D}(A)$ *for all* $t \in (0,T]$, $u \in C((0,T]; \mathcal{D}(A)) \cap C^1((0,T]; X)$, *and* $u'(t) = Au(t) + f(t)$ *for* $t \in (0,T]$.

Finally, we mention the following lemma which is a consequence of [2: Proposition 4.2.3].

Lemma 2.9. *Let A satisfy condition* $(\mathrm{H_A})$. *If* $f \in BC_\beta((0,T]; X)$ *with* $\beta \in (0,1)$ *and if* $Y \in J_\theta(X, \mathcal{D}(A))$ *with* $\theta \in (0,1)$, *then*

$$t \longmapsto t^\alpha \int_0^t e^{(t-s)A} f(s)\, ds \in C([0,T]; Y) \qquad \text{for all } \alpha > \theta + \beta - 1.$$

Remark 2.10. We remark that the results in [2 - 4] often provide stronger statements.

3. Semilinear parabolic equations

3.1 Classical solutions and global existence. Let $(X, \|\cdot\|_X)$ be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator, which satisfies the condition $(\mathrm{H_A})$. Assume that there are two Banach spaces $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ and a map $F : X_1 \to X$ such that

1. $\mathcal{D}(A) \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X$ where $X_1 \in J_{\theta_1}(X_2, \mathcal{D}(A))$ and $X_2 \in J_{\theta_2}(X, \mathcal{D}(A))$ with $\theta_1, \theta_2 \in (0,1)$,
2. $\|F(u) - F(v)\|_X \leq g\big(\|u\|_{X_2} + \|v\|_{X_2}\big)\{\|u\|_{X_1}^{\gamma_1} + \|v\|_{X_1}^{\gamma_1} + 1\}\|u - v\|_{X_1} + g\big(\|u\|_{X_2} + \|v\|_{X_2}\big)\{\|u\|_{X_1}^{\gamma_2} + \|v\|_{X_1}^{\gamma_2} + 1\}\|u - v\|_{X_2}$, where $g : [0,\infty) \to [0,\infty)$ is a continuous map and $\gamma_1, \gamma_2 \geq 0$.

Remark 3.1. If $\theta_1 + \theta_2 < 1$, then it follows that $X_1 \in J_{\theta_1+\theta_2}(X, \mathcal{D}(A))$. To avoid trivial calculations we will assume that $F(0) = 0$.

For convenience we adopt the notation $\eta := \max\big((\gamma_1 + 1)\theta_1, \gamma_2\theta_1\big)$.

Theorem 3.2. *Let $(X, \|\cdot\|_X)$ be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator, which satisfies the condition $(\mathrm{H_A})$. Furthermore, let X_1, X_2 and $F : X_1 \to X$ fulfil the conditions above with $\eta + \theta_2 < 1$. Then for every $u_0 \in X$, which satisfies $\sup_{t>0} \|e^{tA}u_0\|_{X_2} < \infty$, there is a $T(u_0) \in (0,\infty]$ and a unique function $u \in C\big((0,T(u_0)); \mathcal{D}(A)\big) \cap C^1\big((0,T(u_0)); X\big)$ such that the following statements are true:*

1. *$u'(t) = Au(t) + F(u(t))$ for $0 < t < T(u_0)$.*
2. *$\lim_{t\downarrow 0} \|A^{-1}u(t) - A^{-1}u_0\|_X = 0$.*
3. *$u \in BC_{\theta_1}((0,T]; X_1) \cap BC((0,T]; X_2)$ for all $T < T(u_0)$.*
4. *If $T(u_0) < \infty$, then $\limsup_{t\uparrow T(u_0)} \|u(t)\|_{X_1} = \infty$.*

The proof of Theorem 3.2 is as follows: In Proposition 3.9 we show by a fixed point argument that there is a $T > 0$ and a unique $u \in BC_{\theta_1}((0,T]; X_1) \cap BC((0,T]; X_2)$ such that $u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s))\,ds$ with $0 < t \leq T$. Given this function u we consider the initial value problem

$$\left.\begin{aligned} &v'(t) = Av(t) + F(u(t)) \quad (0 < t \leq T) \\ &v(0) = u_0 \end{aligned}\right\}$$

and apply Theorem 2.8. Finally, we derive the test for global existence by a bootstrap argument.

Define for $T > 0$

$$M_T = BC_{\theta_1}\big((0,T]; X_1\big) \cap BC\big((0,T]; X_2\big)$$

and set for $u, v \in M_T$

$$\mu_T(u,v) = \sup_{0<t\leq T} t^{\theta_1}\|u(t) - v(t)\|_{X_1} + \sup_{0<t\leq T} \|u(t) - v(t)\|_{X_2}.$$

By $\|\cdot\|_T$ we denote $\|u\|_T = \mu_T(u,0)$. It follows that $(M_T, \|\cdot\|_T)$ is a Banach space. Observe the following easy consequences of the Lipschitz condition on F.

Proposition 3.3. *Let F satisfy the Lipschitz condition. If $u \in M_T$, then $F(u) \in BC_\eta((0,T];X)$. Moreover, for every $R > 0$ there is a constant $C = C(F,R) > 0$ such that, for all $0 < s \leq T$,*

$$\left\|F(u(s)) - F(v(s))\right\|_X \leq C\{s^{-\eta} + 1\}\, \mu_T(u,v)$$

for all $u, v \in M_T$ satisfying $\|u\|_T, \|v\|_T < R$.

Lemma 3.4. *If $u_0 \in X$ satisfies $\sup_{t>0} \|e^{tA}u_0\|_{X_2} < \infty$, then the function $t \mapsto e^{tA}u_0$ is in M_T for every $T > 0$. In fact, there is a $K = K(u_0) > 0$ such that $\|e^{sA}u_0\|_T \leq K(u_0)$ for all $T > 0$.*

Proof. Note that $s \mapsto e^{tA}u_0 \in C\big((0,\infty);\mathcal{D}(A)\big)$. Moreover, for all $t > 0$ we find that

$$\|e^{tA}u_0\|_{X_1} \leq C\|e^{tA}u_0\|_{\mathcal{D}(A)}^{\theta_1}\|e^{tA}u_0\|_{X_2}^{1-\theta_1} \leq C't^{-\theta_1}\|u_0\|_X^{\theta_1}\|e^{tA}u_0\|_{X_2}^{1-\theta_1} \leq K(u_0)$$

and the proof follows ∎

From now on we assume that $u_0 \in X$ satisfies $\sup_{t>0}\|e^{tA}u_0\|_{X_2} < \infty$. In view of Lemma 3.4 we define for $\delta > 0$

$$M_{T,\delta} = \big\{u \in M_T : \mu_T(u, e^{tA}u_0) \leq \delta\big\}.$$

Note that $\big(M_{T,\delta}; \mu_T(\cdot,\cdot)\big)$ is a complete metric space. Next, define for $u \in M_T$

$$\Gamma(u)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s))\,ds \qquad (0 < t \leq T).$$

Proposition 3.5. *If $T > 0$ and if $u_0 \in X$ satisfies $\sup_{t>0}\|e^{tA}u_0\|_{X_2} < \infty$, then the following statements hold:*

1. *Γ is a well-defined map from M_T into M_T.*

2. *$\|A^{-1}\Gamma(u)(t) - A^{-1}u_0\|_X \to 0$ as $t \downarrow 0$.*

3. *If $u \in M_T$ satisfies $\Gamma(u) = u$, then $u(t) \in \mathcal{D}(A)$ for all $t \in (0,T]$ and $u \in C\big((0,T];\mathcal{D}(A)\big)$ as well as $u \in C^1\big((0,T];X\big)$. Moreover, $u'(t) = Au(t) + F(u(t))$ for $0 < t \leq T$.*

Proof. 1. For $u \in M_T$ we have seen that $F(u) \in BC_\eta((0,T];X)$. Also, note that $X_1 \in J_{\theta_1+\theta_2}(X,\mathcal{D}(A))$ with $\theta_1 + \theta_2 \leq \eta + \theta_2 < 1$. So by Lemma 2.9 we find that

$$t \longmapsto \int_0^t e^{(t-s)A}F(u(s))\,ds \in BC_{\theta_1}((0,T];X_1) \cap BC((0,T];X_2).$$

Together with Lemma 3.4 this implies that $\Gamma(u) \in M_T$.

2. We know that $A^{-1}e^{tA}u_0 \to A^{-1}u_0$ in X as $t \downarrow 0$. On the other hand, we see that $\int_0^t e^{(t-s)A}F(u(s))\,ds \to 0$ in X (even in X_2) as $t \downarrow 0$, which proves the second assertion.

3. Let $\varepsilon > 0$ and write, for $0 \le t \le T - \varepsilon$,

$$v_1(t) = e^{tA}e^{\varepsilon A}u_0 + \int_0^t e^{(t-s)A}F(u(s+\varepsilon))\,ds$$

$$v_2(t) = e^{tA}\int_0^\varepsilon e^{(\varepsilon-s)A}F(u(s))\,ds.$$

According to Theorem 2.8, v_1 is the integral solution of the initial value problem

$$\left.\begin{aligned} &v'(t) = Av(t) + F(u(t+\varepsilon)) \quad (0 \le t \le T-\varepsilon)\\ &v(0) = e^{\varepsilon A}u_0. \end{aligned}\right\}$$

Consequently, Theorem 2.7 implies that $v_1 \in C^{0,1-\theta_1-\theta_2}([0,T-\varepsilon];X_1)$. On the other hand, note that $e^{tA}x \in C^1((0,\infty);\mathcal{D}(A)) \subset C^1((0,\infty);X_1)$ for every $x \in X$. In view of this we find that $v_2(t) \in C^{0,1-\theta_1-\theta_2}([\varepsilon,T-\varepsilon];X_1)$. Hence $u(t) = v_1(t-\varepsilon) + v_2(t-\varepsilon) \in C^{0,1-\theta_1-\theta_2}([2\varepsilon,T];X_1)$, which implies that $F(u(t)) \in C^{0,1-\theta_1-\theta_2}([2\varepsilon,T];X)$. So we can apply Theorem 2.8 to the initial value problem $v'(t) = Av(t) + F(u(t))$ with $v(0) = u_0$, from which the statement follows ∎

Remark 3.6. Note that Lemma 2.9 in fact states that $t \mapsto t^{\theta_1}\int_0^t e^{(t-s)A}F(u(s))\,ds \in C([0,T];X_1)$ as well as $t \mapsto \int_0^t e^{(t-s)A}F(u(s))\,ds \in C([0,T];X_2)$.

Lemma 3.7. *Let $\delta_0 \in (0,1]$. There is a $T = T(\delta_0,u_0) > 0$ such that $\Gamma(e^{sA}u_0) \in M_{T,\delta_0}$.*

Proof. By Lemma 3.4 there is a constant $K = K(u_0) > 0$ such that $\|e^{tA}u_0\|_T \le K(u_0)$ for all $T > 0$. Therefore, Proposition 3.3 implies that there is a constant $C = C(F,K(u_0)) > 0$ such that $\|F(e^{sA}u_0)\|_X \le C(s^{-\eta}+1)$ for all $s > 0$. Since $X_1 \in J_{\theta_1+\theta_2}(X,\mathcal{D}(A))$ we find by Proposition 2.1 and Definition 2.2 that, for $0 < s < t$,

$$\begin{aligned} \left\|e^{(t-s)A}F(e^{sA}u_0)\right\|_{X_1} &\le C\left\|e^{(t-s)A}F(e^{sA}u_0)\right\|_{\mathcal{D}(A)}^{\theta_1+\theta_2}\left\|e^{(t-s)A}F(e^{sA}u_0)\right\|_X^{1-\theta_1-\theta_2}\\ &\le C'\left\|Ae^{(t-s)A}F(e^{sA}u_0)\right\|_X^{\theta_1+\theta_2}\left\|e^{(t-s)A}F(e^{sA}u_0)\right\|_X^{1-\theta_1-\theta_2}\\ &\le C''(t-s)^{-\theta_1-\theta_2}\left\|F(e^{sA}u_0)\right\|_X \end{aligned}$$

which implies that

$$\begin{aligned} t^{\theta_1}\left\|\Gamma(e^{sA}u_0)(t) - e^{tA}u_0\right\|_{X_1} &\le t^{\theta_1}\int_0^t \left\|e^{(t-s)A}F(e^{sA}u_0)\right\|_{X_1}ds\\ &\le C_1\left\{t^{1-\theta_2-\eta} + t^{1-\theta_2}\right\}. \end{aligned}$$

Analogously we find that

$$\left\|\Gamma(e^{sA}u_0)(t) - e^{tA}u_0\right\|_{X_2} \le C_2\left\{t^{1-\theta_2-\eta} + t^{1-\theta_2}\right\}.$$

So, if we choose $T > 0$ such that $(C_1 + C_2)\left\{T^{1-\theta_2-\eta} + T^{1-\theta_2}\right\} \le \delta_0$, then clearly $\mu_T(\Gamma(e^{sA}u_0), e^{tA}u_0) \le \delta_0$ ∎

Lemma 3.8. *If $\delta_0 \in (0,1]$, then there is a $T = T(\delta_0, u_0) > 0$ such that, for all $S \in (0,T]$,*

$$\Gamma : M_{S,1} \to M_{S,\delta_0} \quad \textit{and} \quad \mu_S\big(\Gamma(u),\Gamma(v)\big) \le \frac{1}{2}\mu_S(u,v) \ \textit{ for all } u,v \in M_{S,1}.$$

Proof. Let $S > 0$ and let $u, v \in M_{S,1}$. Note that according to Lemma 3.4

$$\|u\|_S \le \mu_S(u, e^{sA}u_0) + \|e^{sA}u_0\|_S \le 1 + K(u_0).$$

So, by Proposition 3.3, there is a $C = C(u_0) > 0$ (not depending on S) such that

$$\big\|F(u(s)) - F(v(s))\big\|_X \le C(s^{-\eta} + 1)\mu_S(u,v) \qquad (0 < s \le S)$$

for all $u, v \in M_{S,1}$. Thus we find as in the proof of Lemma 3.7 that, for $0 < t \le S$,

$$\begin{aligned} t^{\theta_1}\big\|\Gamma(u)(t) - \Gamma(v)(t)\big\|_{X_1} &\le t^{\theta_1}\int_0^t \big\|e^{(t-s)A}\{F(u(s)) - F(v(s))\}\big\|_{X_1} ds \\ &\le Ct^{\theta_1}\int_0^t (t-s)^{-\theta_1-\theta_2}(s^{-\eta}+1)\, ds\, \mu_S(u,v) \\ &\le C_1\{t^{1-\theta_2-\eta} + t^{1-\theta_2}\}\mu_S(u,v). \end{aligned}$$

Analogously we find that

$$\big\|\Gamma(u)(t) - \Gamma(v)(t)\big\|_{X_2} \le C_2\left\{t^{1-\theta_2-\eta} + t^{1-\theta_2}\right\}\mu_S(u,v).$$

Hence

$$\mu_S\big(\Gamma(u),\Gamma(v)\big) \le (C_1 + C_2)\{S^{1-\theta_2-\eta} + S^{1-\theta_2}\}\mu_S(u,v) \qquad (u, v \in M_{S,1}).$$

So, choose $T_0 > 0$ such that $\mu_{T_0}\big(\Gamma(e^{sA}u_0), e^{tA}u_0\big) \le \frac{\delta_0}{2}$. Then for all $0 < S \le T_0$ and $u \in M_{S,1}$ we find by the preceding that (note that $\mu_S(u, e^{sA}u_0) \le 1$)

$$\begin{aligned} \mu_S\big(\Gamma(u), e^{tA}u_0\big) &\le \mu_S\big(\Gamma(u), \Gamma(e^{sA}u_0)\big) + \mu_S\big(\Gamma(e^{sA}u_0), e^{tA}u_0\big) \\ &\le (C_1 + C_2)\{S^{1-\theta_2-\eta} + S^{1-\theta_2}\} + \frac{\delta_0}{2}. \end{aligned}$$

Thus if we choose $T \in (0, T_0]$ such that $(C_1 + C_2)\{T^{1-\theta_2-\eta} + T^{1-\theta_2}\} \le \frac{\delta_0}{2} \le \frac{1}{2}$ is satisfied, the result follows ∎

Proposition 3.9. *If $u_0 \in X$ satisfies $\sup_{t>0}\|e^{tA}u_0\|_{X_2} < \infty$, then there is a $T = T(u_0) > 0$ for which $\Gamma : M_T \to M_T$ has a unique fixed point u. Moreover, if $u_0 \in \mathcal{D}(A)$, then the fixed point u is an element of $C([0,T]; X_1)$.*

Proof. Fix $\delta_0 \in (0,1)$ and let $T = T(\delta_0, u_0) > 0$ be as in Lemma 3.8. According to Lemma 3.8 the map $\Gamma : M_{T,\delta_0} \to M_{T,\delta_0}$ is a strict contraction, so due to the Banach fixed point theorem there is a unique $u \in M_{T,\delta_0} \subset M_T$ satisfying $\Gamma(u) = u$. Next, assume that $v \in M_T$ satisfies $v \neq u$ as well as $\Gamma(v) = v$. Consider for $0 < S \leq T$ the map $S \mapsto \mu_S(v, e^{tA}u_0)$. By Remark 3.6 the map $S \mapsto \mu_S(v, e^{tA}u_0)$ extends to an increasing continuous function on $[0, T]$, which is 0 for $S = 0$. Since $v \notin M_{T,\delta_0}$, there must be a $\tilde{S} \in (0, T)$ such that

$$\mu_S(v, e^{tA}u_0) \leq \delta_0 \ \text{ for } 0 \leq S \leq \tilde{S} \qquad \text{and} \qquad \mu_S(v, e^{tA}u_0) > \delta_0 \ \text{ for } \tilde{S} < S \leq T.$$

By continuity there is an $\varepsilon > 0$ such that $\mu_S(v, e^{tA}u_0) \leq 1$ for $0 \leq S \leq \tilde{S} + \varepsilon$. However, Lemma 3.8 and the Banach fixed point theorem imply that $\Gamma : M_{\tilde{S}+\varepsilon,1} \to M_{\tilde{S}+\varepsilon,1}$ has a unique fixed point u. Since $u \in M_{\tilde{S}+\varepsilon,1}$ as well as $v \in M_{\tilde{S}+\varepsilon,1}$, it follows that $u = v$ on $[0, \tilde{S} + \varepsilon]$. Hence $\mu_S(v, e^{tA}u_0) = \mu_S(u, e^{tA}u_0) \leq \delta_0$ for $0 \leq S \leq \tilde{S} + \varepsilon$, which is a contradiction. Next assume that $u_0 \in \mathcal{D}(A)$. Since

$$t \longmapsto e^{tA}u_0 \in C([0,\infty); X_1)$$

$$t \longmapsto \int_0^t e^{(t-s)A} F(u(s))\, ds \in C((0,T]; X_1),$$

it is sufficient to show that

$$\left\| \int_0^t e^{(t-s)A} F(u(s))\, ds \right\|_{X_1} \longrightarrow 0 \qquad \text{as } t \downarrow 0.$$

Since $F(u) \in BC_\eta((0,T]; X)$, there is a $C > 0$ such that, for all $0 < t \leq T$,

$$\left\| \int_0^t e^{(t-s)A} F(u(s))\, ds \right\|_{X_1} \leq C \left\{ t^{1-\theta_1-\eta-\theta_2} + t^{1-\theta_1-\theta_2} \right\}$$

(cf. the proof of Lemma 3.7). So if $1 - \theta_1 - \theta_2 - \eta > 0$, then the result follows (note that $\theta_1 + \theta_2 < 1$). Otherwise there must be an $\varepsilon > 0$ such that, because $\eta + \theta_2 < 1$,

$$t \longmapsto t^{\theta_1 - \varepsilon} \int_0^t e^{(t-s)A} F(u(s))\, ds \in C([0,T]; X_1).$$

Hence $u \in BC_{\theta_1-\varepsilon}((0,T]; X_1)$, which implies that $F(u) \in BC_{\eta-\varepsilon}((0,T]; X)$. But in that case we can find a $C > 0$ such that

$$\left\| \int_0^t e^{(t-s)A} F(u(s))\, ds \right\|_{X_1} \leq C \left\{ t^{1-\theta_1-\eta-\theta_2+\varepsilon} + t^{1-\theta_1-\theta_2} \right\} \qquad (0 < t \leq T).$$

We are done if $1 - \theta_1 - \theta_2 - \eta + \varepsilon > 0$. Otherwise we can repeat the argument above and find a $C > 0$ such that

$$\left\| \int_0^t e^{(t-s)A} F(u(s))\, ds \right\|_{X_1} \leq C \left\{ t^{1-\theta_1-\eta-\theta_2+2\varepsilon} + t^{1-\theta_1-\theta_2} \right\} \qquad (0 < t \leq T).$$

Iterating this process the statement follows ∎

Corollary 3.10. *The map $\Gamma : M_T \to M_T$ has at most one fixed point.*

Proof. Let $u, v \in M_T$ with $u \neq v$ be two fixed points of Γ. Define

$$\mathcal{S} = \sup\Big\{K \in (0,T] : u(s) = v(s) \text{ for all } s \in (0,K]\Big\}.$$

In view of Propostion 3.9 and since $u \neq v$ we must have $\mathcal{S} \in (0,T)$. For $0 < t \leq T - \mathcal{S}$ we find that

$$u(t+\mathcal{S}) = e^{tA}u(\mathcal{S}) + \int_0^t e^{(t-\sigma)A}F(u(\sigma+\mathcal{S}))\,d\sigma$$

$$v(t+\mathcal{S}) = e^{tA}v(\mathcal{S}) + \int_0^t e^{(t-\sigma)A}F(v(\sigma+\mathcal{S}))\,d\sigma.$$

However, Proposition 3.5 implies that $u(\mathcal{S}) = v(\mathcal{S}) \in \mathcal{D}(A)$, so due to Proposition 3.9 there must be an $\varepsilon > 0$ such that $u(t+\mathcal{S}) = v(t+\mathcal{S})$ for $0 \leq t \leq \varepsilon$. This is a contradiction ∎

Proof of Theorem 3.2. Define $T(u_0) \in (0,\infty]$ by

$$T(u_0) = \sup\Big\{T > 0 \Big| \Gamma : M_T \to M_T \text{ has a fixed point}\Big\}.$$

According to the preceding there is a unique function

$$u \in C\big((0,T(u_0)];\mathcal{D}(A)\big) \cap C^1\big((0,T(u_0)];X\big),$$

which satisfies statements 1 - 3 of Theorem 3.2. So there remains to prove that

$$T(u_0) < \infty \qquad \Longrightarrow \qquad \limsup_{t\uparrow T(u_0)} \|u(t)\|_{X_1} = \infty.$$

Since we are only interested in what happens near $T(u_0)$ and since $u(t) \in \mathcal{D}(A)$ for $t \in (0,T(u_0))$ we may as well assume that $u_0 \in \mathcal{D}(A)$. The latter implies that $u \in C\big([0,T(u_0));X_1\big)$ and thus $F(u) \in C\big([0,T(u_0));X\big)$. So, let $T(u_0) < \infty$ and assume that $\|u(t)\|_{X_1}$ is uniformly bounded on $[0,T(u_0))$. This implies that $\|F(u(t))\|_X$ is uniformly bounded on $[0,T(u_0))$ and thus by an application of Theorem 2.7 there must be a constant $C > 0$ such that

$$\|u(s) - u(t)\|_{X_1} \leq C|t-s|^{1-\theta_1-\theta_2} \qquad \big(0 \leq s,t < T(u_0)\big).$$

Consequently, $\lim_{t\uparrow T(u_0)} u(t)$ exists in X_1 and thus $u \in C\big([0,T(u_0)];X_1\big)$. Hence we find that $F(u) \in C\big([0,T(u_0)];X\big)$. However, Theorem 2.7 implies in that case that $u(T(u_0)) \in \mathcal{D}(A)$. By Propositon 3.9 there is a $S > 0$ and a unique $w \in C([0,S];X_1)$ such that

$$w(t) = e^{tA}u(T(u_0)) + \int_0^t e^{(t-s)A}F(w(s))\,ds \qquad (0 \leq t \leq S).$$

If we define $\tilde{u} : [0,T(u_0)+S] \to X$ by $\tilde{u}(t) = u(t)$ for $0 \leq t \leq T(u_0)$ and $\tilde{u}(t) = w(t - T(u_0))$ for $T(u_0) < t \leq T(u_0) + S$, then $\tilde{u} \in M_{T(u_0)+S}$ satisfies $\Gamma(\tilde{u}) = \tilde{u}$. This is a contradiction ∎

If F satisfies additional assumptions, then assertion 4 of Theorem 3.2 can be improved by an application of Gronwall's Lemma (for a proof of this lemma we refer to [1]).

Lemma 3.11. *Let $T \in (0,\infty)$, $0 \le \alpha < 1$ and $C_1, C_2 > 0$. If $u : [0,T] \to \mathbb{R}$ is a non-negative and integrable function satisfying*

$$u(t) \le C_1 + C_2 \int_0^t (t-s)^{-\alpha} u(s)\, ds \qquad (0 \le t \le T),$$

then there is a constant $K = K(\alpha, C_2, T) > 0$ such that $0 \le u(t) \le C_1 K$ for all $t \in [0,T]$.

Proposition 3.12. *If F satisfies the Lipschitz condition with $\gamma_1 = 0$ and $\gamma_2 \le 1$, then statement 4 in Theorem 3.2 can be replaced by the following one:*

4'. *If $T(u_0) < \infty$, then $\limsup_{t \uparrow T(u_0)} \|u(t)\|_{X_2} = \infty$.*

Proof. As in the proof of Theorem 3.2 it is sufficient to consider the case $u_0 \in \mathcal{D}(A)$, which implies that $u \in C\big([0, T(u_0)); X_1\big)$. Let $T(u_0) < \infty$. In view of Theorem 3.2 it is enough to show that

$$\sup_{0 \le t < T(u_0)} \|u(t)\|_{X_2} < \infty \qquad \Longrightarrow \qquad \sup_{0 \le t < T(u_0)} \|u(t)\|_{X_1} < \infty.$$

If $\|u(t)\|_{X_2}$ is uniformly bounded on $[0, T(u_0))$, then the additional assumptions on F imply that there is a constant $C > 0$ such that $\|F(u(s))\|_X \le C(\|u(s)\|_{X_1} + 1)$ for $0 \le t < T(u_0)$. So for $0 \le t < T(u_0)$ we find that

$$\begin{aligned}
\|u(t)\|_{X_1} &\le \|e^{tA}u_0\|_{X_1} + \left\| \int_0^t e^{(t-s)A} F(u(s))\, ds \right\|_{X_1} \\
&\le C_1(u_0) + C_2 \int_0^t (t-s)^{-\theta_1-\theta_2} \|F(u(s))\|_X\, ds \\
&\le C_1' + C_2' \int_0^t (t-s)^{-\theta_1-\theta_2} \|u(s)\|_{X_1}\, ds
\end{aligned}$$

and Gronwall's Lemma implies that $\|u(t)\|_{X_1}$ is uniformly bounded on $[0, T(u_0))$ ∎

3.2 Comparison with Von Wahl's and Lunardi's results. In this subsection we will treat some consequences of Theorem 3.2 and we shall indicate how they relate to known results.

Consequence 1: If we set $X_2 = X$, $\gamma_2 = \zeta \ge 1$ and $\gamma_1 = \zeta - 1$, then we arrive at Lunardi's case:

Corollary 3.13. *Let $(X, \|\cdot\|_X)$ be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator, which satisfies the condition* ($\mathrm{H_A}$). *Let $Y \in J_\theta(X, \mathcal{D}(A))$ with $\zeta\theta < 1$ where $\zeta \geq 1$, and let $F : Y \to X$ satisfy*

$$\begin{aligned} &\|F(u) - F(v)\|_X \\ &\quad \leq g(\|u\|_X + \|v\|_X) \\ &\quad\quad \times \left\{ (\|u\|_Y^{\zeta-1} + \|v\|_Y^{\zeta-1} + 1)\|u - v\|_Y + (\|u\|_Y^\zeta + \|v\|_Y^\zeta)\|u - v\|_X \right\}. \end{aligned}$$

Then for every $u_0 \in X$ there is a $T(u_0) \in (0, \infty]$ and a unique function $u \in C\big((0, T(u_0)); \mathcal{D}(A)\big) \cap C^1\big((0, T(u_0)); X\big)$ which satisfies:

1. $u'(t) = Au(t) + F(u(t))$ *for* $0 < t < T(u_0)$.
2. $\lim_{t\downarrow 0} \|A^{-1}u(t) - A^{-1}u_0\|_X = 0$.
3. $u \in BC_\theta((0, T]; Y) \cap BC((0, T]; X)$ *for all* $T < T(u_0)$.
4. *If* $T(u_0) < \infty$, *then* $\limsup_{t\uparrow T(u_0)} \|u(t)\|_Y = \infty$.

Remark that due to condition (L) Lunardi gets a better test for global existence (cf. [2: Proposition 7.2.2]).

Consequence 2: If we set $\gamma_1 = 0$ and $\gamma_2 = 1$, then we arrive at a generalization of Von Wahl's theorem:

Corollary 3.14. *Let $(X, \|\cdot\|_X)$ be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator, which satisfies the condition* ($\mathrm{H_A}$). *Let $X_1 \in J_{\alpha-\beta}(X_2, \mathcal{D}(A))$ with $0 < \beta < \alpha < 1$ and $X_2 \in J_\beta(X, \mathcal{D}(A))$. Let $u_0 \in X$ satisfy $\sup_{t>0} \|e^{tA}u_0\|_{X_2} < \infty$ and assume that $F : X_1 \to X$ satisfies*

$$\begin{aligned} \|F(u) - F(v)\|_X \leq\ & g(\|u\|_{X_2} + \|v\|_{X_2}) \\ & \times \left\{ \|u - v\|_{X_1} + \|u - v\|_{X_2}(\|u\|_{X_1} + \|v\|_{X_1} + 1) \right\}, \end{aligned}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous map. Then there is a $T(u_0) \in (0, \infty]$ and a unique function $u \in C\big((0, T(u_0)); \mathcal{D}(A)\big) \cap C^1\big((0, T(u_0)); X\big)$ which satisfies:

1. $u'(t) = Au(t) + F(u(t))$ *for* $0 < t < T(u_0)$.
2. $\lim_{t\downarrow 0} \|A^{-1}u(t) - A^{-1}u_0\|_X = 0$.
3. $u \in BC_{\alpha-\beta}((0, T]; X_1) \cap BC((0, T]; X_2)$ *for all* $T < T(u_0)$.
4. *If* $T(u_0) < \infty$, *then* $\limsup_{t\uparrow T(u_0)} \|u(t)\|_{X_2} = \infty$.

Note that Von Wahl's result states that $u \in C\big([0, T(u_0)); X_\beta\big)$. However this is due to the fact that $t \mapsto e^{tA}u_0 \in C\big([0, \infty); X_\beta\big)$, if $u_0 \in X_\beta$ with $X_\beta = \mathcal{D}((-A)^\beta)$. The last assertion is due to Proposition 3.12.

Consequence 3: Finally, set $X_1 = X_2$. After Corollary 3.10 we arrive at

Corollary 3.15. *Let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator, which satisfies the condition* ($\mathrm{H_A}$). *Let $u_0 \in \mathcal{D}(A)$ and $Y \in J_\theta(X, \mathcal{D}(A))$ with $0 < \theta < 1$. If $F : Y \to X$ is locally Lipschitz continuous, then there is a $T(u_0) > 0$ and a unique $u \in C([0, T(u_0)); Y) \cap C^1((0, T(u_0)); X) \cap C((0, T(u_0)); \mathcal{D}(A))$ such that:*

1. *$u'(t) = Au(t) + F(u(t))$ for $0 < t < T(u_0)$ and $u(0) = u_0$.*
2. *If $T(u_0) < \infty$, then $\sup_{0 \le t < T(u_0)} \|u(t)\|_Y = +\infty$.*

4. An application to the Cahn-Hilliard equation

Let $n \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with a uniformly C^4-boundary (cf. [2]). Set

$$X = C(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{R} \,|\, f \text{ is continuous}\}$$

endowed with the sup-norm $\|\cdot\|_\infty$, and define the non-densely defined linear operator $A : \mathcal{D}(A) \subset X \to X$ by

$$\mathcal{D}(A) = \Big\{u \in C_0(\overline{\Omega}) : \Delta u = g \text{ with } g \in C_0(\overline{\Omega}) \text{ and } \Delta g \in C(\overline{\Omega})\Big\}$$

and

$$Au = -\Delta^2 u,$$

where $C_0(\overline{\Omega})$ is the subset of $C(\overline{\Omega})$ consisting of the functions which are 0 on the boundary $\partial\Omega$ and where Δu has to be understood in the sense of distributions. In [6] it is proved that A is the generator of a bounded analytic semigroup in X. By Schauder estimates (cf., for example, [1: p. 9]) it follows that $\mathcal{D}(A) \hookrightarrow C^{3+\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. We recall that, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$,

$$C^\alpha(\overline{\Omega}) := \{f \in C(\overline{\Omega}) : [f]_\alpha < \infty\}$$

and

$$C^{k+\alpha}(\overline{\Omega}) := \{f \in C^k(\overline{\Omega}) : \partial^\gamma f \in C^\alpha(\overline{\Omega}),\ |\gamma| = k\},$$

where $C^k(\overline{\Omega})$ is the space of all k times continuously differentiable functions in Ω, whose derivatives are continuously extendable up to the boundary and where $[f]_\alpha$ denotes

$$[f]_\alpha = \sup\Big\{|x-y|^{-\alpha}|f(x) - f(y)| : x, y \in \overline{\Omega},\ x \neq y\Big\}.$$

As usual $C^k(\overline{\Omega})$ and $C^{k+\alpha}(\overline{\Omega})$ are normed by

$$\|f\|_{k,\infty} = \sum_{|\gamma| \le k} \|\partial^\gamma f\|_\infty \qquad \text{and} \qquad \|f\|_{k+\alpha,\infty} = \|f\|_{k,\infty} + \sum_{|\gamma| = k} [\partial^\gamma f]_\alpha,$$

respectively. Finally, we note that $C^\beta(\overline{\Omega}) = \big(C^\alpha(\overline{\Omega}), C(\overline{\Omega}), \frac{\beta}{\alpha}\big)$ for $0 < \beta < \alpha$, if Ω has a uniformly C^α boundary (cf. [2: pp. 8 and 13]).

As an example we consider the following generalized Cahn-Hilliard equation with Dirichlet boundary conditions:

$$\textbf{(GCH)}\quad \begin{cases} u_t = -\Delta^2 u + F(u) \text{ for } x \in \overline{\Omega} \text{ and } t > 0 \\ u(x,t) = \Delta u(x,t) = 0 \text{ for } x \in \partial\Omega \text{ and } t > 0 \\ u(0,x) = u_0(x) \in C(\overline{\Omega}). \end{cases}$$

We assume that the map $F : X_\delta \to C(\overline{\Omega})$ satsifies

$$\begin{aligned} \|F(u) - F(v)\|_X &\le g\big(\|u\|_X + \|v\|_X\big) \\ &\qquad \times \Big\{ \|u - v\|_{X_\delta} + \big(\|u\|_{X_\delta} + \|v\|_{X_\delta} + 1\big)\|u - v\|_X \Big\}, \end{aligned}$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and where X_δ denotes the space $C^{3+\delta}(\overline{\Omega})$. Since $X_\delta \in J_{\frac{3+\delta}{3+\alpha}}(X, \mathcal{D}(A))$ with $\alpha \in (\delta, 1)$, we can apply Corollary 3.13 and Proposition 3.12, which implies

Theorem 4.1. *For every $u_0 \in C(\overline{\Omega})$ there is a $T(u_0) > 0$ and a unique $u \in C^1\big((0,T(u_0)); C(\overline{\Omega})\big) \cap C\big((0,T(u_0)); \mathcal{D}(A)\big)$ such that:*

1. $u_t = -\Delta^2 u + F(u)$ *for all* $x \in \overline{\Omega}$ *and* $0 < t < T(u_0)$.
2. $u(x,t) = \Delta u(x,t) = 0$ *for* $x \in \partial\Omega$ *and* $0 < t < T(u_0)$.
3. $\|A^{-1}u(t,\cdot) - A^{-1}u_0(\cdot)\|_\infty \to 0$ *as* $t \downarrow 0$.
4. $\sup_{0<t<T} t^\theta \|u(t,\cdot)\|_{3+\delta,\infty} < \infty$ *and* $\sup_{0<t<T} \|u(t,\cdot)\|_\infty < \infty$ *for every* $T < T(u_0)$ *and* $\theta \in \left(\frac{3+\delta}{4}, 1\right)$.
5. *If* $T(u_0) < \infty$, *then* $\limsup_{t \uparrow T(u_0)} \|u(t,\cdot)\|_\infty = \infty$.

An example of such an F is

$$F(u) = \left| \sum_{i,j,k} a_{i,j,k}(u) \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} \right|^\alpha + \left| \sum_{i,j} b_{i,j}(u) \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^\beta + \left| \sum_i c_i(u) \frac{\partial u}{\partial x_i} \right|^\gamma,$$

where $a_{i,j,k}, b_{i,j}$ and c_i are locally Lipschitz continuous and where $1 \le \alpha < \frac{4}{3}$, $1 \le \beta < 2$ and $1 \le \gamma < 4$. If we take $F(u) = \Delta f(u) = f'(u)\Delta u + f''(u)|\nabla u|^2$ with $f \in C^2(\mathbb{R}; \mathbb{R})$, then we are in the situation of the original Cahn-Hilliard equation with Dirichlet boundary conditions:

$$\textbf{(CH)}\quad \begin{cases} u_t = -\Delta^2 u + \Delta f(u) \text{ for } x \in \overline{\Omega} \text{ and } t > 0 \\ u(x,t) = \Delta u(x,t) = 0 \text{ for } x \in \partial\Omega \text{ and } t > 0 \\ u(0,x) = u_0(x) \in C(\overline{\Omega}). \end{cases}$$

So, if f'' is locally Lipschitz continuous, then Theorem 4.1 implies the existence of a local solution for problem (CH) (indeed, one easily verifies that F satisfies the above Lipschitz condition). As in Von Wahl [9] and Temam [8] we shall give sufficient conditions for f such that problem (CH) has a global solution in space dimension 3.

We consider the case $n = 3$ and assume that f is a twice continuously differentiable function, such that f'' is locally Lipschitz continuous, $f(0) = 0$ and $\int_0^t f(s)\,ds \geq 0$ for all $t \in \mathbb{R}$. Moreover, we assume that

$$|f'(u)| \leq C(|u|^q + 1) \;\; (1 \leq q < 4) \quad \text{and} \quad |f''(u)| \leq C(|u|^r + 1) \;\; (1 \leq r < 3).$$

Lemma 4.2. *If u satisfies condition* (CH) *on $(0, T(u_0))$, then $t \mapsto \| |\nabla u(t)| \|_2^2$ and $t \mapsto \|\Delta u(t)\|_2^2$ are continuously differentiable on $(0, T(u_0))$, where $\|\cdot\|_2$ denotes the norm on $L^2(\Omega)$. Moreover, they satisfy for $t \in (0, T(u_0))$*

$$\frac{d}{dt}\big\||\nabla u(t)|\big\|_2^2 = -2\int_\Omega \Delta u(t)\,\frac{d}{dt}u(t)\,dx \quad \text{and} \quad \frac{d}{dt}\|\Delta u(t)\|_2^2 = 2\int_\Omega \Delta^2 u(t)\,\frac{d}{dt}u(t)\,dx.$$

Proof. From $u \in C^1\big((0, T(u_0)); C(\overline{\Omega})\big) \cap C\big((0, T(u_0)); \mathcal{D}(A)\big)$ it follows that

$$\left.\begin{aligned} &\int_\Omega \Delta u(t)\,\frac{u(t+h) - u(t)}{h}\,dx \\ &\int_\Omega \Delta u(t+h)\,\frac{u(t+h) - u(t)}{h}\,dx \end{aligned}\right\} \longrightarrow \int_\Omega \Delta u(t)\,\frac{d}{dt}u(t)\,dx$$

as $h \to 0$. On the other hand, Green's formula and the boundary conditions imply that

$$\begin{aligned} &\frac{\big\||\nabla u(t+h)|\big\|_2^2 - \big\||\nabla u(t)|\big\|_2^2}{h} \\ &= -\int_\Omega \Delta u(t+h)\,\frac{u(t+h) - u(t)}{h}\,dx - \int_\Omega \Delta u(t)\,\frac{u(t+h) - u(t)}{h}\,dx \\ &\to -2\int_\Omega \Delta u(t)\,\frac{d}{dt}u(t)\,dx \end{aligned}$$

as $h \downarrow 0$. The second part of the lemma is proved analogously ∎

Since we are only interested in what happens near $T(u_0)$, we assume again that $u_0 \in \mathcal{D}(A)$ and thus $u \in C^1\big([0, T(u_0)) : C(\overline{\Omega})\big) \cap C\big([0, T(u_0)); \mathcal{D}(A)\big)$.

Lemma 4.3. *There is a constant $C > 0$ such that $\|u(t)\|_2 \leq C$ and $\big\||\nabla u(t)|\big\|_2 \leq C$ for all $t \in [0, T(u_0))$.*

Proof. First remark that it is sufficient to prove that $\big\||\nabla u(t)|\big\|_2$ is uniformly bounded. Indeed, we can apply Poincaré's inequality, since $u(t) \in H_0^1(\Omega)$ for $t \in [0, T(u_0))$. Next define

$$J(u) = \frac{1}{2}\big\||\nabla u|\big\|_2^2 + \int_\Omega g(u)\,dx \qquad \text{with } g(s) = \int_0^s f(\sigma)\,d\sigma.$$

In view of Lemma 4.2, $t \mapsto J(u(t))$ is continuously differentiable on $[0, T(u_0))$ and we find that

$$\begin{aligned} \frac{d}{dt}J(u) &= -\int_\Omega \Delta u\, u_t\,dx + \int_\Omega f(u)\,u_t\,dx \\ &= \langle -\Delta u + f(u), u_t\rangle \\ &= \langle -\Delta u + f(u), -\Delta^2 u + \Delta f(u)\rangle \\ &= \int_{\partial\Omega} \frac{\partial(-\Delta u + f(u))}{\partial\nu}\big(-\Delta u + f(u)\big)\,d\Gamma - \int_\Omega \big|\nabla\big(-\Delta u + f(u)\big)\big|^2\,dx \\ &= -\big\||\nabla(-\Delta u + f(u))|\big\|_2^2 \end{aligned}$$

(note that $-\Delta u + f(u) = 0$ on $\partial\Omega$, for $u = \Delta u = 0$ on $\partial\Omega$ and $f(0) = 0$). Hence

$$\frac{d}{dt}J(u) + \big\|\,|\nabla(-\Delta u + f(u))|\,\big\|_2^2 = 0,$$

which implies that $J(u(t)) \leq J(u(0))$ for all $t \in [0, T(u_0))$. Since by assumption $\int_\Omega g(u)\,dx \geq 0$, we find that $\big\|\,|\nabla u(t)|\,\big\|_2^2 \leq 2J(u(0))$ for all $t \in [0, T(u_0))$ ∎

Theorem 4.4. *If f satisfies the above conditions, then problem* (CH) *has a global solution.*

Proof. Due to the regularity of the Dirichlet problem and the biharmonic Dirichlet problem there is a constant $C > 0$ only depending on Ω such that $\|u\|_{H^4} \leq C\|\Delta^2 u\|_2$ and $\|u\|_{H^2} \leq C\|\Delta u\|_2$ for all $u \in \mathcal{D}(A)$. Since $H^2(\Omega)$ is continuously imbedded in $C(\Omega)$ for $n = 3$, it is sufficient to show that $\|\Delta u(t)\|_2$ is uniformly bounded on $[0, T(u_0))$. If we multiply the equation by $\Delta^2 u(t)$ and integrate over Ω, we find by Lemma 4.2

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta u(t)\|_2^2 + \|\Delta^2 u(t)\|_2^2 &= \langle \Delta f(u(t)), \Delta^2 u(t)\rangle \\ &\leq \|\Delta f(u(t))\|_2\,\|\Delta^2 u(t)\|_2 \\ &\leq \frac{1}{2}\|\Delta f(u(t))\|_2^2 + \frac{1}{2}\|\Delta^2 u(t)\|_2^2\end{aligned}$$

hence

$$\frac{d}{dt}\|\Delta u(t)\|_2^2 + \|\Delta^2 u(t)\|_2^2 \leq \|\Delta f(u(t))\|_2^2.$$

Next we find that

$$\begin{aligned}\|\Delta f(u(t))\|_2 &\leq \|f'(u)\|_\infty\|\Delta u\|_2 + \|f''(u)\|_\infty\|\,|\nabla u|\,\|_4^2 \\ &\leq C(\|u\|_\infty^q + 1)\|\Delta u\|_2 + C(\|u\|_\infty^r + 1)\|\,|\nabla u|\,\|_4^2.\end{aligned}$$

Interpolation, the Sobolev embedding theorem and Agmon's inequality imply for $u \in \mathcal{D}(A)$ that

$$\begin{aligned}\|\Delta u\|_2 &\leq C\|\,|\nabla u|\,\|_2^{\frac{2}{3}}\|\Delta^2 u\|_2^{\frac{1}{3}} \\ \|\,|\nabla u|\,\|_4 &\leq C\|u\|_{H^{1+\frac{3}{4}}} \leq C\|\,|\nabla u|\,\|_2^{\frac{3}{4}}\|\Delta^2 u\|_2^{\frac{1}{4}} \\ \|u\|_\infty &\leq C\|\,|\nabla u|\,\|_2^{\frac{5}{6}}\|\Delta^2 u\|_2^{\frac{1}{6}}.\end{aligned}$$

Since $\|\,|\nabla u(t)|\,\|_2$ is uniformly bounded on $[0, T(u_0))$, we see that

$$\|\Delta f(u(t))\|_2 \leq C_1 + C_2\|\Delta^2 u(t)\|_2^{\frac{1}{3}+\frac{q}{6}} + C_3\|\Delta^2 u(t)\|_2^{\frac{1}{2}+\frac{r}{6}} \leq C_1 + C_4\|\Delta^2 u(t)\|_2^{\alpha}$$

with $0 < \alpha < 1$. Then Young's inequality implies that there is a constant $K > 0$ such that

$$\|\Delta f(u(t))\|_2^2 \leq K + \frac{1}{2}\|\Delta^2 u(t)\|_2^2.$$

Hence

$$\frac{d}{dt}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\Delta^2 u(t)\|_2^2 \leq K.$$

So $\|\Delta u(t)\|_2$ must be uniformly bounded on $[0, T(u_0))$ ∎

Remark 4.5. 1. If $n = 2$, one can show by the same argument that there is also global existence if $|f'(u)| \leq C(|u|^q + 1)$ and $|f''(u)| \leq C(|u|^q + 1)$ with $q \geq 1$. Indeed, if $n = 2$, then for every $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that, for all $u \in \mathcal{D}(A)$, $\|u\|_\infty \leq C\| \, |\nabla u| \, \|_2^{1-\varepsilon} \|\Delta^2 u\|_2^\varepsilon$.

2. A closer inspection of Lemma 4.3 shows that the condition $f(0) = 0$ can be dropped. Also note that our conditions on f are slightly weaker than those in [9].

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