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On Ill-Posedness Measures and Space Change in Sobolev Scales

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Abstract. The degree of ill-posedness of a linear inverse problem is an important knowledge base to select appropriate regularization methods for the stable approximate solution of such a problem. In this paper, we consider ill-posedness measures for a linear ill-posed operator equation $Ax = y$, where the compact linear operator $A : X \to Y$ maps between infinite dimensional Hilbert spaces. Using the decay rate of singular values of A tending to zero we define an interval of ill-posedness and motivate its meaning by considering lower and upper bounds for the rates of the condition numbers occurring in the numerical solution process of the discretized problem. An equivalent interval information is obtained when compactness measures as ε-entropy or ε-capacity are exploited alternatively. For the specific case $X := L^2(0,1)$, the space change problem of shifting the space X along a Sobolev scale is treated. In detail, we study the change of the interval of ill-posedness if the solutions are restricted to the Sobolev space $W_2^1[0,1]$. The results of these considerations are a warning to characterize the ill-posedness of a problem superficial. Moreover, the interdependences between ill-posedess measures, embedding operators, Hilbert and Sobolev scales are discussed.

Keywords: *Ill-posed problems, compact linear operators, ill-posedness measures, degree of ill-posedness, interval of ill-posedness, ε-capacity, Hilbert scale, Sobolev scale, embedding operators*

AMS subject classification: 65 J 20, 65 R 30, 46 E 35, 47 B 38, 47 B 06

1. Introduction

Let $A : X \to Y$ be a *compact linear injective* operator mapping between infinite dimensional Hilbert spaces X and Y with corresponding inner products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$ and norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. Throughout this paper, we always consider *separable* Hilbert spaces. Moreover, A is assumed to be non-degenerate (of infinite rank), i.e., the range $R(A)$ is infinite dimensional. In Sections 3 and 4 of this paper concerning the space change problem, the particular case $X = L^2(0,1)$ will be treated. Now we consider the operator equation

$$Ax = y \qquad (x \in X,\ y \in Y). \tag{1.1}$$

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Since the operator A is compact, it has a non-closed range in Y, i.e., we have $R(A) \neq \overline{R(A)}$, the inverse operator A^{-1} is unbounded, and the equation (1.1) is *ill-posed*. In the paper [21], Nashed calls this situation *ill-posedness of type II*. In the sequel we denote by

$$\left\{s_i(A);\, u_i(A);\, v_i(A)\right\}_{i=1}^{\infty}$$

the singular system (cf., e.g., [1: p. 63]) of the compact linear operator A, where $\{s_i(A)\}_{i=1}^{\infty}$ is the ordered sequence

$$\|A\|_{\mathcal{L}(X,Y)} = s_1(A) \geq s_2(A) \geq \ \dots\ \geq s_i(A) \geq s_{i+1}(A) \geq \ \dots\ \to 0$$

of positive singular values of A tending to zero as $i \to \infty$, $\{u_i(A)\}_{i=1}^{\infty} \subset X$ expresses an associated complete orthornormal eigensystem in X, and $\{v_i(A)\}_{i=1}^{\infty} \subset Y$ a complete orthonormal eigensystem in the closed subspace $\overline{R(A)}$ of Y, such that

$$A\, u_i(A) = s_i(A)\, v_i(A) \quad \text{and} \quad A^*\, v_i(A) = s_i(A)\, u_i(A) \qquad (i \in \mathbb{N}).$$

To evaluate the strength of ill-posedness of equation (1.1) by introducing an appropriate concept of *ill-posedness measures*, it can be recommended to consider the *decay rate* of the singular values $s_i(A)$ of the operator A for measuring the *smoothing properties* of A. As a consequence of ill-posedness, difficulties (instability effects) occur in the process of the approximate numerical solution of equation (1.1). Different ill-posedness levels, however, cause different levels of instability. Following the ideas of Liu, Guerrier and Bernard in [18] the numerical solution of (1.1) can be considered as a numerical realization of the constrained least-squares solution minimizing

$$\|Ax - y\|_Y \to \min \qquad \text{subject to } x \in X_N, \tag{1.2}$$

where X_N is an N-dimensional subspace of $X = X_N \oplus (X_N)^{\perp}$. For obtaining a good approximation by the extremal problem (1.2), an upper bound condition

$$B_{sup}((X_N)^{\perp}) := \sup_{x \in (X_N)^{\perp} \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X} < \varepsilon$$

should be satisfied, whereas stability is ensured whenever we have a lower bound condition

$$B_{inf}(X_N) := \inf_{x \in X_N \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X} \geq \varepsilon.$$

Often one uses the condition number

$$\operatorname{cond}(X_N) := \frac{B_{sup}(X_N)}{B_{inf}(X_N)},$$

since the relative error of the least-squares solution according to (1.2) is equal to the product of the relative error in the data y and the condition number $\operatorname{cond}(X_N)$. If we choose

$$X_N := \operatorname{span}(u_1(A), \dots, u_N(A))$$

with elements $u_i(A)$ from the singular system of A associated with the N largest singular values $s_i(A)$, then we have for $\varepsilon > 0$ given sufficiently small

$$B_{sup}((X_N)^\perp) = s_{N+1}(A) < \varepsilon \leq s_N(A) = B_{inf}(X_N)$$

when N is selected appropriately. For the condition number we then have $\mathrm{cond}(X_N) = \frac{s_1(A)}{s_N(A)}$. Provided that for $0 < \nu < \infty$ a decay rate $s_i(A) \sim i^{-\nu}$ of the singular values is assumed, a condition number rate

$$\mathrm{cond}(X_N) \sim N^\nu \tag{1.3}$$

is the consequence. Hence, large values of ν correspond to a fast decay rate of the singular values $s_i(A)$ and to a strong form of ill-posedness expressed by a high degree of instability in the numerical solution process. Note that throughout this paper $s_i(A) \sim i^{-\nu}$ denotes that there exist positive constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}\, i^{-\nu} \leq s_i(A) \leq \overline{c}\, i^{-\nu}. \tag{1.4}$$

Another condition number approach is for example mentioned by Wing in [32]. In that paper, discrete N-by-N-systems

$$\widehat{A}\widehat{x} = \widehat{y} \tag{1.5}$$

are considered to solve equation (1.1) approximately. Using orthonormal systems $\{\varphi_i\}_{i=1}^\infty \subset X$ and $\{\psi_i\}_{i=1}^\infty \subset Y$ a matrix $\widehat{A}$ of the form $\widehat{A} = \left((A\varphi_i, \psi_j)_Y\right)_{j=1,2,\ldots,N}^{i=1,2,\ldots,N}$ occurs in (1.5) when the Galerkin method is applied. In such a case a condition number

$$\gamma_N(\widehat{A}) := \|\widehat{A}\|\|\widehat{A}^{-1}\|$$

using spectral norms $\|\cdot\|$ helps to measure the stability of the discretized system. We always have an inequality

$$\gamma_N(\widehat{A}) \geq \frac{c}{s_N(A)} \tag{1.6}$$

with a constant c which is independent of N. Consequently, (1.4) then yields

$$\gamma_N(\widehat{A}) \geq \tilde{c}\, N^\nu.$$

A rapidly decreasing sequence of singular values of A implies also an essential instability of N-by-N systems approximating (1.1).

There are many cases of ill-posed linear operator equations (1.1) of great practical relevance, where (1.4) with fixed ν is satisfied. Then we can compute ν as limit

$$\nu = \lim_{i\to\infty} \frac{-\log s_i(A)}{\log i} \tag{1.7}$$

depending on the singular value sequence $\{s_i(A)\}_{i=1}^\infty$. The base of the logarithm in (1.7) can be chosen arbitrarily. For example, the classical Abel integral equation leads to

$\nu = \frac{1}{2}$. On the other hand, the problem of finding the k-th derivative ($k = 1, 2, \ldots$) of a function with appropriate homogeneous initial conditions leads to $\nu = k$. Then various condition number rates ν in (1.3) suggest a classification of *mildly, moderately* and *severely ill-posed* problems according to Wahba [31] (see also [9: Section 2.2.2]) whenever ν is small (e.g., if $\nu < 1$), moderate (e.g., if $1 \le \nu < \infty$) and infinite (if no finite power ν exists), respectively.

Unfortunately, it is not always possible to characterize the decay rate of singular values $s_i(A)$ and consequently the ill-posedness of (1.1) by a single constant ν. However, if (1.1) is considered for example as a linear Fredholm integral equation of the first kind

$$\int_0^1 K(t,\tau)\,x(\tau)\,d\tau = y(t) \qquad (0 \le t \le 1) \tag{1.8}$$

with $X = Y = L^2(0,1)$ and a quadratically integrable kernel $K \in L^2((0,1) \times (0,1))$, some given degree of kernel smoothness can imply an inequality of the form

$$s_i(A) \le \frac{\bar{c}}{i^\nu} \tag{1.9}$$

(for details cf., e.g., Chang [3] and Ha [8]). Therefore, it is motivated to call the supremum of all values ν satisfying (1.9) the *degree of ill-posedness* of (1.1). Note that this supremum can be expressed as a lower limit:

$$\liminf_{i\to\infty} \left(\frac{-\log s_i(A)}{\log i}\right) = \sup\left\{\nu :\ s_i(A) = O(i^{-\nu}) \text{ as } i \to \infty\right\}.$$

Definition 1.1. We call the equation (1.1) *ill-posed of degree* $\underline{\mu}(A)$, where $0 \le \underline{\mu}(A) \le \infty$, if we have for the singular values $s_i(A)$ of the non-degenerate compact linear injective operator A

$$\underline{\mu}(A) = \liminf_{i\to\infty} \left(\frac{-\log s_i(A)}{\log i}\right).$$

If $\underline{\mu}(A) = \infty$, we call the problem *severely ill-posed*.

If, for example, in equation (1.8) the kernel $K = K(t,\tau)$ and its partial derivatives $\frac{\partial^\alpha K}{\partial t^\alpha}$ are continuous in t for almost all τ and all $\alpha = 1, 2, \ldots, k-1$, where moreover

$$\frac{\partial^k K(t,\tau)}{\partial t^k} = \int_0^t f(\theta,\tau)\,d\theta + g(\tau)$$

with f quadratically integrable and g summable, then we obtain $\underline{\mu}(A) \ge k + \frac{3}{2}$ (cf. [32: Lemma 3.3]). For Hilbert-Schmidt operators there holds $\underline{\mu}(A) \ge \frac{1}{2}$.

The degree of ill-posedness $\underline{\mu}(A)$ expresses the slowest decay rate asymptotics of all singular value subsequences. In the present paper, we focus our attention in particular to the case of a *finite* degree of ill-posedness.

By the authors' knowledge there are no assertions on the correspondence between properties of the kernel K of an integral equation (1.8) and inequalities of the form $\frac{c}{i^\nu} \le s_i(A)$. On the other hand, as the following example will show, the non-increasing sequence $\{s_i\}_{i=1}^\infty$ of singular values can possess a subsequence $\{s_{i_j}\}_{j=1}^\infty$ tending to zero faster than expressed by the decay rate $\underline{\mu}$.

Example 1.2. Consider, for $j = 1, 2, \ldots$, the singular value sequence $s_i = s_i(A)$ with

$$s_i := \frac{1}{10^{2^j}} \qquad \text{if } i = 10^{2^{(j-1)}}, 10^{2^{(j-1)}} + 1, \ldots, 10^{2^j} - 1.$$

Then we have subsequences $i_j := 10^{2^{(j-1)}}$ with $s_{i_j} = \frac{1}{i_j^2}$ and $i_j := 10^{2^j} - 1$ with $s_{i_j} \sim \frac{1}{i_j}$. For this singular value sequence we obtain

$$\underline{\mu}(A) = \liminf_{i\to\infty} \frac{-\lg s_i}{\lg i} = 1 < 2 = \limsup_{i\to\infty} \frac{-\lg s_i}{\lg i}.$$

On the other hand, let for a singular value sequence

$$s_i := \begin{cases} 1 & \text{if } i = 1, 2, \ldots, 99 \\ \frac{1}{10^{2^{2^j}}} & \text{if } i = 10^{2^{2^{(j-1)}}}, 10^{2^{2^{(j-1)}}} + 1, \ldots, 10^{2^{2^j}} - 1 \quad (j \in \mathbb{N}). \end{cases}$$

Then we have $s_{i_j} \sim \frac{1}{i_j}$ for the subsequence $i_j := 10^{2^{2^j}} - 1$, consequently $\underline{\mu}(A) = \liminf_{i\to\infty} \frac{-\lg s_i}{\lg i} = 1$. However, the subsequence $i_j := 10^{2^{2^{(j-1)}}}$ implies $\frac{-\lg s_{i_j}}{\lg i_j} = \frac{\lg 10^{2^{2^j}}}{\lg 10^{2^{2^{(j-1)}}}} = 2^{2^{(j-1)}} \to \infty$ as $j \to \infty$. Hence, $\limsup_{i\to\infty} \frac{-\lg s_i}{\lg i} = \infty$.

Definition 1.3. We call the finite or infinite interval

$$[\underline{\mu}(A), \overline{\mu}(A)] := \left[\liminf_{i\to\infty} \frac{-\log s_i(A)}{\log i}, \limsup_{i\to\infty} \frac{-\log s_i(A)}{\log i}\right]$$

the *interval of ill-posedness* of the ill-posed linear operator equation (1.1).

Note that $\overline{\mu}(A)$ can be rewritten as

$$\overline{\mu}(A) = \inf\left\{\nu : i^{-\nu} = O(s_i(A)) \text{ as } i \to \infty\right\},$$

hence this value expresses the fastest decay rate asymptotics of all singular value subsequences. Moreover, it should be mentioned that an interval of ill-posedness $[\underline{\mu}(A), \overline{\mu}(A)]$ with $\underline{\mu} > 0$ and $\overline{\mu} < \infty$ does not imply an inequality of the form

$$\frac{\underline{c}}{i^{\overline{\mu}}} \le s_i(A) \le \frac{\overline{c}}{i^{\underline{\mu}}} \tag{1.10}$$

with positive finite constants $\underline{c}$ and $\overline{c}$. Namely, for fixed $0 < \nu < \infty$ the singular value sequence $s_i := \frac{1}{i^\nu \ln(i+1)}$ (and respectively $s_i := \frac{\ln(i+1)}{i^\nu}$) with $\underline{\mu} = \overline{\mu} = \nu$ cannot satisfy the left (right) inequality of (1.10). However, for $0 < \varepsilon < \underline{\mu}(A) \le \overline{\mu}(A) < \infty$ we always have an inequality of the form

$$\frac{\underline{c}}{i^{\overline{\mu}+\varepsilon}} \le s_i(A) \le \frac{\overline{c}}{i^{\underline{\mu}-\varepsilon}}.$$

The interval of ill-posedness of Definition 1.3 may serve as an appropriate *ill-posedness measure* of equation (1.1). Both the left and the right end of the interval of ill-posedness are important to characterize the ill-posedness of (1.1). The left end expresses the lower bound and the right end expresses the upper bound of power rates ν such that in both cases there exist subsequences of Ansatz dimensions $N_j \to \infty$ as $j \to \infty$ in the numerical solution process, where $\mathrm{cond}(X_{N_j}) \sim (N_j)^\nu$. On the other hand, the one-sided degree of ill-posedness of Definition 1.1 characterizes the ill-posedness behaviour of (1.1) too superficial. A similar situation is mentioned by Dicken and Maaß in Remark 3.1 of [4], where not a single value, but a pair of characteristics (smoothing order and a smoothing potential) are introduced to describe the smoothing properties of an operator A. Note, however, that in [4] as well as in the frameworks of Louis [19: p. 49] and Natterer (cf., e.g., [23]) a further approach for classifying linear ill-posed problems is used, where both the smoothing properties of A and the smoothness of solutions are taken into consideration. For some remarks on cross connections between these approaches and our ideas we refer to Section 4 of this paper.

Note that the problem of measuring the ill-posedness also occurs for *ill-posed operator equations*

$$F(x) = y \qquad (x \in D(F) \subset X,\ y \in Y) \tag{1.11}$$

with *nonlinear operators* $F : D(F) \subset X \to Y$ and domain $D(F)$. Based on the papers [5] of Engl, Kunisch and Neubauer and [28] of Seidman and Vogel the analysis and regularization of such nonlinear ill-posed problems became of great interest in the recent literature (for some overview cf. Chapter 10 of the book [6] of Engl, Hanke and Neubauer). In correspondence with the nonlinearity of the operator F the strength of ill-posedness of problems (1.11) may change when the solution point $x \in D(F)$ changes. Preferably, the Fréchet derivative $A := F'(x_0)$ can be used for measuring the local ill-posedness behaviour of the nonlinear equation (1.11) (cf. Hofmann and Scherzer [11] and [12]).

The further sections of this paper are organized as follows: In Section 2, alternative ill-posedness measures based on specific measures of compactness are compared. It is shown that the interval of ill-posedness of Definition 1.3 is able to express all the information included in these alternative measures provided that $\underline{\mu}(A)$ and $\overline{\mu}(A)$ are both finite and positive values. On the other hand, the alternative compactness measures give some additional motivation for the utility of our ill-posedness interval and help us to interpret the bounds of this interval from another point of view. In Section 3, for $X := L^2(0,1)$ we are dealt with the space change problem of (1.1). That means, we ask for the transformation of the interval of ill-posedness when the solutions are restricted to the Hilbertian Sobolev space $Z := W_2^1[0,1]$. The space change problem becomes of interest if we intend to improve assertions on convergence in $L^2(0,1)$, for example concerning regularized solutions of (1.1), to the stronger topology in the Sobolev space $W_2^1[0,1]$. It will be shown that the values on both ends of the interval of ill-posedness at least grow by one. However, it may happen that the interval length increases when the space X is changed. So we give examples with an interval of ill-posedness of length zero in $L^2(0,1)$ and positive or infinite length in $W_2^1[0,1]$. We can consider such situations as a warning that a regularization strategy which works well in $L^2(0,1)$ is not necessarily appropriate for finding solutions in $W_2^1[0,1]$. On the other hand, by using Hilbert scales generated by the selfadjoint operator A^*A, in Section 3 we can also formulate sufficient conditions for the well-behaved extremal case that the interval of ill-posedness is shifted to the right by the increment one. As the examples presented and the general considerations on Sobolev scales in the context of the space change problem in Section 4 show, this advantageous situation occurs when corresponding parts of the generated Hilbert scales are closely related to scales of Sobolev spaces.

2. Alternative ill-posedness measures

In this section we are going to compare the measures of *ε-entropy*, *ε-capacity* and *number of degrees of freedom* comprehensively studied by Prosser in [26] and mentioned as ill-posedness measures by Bertero and Boccacci in [2] with respect to the degree and interval of ill-posedness introduced here by the above definitions. The basic ideas of these concepts are due to Kolmogorov and Tihomirov (see, e.g., [16]). We consider the unit ball

$$B_1 := \{x \in X : \|x\|_X \le 1\}$$

in X. Then in view of the compactness of A the image set

$$S_1 := AB_1 = \{y \in Y : y = Ax,\ x \in B_1\}$$

is a compact ellipsoid in Y. The most convincing ill-posedness measure in this context is the *ε-capacity*

$$C_1(\varepsilon, A) := \log_2 M_1(\varepsilon, A),$$

where $M_1(\varepsilon, A)$ is the largest number of ε-distinguishable elements $y_1, y_2, \ldots, y_{M_1}$ in S_1 satisfying the conditions $\|y_i - y_j\|_Y > \varepsilon$ for all $i \ne j$. The values C_1 as a function of ε express the maximum number of information (bits), which (restricted to the unit ball) can be recovered by solving the inverse problem (1.1) for given noisy data with noise level ε. To get an asymptotic ε-classification, along the ideas of Prosser one searches for the infimum of all ν which allow an inequality of the form $C_1(\varepsilon, A) \le \frac{\bar{c}}{\varepsilon^\nu}$. Thus one can define an *order of growth*

$$\rho(A) := \limsup_{\varepsilon \to 0} \frac{\log C_1(\varepsilon, A)}{\log 1/\varepsilon}.$$

As we will show, $\rho(A)$ and our *degree of ill-posedness* $\underline{\mu}(A)$ are inverse values. To recognize this fact, however, we have to consider two more auxiliary measures.

A second compactness measure, not so easy to interpret with respect to ill-posedness, but closely related to the ε-capacity, is the *ε-entropy* defined as

$$H_1(\varepsilon, A) := \log_2 N_1(\varepsilon, A)$$

with $N_1(\varepsilon, A)$ the minimum number of open balls with centres in S_1 and radius ε covering S_1, i.e., the union of these $N_1(\varepsilon)$ balls includes S_1. We have (cf. [26: Lemma 1])

$$H_1(2\varepsilon, A) \le C_1(\varepsilon, A) \le H_1(\varepsilon, A). \tag{2.1}$$

Moreover, we can consider a measure similar to the singular value decay rates discussed in Section 1, namely

$$K_1(\varepsilon, A) := \max\{i : s_i(A) \geq \varepsilon\},$$

which is called in [2] the *number of degrees of freedom* with an *order of growth*

$$\lambda(A) := \limsup_{\varepsilon \to 0} \frac{\log K_1(\varepsilon, A)}{\log 1/\varepsilon}.$$

For this measure K_1 we have inequalites of the form (cf. [26: Theorem 3])

$$K_1(2\varepsilon, A) \leq H_1(\varepsilon, A) \leq K_1(\varepsilon/2, A) \log_2 \frac{4\sqrt{K_1(\varepsilon, A)}}{\varepsilon}. \tag{2.2}$$

In [26] it is shown that $\lambda(A) = \rho(A)$. The papers [2] and [26], indeed, do not discuss the corresponding *lower limits* of $\frac{\log C_1(\varepsilon,A)}{\log 1/\varepsilon}$ and $\frac{\log K_1(\varepsilon,A)}{\log 1/\varepsilon}$ as $\varepsilon \to 0$. However, one cannot exclude in general the case that there exist sequences $\{\varepsilon_i\}_{i=1}^{\infty}$ tending to zero as $i \to \infty$ such that $\lim_{i\to\infty} \frac{\log C_1(\varepsilon_i,A)}{\log 1/\varepsilon_i} < \rho(A)$ exists. Consequently, we have the problem of requiring not only a single value, but a whole interval for the alternative ill-posedness measures ε-entropy, ε-capacity and number of degrees of freedom in the same manner as for the singular value decay rate. The following theorem formulates the corresponding cross connections concerning lower and upper bounds of such intervals.

Theorem 2.1. *Let $A : X \to Y$ be a compact linear injective operator of infinite rank and $\{s_i(A)\}$ the monotonically non-increasing sequence of singular values of A with*

$$s_1(A) \geq s_2(A) \geq \dots \geq s_i(A) \geq \dots > 0,$$

where $s_i(A) \to 0$ as $i \to \infty$. Then we have the limit equations

$$\underline{\mu}(A) := \liminf_{i\to\infty} \left(\frac{-\log s_i(A)}{\log i} \right) = \liminf_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log K_1(\varepsilon, A)} \right) \tag{2.3}$$

$$\liminf_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log K_1(\varepsilon, A)} \right) = \liminf_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log H_1(\varepsilon, A)} \right) = \liminf_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log C_1(\varepsilon, A)} \right) \tag{2.4}$$

and

$$\overline{\mu}(A) := \limsup_{i\to\infty} \left(\frac{-\log s_i(A)}{\log i} \right) = \limsup_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log K_1(\varepsilon, A)} \right) \tag{2.5}$$

$$\limsup_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log K_1(\varepsilon, A)} \right) = \limsup_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log H_1(\varepsilon, A)} \right) = \limsup_{\varepsilon\to 0} \left(\frac{-\log \varepsilon}{\log C_1(\varepsilon, A)} \right). \tag{2.6}$$

If one of the lower limits in the equations (2.3) and (2.4) is equal to ∞, then all other lower limits in these equations also attain the improper value ∞. The same assertion is valid for the upper limits in the equations (2.5) and (2.6).

Proof. First we prove (2.3) and (2.5) by dividing the interval $(0, s_1(A)]$ into subintervals $(s_{i+1}(A), s_i(A)]$ $(i \in \mathbb{N})$. On each subinterval $\varepsilon \in (s_{i+1}(A), s_i(A)]$ we have

$$K_1(\varepsilon, A) = i \qquad \text{and} \qquad -\log s_i(A) \le -\log\varepsilon \le -\log s_{i+1}(A),$$

consequently,

$$\frac{-\log s_i(A)}{\log i} \le \frac{-\log\varepsilon}{\log K_1(\varepsilon, A)} \le -\frac{\log s_{i+1}(A)}{\log(i+1)}\,\frac{\log(i+1)}{\log i} \qquad \big(\varepsilon \in (s_{i+1}(A), s_i(A)]\big). \quad (2.7)$$

Taking into consideration that $\lim_{i\to\infty} \frac{\log(i+1)}{\log i} = 1$ we can apply lower and upper limits as $i \to \infty$ and $\varepsilon \to 0$, respectively, to the inequalites of (2.7). This yields the equations (2.3) and (2.5). The missing proof of equations (2.4) and (2.6) immediately follows from the inequalities (2.1) and (2.2). The corresponding technique used in [26: Corollary 4] applies in the lower and in the upper limit case ∎

The equations of Theorem 2.1 indicate that the lower and upper bounds of the interval of ill-posedness $[\underline{\mu}(A), \overline{\mu}(A)]$ represent all information expressed by the asymptotic behaviour of $C_1(\varepsilon, A)$, $H_1(\varepsilon, A)$ and $K_1(\varepsilon, A)$ as $\varepsilon \to 0$. In particular, if the occurring lower (upper) limits are positive finite values, then we have equations of the form

$$\limsup_{\varepsilon\to 0}\left(\frac{\log C_1(\varepsilon, A)}{\log 1/\varepsilon}\right) = \frac{1}{\underline{\mu}(A)} \qquad \text{and} \qquad \liminf_{\varepsilon\to 0}\left(\frac{\log C_1(\varepsilon, A)}{\log 1/\varepsilon}\right) = \frac{1}{\overline{\mu}(A)}.$$

3. The space change problem

Now we are going to compare the ill-posed basic problem (1.1), where we set in the sequel $X := L^2(0,1)$ with the modified operator equation

$$\widetilde{A}x = y \qquad (x \in Z := W_2^1[0,1],\ y \in Y). \quad (3.1)$$

In this equation, the operator $\widetilde{A}$ expresses the restriction of A from $L^2(0,1)$ to the Hilbertian Sobolev space $W_2^1[0,1]$ of $L^2(0,1)$-functions x possessing a generalized derivative $x' \in L^2(0,1)$ with norm

$$\|x\|_{W_2^1[0,1]} := \left\{\|x\|^2_{L^2(0,1)} + \|x'\|^2_{L^2(0,1)}\right\}^{\frac{1}{2}}.$$

Consequently,

$$\widetilde{A} := A\,\mathcal{E}^{L^2(0,1)}_{W_2^1[0,1]} :\ Z = W_2^1[0,1] \to Y \quad (3.2)$$

is the *composition operator* of A and the *embedding operator*

$$\mathcal{E}^{L^2(0,1)}_{W_2^1[0,1]} :\ W_2^1[0,1] \to L^2(0,1)$$

from $W_2^1[0,1]$ into $L^2(0,1)$. The question under consideration in the following is the

Space change problem: *How does the interval of ill-posedness change if equation* (1.1) *is replaced by equation* (3.1) ?

Now the singular system $\{\hat{s}_i\,;\,\hat{u}_i\,;\,\hat{v}_i\}_{i=1}^{\infty}$ of the compact embedding operator $\mathcal{E}_{W_2^1[0,1]}^{L^2(0,1)}$ can be explicitly verified as $\{\hat{s}_1;\hat{u}_1;\hat{v}_1\} = \{1;1;1\}$ and $\{\hat{s}_i;\hat{u}_i;\hat{v}_i\}_{i=2}^{\infty}$ given by

$$\left\{\frac{1}{\sqrt{1+(i-1)^2\pi^2}}\,;\,\sqrt{\frac{2}{1+(i-1)^2\pi^2}}\cos\left((i-1)\pi t\right)\,;\,\sqrt{2}\cos\left((i-1)\pi t\right)\right\}_{i=2}^{\infty}.$$

That means $s_i(\mathcal{E}_{W_2^1[0,1]}^{L^2(0,1)}) \sim \frac{1}{i}$. Nevertheless one cannot easily conclude that the composition operator $\widetilde{A}$ has singular values $s_i(\widetilde{A}) \sim \frac{1}{i^{\nu+1}}$ if $s_i(A) \sim \frac{1}{i^{\nu}}$, since the interplay of eigensystems of the factor operators A and $\mathcal{E}_{W_2^1[0,1]}^{L^2(0,1)}$ is a priori unknown. Therefore, it *cannot be expected* in general that

$$[\underline{\mu}(\widetilde{A})\,,\,\overline{\mu}(\widetilde{A})] = [\underline{\mu}(A)+1\,,\,\overline{\mu}(A)+1]. \tag{3.3}$$

First we give an example, where the composition operator $\widetilde{A}$ possesses an interval of ill-posedness of positive length, although we have $\underline{\mu}(A) = \overline{\mu}(A)$ for the operator A. For simplicity, instead of the operator product (3.2) we are in the following always dealt with the composition operator

$$\widetilde{A} = A\,\widetilde{\mathcal{E}} \qquad \text{with } \widetilde{\mathcal{E}} : W_2^1[0,1] \to L^2(0,1), \tag{3.4}$$

where we define

$$\left\{s_i(\widetilde{\mathcal{E}}) := \frac{1}{i}\,;\,u_i(\widetilde{\mathcal{E}}) := \hat{u}_i\,;\,v_i(\widetilde{\mathcal{E}}) := \hat{v}_i\right\}_{i=1}^{\infty}$$

for the singular system of $\widetilde{\mathcal{E}}$. Since $\widetilde{\mathcal{E}}$ and $\mathcal{E}_{W_2^1[0,1]}^{L^2(0,1)}$ are spectrally equivalent operators with the same eigenelements, the asymptotics results under consideration here are identical.

Example 3.1. Let $n \geq 2$ be a given fixed integer. We assume that $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is the sequence of singular values of both the operator A and the operator $\widetilde{\mathcal{E}}$ and subdivide this sequence into blocks

$$\left\{1, \dots, \frac{1}{10^{n^0}-1}\,;\,\frac{1}{10^{n^0}}, \dots, \frac{1}{10^{n^1}-1}\,;\,\frac{1}{10^{n^1}}, \dots, \frac{1}{10^{n^2}-1}\,;\,\dots\right\}.$$

Moreover, we assume that for any block $\{s_m(\widetilde{\mathcal{E}}), \dots, s_{m+l}(\widetilde{\mathcal{E}})\} := \{\frac{1}{10^{n^{(k-1)}}}, \dots, \frac{1}{10^{n^k}-1}\}$ with $l = 10^{n^k} - 10^{n^{(k-1)}} - 1$ the eigenelements $u_i(A)$ of A and $\hat{v}_j$ of $\widetilde{\mathcal{E}}$ coincide, but they are arranged in reverse order inside the block, i.e., $u_m(A) = \hat{v}_{m+l}$, $u_{m+1}(A) = \hat{v}_{m+l-1}, \dots, u_{m+l}(A) = \hat{v}_m$. Then the singular values of the composition operator $\widetilde{A} = A\,\widetilde{\mathcal{E}}$ have the product form $s_i(\widetilde{A}) = s_{i_1}(\widetilde{\mathcal{E}})s_{i_2}(A)$ with appropriate indices i_1 and i_2. The blocks mentioned above remain stable in the ordered $s_i(\widetilde{A})$-sequence, and inside of the blocks we have first elements that form the most rapidly decreasing subsequence $s_{i_j}(\widetilde{A})$ for $i_j := 10^{n^j}$ $(j = 0,1,2,\dots)$, which is given by

$$s_{i_j}(\widetilde{A}) = \frac{1}{10^{n^j}(10^{n^{(j+1)}}-1)} = \frac{1}{i_j((i_j)^n-1)}.$$

Hence,

$$\limsup_{i\to\infty} \frac{-\lg s_i(\widetilde{A})}{\lg i} = 1 + n.$$

The last elements of any block, however, form the most slowly decreasing sequence with $i_j := 10^{n^j} - 1$ $(j = 0, 1, 2, \dots)$ realizing the lower limit

$$\liminf_{i\to\infty} \frac{-\lg s_i(\widetilde{A})}{\lg i} = 2.$$

Consequently, for this example $\underline{\mu}(A) = \overline{\mu}(A) = 1$ is transformed by the space change from $X := L^2(0,1)$ to $Z := W_2^1[0,1]$ to the interval of ill-posedness $[\underline{\mu}(\widetilde{A}), \overline{\mu}(\widetilde{A})] = [2, n]$. It should be noted that we can also modify the example such that an infinite interval of ill-posedness $[2, \infty)$ is obtained for $\widetilde{A}$. Then the blocks have to be formed with bounds $10^{2^{2^j}}$ instead of 10^{n^j} (see the second sequence of Example 1.2).

General assertions on the ill-posedness interval transformation for the space change problem can be derived from the fact that the singular values of compact operators are *multiplicative* in the sense of the following lemma (for the proof see, e.g., König [14: p. 70]). Propositions on ill-posedness measures of operator products are also presented in [26: Theorem 6].

Lemma 3.2. *Let $T : H_1 \to H_2$ and $S : H_2 \to H_3$ denote compact linear operators between Hilbert spaces H_1, H_2 and H_3. Then we have for the singular values of the compact linear composite operator $ST : H_1 \to H_3$*

$$s_{i+j-1}(ST) \le s_i(S)\, s_j(T) \qquad (i, j = 1, 2, \dots). \tag{3.5}$$

If $\overline{\mu}(A)$ of equation (1.1) is a finite number, then the space change with respect to admissible solutions from $L^2(0,1)$ to $W_2^1[0,1]$ is connected with a growth of both bounds of the interval of ill-posedness at least by one.

Theorem 3.3. *Let for the interval of ill-posedness $[\underline{\mu}(A), \overline{\mu}(A)]$ of the operator equation (1.1) hold $0 \le \underline{\mu}(A) \le \overline{\mu}(A) < \infty$. Then for the interval of ill-posedness $[\underline{\mu}(\widetilde{A}), \overline{\mu}(\widetilde{A})]$ of the changed equation (3.1) we obtain*

$$\underline{\mu}(\widetilde{A}) \ge \underline{\mu}(A) + 1 \qquad \text{and} \qquad \overline{\mu}(\widetilde{A}) \ge \overline{\mu}(A) + 1. \tag{3.6}$$

On the other hand, the conditions $\underline{\mu}(A) = \infty$ and $\overline{\mu}(A) = \infty$ imply $\underline{\mu}(\widetilde{A}) = \infty$ and $\overline{\mu}(\widetilde{A}) = \infty$, respectively.

Proof. From formula (3.5) for $i := k$ and $j := k+1$ we get $s_{2k}(\widetilde{A}) \le s_k(A)\,\hat{s}_{k+1} \le \frac{s_k(A)}{k}$ and

$$\frac{-\log s_{2k}(\widetilde{A})}{\log(2k)} \ge \frac{-\log s_k(A)}{\log k}\,\frac{\log k}{\log 2k} + \frac{\log k}{\log 2k}. \tag{3.7}$$

Taking into consideration that $\lim_{k\to\infty} \frac{\log k}{\log(2k)} = 1$ we now apply the lower and the upper limit to the inequality (3.7). This yields

$$\liminf_{k\to\infty} \frac{-\log s_{2k}(\widetilde{A})}{\ln(2k)} \ge \underline{\mu}(A) + 1\,; \qquad \limsup_{k\to\infty} \frac{-\log s_{2k}(\widetilde{A})}{\ln(2k)} \ge \overline{\mu}(A) + 1. \tag{3.8}$$

On the other hand, by setting $i := k+1$ and $j := k+1$ in formula (3.5) we obtain $s_{2k+1}(\widetilde{A}) \le s_{k+1}(A)\,\hat{s}_{k+1} \le \frac{s_k(A)}{k}$ and

$$\liminf_{k\to\infty} \frac{-\log s_{2k+1}(\widetilde{A})}{\ln(2k+1)} \ge \underline{\mu}(A)+1\,; \qquad \limsup_{k\to\infty} \frac{-\log s_{2k+1}(\widetilde{A})}{\ln(2k+1)} \ge \overline{\mu}(A)+1. \tag{3.9}$$

A combination of the inequalities (3.8) and (3.9) implies (3.6) and the assertions of this theorem concerning the case of infinite values $\underline{\mu}(A)$ and $\overline{\mu}(A)$ ∎

Theorem 3.3 provides us with lower bounds of $\underline{\mu}(\widetilde{A})$ and $\overline{\mu}(\widetilde{A})$ if $\underline{\mu}(A)$ and $\overline{\mu}(A)$ are given. As mentioned at the end of Example 3.1 corresponding a priori upper bounds at least for the upper limit do not exist, since $\overline{\mu}(\widetilde{A}) = \infty$ may hold even if $\overline{\mu}(A) < \infty$. We note that there also exist operators A such that $\underline{\mu}(\widetilde{A}) > \underline{\mu}(A)+1$, i.e., for which the increment of the degree of ill-posedness for the space change problem is greater than one and can even be infinity as the following example shows.

Example 3.4. Let $\{w_j\}_{j=1}^{\infty}$ denote an orthonormal system in the Hilbert space Y and let define $A: L^2(0,1) \to Y$ according to $A\,\hat{v}_i = \lambda_i(A)\,w_i \quad (i=1,2,\ldots)$, where

$$\lambda_i(A) = \begin{cases} \frac{1}{\lg i} & \text{if } i = 10^k \quad (k=1,2,\ldots) \\ \frac{1}{10^i} & \text{if } i \text{ is chosen otherwise} \end{cases}$$

are the unordered singular values of A. We reorder this sequence $\{\lambda_i(A)\}_{i=1}^{\infty}$ according to its magnitude and obtain the ordered sequence $\{s_i(A)\}_{i=1}^{\infty}$ satisfying the inequalities $\frac{1}{i} \le s_i(A) \le \frac{2}{i}$. Consequently, we have $\mu(A) := \underline{\mu}(A) = \overline{\mu}(A) = 1$. Furthermore, for the composite operator $\widetilde{A} = A\widetilde{\mathcal{E}}$ with $\widetilde{\mathcal{E}} : W_2^1[0,1] \to L^2(0,1)$ given in (3.4) we find $s_i(\widetilde{A}) \sim \frac{1}{i\,10^i}$ and $\mu(\widetilde{A}) := \underline{\mu}(\widetilde{A}) = \overline{\mu}(\widetilde{A}) = \infty$. Hence, the increment of the degree of ill-posedness for the space change problem is here infinity.

Now it seems to be of some interest to find classes of operators A such that (3.3) holds. For the space change problem this equation characterizes the situation of a minimal growth of $\underline{\mu}$ and $\overline{\mu}$ by one. We restrict our considerations here in the following to the case

$$0 < \mu(A) := \underline{\mu}(A) = \overline{\mu}(A) < \infty. \tag{3.10}$$

Provided that (3.10) is satisfied we construct for the given operator A a Hilbert scale $(H_r)_{r\in\mathbb{R}}$ in the sense of Krein and Petunin [15] (see also Baumeister [1: Section 5.1] and Neubauer [25]) by considering the injective linear compact self-adjoint operator

$$B := (A^*A)^{\frac{1}{2\mu(A)}}$$

in $X = L^2(0,1)$ with the singular system

$$\left\{ s_i(B) = s_i(A)^{\frac{1}{\mu(A)}}\,;\ u_i(B) = u_i(A)\,;\ v_i(B) = u_i(A) \right\}$$

and

$$\mu(B) := \underline{\mu}(B) = \overline{\mu}(B) = \lim_{i\to\infty} \frac{-\log s_i(B)}{\log i} = 1.$$

In the obvious way we define for any real r the Hilbert spaces of the scale

$$H_r := \{x : \|x\|_r := \|B^{-r}x\|_X < \infty\}.$$

Then we have for all $x \in X := L^2(0,1)$

$$\|Ax\|_Y = \|B^{\mu(A)}x\|_X = \|x\|_{-\mu(A)}. \tag{3.11}$$

If we consider the embedding operator $\mathcal{E}_{L^2(0,1)}^{H_{-\mu(A)}} : H_0 = L^2(0,1) \to H_{-\mu(A)}$, then we can rewrite (3.11) as

$$\|Ax\|_Y = \|\mathcal{E}_{L^2(0,1)}^{H_{-\mu(A)}} x\|_{-\mu(A)}.$$

By the definition of our Hilbert scale $(H_r)_{r\in\mathbb{R}}$ it immediately follows that for $t < r$ the embedding operator $\mathcal{E}_{H_r}^{H_t} : H_r \to H_t$ is compact and possesses the singular system

$$\{s_i(B)^{r-t}\,;\ s_i(B)^r u_i(A)\,;\ s_i(B)^t u_i(A)\}.$$

Now we come back to the restriction operator $\widetilde{A}$ according to (3.2). From formula (3.11) we obtain for all $\tilde{x} \in Z := W_2^1[0,1]$

$$\|\widetilde{A}\,\tilde{x}\|_Y = \|\mathcal{E}_{W_2^1[0,1]}^{H_{-\mu(A)}}\,\tilde{x}\|_{-\mu(A)}.$$

The following Lemma 3.5 implies that the operator $\widetilde{A} : W_2^1[0,1] \to Y$ and the embedding operator $\mathcal{E}_{W_2^1[0,1]}^{H_{-\mu(A)}} : W_2^1[0,1] \to H_{-\mu(A)}$ have the same singular values:

$$s_i(\widetilde{A}) = s_i(\mathcal{E}_{W_2^1[0,1]}^{H_{-\mu(A)}}) \qquad (i = 1,2,\dots). \tag{3.12}$$

Lemma 3.5 ([9: Lemma 2.46]). *Let $T : H_1 \to H_2$ denote a compact linear operator between Hilbert spaces H_1 and H_2. Moreover, let H_1 be a subset of the Hilbert space H_3 and $\mathcal{E} : H_1 \to H_3$ denote the compact embedding operator from H_1 into H_3. If we then have inequalities of the form*

$$c_1\,\|x\|_{H_3} \le \|Tx\|_{H_2} \le c_2\,\|x\|_{H_3} \qquad \text{for all } x \in H_1 \tag{3.13}$$

with constants $0 < c_1 \le c_2 < \infty$, then the singular values $s_i(T)$ of T and $s_i(\mathcal{E})$ of $\mathcal{E}$ satisfy the condition

$$c_1\, s_i(\mathcal{E}) \le s_i(T) \le c_2\, s_i(\mathcal{E}) \qquad (i = 1,2,\dots).$$

The compactness of the embedding operator $\mathcal{E}_{W_2^1[0,1]}^{H_{-\mu(A)}}$ follows from the decomposition of this operator into a pair of compact factors $\mathcal{E}_{W_2^1[0,1]}^{H_{-\mu(A)}} = \mathcal{E}_{H_0}^{H_{-\mu(A)}}\,\mathcal{E}_{W_2^1[0,1]}^{H_0}$.

For the next considerations we assume that the space

$$H_1 = \{x = B\hat{x} :\ \hat{x} \in X := L^2(0,1)\}$$

is continuously embedded into $W_2^1[0,1]$, i.e., we have a constant $0 < c < \infty$ such that

$$H_1 \subset W_2^1[0,1] \quad \text{and} \quad \|B\hat{x}\|_{W_2^1[0,1]} \le c\|\hat{x}\|_{L^2(0,1)} \qquad \text{for all } \hat{x} \in L^2(0,1). \tag{3.14}$$

Provided that (3.14) holds, we have a composition operator

$$\mathcal{E}_{H_1}^{H_{-\mu(A)}} = \mathcal{E}_{W_2^1[0,1]}^{H_{-\mu(A)}} \mathcal{E}_{H_1}^{W_2^1[0,1]}, \tag{3.15}$$

where $\mathcal{E}_{H_1}^{W_2^1[0,1]} \in \mathcal{L}(H_1, W_2^1[0,1])$ is a bounded linear operator and the other two embedding operators in formula (3.15) are compact linear operators. For the validity of (3.14) it is necessary that all elements $u_i := u_i(A) = u_i(B)$ of the eigensystem of both operators A and B belong to $W_2^1[0,1]$. Moreover, the inequality in (3.14) can be written as

$$\left\|\sum_{i=1}^{\infty} s_i(B)(\hat{x}, u_i)_{L^2} u_i'\right\|_{L^2} \le \hat{c}\left\|\sum_{i=1}^{\infty}(\hat{x}, u_i)_{L^2} u_i\right\|_{L^2} \qquad \text{for all } \hat{x} \in L^2 \tag{3.16}$$

with $L^2 = L^2(0,1)$. For $H_1 \subset W_2^1[0,1]$ we can interpret this inequality as follows: We consider the operator $\widetilde{B} : L^2(0,1) \to W_2^1[0,1]$, where $B = \mathcal{E}_{W_2^1[0,1]}^{L^2(0,1)} \widetilde{B}$. Moreover, we consider the in $L^2(0,1)$ unbounded and densely defined operator $Dx := x'$. Then with the inner product $(\cdot,\cdot)_{L^2(0,1)}$ in $L^2(0,1)$ we have

$$\|\widetilde{B}x\|^2_{W_2^1[0,1]} = \|Bx\|^2_{L^2(0,1)} + \|DBx\|^2_{L^2(0,1)} = \big(B(I + D^*D)Bx, x\big)_{L^2(0,1)}.$$

If and only if the linear operator $Q := B(I + D^*D)B$ mapping in $L^2(0,1)$ is bounded (and thus defined on the whole space $L^2(0,1)$), then the inequality in (3.14) is satisfied. One can easily verify that Q is bounded if and only if $DB \in \mathcal{L}\big(L^2(0,1), L^2(0,1)\big)$, i.e., the composite operator DB is bounded or in other words B is 'smoothing' enough in order to compensate differentiation. As (3.16) shows, the inequality $\|u_i'\|_{L^2(0,1)} \le \frac{\hat{c}}{s_i(B)}$ represents a necessary condition for the inequality in (3.14). Note that we have a closed range $R(\widetilde{B})$ forming a closed subspace in $W_2^1[0,1]$ if and only if Q is positive definite and Q^{-1} is continuous in $L^2(0,1)$. Such a situation, where even a proportionality

$$\|u_i'\|_{L^2(0,1)} \sim \frac{1}{s_i(B)} \sim i \tag{3.17}$$

is satisfied, will occur in Example 3.8 below (see also the discussions of Louis in [19: p. 50]). In general, it seems to be typical for a condition (3.17) that the oscillation frequency of the eigenelements u_i *monotonically* grows with the index i.

Since not only the singular values $s_i(T)$ of a compact linear operator $T \in \mathcal{L}(U,V)$ mapping between Hilbert spaces U and V are multiplicative, but also the *approximation numbers* $a_i(T)$ of a bounded linear operator $T \in \mathcal{L}(U,V)$ defined by

$$a_i(T) := \inf\left\{\|T - T_i\|_{\mathcal{L}(U,V)},\ T_i \in \mathcal{L}(U,V),\ \dim(R(T_i)) < i\right\} \qquad (i = 1,2,\ldots),$$

where $a_1(T) = \|T\|_{\mathcal{L}(U,V)}$, we can replace s_i by a_i in formula (3.5). It holds $a_i(T) = s_i(T)$ for all i whenever the operator T is compact (cf. [14: p. 69]). Then this multiplicativity of approximation numbers yields in particular the following lemma.

Lemma 3.6. *Let denote $T : U \to V$ a bounded linear operator and $S : V \to W$ a compact linear operators between Hilbert spaces U, V and W. Then we have for the singular values of the compact composite operator $ST : U \to W$*

$$s_i(ST) \leq \|T\|_{\mathcal{L}(U,V)}\, s_i(S) \qquad (i = 1, 2, ...).$$

Applying this lemma to formulae (3.12) – (3.15) we obtain

$$s_i(A)^{\frac{\mu(A)+1}{\mu(A)}} \leq c\, s_i(\widetilde{A}).$$

In combination with formula (3.6) this immediately yields

Theorem 3.7. *If the operator $A : L^2(0,1) \to Y$ satisfies conditions* (3.10) *and* (3.14), *then we have for the 'space change' operator $\widetilde{A} : Z = W_2^1[0,1] \to Y$*

$$\mu(\widetilde{A}) := \underline{\mu}(\widetilde{A}) = \overline{\mu}(\widetilde{A}) = \mu(A) + 1,$$

i.e. in that case the degree of ill-posedness grows exactly by one if we change the solution space of equation (1.1) *from $L^2(0,1)$ to $W_2^1[0,1]$.*

Now we consider an example where Theorem 3.7 applies.

Example 3.8. Let, for fixed $0 < \nu < \infty$, $A \in \mathcal{L}\big(L^2(0,1), L^2(0,1)\big)$ be a selfadjoint compact linear operator with the singular system

$$\Big\{ s_i(A) := i^{-\nu}\,;\; u_i(A) := \sqrt{2}\sin(i\pi t)\,;\; v_i(A) := \sqrt{2}\sin(i\pi t) \Big\}_{i=1}^{\infty}.$$

Consequently, we have a condition (3.10) with a degree of ill-posedness $\mu(A) = \nu$ according to equation (1.1). Then the corresponding singular system of B is

$$\Big\{ s_i(B) := i^{-1}\,;\; u_i(B) := \sqrt{2}\sin(i\pi t)\,;\; v_i(B) := \sqrt{2}\sin(i\pi t) \Big\}_{i=1}^{\infty}.$$

In [24: p. 46] the associated Hilbert scale $(H_r)_{r\in\mathbb{R}}$ is characterized for positive values r by the formula

$$H_r = \left\{ x \in W_2^r[0,1] : x^{(2l)}(0) = x^{(2l)}(1) = 0 \quad \left(l = 0, 1, ..., \left[\frac{r}{2} - \frac{1}{4}\right] \right) \right\}, \tag{3.18}$$

where $[a]$ denotes the largest integer that does not exceed the real number a. The space $W_2^r[0,1]$ is the usual Hilbertian Sobolev space of order r (see, e.g., Wloka [33]), and $\|x\|_k := \pi^{-1}\|x^{(k)}\|_{L^2(0,1)}$ forms a norm of H_k using the generalized k-th derivative of x if k is a positive integer. That means, we have in our example $H_1 = \overset{\circ}{W}{}_2^1[0,1]$, where the spaces $\overset{\circ}{W}{}_2^r[0,1] = \overline{C_0^\infty[0,1]}^{W_2^r[0,1]}$ for $r > 0$ are obtained by the completion with respect to the $W_2^r[0,1]$-norm of the set of $C^\infty[0,1]$-functions with zero boundary conditions imposed on both ends of the interval $[0,1]$ and on derivatives of arbitrary order. Now conditions (3.14) and (3.16) with $\hat{c} = 1$ and even a condition (3.17) are fulfilled. Namely, it is well-known that $H_1 = \overset{\circ}{W}{}_2^1[0,1]$ is a closed subspace of $W_2^1[0,1]$. Moreover, the functions $s_i(B)u_i' = \sqrt{2}\cos(i\pi t)$ $(i = 1, 2, \ldots)$ form an orthonormal system in $L^2(0,1)$. By Theorem 3.7 the operator $\widetilde{A} : W_2^1[0,1] \to L^2(0,1)$ in our example provides a degree of ill-posedness $\mu(\widetilde{A}) = \nu + 1$ to equation (3.1).

If we compare the situations of Examples 3.1 and 3.8, then Theorem 3.7 is not applicable to the first one, since in that the eigenfunctions are *reordered*. Namely, in Example 3.1 the oscillation frequency j of the eigenelements $u_i(A)$, which are functions of the form $\sqrt{2}\cos(j\pi t)$ $(0 \le t \le 1)$, is not monotone with respect to i. This leads to an interval of ill-posedness of positive length for $\widetilde{A}$, although A satisfies condition (3.10). In general, for the space change problem, increment values $\underline{\mu}(\widetilde{A}) - \underline{\mu}(A)$ and $\overline{\mu}(\widetilde{A}) - \overline{\mu}(A)$ substantial larger than one must be expected if the smoothing character of A is rather *turbulent*. Turbulence denotes in this context that the behaviour of the oscillation frequencies of the eigenelements $u_i(A)$ according to the i-th largest singular value $s_i(A)$ is far from a proportionality to the index number i.

4. On Hilbert scales and Sobolev scales

In the framework of some authors (see, e.g., Natterer [22, 23], Neubauer [24], Louis [19], Mair [20], and Tautenhahn [13, 29]) the investigations use the assumption that for a fixed value $0 < \nu < \infty$ the *ill-posedness* of an ill-posed equation (1.1) *is measured* by that value ν satisfying the inequality

$$c_1 \|x\|_{-\nu} \le \|Ax\|_Y \le c_2 \|x\|_{-\nu} \qquad \text{for all } x \in X, \tag{4.1}$$

where $0 < c_1 \le c_2 < \infty$ are constants. Here, for $H_0 := X$, $\|\cdot\|_r$ is the norm associated with an appropriately chosen Hilbert scale $(H_r)_{r\in\mathbb{R}}$. Inequalities of the form (4.1) in Hilbert scales are very interesting in the context of this paper, since they allow us to apply Lemma 3.5 (cf. formula (3.13)). Namely, by the assertion of this lemma we obtain spectral equivalence of the operator A and of the embedding operator $\mathcal{E}_{H_0}^{H_{-\nu}}$ between different Hilbert scale elements.

Example 4.1. For $X := L^2(0,1)$, consider the Hilbert scale

$$H_r := \left\{x : \|x\|_r := \left\|(J^*J)^{-\frac{r}{2}}x\right\|_X < \infty\right\} \tag{4.2}$$

generated by the compact linear integral operator

$$(Jx)(s) := \int_0^s x(t)\,dt \qquad (0 \le s \le 1), \tag{4.3}$$

to which the differentiation operator $(Dy)(s) := \frac{dy(t)}{dt}$ $(0 \le s \le 1)$ is a left inverse. The singular values of the operator $J : L^2(0,1) \to L^2(0,1)$ defined by (4.3) are $s_i(J) = \frac{2}{(2i-1)\pi}$ $(i = 1, 2, \ldots)$. Evidently, an assumption (4.1) on A concerning our example implies $s_i(A) \sim s_i(\mathcal{E}_{L^2(0,1)}^{H_{-\nu}}) \sim i^{-\nu}$.

As the following lemma will show, the spaces H_r of the Hilbert scale (4.2) of Example 4.1 are closely related with the Sobolev scale $(W_2^r[0,1])_{r\in\mathbb{R}}$ of Hilbertian Sobolev spaces.

Lemma 4.2. ([7: Lemma 8]). *For the Hilbert scale* (4.2) *we have*

$$\left. \begin{aligned} H_\beta &= W_2^\beta[0,1] \quad (0 \le \beta < \tfrac{1}{2}) \\ H_{\frac{1}{2}} &= \left\{ x \in W_2^{\frac{1}{2}}[0,1] : \int_0^1 (1-t)^{-1} x^2(t)\, dt < \infty \right\} \\ H_\beta &= \left\{ x \in W_2^\beta[0,1] : x(1) = 0 \right\} \quad (\tfrac{1}{2} < \beta \le 1) \end{aligned} \right\} \tag{4.4}$$

and for $\beta \in [0,1]$, $\beta \neq \frac{1}{2}$,

$$\|x\|_\beta \sim \|x\|_{W_2^\beta[0,1]} \qquad \textit{for all } x \in H_\beta.$$

For negative exponents the spaces $W_2^{-r}[0,1] := (W_2^r[0,1])'$ $(r > 0)$ are defined as dual spaces (cf., e.g., Schechter [27]). Neubauer has shown in [25] (see also Theorem 9.3 in [15: p. 151]) that finite parameter intervals of this Sobolev scale with parameters $r \in [-r_0, r_0]$ $(0 < r_0 < \infty)$ represent parts (sections) of Hilbert scales, but the complete Sobolev scale $(W_2^r[0,1])_{r\in\mathbb{R}}$ is not a Hilbert scale. For fixed positive values ν inequalities of the form

$$c_1 \|x\|_{W_2^{-\nu}[0,1]} \le \|Ax\|_Y \le c_2 \|x\|_{W_2^{-\nu}[0,1]} \qquad \text{for all } x \in L^2(0,1) \tag{4.5}$$

could serve as an alternative to (4.1). In the notation of Lavrent'ev (cf. [17]) such operator equations (1.1) satisfying (4.5) are called *weakly ill-posed*. From (4.5) we obtain by Lemma 3.5 that A is spectrally equivalent to the embedding operator $\mathcal{E}_{L^2(0,1)}^{W_2^{-\nu}[0,1]}$. Regarding the results of [14: p. 186] on such embedding operators it follows

$$s_i(A) = s_i(\mathcal{E}_{L^2(0,1)}^{W_2^{-\nu}[0,1]}) \sim i^{-\nu}.$$

Inequalities of the form (4.1) subject to the Hilbert scale (4.2) and of the form (4.5) imply conditions (1.4) and (3.10). Hence in both cases the equation (1.1) is ill-posed of degree $\mu(A) := \underline{\mu}(A) = \nu$ in the sense of Definition 1.1 and the interval of ill-posedness has a length equal to zero. Note that vice versa (1.4) in general does imply neither a condition (4.1), nor a condition (4.5).

Now we come back to the space change problem. Since finite parts of the Sobolev scale $(W_2^r[0,1])_{r\in\mathbb{R}}$ have Hilbert scale properties, we have for the embedding operator $\mathcal{E}_{W_2^1[0,1]}^{W_2^{-\nu}[0,1]}$ the singular value behaviour $s_i(\mathcal{E}_{W_2^1[0,1]}^{W_2^{-\nu}[0,1]}) \sim i^{-(\nu+1)}$. This yields the following theorem.

Theorem 4.3. *Let for* $0 < \mu(A) =: \nu < \infty$ *hold an inequality* (4.5). *Then for the 'space change' operator* $\widetilde{A}$ *according to* (3.2) *and for the degree of ill-posedness of equation* (3.1) *we have the equation*

$$\mu(\widetilde{A}) := \underline{\mu}(\widetilde{A}) = \overline{\mu}(\widetilde{A}) = \mu(A) + 1 = \nu + 1.$$

Unfortunately, a condition (4.5) is difficult to prove, since the dual spaces $W_2^{-r}[0,1]$ are difficult to characterize. On the other hand, by using Fourier transforms we can obtain inequalities of the form

$$c_1 \|x\|_{H^{-\nu}[0,1]} \le \|Ax\|_Y \le c_2 \|x\|_{H^{-\nu}[0,1]} \qquad \text{for all } x \in L^2(0,1) \tag{4.6}$$

in some situations, where $H^r[0,1]$ $(r \in \mathbb{R})$ are the spaces of restrictions on $[0,1]$ of *tempered distributions* x with a finite Fourier norm

$$\|x\|_{H^r(\mathbb{R})} := \sqrt{\int_{\mathbb{R}} (1+\omega^2)^r \, |\hat{x}(\omega)|^2 \, d\omega} < \infty.$$

In that context, $\hat{x}$ denotes the Fourier transform of x. It is well-known (see [33: p. 99]) that $H^r[0,1] \cong W_2^r[0,1]$ whenever $r \ge 0$. On the other hand, the spaces $H^{-r}[0,1]$ $(r > 0)$ with negative exponents can be interpreted as dual spaces of $\overset{\circ}{W}{}_2^r[0,1]$. Note that $\overset{\circ}{W}{}_2^r[0,1] \cong W_2^r[0,1]$ if $0 < r < \frac{1}{2}$. Up to now it is an open question for the authors for what values $r \ge \frac{1}{2}$ the norms $\|x\|_{W_2^{-r}[0,1]}$ and $\|x\|_{H^{-r}[0,1]}$ are equivalent for $x \in L^2(0,1)$. Therefore, it is not quite clear that Theorem 4.3 also holds if condition (4.5) is replaced by (4.6), because we cannot characterize the singular values of the embedding operator $\mathcal{E}_{W_2^1[0,1]}^{H^{-\nu}[0,1]}$. This question is also related to the problem whether the spaces $\overset{\circ}{W}{}_2^r[0,1]$ form a part of a Hilbert scale in the parameter interval $r \in [0, r_0]$, $r_0 > 0$. Following the ideas of Triebel [30: Section 4.3.2/p. 317 - 319] there seems to be some doubt that such an assertion holds if $r_0 \ge \frac{1}{2}$.

Finally, we discuss the problem whether the assertion of Theorem 4.3 remains valid if (4.5) is replaced by an inequality (4.1) where $\|\cdot\|_r$ are norms corresponding to Hilbert scales $(H_r)_{r\in\mathbb{R}}$ subject to $X = H_0 = L^2(0,1)$ as introduced in Example 3.8 (see formula (3.18)) and in Example 4.1 (see formula (4.4) of Lemma 4.2). Note that H_1 in both examples represents a closed subspace of $W_2^1[0,1]$. On the other hand, the orthogonal complement $H_1^\perp = W_2^1[0,1] \ominus H_1$ with respect to the standard inner product in $W_2^1[0,1]$ is finite-dimensional in both examples. Namely, $(\overset{\circ}{W}{}_2^1[0,1])^\perp$ is two-dimensional, whereas $\big(\{x \in W_2^1[0,1] : x(1) = 0\}\big)^\perp$ is a one-dimensional subspace of $W_2^1[0,1]$. We will exploit the following lemma.

Lemma 4.4. *Let V and W be a pair of infinite dimensional Hilbert spaces. The space $V = U_1 \oplus U_2$ is assumed to be the orthogonal sum of the subspaces U_1 and U_2. We consider a compact linear operator $S : U_1 \to W$ with infinite dimensional range* $\dim(R(S)) = \infty$ *and sigular values satisfying the inequality $s_i(S) \le C s_{i+1}(S)$ for $i = 1, 2, \ldots$ and a positive constant C. Moreover, we consider a compact linear operator $T : U_2 \to W$ with finite-dimensional range* $\dim(R(T)) < \infty$ *and a compact linear sum operator $S + T : V \to W$ defined by $(S+T)x = Sx_1 + Tx_2$ for $x = x_1 + x_2$, where $x_1 \in U_1$ and $x_2 \in U_2$. Then for the singular values*

$$s_i(S+T) \sim s_i(S)$$

holds.

Proof. The assertion of this lemma is an immediate consequence of the approximation numbers inequality (cf. condition (b) of Definition 1.d.13 in [14])

$$s_{i+j-1}(S+T) = a_{i+j-1}(S+T) \leq a_i(S) + a_j(T)$$

and the fact that $a_j(T)$ is zero for sufficiently large integers j. Let $a_j(T) = 0$ for $j > m_0$. Then we have for sufficiently large i

$$s_i(S+T) \leq s_{i-m_0}(S) + a_{m_0+1}(T) = s_{i-m_0}(S)$$

and

$$s_{i+m_0}(S) \leq s_i(S+T) + a_{m_0+1}(-T) = s_i(S+T).$$

Consequently,

$$\frac{1}{C^{m_0}} s_i(S) \leq s_{i+m_0}(S) \leq s_i(S+T) \leq s_{i-m_0}(S) \leq C^{m_0} s_i(S)$$

and $s_i(S+T) \sim s_i(S)$ ∎

Let, for the Hilbert scale $(H_r)_{r\in\mathbb{R}}$, the follwing conditions be satisfied:

$$H_0 := L^2(0,1) \text{ and } H_1 \text{ is a closed subspace of } W_2^1[0,1] \tag{4.7}$$

$$H_1^{\perp} = W_2^1[0,1] \ominus H_1 \text{ is finite-dimensional} \tag{4.8}$$

$$s_i(\mathcal{E}_{H_1}^{L^2(0,1)}) \sim \frac{1}{i}. \tag{4.9}$$

If we then set in the above lemma $V := W_2^1[0,1]$, $U_1 := H_1$, $U_2 = H_1^{\perp}$, $W := H_{-\nu}$, and for S and $S+T$ the embedding operators $S := \mathcal{E}_{H_1}^{H_{-\nu}}$ and $S+T := \mathcal{E}_{W_2^1[0,1]}^{H_{-\nu}}$, then the singular values $s_i(\mathcal{E}_{W_2^1[0,1]}^{H_{-\nu}})$ and $s_i(\mathcal{E}_{H_1}^{H_{-\nu}})$ are both proportional to $i^{-(\nu+1)}$. Namely, we recognize this proportionality for the second embedding operator to be a Hilbert scale property.

By applying Lemma 4.4, Theorem 4.3 remains also valid if (4.5) is replaced by (4.1) subject to a Hilbert scale $(H_r)_{r\in\mathbb{R}}$ satisfying conditions (4.7) – (4.9). Note that the Hilbert scales of Example 3.8 and of Example 4.1 both satisfy such conditions.

Example 4.5. In the recent paper [7] of Yamamoto and Gorenflo it is proven that for any fixed ν from the interval $(0,1]$ a condition (4.1) subject to the Hilbert scale $(H_r)_{r\in\mathbb{R}}$ of Example 4.1 defined by (4.2) - (4.3) is satisfied for the generalized Abel integral operator $A : X = L^2(0,1) \to Y = L^2(0,1)$ of the form

$$[A\,x](t) := \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} K(t,\tau)\, x(\tau)\, d\tau \qquad (0 \leq t \leq 1)$$

where a continuous kernel $K = K(t,\tau)$ for $0 \leq \tau \leq t \leq 1$ with $K(\tau,\tau) = 1$ for $0 \leq \tau \leq 1$ and the existence of a decreasing function $\kappa \in L^2(0,1)$ are assumed such that $|\frac{\partial K}{\partial \tau}(t,\tau)| \le \kappa(\tau)$ for $0 < \tau \le t \le 1$ (cf. [7: Theorem 1]). It is also mentioned that in the subcase $0 < \nu < \frac{1}{2}$ there holds even a condition (4.6). From the above results it follows that we have for such an Abel integral operator

$$\mu(A) := \underline{\mu}(A) = \overline{\mu}(A) = \nu \qquad (0 < \nu \le 1).$$

If we change the solution space from $X := L^2(0,1)$ to $Z := W_2^1[0,1]$, then the degree of ill-posedness grows exactly by one. That means, we have for $\widetilde{A} : Z = W_2^1[0,1] \to Y = L^2(0,1)$ the equation

$$\mu(\widetilde{A}) := \underline{\mu}(\widetilde{A}) = \overline{\mu}(\widetilde{A}) = \nu + 1 \qquad (0 < \nu \le 1).$$

All the considerations of this section show that requirements of the form (4.1) are very specific for the operator A. According to the ideas of the previous sections of this paper the class of problems under consideration could be substantially enlarged if lower and upper bound exponents $0 < \nu_2 \le \nu_1 < \infty$ are taken into account such that requirements of the form

$$c_1 \|x\|_{-\nu_1} \le \|Ax\|_Y \le c_2 \|x\|_{-\nu_2} \qquad \text{for all } x \in L^2(0,1)$$

can be introduced. Then intervals of ill-posedness of positive lengt will occur again for both the operator A and the operator $\widetilde{A}$.

5. Conclusions

In the traditional framework of linear inverse problems $Ax = y$ with compact operators $A : X \to Y$ using infinite dimensional Hilbert spaces X and Y, the degree of ill-posedness of such problems is obviously characterized by a single real number ν. This number expresses the decay rate to zero of the singular values of A, $s_i(A) \sim i^{-\nu}$, and consequently the power rate N^ν of condition numbers for discretized linear systems using N-dimensional vectors to find numerical approximate solutions. Frequently, however, the singular values are not proportional to a fixed power rate $i^{-\nu}$, but satisfy an estimate of the form $c_1 i^{-\overline{\mu}} \le s_i(A) \le c_2 i^{-\underline{\mu}}$. Then in the context of numerical solutions there exist sequences of finite-dimensional subspaces with slowly growing condition numbers proportional to $N^{\underline{\mu}}$ and other subspaces with rapidly growing condition numbers proportional to $N^{\overline{\mu}}$. To characterize this behaviour, we introduce $[\underline{\mu}, \overline{\mu}]$ as the interval of ill-posedness. An equivalent interval information can also be obtained by alternative ill-posedness measures as ε-entropy, ε-capacity and number of degrees of freedom. The knowledge of the interval of ill-posedness helps to select convenient regularization strategies for solving the inverse problems in a stable manner.

It is of some interest to study the change of the interval of ill-posedness if the solution space $X = L^2(0,1)$ of $Ax = y$ is replaced by $W_2^1[0,1]$ with a 'stronger' topology, whereas Y does not change. For this space shift along a Sobolev scale with a step length one, it is proven that the left bound as well as the right bound of the interval of ill-posedness both grow at least by one. However, there exist operators A, where the interval of ill-posedness is shifted much more and where the length of the interval is growing up to infinity. So we have given examples with an interval of ill-posedness of length zero in $L^2(0,1)$ and positive or infinite length in $W_2^1[0,1]$. Such situations are presented as a warning that a regularization strategy which works well in $L^2(0,1)$ is not necessarily appropriate for finding solutions in $W_2^1[0,1]$.

On the other hand, using Hilbert scales we have formulated sufficient conditions and we have given examples for the situation easy to survey, where a singular value rate $s_i \sim i^{-\nu}$ in $L^2(0,1)$ corresponds to a rate $s_i \sim i^{-(\nu+1)}$ in $W_2^1[0,1]$. In general this seems to be the case when strongly oscillating eigenfunctions belong to small singular values s_i and when the level of oscillation is monotone with respect to the number i.

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