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Generalized golden ratios of ternary alphabets

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Abstract. Expansions in noninteger bases often appear in number theory and probability theory, and they are closely connected to ergodic theory, measure theory and topology. For two-letter alphabets the golden ratio plays a special role: in smaller bases only trivial expansions are unique, whereas in greater bases there exist nontrivial unique expansions. In this paper we determine the corresponding critical bases for all three-letter alphabets and we establish the fractal nature of these bases in dependence on the alphabets.

Keywords. Golden ratio, ternary alphabet, unique expansion, noninteger base, beta-expansion, greedy expansion, lazy expansion, univoque sequence, Sturmian sequences

1. Introduction

Since the appearance of Rényi's β-expansions [13] many works have been devoted to expansions in noninteger bases. Much research was stimulated by the discovery of Erdős, Horváth and Joó [5] who proved the existence of many real numbers $1 < q < 2$ for which only one sequence (c_i) of zeroes and ones satisfies the equality

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = 1.$$

The set of such "univoque" bases has a fractal nature; see, e.g., [6], [8], [10], where arbitrary bases $q > 1$ are also considered.

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In contrast to the integer case, in a given noninteger base $q > 1$ a real number x may have sometimes many different expansions of the form

$$\pi_q(c) := \sum_{i=1}^{\infty} \frac{c_i}{q^i} = x \tag{1.1}$$

with integer "digits" satisfying $0 \le c_i < q$ for every i. On the other hand, the set of numbers x having a unique expansion has many unexpected topological and combinatorial properties, depending on the value of q; see, e.g., Daróczy and Kátai [1], de Vries [2], [3], Glendinning and Sidorov [7], and [4].

Given a finite alphabet $A = \{a_1, \dots, a_J\}$ of real numbers $a_1 < \cdots < a_J$ and a real number $q > 1$, by an expansion of a real number x we mean a sequence (c_i) of numbers $c_i \in A$ satisfying (1.1). The expansions of

$$x_1 := \sum_{i=1}^{\infty} \frac{a_1}{q^i} \quad \text{and} \quad x_2 := \sum_{i=1}^{\infty} \frac{a_J}{q^i}$$

are always unique; they are called the *trivial* unique expansions.

For two-letter alphabets $A = \{a_1, a_2\}$ the golden ratio $p := (1+\sqrt{5})/2$ plays a special role: there exist nontrivial unique expansions in base q if and only if $q > p$.

The purpose of this paper is to determine analogous critical bases for each ternary alphabet $A = \{a_1, a_2, a_3\}$. Our main tool is a lexicographic characterization of unique expansions, given in [12], which generalized to arbitrary finite alphabets a theorem of Parry [11] and its various extensions [1], [5], [6], [9].

By a normalization it suffices to consider the alphabets $A_m := \{0, 1, m\}$ with $m \ge 2$. Our main result is the following:

Theorem 1.1. *There exists a continuous function $p : [2, \infty) \to \mathbb{R}$, $m \mapsto p_m$, satisfying*

$$2 \le p_m \le P_m := 1 + \sqrt{\frac{m}{m-1}}$$

for all m such that the following properties hold true:

(a) *for each $m \ge 2$, there exist nontrivial univoque expansions if $q > p_m$ and there are no such expansions if $q < p_m$;*
(b) *$p_m = 2$ if and only if $m = 2^k$ for some positive integer k;*
(c) *the set $C := \{m \ge 2 \ : \ p_m = P_m\}$ is a Cantor set, i.e., an uncountable closed set having neither interior nor isolated points; its smallest element is $1 + x \approx 2.3247$ where x is the first Pisot number, i.e., the positive root of the equation $x^3 = x + 1$;*
(d) *each connected component (m_d, M_d) of $[2, \infty) \setminus C$ has a point μ_d such that p is strictly decreasing in $[m_d, \mu_d]$ and strictly increasing in $[\mu_d, M_d]$.*

Moreover, we will determine explicitly the function p and the numbers m_d, M_d, μ_d, and we will also determine those m for which there exist nontrivial univoque sequences in base p_m (Remark 5.12).

Since the proofs are rather technical let us explain how we arrived at the above results and at the particular constructions in the proof. Given a real number $m \geq 2$ and a base $1 < q \leq (2m-1)/(m-1)$, it follows directly from the definition of the expansions that a sequence (c_i) on the alphabet $\{0, 1, m\}$ is the unique expansion of

$$x := \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

if and only if the following four conditions are satisfied (Lemma 5.1):

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < 1 \qquad \text{whenever } c_n = 0;$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < m-1 \qquad \text{whenever } c_n = 1;$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} > \frac{m}{q-1} - 1 \qquad \text{whenever } c_n = 1;$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} > \frac{m}{q-1} - (m-1) \qquad \text{whenever } c_n = m.$$

Then we say that (c_i) is a *univoque sequence*. Using a computer program we have found univoque sequences for many particular values of m for small values $q > 2$ containing only two different digits. Trying to find an explanation, we proved that if q is sufficiently close to one, namely if $1 < q \leq P_m := 1 + \sqrt{m/(m-1)}$, then no sequence satisfying these conditions (except the trivial sequence 0^∞) can contain infinitely many zero digits (Lemma 5.3). Since after removing a finite number of initial elements a univoque sequence remains univoque, it follows that if there exists a nontrivial univoque sequence in some base $1 < q \leq P_m$, then there also exists a nontrivial univoque sequence in this base which only contains the digits 1 and m. Assuming that there are such sequences in some base $1 < q \leq P_m$, this allows us to investigate two-digit sequences instead of more complicated three-digit sequences.

In the next stage we made an extensive computer research in order to find such univoque sequences. For most integer values of $m = 2, 3, \ldots, 65536$ we have found essentially one such sequence, namely the periodic sequence $(m^{h_1}1)^\infty$ with $h_1 = [\log_2 m]$. Using the above characterization it is easy to see that this sequence can be univoque in a base q only if $q > p_m := \max\{p'_m, p''_m\}$ where p'_m and p''_m are defined by the equations

$$\pi_{p'_m}((m^{h_1}1)^\infty) = m - 1 \quad \text{and} \quad \pi_{p''_m}((m^{h_1}1))^\infty) = \frac{m}{p''_m - 1} - 1,$$

and one can prove that the condition $q > p_m$ is also *sufficient*.

However, there were seven exceptional integer values: 5, 9, 130, 258, 2051, 4099, 32772, for which we have found only univoque sequences of a more complicated form, for instance $(m^2 1 m^2 1 m 1)^\infty$ for $m = 5$ and $(m^3 1 m^2 1)^\infty$ for $m = 9$ (see Example 5.6).

Each such sequence provided a univoque sequence in some base $1 < q < P_m$ also for small perturbations of the integer digit m. In this way we could also cover many real numbers $m \in [2, 65536]$ but not all of them.

In order to find nontrivial univoque sequences in bases $1 < q \le P_m$ for *each* real number $m \in [2, \infty)$, we have generalized the structure of the above sequences. This led to the notion of *admissible sequences*. It turned out that each admissible sequence $d \neq 1^\infty$ provides a nontrivial univoque sequence in every base $q > P_m$ for real digits m belonging to a precisely defined interval I_d, and that the intervals I_d provide a disjoint covering of $[2, \infty)$ (Lemmas 5.4 and 5.13(a), (b)). The other properties mentioned in Theorem 1.1 were obtained by a closer investigation of the admissible sequences d and the corresponding intervals I_d (Lemma 5.13 (c)).

In Section 2, we review some basic facts about expansions and we also give some new results. In Sections 3–4 we introduce the class of *admissible sequences* and we clarify their structure and their basic properties. As a byproduct, we obtain a new characterization of Sturmian sequences (Remark 3.7). These results allow us to determine in Section 5 the critical bases for all ternary alphabets.

2. Some results on arbitrary alphabets

Throughout this section we consider a fixed finite alphabet $A = \{a_1, \dots, a_J\}$ of real numbers $a_1 < \cdots < a_J$. Given a real number $q > 1$, by an expansion of a real number x we mean a sequence $c = (c_i)$ of numbers $c_i \in A$ satisfying the equality

$$\pi_q(c) := \sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

The real number $\pi_q(c)$ is called the *value of the expansion c in base q*. In order to have an expansion, x must belong to the interval $\left[\frac{a_1}{q-1}, \frac{a_J}{q-1}\right]$. Conversely, we recall from [12] the following results:

Theorem 2.1. *Every $x \in \left[\frac{a_1}{q-1}, \frac{a_J}{q-1}\right]$ has at least one expansion in base q if and only if*

$$1 < q \le Q_A := 1 + \frac{a_J - a_1}{\max_{j>1}\{a_j - a_{j-1}\}} \ (\le J). \tag{2.1}$$

A sequence $(c_i) \in A^\infty$ is called *univoque* in base q if

$$x := \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

has no other expansion in this base. The constant sequences $(a_1)^\infty$ and $(a_J)^\infty$ are univoque in every base q; they are called the *trivial unique expansions*. We also recall from [12] the following characterization of unique expansions:

Theorem 2.2. *Assume* (2.1). *An expansion* (c_i) *is unique in base* q *if and only if the following conditions are satisfied:*

$$\sum_{i=1}^{\infty} \frac{c_{n+i} - a_1}{q^i} < a_{j+1} - a_j \quad \textit{whenever } c_n = a_j < a_J;$$
$$\sum_{i=1}^{\infty} \frac{a_J - c_{n+i}}{q^i} < a_j - a_{j-1} \quad \textit{whenever } c_n = a_j > a_1.$$

Proof of the sufficiency. We have to show that if (d_i) is another sequence in A^∞ then it represents a different sum. Let $n \geq 1$ be the first index such that $c_n \neq d_n$. If $c_n < d_n$, then writing $c_n = a_j$ we have $a_j < a_J$, so that

$$\sum_{i=1}^{\infty} \frac{d_i}{q^i} - \sum_{i=1}^{\infty} \frac{c_i}{q^i} \geq \frac{a_{j+1} - a_j}{q^n} + \sum_{i=n+1}^{\infty} \frac{a_1 - c_i}{q^i} > 0$$

by our assumption. If $c_n > d_n$, then writing $c_n = a_j$ we have $a_j > a_1$, so that

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} - \sum_{i=1}^{\infty} \frac{d_i}{q^i} \geq \frac{a_j - a_{j-1}}{q^n} + \sum_{i=n+1}^{\infty} \frac{c_i - a_J}{q^i} > 0$$

by our second assumption.

Proof of the necessity. If the first condition is not satisfied for some $c_n = a_j < a_J$, then by Theorem 2.1 there exists another expansion beginning with $c_1 \cdots c_{n-1} a_{j+1}$. If the second condition is not satisfied for some $c_n = a_j > a_1$, then by Theorem 2.1 there exists another expansion beginning with $c_1 \cdots c_{n-1} a_{j-1}$. □

Let us mention some consequences of this characterization.

Corollary 2.3. *For every given set* $C \subset A^\infty$ *there exists a number* $1 \leq q_C \leq Q_A$ *such that*

$$q > q_C \Rightarrow \textit{every sequence } c \in C \textit{ is univoque in base } q;$$
$$1 < q < q_C \Rightarrow \textit{not every sequence } c \in C \textit{ is univoque in base } q.$$

Proof. If $C = \emptyset$, then we may choose $q_C = 1$. If C is nonempty, then for each sequence $c \in C$, each condition of Theorem 2.2 is equivalent to an inequality of the form $q > q_\alpha$. Since we consider only bases q satisfying (2.1), we may assume that $q_\alpha \leq Q_A$ for every α. Then $q_C := \max\{1, \sup q_\alpha\}$ has the required properties. □

Definition 2.4. The number q_C is called the *critical base* of C. If $C = \{c\}$ is a one-point set, then $q_c := q_C$ is also called the critical base of the sequence c.

Remark 2.5. If C is a nonempty finite set of eventually periodic sequences, then the supremum $\sup q_\alpha$ in the above proof is actually a maximum. In this case not all sequences $c \in C$ are univoque in base $q = q_C$.

Example 2.6. Consider the ternary alphabet $A = \{0, 1, 3\}$ and the periodic sequence $(c_i) = (31)^\infty$. By the periodicity of (c_i) we have for each n either $c_n = 3$ and $(c_{n+i}) = (13)^\infty$ or $c_n = 1$ and $(c_{n+i}) = (31)^\infty$. According to the preceding remark Theorem 2.2 contains only three conditions on q. For $c_n = 3$ we have the condition

$$\sum_{i=1}^{\infty} \frac{3 - c_{n+i}}{q^i} < 2 \Leftrightarrow \frac{2q}{q^2 - 1} < 2,$$

while for $c_n = 1$ we have the following two conditions:

$$\sum_{i=1}^{\infty} \frac{3 - c_{n+i}}{q^i} < 1 \Leftrightarrow \frac{2}{q^2 - 1} < 1$$

and

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < 2 \Leftrightarrow \frac{3}{q - 1} - \frac{2}{q^2 - 1} < 2.$$

They are approximatively equivalent to the inequalities $q > 1.61803$, $q > 1.73205$ and $q > 2.18614$ respectively, so that $q_c \approx 2.18614$.

It is well-known that for the alphabet $A = \{0, 1\}$ there exist nontrivial univoque sequences in base q if and only if $q > (1 + \sqrt{5})/2$. There exists a "generalized golden ratio" for every alphabet:

Corollary 2.7. *There exists a number* $1 < G_A \le Q_A$ *such that*

$$q > G_A \Rightarrow \text{there exist nontrivial univoque sequences;}$$
$$1 < q < G_A \Rightarrow \text{there are no nontrivial univoque sequences.}$$

Proof. If a sequence is univoque in some base, then it is also univoque in every larger base. If there exists a base satisfying (2.1) in which there exist nontrivial univoque sequences, then the infimum of such bases satisfies the requirements for G_A, except perhaps the strict inequality $G_A > 1$. Otherwise we may choose $G_A := Q_A$.

To show that $G_A > 1$, we prove that if $q > 1$ is sufficiently close to one, then the only univoque sequences are a_1^∞ and a_J^∞. We show that it suffices to choose $q > 1$ so small that the following three conditions are satisfied:

$$\frac{a_J - a_1}{q - 1} \ge a_{j+1} - a_{j-1}, \qquad j = 2, \dots, J - 1, \tag{2.2}$$

$$\frac{a_j - a_1}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} \ge (a_2 - a_1) + \frac{a_j - a_{j-1}}{q}, \qquad j = 2, \dots, J, \tag{2.3}$$

$$\frac{a_J - a_j}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} \ge (a_J - a_1) + \frac{a_{j+1} - a_j}{q}, \qquad j = 1, \dots, J - 1. \tag{2.4}$$

The proof consists of three steps. Let (c_i) be a univoque sequence in base q.

If $c_n = a_j$ for some n and $1 < j < J$, then the conditions of Theorem 2.2 imply that

$$\sum_{i=1}^{\infty} \frac{c_{n+i} - a_1}{q^i} < a_{j+1} - a_j \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{a_J - c_{n+i}}{q^i} < a_j - a_{j-1}.$$

Taking their sum we conclude that

$$\frac{a_J - a_1}{q - 1} < a_{j+1} - a_{j-1},$$

which contradicts (2.2). This proves that $c_n \in \{a_1, a_J\}$ for every n.

If $c_n = a_1$ and $c_{n+1} = a_j > a_1$ for some n, then applying Theorem 2.2 we obtain

$$\sum_{i=1}^{\infty} \frac{c_{n+i} - a_1}{q^i} < a_2 - a_1 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{a_J - c_{n+i+1}}{q^i} < a_j - a_{j-1}.$$

Dividing the second inequality by q and adding the result to the first one we obtain

$$\frac{a_j - a_1}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} < (a_2 - a_1) + \frac{a_j - a_{j-1}}{q},$$

which contradicts (2.3). This proves that $c_n = a_1$ implies $c_{n+1} = a_1$ for every n.

Finally, if $c_n = a_J$ and $c_{n+1} = a_j < a_J$ for some n, then Theorem 2.2 yields

$$\sum_{i=1}^{\infty} \frac{a_J - c_{n+i}}{q^i} < a_J - a_{J-1} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{c_{n+i+1} - a_1}{q^i} < a_{j+1} - a_j.$$

Dividing the second inequality by q and adding the result to the first one we now obtain

$$\frac{a_J - a_j}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} < (a_J - a_{J-1}) + \frac{a_{j+1} - a_j}{q},$$

which contradicts (2.4). This proves that $c_n = a_J$ implies $c_{n+1} = a_J$ for every n. □

Definition 2.8. The number G_A is called the *critical base* of the alphabet A.

The following invariance properties of critical bases readily follow from the definitions; they will simplify our proofs.

Lemma 2.9. *The critical base does not change if we replace the alphabet A by:*

- $b + A = \{b + a_j \mid j = 1, \dots, m\}$ *for some real number b;*
- $dA = \{da_j \mid j = 1, \dots, m\}$ *for some nonzero real number d;*
- *the* conjugate alphabet $A' := \{a_m + a_1 - a_j \mid j = 1, \dots, m\}$.

Proof. First we note that $Q_A = Q_{b+A} = Q_{dA} = Q_{A'}$. Fix a base $1 < q \le Q_A$ and a sequence (c_i) of real numbers. It follows from the definitions that the following properties are equivalent:

- (c_i) is an expansion of x for the alphabet A;
- $(b + c_i)$ is an expansion of $x + \frac{b}{q-1}$ for the alphabet $b + A$;
- (dc_i) is an expansion of dx for the alphabet dA;
- $(a_m + a_1 - c_i)$ is an expansion of $\frac{a_m + a_1}{q-1} - x$ for the alphabet A'.

Hence if one of these expansions is unique, then the others are unique as well. □

3. Admissible sequences

This section contains some preliminary technical results.

Definition 3.1. A sequence $d = (d_i) = d_1 d_2 \cdots$ of zeroes and ones is *admissible* if

$$0d_2d_3 \cdots \leq (d_{n+i}) \leq d_1d_2d_3 \cdots \tag{3.1}$$

for all $n = 0, 1, \ldots$.

Examples 3.2.

- The trivial sequences 0^∞ and 1^∞ are admissible.
- More generally, the sequences $(1^N0)^\infty$ $(N = 1, 2, \ldots)$ and $(10^N)^\infty$ $(N = 0, 1, \ldots)$ are admissible.
- The sequence $(11010)^\infty$ is also admissible.
- The (not purely periodic) sequence 10^∞ is admissible.

In order to clarify the structure of admissible sequences we give an equivalent recursive definition. Given a sequence $h = (h_i)$ of positive integers, starting with

$$S_h(0, 1) := 1 \quad \text{and} \quad S_h(0, 0) := 0$$

we define the blocks $S_h(j, 1)$ and $S_h(j, 0)$ for $j = 1, 2, \ldots$ by the recursive formulae

$$S_h(j, 1) := S_h(j-1, 1)^{h_j} S_h(j-1, 0),$$
$$S_h(j, 0) := S_h(j-1, 1)^{h_j-1} S_h(j-1, 0).$$

Observe that $S_h(j, 1)$ and $S_h(j, 0)$ depend only on $h_1, \ldots, h_j$, so that they can also be defined for every finite sequence $h = (h_j)$ of length $\geq j$. We also note that $S_h(j, 0) = S_h(j-1, 0)$ whenever $h_j = 1$.

Let us denote by ℓ_j the length of $S_h(j, 1)$, and set furthermore $\ell_{-1} := 0$. Then the length of $S_h(j, 0)$ is equal to $\ell_j - \ell_{j-1}$. We observe that ℓ_j tends to infinity as $j \to \infty$.

If the sequence $h = (h_j)$ is given, we often omit the subscript h and we write simply $S(j, 1)$ and $S(j, 0)$.

Let us mention some properties of these blocks that we use in what follows. Given two finite blocks A and B we write for brevity

- $A \to B$ or $B = \cdots A$ if B ends with A;
- $A < B$ or $A \cdots < B \cdots$ if $Aa_1a_2 \cdots < Bb_1b_2 \cdots$ lexicographically for any sequences (a_i) and (b_i) of zeroes and ones;
- $A \leq B$ or $A \cdots \leq B \cdots$ if $A < B$ or $A = B$.

Lemma 3.3. *For any given sequence $h = (h_j)$ the blocks $S(j, 1)$ and $S(j, 0)$ have the following properties:*

(a) *We have*

$$S(j,1) = 1S(1,0)\cdots S(j,0) \qquad \text{for all } j \ge 0; \tag{3.2}$$
$$S(0,0)\cdots S(j-1,0) \to S(j,1) \qquad \text{for all } j \ge 1; \tag{3.3}$$
$$S(0,0)\cdots S(j-1,0) \to S(j,0) \qquad \text{whenever } h_j \ge 2; \tag{3.4}$$
$$S(j,0) < S(j,1) \qquad \text{for all } j \ge 0. \tag{3.5}$$

(b) *If* $A_j \to S(j,1)$ *for some nonempty block* A_j*, then* $A_j \le S(j,1)$.
(c) *If* $B_j \to S(j,0)$ *for some nonempty block* B_j*, then* $B_j \le S(j,0)$.
(d) *The finite sequence* $S(j,1)S(j,0)$ *is obtained from* $S(j,0)S(j,1)$ *by changing one block* 10 *to* 01.

Proof. (a) *Proof of* (3.2). For $j=0$ we have $S(j,1)=1$ by definition. If $j \ge 1$ and the identity is true for $j-1$, then the identity for j follows by using the equality $S(j,1) = S(j-1,1)S(j,0)$ coming from the definition of $S(j,1)$ and $S(j-1,1)$.

Proof of (3.3) *and* (3.4). For $j=1$ we have $S(0,0)=0$ and $S(1,0)=1^{h_1-1}0$, so that $S(0,0) \to S(1,0) \to S(1,1)$. (The condition $h_1 \ge 2$ is not needed here.) Proceeding by induction, if (3.3) holds for some $j \ge 1$, then both hold for $j+1$ because

$$S(0,0)\cdots S(j-1,0)S(j,0) \to S(j,1)S(j,0) \to S(j+1,1),$$

and in case $h_{j+1} \ge 2$ we also have $S(j,1)S(j,0) \to S(j+1,0)$.

Proof of (3.5). The case $j=0$ is obvious because the left side begins with 0 and the right side begins with 1. If $j \ge 1$ and (3.5) holds for $j-1$, then we deduce from the inequality $S(j-1,0)\cdots < S(j-1,1)\cdots$ that

$$S(j,0)\cdots = S(j-1,1)^{h_j-1}S(j-1,0)\cdots < S(j-1,1)^{h_j}\cdots.$$

Since $S(j,1)$ begins with $S(j-1,1)^{h_j}$, this implies (3.5) for j.

(b) We may assume that $A_j \ne S(j,1)$; this excludes the case $j=0$ when we have necessarily $A_0 = S(0,1) = 1$. For $j=1$ we have $S(j,1) = 1^{h_1}0$ and $A_j = 1^t0$ with some integer $0 \le t < h_1$, and we conclude by observing that $1^t0\cdots < 1^{h_1}\cdots$.

Now let $j \ge 2$ and assume that the result holds for $j-1$. Using the equality $S(j,1) = S(j-1,1)^{h_j}S(j-1,0)$ we distinguish three cases.

If $A_j \to S(j-1,0)$, then we have the implications

$$\begin{aligned} A_j \to S(j-1,0) &\Rightarrow A_j \to S(j-1,1) \text{ and } A_j \ne S(j-1,1)\\ &\Rightarrow A_j\cdots < S(j-1,1)\cdots\\ &\Rightarrow A_j\cdots < S(j,1)\cdots. \end{aligned}$$

If $A_j = A_{j-1}S(j-1,1)^tS(j-1,0)$ for some $0 \le t < h_j$, $A_{j-1} \to S(j-1,1)$ and $A_{j-1} \ne S(j-1,1)$, then

$$\begin{aligned} A_{j-1}\cdots < S(j-1,1)\cdots &\Rightarrow A_j\cdots < S(j-1,1)\cdots\\ &\Rightarrow A_j\cdots < S(j,1)\cdots. \end{aligned}$$

Finally, if $A_j = S(j-1,1)^t S(j-1,0)$ for some $0 \le t < h_j$, then by (3.5),

$$A_j \cdots < S(j-1,1)^{t+1} \cdots \quad \text{and therefore} \quad A_j \cdots < S(j,1) \cdots .$$

(c) We proceed by induction. The case $j = 0$ is obvious because then we have necessarily $B_0 = S(0,0) = 0$. Let $j \ge 1$ and assume that the property holds for $j-1$ instead of j. If $h_j > 1$, then the case of j follows by applying part (b) with h_j replaced by $h_j - 1$. If $h_j = 1$, then we have $S(j,0) = S(j-1,0)$ and applying (b) we conclude that

$$B_j \to S(j,0) \;\Rightarrow\; B_j \to S(j-1,0) \;\Rightarrow\; B_j \le S(j-1,0) = S(j,0).$$

(d) The assertion is obvious for $j = 0$ because $S(0,1) = 1$ and $S(0,0) = 0$. Proceeding by induction, let $j \ge 1$ and assume that the result holds for $j-1$. Comparing the expressions

$$S(j,1)S(j,0) = S(j-1,1)^{h_j} S(j-1,0) S(j-1,1)^{h_j - 1} S(j-1,0),$$
$$S(j,0)S(j,1) = S(j-1,1)^{h_j - 1} S(j-1,0) S(j-1,1)^{h_j} S(j-1,0),$$

we see that $S(j,0)S(j,1)$ is obtained from $S(j,1)S(j,0)$ by changing the first block $S(j-1,1)S(j-1,0)$ to $S(j-1,0)S(j-1,1)$. □

The following corollary of Lemma 3.3(d) will not be used in this paper but it can be useful in similar investigations.

Corollary 3.4. *Let (a_k) be a sequence of zeroes and ones, containing both infinitely many zeroes and ones, and j a nonnegative integer. Then*

$$\pi_q(S(j,0)S(j,a_1)S(j,a_2)\cdots) < \pi_q(S(j,a_1)S(j,a_2)\cdots)$$
$$< \pi_q(S(j,1)S(j,a_1)S(j,a_2)\cdots)$$

for every base $q > 1$.

Proof. We prove the first inequality; the proof of the second one is similar. It is sufficient to show that the sequence $S(j,0)S(j,a_1)S(j,a_2)\cdots$ is obtained from $S(j,a_1)S(j,a_2)\cdots$ by changing certain (infinitely many) blocks $S(j,1)S(j,0)$ to $S(j,0)S(j,1)$. Indeed, each such change is equivalent to the change of a block 10 to 01 by Lemma 3.3(d), and therefore decreases the value of the expansion because if the block 10 figures on the ith and $(i+1)$th places, then

$$\frac{1}{q^i} + \frac{0}{q^{i+1}} > \frac{0}{q^i} + \frac{1}{q^{i+1}}.$$

Equivalently, we show that $0, a_1, a_2 \ldots$ is obtained from $a_1, a_2 \ldots$ by changing certain blocks 10 to 01. If $(a_k) = 0^{n_0}1^{n_1}0^{n_2}1^{n_3}0^{n_4}\cdots$ with a nonnegative integer n_0 and positive integers $n_1, n_2 \ldots$, then we obtain

- $00^{n_0}1^{n_1}0^{n_2-1}1^{n_3}0^{n_4}\cdots$ from $0^{n_0}1^{n_1}0^{n_2}1^{n_3}0^{n_4}\cdots$ by n_1 such changes,
- $00^{n_0}1^{n_1}0^{n_2}1^{n_3}0^{n_4-1}\cdots$ from $00^{n_0}1^{n_1}0^{n_2-1}1^{n_3}0^{n_4}\cdots$ by n_3 such changes,

and so on. □

The following lemma is a partial converse of (3.3).

Lemma 3.5. *If A is a block of length ℓ_{N-1} in some sequence $S(N,a_1)S(N,a_2)\cdots$ with $N \ge 1$ and $(a_i) \subset \{0,1\}$, then $A \ge S(0,0)\cdots S(N-1,0)$. Furthermore, we have $A = S(0,0)\cdots S(N-1,0)$ if and only if $A \to S(N,a_i)$ with some $a_i = 1$.*

Proof. The case $N = 1$ is obvious because then $S(0,0) = 0$ implies that $A = 0$, and $S(1,1) = 1^{h_1}0$ ends with 0.

Now let $N \ge 2$ and assume by induction that the result holds for $N-1$. Writing $A = BC$ with a block B of the same length as $S(0,0)\cdots S(N-2,0)$ and applying the induction hypothesis to B in the sequence

$$S(N,a_1)S(N,a_2)\cdots = (S(N-1,1)^{h_{N-1+a_i}}S(N-1,0))$$

we find that $B \to S(N-1,1)$ for one of the blocks on the right side and thus $B = S(0,0)\cdots S(N-2,0)$. Then it follows from our assumption that C has the same length as $S(N-1,0)$ and $C \le S(N-1,0)$. Since $S(N-1,0) < S(N-1,1)$, the block containing B must be followed by a block $S(N-1,0)$. We conclude that $C = S(N-1,0)$ and therefore $A = BC = S(0,0)\cdots S(N-1,0)$ and

$$A \to S(N-1,1)^{h_{N-1+a_i}}S(N-1,0) = S(N,a_i)$$

for some $a_i = 1$. □

Lemma 3.6. *A sequence $d = (d_i)$ is admissible if and only if one of the following three conditions is satisfied:*

- *$d = 0^\infty$;*
- *$d = S_h(N,1)^\infty$ with some nonnegative integer N and a finite sequence $h = (h_1,\ldots,h_N)$ of positive integers;*
- *there exists an infinite sequence $h = (h_i)$ of positive integers such that d begins with $S_h(N,1)$ for every $N = 0,1,\ldots$.*

Proof. It follows from the definition that $d_1 = 1$ for all admissible sequences other than 0^∞. In the following we consider only admissible sequences beginning with $d_1 = 1$. We omit the subscript h for brevity.

Let $d = (d_i)$ be an admissible sequence. Setting $d_i^0 := d_i$ we have

$$d = S(0,d_1^0)S(0,d_2^0)\cdots$$

with the admissible sequence (d_i^0).

Proceeding by recurrence, assume that

$$d = S(j,d_1^j)S(j,d_2^j)\cdots$$

for some integer $j \ge 0$ with an admissible sequence (d_i^j) and positive integers $h_1,\ldots,h_j$. (We need no such positive integers for $j = 0$.)

If $(d_i^j) = 1^\infty$, then $d = S(j,1)^\infty$. Otherwise there exists a positive integer h_{j+1} such that d begins with $S(j,1)^{h_{j+1}}S(j,0)$. Since the sequence (d_i^j) is admissible, we have

$$0d_2^jd_3^j\cdots \le d_{n+1}^jd_{n+2}^j\cdots \le d_1^jd_2^j\cdots$$

for all $n = 0, 1, \dots$. Since the map $(c_i) \mapsto (S(j, c_i))$ preserves the lexicographic ordering by (3.5), it follows that

$$S(j,0)S(j,d_2^j)S(j,d_3^j)\cdots \le S(j,d_{n+1}^j)S(j,d_{n+2}^j)\cdots \le S(j,d_1^j)S(j,d_2^j)\cdots$$

for all $n = 0, 1, \dots$. Thanks to the definition of h_{j+1} we conclude that

$$S(j,0)S(j,1)^{h_{j+1}-1}S(j,0)\cdots \le S(j,d_{n+1}^j)S(j,d_{n+2}^j)\cdots \le S(j,1)^{h_{j+1}}S(j,0)\cdots$$

for all $n = 0, 1, \dots$. This implies that each block $S(j, 0)$ is followed by at least $h_{j+1} - 1$ and at most h_{j+1} consecutive blocks $S(j, 1)$, so that

$$d = S(j+1, d_1^{j+1})S(j+1, d_2^{j+1})\cdots$$

for a suitable sequence (d_i^{j+1}) of zeroes and ones. The admissibility of (d_i^j) can then be rewritten in the form

$$\begin{aligned} S(j,0)S(j+1,0)S(j+1,d_2^{j+1})S(j+1,d_3^{j+1})\cdots, \\ \le S(j,d_{n+1}^j)S(j,d_{n+2}^j)\cdots \\ \le S(j+1,1)S(j+1,d_2^{j+1})S(j+1,d_3^{j+1})\cdots \end{aligned} \tag{3.6}$$

for $n = 0, 1, \dots$.

We claim that the sequence (d_i^{j+1}) is also admissible. We have $d_1^{j+1} = 1$ by the definition of h_{j+1}. It remains to show that

$$\begin{aligned} S(j+1,0)S(j+1,d_2^{j+1})S(j+1,d_3^{j+1})\cdots \\ \le S(j+1,d_{k+1}^{j+1})S(j+1,d_{k+2}^{j+1})S(j+1,d_{k+3}^{j+1})\cdots \\ \le S(j+1,1)S(j+1,d_2^{j+1})S(j+1,d_3^{j+1})\cdots \end{aligned}$$

for $k = 0, 1, \dots$.

The second inequality is a special case of the second inequality of (3.6). The first inequality is obvious for $k = 0$. For $k \ge 1$ it is equivalent to

$$\begin{aligned} S(j,0)S(j+1,0)S(j+1,d_2^{j+1})S(j+1,d_3^{j+1})\cdots \\ \le S(j,0)S(j+1,d_{k+1}^{j+1})S(j+1,d_{k+2}^{j+1})S(j+1,d_{k+3}^{j+1})\cdots \end{aligned}$$

and this is a special case of the first inequality of (3.6) because $S(j+1, d_k^{j+1})$ ends with $S(j, 0)$.

It follows from the above construction that (d_i) has one of the two forms specified in the statement of the lemma.

Turning to the proof of the converse statement, first we observe that if d begins with $S(N, 1)$ for some sequence $h = (h_i)$ and for some integer $N \ge 1$, then

$$d_n \cdots d_{\ell_N} < d_1 \cdots d_{\ell_N - n+1} \quad \text{for } n = 2, \dots, \ell_N; \tag{3.7}$$

this is just a reformulation of Lemma 3.3(b).

If $d_1d_2\cdots$ begins with $S(N,1)$ for all N, then the second inequality of (3.1) follows for all $n \geq 1$ by using the relation $\ell_N \to \infty$. Moreover, the inequality is strict. For $n = 0$ we clearly have equality.

If $d = S(N,1)^\infty$ for some $N \geq 0$, then d is ℓ_N-periodic so that the second inequality of (3.1) follows from (3.7) for all n, except if n is a multiple of ℓ_N; we get strict inequalities in these cases. If n is a multiple of ℓ_N, then we obviously have equality again.

It remains to prove the first inequality of (3.1). If $d = S(N,1)^\infty$ for some $N \geq 0$, then we deduce from Lemma 3.5 that either

$$(d_{n+i}) > S(0,0)\cdots S(N-1,0) \quad \text{or} \quad (d_{n+i}) = S(0,0)\cdots S(N-1,0)S(N,1)^\infty.$$

Since

$$0d_2d_3\cdots = S(0,0)\cdots S(N-1,0)S(N,0)S(N,1)^\infty,$$

we conclude in both cases the strict inequalities

$$(d_{n+i}) > 0d_2d_3\cdots.$$

If $d_1d_2\cdots$ begins with $S(N,1)$ for all N, then

$$0d_2d_3\cdots = S(0,0)S(1,0)\cdots S(N,0)\cdots \leq (d_{n+i})$$

by Lemma 3.5. □

Remark 3.7. The end of the proof also shows that in case $d = 0^\infty$ or $d = S_h(N,1)^\infty$ the inequalities (3.1) are not strict for all $n \geq 1$. The same is true if d is defined by an infinite sequence $h = (h_i)$ with at most finitely many $h_i > 1$. Indeed, in this case $d = S(N-1,1)S(N,1)^\infty$ for some $N \geq 1$, so that

$$0d_2d_3\cdots = S(0,0)\cdots S(N-1,0)S(N,1)^\infty = (d_{n+i})$$

for infinitely many n by (3.2) and (3.3).

On the other hand, if d is defined by an infinite sequence $h = (h_i)$ and $h_i \geq 2$ for infinitely many n, then the inequalities (3.1) are strict for all $n \geq 1$. We already know from the above proof that the second inequality is always strict. Assume on the contrary that $0d_2d_3\cdots = (d_{n+i})$ for some n. Since $0d_2d_3\cdots = S(0,0)S(1,0)\cdots$ in this case, it follows that (d_{n+i}) begins with $S(0,0)\cdots S(N-1,0)$ for every $N \geq 1$. Writing d in the form $S(N,a_1)S(N,a_2)\cdots$ and using Lemma 3.5 we conclude that

$$n + |S(0,0)| + \cdots + |S(N-1,0)| \geq |S(N,1)|$$

for every N, where $|w|$ means the length of the word w. This, however, is impossible because $|S(N+1,1)| - |S(N,1)| \geq |S(N,0)|$ for all N, and $|S(N+1,1)| - |S(N,1)| \geq |S(N,0)| + 1$ whenever $h_N \geq 2$, i.e., for infinitely many N.

We have obtained in this way a new characterization of Sturmian sequences: a sequence s is Sturmian if and only if $1s$ is an admissible sequence defined by an infinite sequence $h = (h_i)$ such that $h_i \geq 2$ for infinitely many i.

Definition 3.8. We say that an admissible sequence d is *of finite type* if $d = 0^\infty$ or if $d = S_h(N,1)^\infty$ with some nonnegative integer N and a finite sequence $h = (h_1, \dots, h_N)$ of positive integers. Otherwise it is said to be *of infinite type*.

Lemma 3.9. *Let $d = (d_i) \neq 1^\infty$ be an admissible sequence.*

(a) *If $(d_i) = S(N,1)^\infty$ (then $N \geq 1$ because $d \neq 1^\infty$) and $(d_i') = (d_{i+1+\ell_N-\ell_{N-1}})$, then*

$$(d_{n+i}') \geq (d_i') > (d_{1+i}) \quad \text{whenever } d_n' = 0.$$

Moreover,

$$(d_i') = S(1,0)\cdots S(N-1,0)S(N,1)^\infty, \tag{3.8}$$

$$(d_{1+i}) = S(1,0)\cdots S(N-1,0)S(N,0)S(N,1)^\infty. \tag{3.9}$$

(b) *In the other cases the sequence $(d_i') := (d_{1+i})$ satisfies*

$$(d_{n+i}') \geq (d_i') \quad \text{whenever } d_n' = 0.$$

(c) *We have $d' = d$ if and only if $d = (1^{k-1}0)^\infty$ for some positive integer k, i.e., $d = 0^\infty$ or $d = S(N,1)^\infty$ with $N = 1$.*

Proof. (a) The first inequality follows from Lemma 3.5; the proof also shows that we have equality if and only if n is a multiple of ℓ_N.

The relations (3.2) and (3.3) of Lemma 3.3 imply (3.8)–(3.9) and they imply the second inequality because $S(N,0) < S(N,1)$.

(b) The case $(d_i) = 0^\infty$ is obvious. Otherwise (d_i) begins with $S(N,1)$ for all $N \geq 0$ and $\ell_N \to \infty$, so that we deduce from the relation (3.2) of Lemma 3.3 the equality

$$0d_2d_3\cdots = S(0,0)S(1,0)\cdots.$$

On the other hand, it follows from Lemma 3.5 that for any $n \geq 0$ we have

$$(d_{n+i}') \geq S(0,0)\cdots S(N-1,0)S(N,0)^\infty \quad \text{for every } N \geq 0.$$

This implies that

$$(d_{n+i}') \geq 0d_2d_3\cdots \quad \text{for every } n \geq 0.$$

If $d_n' = 0$, then we conclude that

$$d_n'd_{n+1}'d_{n+2}'\cdots \geq 0d_2d_3\cdots,$$

which is equivalent to the required inequality

$$d_{n+1}'d_{n+2}'\cdots \geq d_2d_3\cdots.$$

(c) It follows from the above proof that $d = d'$ if and only if $d = 0^\infty$ or $d = S(N,1)^\infty$ for some integer $N \geq 1$ and h such that $\ell_{N-1} = 1$. These conditions are equivalent to $d = (1^{k-1}0)^\infty$ for some positive integer k. □

Example 3.10. By Lemma 3.6 all admissible sequences $d \neq 0^\infty$ are defined by a finite or infinite sequence $h = (h_j)$. If we add the symbol ∞ to the end of each finite sequence (h_j), then the map $d \mapsto h$ is increasing with respect to the lexicographic orders of sequences. It follows that if $d = S(N,1)^\infty$ is an admissible sequence of finite type with $N \geq 1$ (i.e., $d \neq 1^\infty$) and $h_1, \dots, h_N \geq 1$, then there exists a smallest admissible sequence $\hat{d} > d$. It is of infinite type, corresponding to the infinite sequence $h = h_1 \cdots h_{N-1} h_N^+ 1^\infty$ with $h_N^+ := 1 + h_N$. Observe that $\hat{d} = S(N-1,1)d$ and hence $\hat{d}' = d'$ and $\hat{d} = 1d'$.

For $d = 0^\infty$, there exists a smallest admissible sequence $\hat{d} > d$, too. It is also of infinite type: $\hat{d} = 10^\infty$, corresponding to $h = (1,1,\dots)$, and $\hat{d}' = d' = 0^\infty$.

Lemma 3.11. *If $d = (d_i)$ is an admissible sequence of finite type, then no sequence (c_i) of zeroes and ones satisfies*

$$0d_2d_3 \cdots < (c_{n+i}) < d_1d_2d_3 \cdots \quad \textit{for all } n = 1, 2, \dots.$$

Proof. The case $d = 0^\infty$ is obvious because then $0d_2d_3 \cdots = d_1d_2d_3 \cdots$. The case $d = 1^\infty$ is obvious too, because then (c_i) cannot have any zero digit by the first condition, while $(c_i) = 1^\infty$ does not satisfy the second condition. We may therefore assume that $d = S(N,1)^\infty$ for some $N \geq 1$ and some $h = (h_i)$. Then our assumption takes the form

$$S(0,0) \cdots S(N-1,0) S(N,0) S(N,1)^\infty < (c_{n+i}) < S(N,1)^\infty. \tag{3.10}$$

First step: the sequence (c_i) cannot end with $S(K,0)^\infty$ for any $0 \leq K \leq N$.

This is true if $S(K,0) = 0$ because $0^\infty \leq 0d_2d_3 \cdots$.

Otherwise we have $K \geq 1$ and there exists $1 \leq M \leq K$ such that $h_M \geq 2$ and $h_{M+1} = \cdots = h_K = 1$. Then we have

$$S(M,0) = S(M+1,0) = \cdots = S(K,0)$$

and (see (3.4))

$$S(0,0) \cdots S(M-1,0) \to S(M,0) = S(K,0).$$

Therefore in case (c_i) ends with $S(K,0)^\infty$ there exists n such that

$$\begin{aligned}(c_{n+i}) &= S(0,0) \cdots S(M-1,0) S(K,0)^\infty = S(0,0) \cdots S(K-1,0) S(K,0)^\infty \\ &\leq S(0,0) \cdots S(N-1,0) S(N,0)^\infty < S(0,0) \cdots S(N-1,0) S(N,0) S(N,1)^\infty,\end{aligned}$$

contradicting the first inequality of (3.10).

Second step: the sequence (c_i) ends with $S(N,c_1^N) S(N,c_2^N) \cdots$ for a suitable sequence $(c_j^N) \subset \{0,1\}$.

We have $(c_i) = S(0,c_1^0) S(0,c_2^0) \cdots$ with $c_i^0 := c_i$. Now let $1 \leq M \leq N$ and assume by induction that (c_i) ends with $S(M-1,c_1^{M-1}) S(M-1,c_2^{M-1}) \cdots$ for a suitable sequence $(c_j^{M-1}) \subset \{0,1\}$.

Since $S(N,1)$ begins with $S(M,1) = S(M-1,1)^{h_M} S(M-1,0)$, by (3.10) each block $S(M-1, c_j^{M-1})$ is followed by at most h_M consecutive blocks $S(M-1,1)$. On the other hand, since the first expression in (3.10) begins with

$$S(0,0)\cdots S(M-2,0)S(M-1,0)S(M-1,1)^{h_M-1}S(M-1,0)$$

and since (see (3.3))

$$S(0,0)\cdots S(M-2,0) \to S(M-1,1)$$

(for $M = 1$ the block $S(0,0)\cdots S(M-2,0)$ is empty by definition), each block $S(M-1,1)S(M-1,0)$ in $(S(M-1, c_j^{M-1}))$ is followed by at least $h_M - 1$ consecutive blocks $S(M-1,1)$.

Since (c_i) cannot end with $S(M-1,0)^\infty$ by the first step, we conclude that (c_i) ends with $(S(M, c_j^M))$ for a suitable sequence $(c_j^M) \subset \{0,1\}$.

Third step: the sequence (c_i) ends with $S(N,1)S(N,0)S(N,a_1)S(N,a_2)\cdots$ for a suitable sequence $(a_j) \subset \{0,1\}$.

Indeed, in view of the first two steps it suffices to observe that (c_i) cannot end with $S(N,1)^\infty$ by the second condition of (3.10).

Fourth step. Using the relation $S(0,0)\cdots S(N-1,0) \to S(N,1)$ (see (3.3)) we deduce from the preceding step that (c_i) ends with

$$S(0,0)\cdots S(N-1,0)S(N,0)(S(N,a_j)) \le S(0,0)\cdots S(N-1,0)S(N,0)S(N,1)^\infty,$$

contradicting the first condition in (3.10) again. □

Lemma 3.12. *If $d = (d_i) \ne 1^\infty$ is an admissible sequence of finite type, then no sequence (c_i) of zeroes and ones satisfies*

$$0(d_i') < (c_{n+i}) < 1(d_i') \quad \textit{for all } n = 1, 2, \ldots.$$

Proof. If $d = 0^\infty$, then $d' = 0^\infty$ and our hypothesis takes the form $0^\infty < (c_{n+i}) < 10^\infty$. Such a sequence cannot have digits 1 by the second condition, but it cannot be 0^∞ either by the first condition. It remains to consider the case where $d = S(N,1)^\infty$ for some $N \ge 1$ and $h = (h_1, \ldots, h_N)$. Then by Lemmas 3.3 and 3.9 our hypothesis may be written in the form

$$S(0,0)\cdots S(N-1,0)S(N,1)^\infty < (c_{n+i}) < S(N-1,1)S(N,1)^\infty. \tag{3.11}$$

Using (3.11) instead of (3.10), we may repeat the proof of the preceding proposition by keeping $h_1, \ldots, h_{N-1}$ but changing h_N to $h_N + 1$. At the end of the third step we find that a sequence (c_i) satisfying (3.11) must end with

$$S_+(N,1)S_+(N,0)S_+(N,a_1)S_+(N,a_2)\cdots$$

for a suitable sequence $(a_j) \subset \{0, 1\}$, where we use the notation

$$S_+(N, 1) := S(N-1, 1)^{h_N+1} S(N-1, 0),$$
$$S_+(N, 0) := S(N, 1) = S(N-1, 1)^{h_N} S(N-1, 0).$$

Since

$$\begin{aligned} S_+(N, 1)S_+(N, 0)S_+(N, a_1)S_+(N, a_2)\cdots &\ge S_+(N, 1)S_+(N, 0)^\infty \\ &= S(N-1, 1)S(N, 1)^\infty, \end{aligned}$$

this contradicts the second inequality of (3.11). □

4. m-admissible sequences

Throughout this section we fix an admissible sequence $d = (d_i) \neq 1^\infty$ and we define the sequence $d' = (d'_i)$ as in Lemma 3.9. Furthermore, for any given real number $m > 1$ we denote by $\delta = (\delta_i)$ and $\delta' = (\delta'_i)$ the sequences obtained from d and d' by the substitutions $1 \to m$ and $0 \to 1$. We define the numbers $p'_m, p''_m > 1$ by the equations

$$\sum_{i=1}^{\infty} \frac{\delta_i}{(p'_m)^i} = m - 1 \tag{4.1}$$

and

$$\sum_{i=1}^{\infty} \frac{m - \delta'_i}{(p''_m)^i} = 1 \tag{4.2}$$

and we put $p_m := \max\{p'_m, p''_m\}$.

Introducing the *conjugate* of δ by the formula $\overline{\delta'_i} := m - \delta'_i$ we may also write (4.1) and (4.2) in the more economical form

$$\pi_{p'_m}(\delta) = m - 1 \quad \text{and} \quad \pi_{p''_m}(\overline{\delta'}) = 1.$$

Let us also introduce the number

$$P_m := 1 + \sqrt{\frac{m}{m-1}}.$$

A direct computation shows that $P_m > 1$ can also be defined by any of the following equivalent conditions:

$$(P_m - 1)^2 = \frac{m}{m-1}; \tag{4.3}$$

$$\frac{m}{P_m} + \frac{1}{P_m}\left(\frac{m}{P_m - 1} - 1\right) = m - 1; \tag{4.4}$$

$$(m-1)P_m - m = \frac{m}{P_m - 1} - 1; \tag{4.5}$$

$$\frac{m}{P_m - 1} - (m-1) = \frac{1}{P_m}. \tag{4.6}$$

We begin by investigating the dependence of P_m, p'_m and p''_m on m. The following two lemmas establish in particular Theorem 1.1(b).

Lemma 4.1. (a) *The function* $m \mapsto P_m$ *is continuous and strictly decreasing in* $(1, \infty)$.
(b) *The function* $m \mapsto p'_m - P_m$ *is continuous and strictly decreasing in* $(1, \infty)$*, and it has a unique zero* m_d.
(c) *The function* $m \mapsto p''_m - P_m$ *is continuous and strictly increasing in* $(1, \infty)$*, and it has a unique zero* M_d.
(d) *The function* $m \mapsto p'_m - p''_m$ *is continuous and strictly decreasing in* $(1, \infty)$*, and it has a unique zero* μ_d.
(e) *The function* $m \mapsto p_m$ *is continuous in* $(1, \infty)$*, strictly decreasing in* $(1, \mu_d]$ *and strictly increasing in* $[\mu_d, \infty)$*, so that it has a strict global minimum in* μ_d.

Proof. (a) A straightforward computation shows that P is infinitely differentiable in $(1, \infty)$ and

$$P'(m) = \frac{-1}{2(m-1)\sqrt{m(m-1)}} < 0 \quad \text{for all } m > 1.$$

(b) Since $\delta_i = 1 + (m-1)d_i$, we may rewrite (4.1) in the form

$$\frac{1}{m-1} + (p'_m - 1)\sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^i} = p'_m - 1. \tag{4.7}$$

Applying the implicit function theorem it follows that the function $m \mapsto p'_m$ is C^∞.

Differentiating the last identity with respect to m, denoting the derivatives by dots and setting

$$A := 1 + (p'_m - 1)\sum_{i=1}^{\infty} \frac{d_i}{i(p'_m)^{i+1}} - \sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^i},$$

we get

$$A\dot{p}'_m = \frac{-1}{(m-1)^2}.$$

Differentiating (4.3) we find that the right side is equal to $2(P_m - 1)\dot{P}_m$, so that

$$A\dot{p}'_m = 2(P_m - 1)\dot{P}_m.$$

Since $\dot{P}_m < 0$ and $2(P_m - 1) > 1$, it suffices to show that $A \in (0, 1)$. Indeed, then we will have $\dot{p}'_m / \dot{P}_m > 1$ and therefore $\dot{p}'_m < \dot{P}_m$ (< 0).

The inequality $A > 0$ follows by using (4.7):

$$A = (p'_m - 1)\sum_{i=1}^{\infty} \frac{d_i}{i(p'_m)^{i+1}} + \frac{1}{(m-1)(p'_m - 1)} > 0,$$

while the proof of $A < 1$ is straightforward:

$$A \le 1 + (p'_m - 1) \sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^{i+1}} - \sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^i}$$
$$= 1 - \frac{1}{p'_m} \sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^i} < 1.$$

It remains to show that $p'_m - P_m$ changes sign in $(1, \infty)$. It is clear from the definition that

$$\lim_{m \searrow 1} P_m = \infty \quad \text{and} \quad \lim_{m \to \infty} P_m = 2. \tag{4.8}$$

Furthermore, using the equality $d_1 = 1$ it follows from (4.7) that

$$\frac{1}{m-1} \le p'_m - 1 \le 1 + \frac{1}{m-1};$$

hence

$$\lim_{m \searrow 1} p'_m = \infty \quad \text{and} \quad \lim_{m \to \infty} p'_m = 1. \tag{4.9}$$

We infer from (4.8)–(4.9) that $\lim_{m\to\infty} p'_m - P_m = -1 < 0$. The proof is completed by observing that

$$p'_m - P_m \ge \frac{1}{m-1} - 1 - \sqrt{\frac{m}{m-1}} \to \infty > 0 \quad \text{as } m \searrow 1.$$

(c) We may rewrite (4.2) in the form

$$\sum_{i=1}^{\infty} \frac{1 - d'_i}{(p''_m)^i} = \frac{1}{m-1}. \tag{4.10}$$

Applying the implicit function theorem it follows from (4.10) that the function $m \mapsto p''_m$ is C^∞.

The last identity also shows that the function $m \mapsto p''_m$ is strictly increasing. Using (a) we conclude that the function $m \mapsto p''_m - P_m$ is strictly increasing, too.

It remains to show that $p''_m - P_m$ changes sign in $(1, \infty)$. Since $d \ne 1^\infty$, there exists an index k such that $d'_k = 0$. Therefore we deduce from (4.10) the inequalities

$$\frac{1}{(p''_m)^k} \le \frac{1}{m-1} \le \frac{1}{p''_m - 1}$$

and hence

$$\lim_{m \searrow 1} p''_m = 1 \quad \text{and} \quad \lim_{m \to \infty} p''_m = \infty. \tag{4.11}$$

We conclude from (4.8) and (4.11) that

$$\lim_{m \searrow 1} (p''_m - P_m) = -\infty < 0 \quad \text{and} \quad \lim_{m \to \infty} (p''_m - P_m) = \infty > 0.$$

(d) The proof of (b) and (c) shows that $m \mapsto p'_m$ is continuous and strictly decreasing and $m \mapsto p''_m$ is continuous and strictly increasing; hence the function $m \mapsto p'_m - p''_m$ is continuous and strictly decreasing. It remains to observe that $p'_m - p''_m$ changes sign in $(1, \infty)$ because (4.9) and (4.11) imply that

$$\lim_{m \searrow 1} (p'_m - p''_m) = \infty > 0 \quad \text{and} \quad \lim_{m \to \infty} (p'_m - p''_m) = -\infty < 0.$$

(e) This follows from the definition $p_m := \max\{p'_m, p''_m\}$ and from the fact that $m \mapsto p'_m$ is continuous and strictly decreasing and $m \mapsto p''_m$ is continuous and strictly increasing. ◻

The first part of the following lemma is a variant of a similar result in [6].

Lemma 4.2. *We consider expansions in some base $q > 1$ on some alphabet $\{a, b\}$ with $a < b$.*

(a) *Let (c_i) be an expansion of some number $s \le b - a$. If*

$$c_{n+1}c_{n+2}\cdots \le c_1c_2\cdots \quad \textit{whenever } c_n = a,$$

then

$$\frac{c_{n+1}}{q^{n+1}} + \frac{c_{n+2}}{q^{n+2}} + \cdots \le \frac{s}{q^n} \quad \textit{whenever } c_n = a.$$

Moreover, the inequality is strict if the sequence (c_i) is infinite and $(c_{n+i}) \ne (c_i)$.

(b) *Let $c = (c_i)$ and $d = (d_i)$ be two expansions. If $q \ge 2$, then*

$$(c_i) \le (d_i) \;\Rightarrow\; \pi_q(c) \le \pi_q(d).$$

Moreover, if $q > 2$, then

$$(c_i) < (d_i) \;\Leftrightarrow\; \pi_q(c) < \pi_q(d).$$

Proof. (a) Starting with $k_0 := n$ we define by recurrence a sequence of indices $k_0 < k_1 < \cdots$ satisfying for $j = 1, 2, \ldots$ the conditions

$$c_{k_{j-1}+i} = c_i \quad \text{for } i = 1, \ldots, k_j - k_{j-1} - 1, \qquad c_{k_j} < c_{k_j - k_{j-1}}.$$

If we obtain an infinite sequence, then we have

$$\begin{aligned}\sum_{i=n+1}^{\infty} \frac{c_i}{q^i} &= \sum_{j=1}^{\infty} \sum_{i=1}^{k_j - k_{j-1}} \frac{c_{k_{j-1}+i}}{q^{k_{j-1}+i}} \le \sum_{j=1}^{\infty} \Big(\Big(\sum_{i=1}^{k_j - k_{j-1}} \frac{c_i}{q^{k_{j-1}+i}} \Big) - \frac{b-a}{q^{k_j}} \Big) \\ &\le \sum_{j=1}^{\infty} \left(\frac{s}{q^{k_{j-1}}} - \frac{b-a}{q^{k_j}} \right) \le \sum_{j=1}^{\infty} \left(\frac{s}{q^{k_{j-1}}} - \frac{s}{q^{k_j}} \right) = \frac{s}{q^n}.\end{aligned}$$

Otherwise we have $(c_{k_N+i}) = (c_i)$ after a finite number of steps (we do not exclude the possibility that $N = 0$), and we may conclude as follows:

$$\begin{aligned}\sum_{i=n+1}^{\infty} \frac{c_i}{q^i} &= \left(\sum_{j=1}^{N} \sum_{i=1}^{k_j-k_{j-1}} \frac{c_i}{q^{k_{j-1}+i}}\right) + \sum_{i=1}^{\infty} \frac{c_{k_N+i}}{q^{k_N+i}}\\ &\le \sum_{j=1}^{N}\left(\left(\sum_{i=1}^{k_j-k_{j-1}} \frac{c_i}{q^{k_{j-1}+i}}\right) - \frac{b-a}{q^{k_j}}\right) + \sum_{i=1}^{\infty} \frac{c_i}{q^{k_N+i}}\\ &\le \sum_{j=1}^{N}\left(\frac{s}{q^{k_{j-1}}} - \frac{b-a}{q^{k_j}}\right) + \frac{s}{q^{k_N}}\\ &\le \sum_{j=1}^{N}\left(\frac{s}{q^{k_{j-1}}} - \frac{s}{q^{k_j}}\right) + \frac{s}{q^{k_N}} = \frac{s}{q^n}.\end{aligned}$$

The last property follows from the above proof.

(b) If $c < d$, then let n be the first integer for which $c_n < d_n$. Then $c_i = d_i$ for $i < n$, $d_n - c_n = b - a$, and $d_i - c_i \ge a - b$ for $i > n$, so that

$$\pi_q(d) - \pi_q(c) \ge \frac{b-a}{q^n} - \sum_{i=n+1}^{\infty} \frac{b-a}{q^i} = \frac{b-a}{q^n} - \frac{b-a}{q^n(q-1)} \ge 0.$$

Moreover, in case $q > 2$ the last inequality is strict. □

Now we investigate the mutual positions of m_d, M_d and μ_d.

Lemma 4.3. (a) *If d is of finite type, then $m_d < \mu_d < M_d$, and $p_m < P_m$ for all $m_d < m < M_d$. Furthermore, $p_m \ge 2$ for all $m \in (1, \infty)$ with equality if and only if $d = (1^{k-1}0)^\infty$ and $m = 2^k$ for some positive integer k.*

(b) *In the other cases we have $m_d = \mu_d = M_d$ and $p_m \ge p_{\mu_d} = P_{\mu_d} > 2$ for all $m \in (1, \infty)$.*

Proof. (a) In view of Lemma 4.1 the first assertion will follow if we show that $p_m < P_m$ for $m := \mu_d$. Note that $p_m = p'_m = p''_m$ in this case.

If $d = 0^\infty$, then $d' = 0^\infty$ and therefore

$$m - 1 = \pi_{p'_m}(\delta) = \pi_{p''_m}(\delta') = \frac{m}{p''_m - 1} - 1 = \frac{m}{p_m - 1} - 1.$$

It follows that $p_m = 2$ and therefore $P_m = 1 + \sqrt{m/(m-1)} > p_m$.

In the other cases, using the relations (3.8)–(3.9) of Lemma 3.9 we have

$$\begin{aligned}m - 1 &= \sum_{i=1}^{\infty} \frac{\delta_i}{p_m^i} = \frac{m}{p_m} + \frac{1}{p_m}\sum_{i=1}^{\infty} \frac{\delta_{i+1}}{p_m^i}\\ &< \frac{m}{p_m} + \frac{1}{p_m}\sum_{i=1}^{\infty} \frac{\delta'_i}{p_m^i} = \frac{m}{p_m} + \frac{1}{p_m}\left(\frac{m}{p_m - 1} - 1\right).\end{aligned}$$

In this computation the crucial inequality follows from Lemmas 3.9 and 4.2(a). Indeed, writing $d = S(N,1)^\infty$, in view of the relations (3.8)–(3.9) of Lemma 3.9 the inequality is equivalent to

$$\pi_{p'_m}((\delta_{\ell_{N-1}+i})) < \pi_{p'_m}(\delta),$$

and this inequality follows from Lemma 4.2(a) with $c = \delta$, $q = p'_m$ and $n = \ell_{N-1}$. (The hypotheses of the lemma are satisfied because d is an admissible sequence.)

Using (4.4) we conclude that $p_m < P_m$ indeed.

Furthermore, for $m := \mu_d$ we deduce from the equalities

$$\pi_{p_m}(\delta) = m - 1 \quad \text{and} \quad \pi_{p_m}(\overline{\delta'}) = 1$$

that

$$\sum_{i=1}^{\infty} \frac{m - \delta'_i + \delta_i}{p_m^i} = m.$$

It follows that $p_m \geq 2$ if and only if

$$\sum_{i=1}^{\infty} \frac{m - \delta'_i + \delta_i}{2^i} \geq m,$$

which is equivalent to the inequality

$$\pi_2(\delta') \leq \pi_2(\delta).$$

Since $\delta' \leq \delta$ by Lemma 3.9, this is satisfied by a well-known property of dyadic expansions.

The proof also shows that we have equality if and only if $\delta' = \delta$. By Lemma 3.9(c) this is equivalent to $d = (1^{k-1}0)^\infty$ for some positive integer k. In this case we infer from the equalities

$$\frac{m}{p'_m - 1} - \frac{m-1}{(p'_m)^k - 1} = m - 1$$

and

$$\frac{m}{p''_m - 1} - \frac{m-1}{(p''_m)^k - 1} = \frac{m}{p''_m - 1} - 1$$

that $p'_m = p''_m = m^{1/k} = 2$.

Since by Lemma 4.1, p_m has a global strict minimum at $m = \mu_d$, we have $p_m > 2$ for all other values of m.

(b) Putting $m = \mu_d$ and repeating the first part of the proof of (a), by Lemma 3.9 we now have equality instead of strict inequality; using (4.4) we conclude that $p_m = P_m$ and so $p_m = p'_m = p''_m = P_m$. Applying Lemma 4.1 we conclude that $m_d = \mu_d = M_d$. □

5. Univoque sequences in small bases

In this section we determine the generalized golden ratio for every ternary alphabet $A = \{a_1, a_2, a_3\}$. Putting

$$m := \max\left\{\frac{a_3 - a_1}{a_2 - a_1}, \frac{a_3 - a_1}{a_3 - a_2}\right\}$$

we will show that

$$2 \le G_A \le P_m := 1 + \sqrt{\frac{m}{m-1}}.$$

Moreover, we will give an exact expression of G_A for each m and we will determine the values of m for which $G_A = 2$ or $G_A = P_m$.

By Lemma 2.9 we may restrict ourselves without loss of generality to the case of the alphabets $A_m = \{0, 1, m\}$ with $m \ge 2$. Condition (2.1) takes the form

$$1 < q \le \frac{2m-1}{m-1};$$

under this assumption, which we make henceforth, the results of the preceding section apply. In what follows we fix a real number $m \ge 2$ and we consider expansions in bases $q > 1$ with respect to the ternary alphabet $A_m := \{0, 1, m\}$.

One of our main tools will be Theorem 2.2, which now takes the following special form:

Lemma 5.1. *An expansion (c_i) is unique in base q for the alphabet A_m if and only if the following conditions are satisfied:*

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < 1 \qquad \textit{whenever } c_n = 0; \tag{5.1}$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < m-1 \qquad \textit{whenever } c_n = 1; \tag{5.2}$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} > \frac{m}{q-1} - 1 \qquad \textit{whenever } c_n = 1; \tag{5.3}$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} > \frac{m}{q-1} - (m-1) \qquad \textit{whenever } c_n = m. \tag{5.4}$$

Corollary 5.2. *We have $G_{A_m} \ge 2$.*

Proof. Let (c_i) be a univoque sequence in some base $1 < q \le 2$. We infer from (5.2) and (5.3) that $c_n \ne 1$ for every n. Since $m \ge q$, we conclude from (5.1) that each 0 digit is followed by another 0 digit. Therefore condition (5.4) implies that each m digit is followed by another m digit. For otherwise the left-hand side of (5.4) would be zero, while the right-hand side is greater than zero. Hence (c_i) must be equal to 0^∞ or m^∞. □

Lemma 5.3. *If (c_i) is a nontrivial univoque sequence in some base $1 < q \le P_m$, then (c_i) contains at most finitely many zero digits.*

Proof. Since a univoque sequence remains univoque in every larger base, we may assume that $q = P_m$. It suffices to prove that (c_i) does not contain any block of the form $m0$ or 10.

(c_i) does not contain any block of the form $m0$. If $c_n = m$ and $c_{n+1} = 0$ for some n, then we deduce from Lemma 5.1 that

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{P_m^i} > \frac{m}{P_m - 1} - (m-1) \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{c_{n+i+1}}{P_m^i} < 1.$$

Hence

$$\frac{m}{P_m - 1} - (m-1) < \sum_{i=1}^{\infty} \frac{c_{n+i}}{P_m^i} = \frac{1}{P_m} \sum_{i=1}^{\infty} \frac{c_{n+i+1}}{P_m^i} < \frac{1}{P_m},$$

contradicting condition (4.6) on P_m.

(c_i) does not contain any block of the form 10. If $c_n = 1$ and $c_{n+1} = 0$ for some n, then an application of Lemma 5.1 shows that

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{P_m^i} > \frac{m}{P_m - 1} - 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{c_{n+i+1}}{P_m^i} < 1.$$

Since $m \ge 2$, these inequalities imply those of the preceding step, contradicting again our condition on P_m. □

Next we select a particular admissible sequence for each given m. Given an admissible sequence $d \ne 1^\infty$ we set

$$I_d := \begin{cases} [m_d, M_d) & \text{if } m_d < M_d, \\ \{m_d\} & \text{if } m_d = M_d. \end{cases} \tag{5.5}$$

Lemma 5.4. *Given a real number $m \ge 2$ there exists a lexicographically largest admissible sequence $d = (d_i)$ such that using the notation of the preceding section we have*

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \le m - 1. \tag{5.6}$$

Furthermore, $d \ne 1^\infty$ and $m \in I_d$.

Remark 5.5. The lemma and its proof remain valid for all $m \ge (1+\sqrt{5})/2$.

Proof. The sequence $d = 0^\infty$ always satisfies (5.6) because using (4.3) we have

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} = \frac{1}{P_m - 1} = \sqrt{\frac{m-1}{m}} \le m - 1;$$

the last inequality is equivalent to $m \ge (1+\sqrt{5})/2$. If it is not the only such admissible sequence, then applying the monotonicity of the map $d \mapsto h$ mentioned in Example 3.10

we obtain the existence of a lexicographically largest finite or infinite sequence h such that the corresponding admissible sequence $d = (d_i)$ satisfies (5.6).

We have $d \neq 1^\infty$ because the sequence $d = 1^\infty$ does not satisfy (5.6): using (4.3) again we have

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} = \frac{m}{P_m - 1} = \sqrt{(m-1)m} > m - 1.$$

It remains to prove that $m \in I_d$. We distinguish three cases.

(a) If (d_i) is defined by an infinite sequence (h_j), then we already know that $p_m = p'_m = p''_m$ and

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \leq m - 1.$$

It remains to show the reverse inequality

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \geq m - 1. \tag{5.7}$$

It follows from the definition of (δ_i) that if we denote by (δ_i^N) the sequence associated with the admissible sequence defined by the sequence $h := h_1, \dots, h_{N-1}, h_N + 1, 1, 1, \dots,$ then

$$\sum_{i=1}^{\infty} \frac{\delta_i^N}{P_m^i} > m - 1.$$

Since both (d_i) and (d_i^N) begin with $S(N-1, 1)^{h_N}$ and since the length of this block tends to infinity, letting $N \to \infty$ we deduce (5.7).

(b) If $(d_i) = S(N, 1)^\infty$ for some $N \geq 1$, then

$$\begin{aligned}(e_i) &:= S(N-1, 1)^{h_N+1} S(N-1, 0) [S(N-1, 1)^{h_N} S(N-1, 0)]^\infty \\ &= S(N-1, 1) S(N, 1)^\infty\end{aligned}$$

does not satisfy (5.6), so that

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{P_m^i} > m - 1$$

where (ε_i) is obtained from (e_i) by the usual substitutions $1 \to m$ and $0 \to 1$.

Observe that now $e_1 e_2 \cdots = 1 d'_1 d'_2 \cdots$ and therefore (using the notation of the introduction)

$$m - 1 < \pi_{P_m}(\varepsilon) = \frac{m}{P_m} + \frac{1}{P_m} \pi_{P_m}(\delta').$$

It follows that

$$\pi_{P_m}(\delta') > (m-1)P_m - m = \frac{m}{P_m - 1} - 1,$$

which is equivalent to $\pi_{P_m}(\overline{\delta'}) < 1$. Since $\pi_{p''_m}(\overline{\delta'}) = 1$ by the definition of p''_m, we conclude that $P_m > p''_m$.

Finally, since $\pi_{P_m}(\delta) \le m-1 = \pi_{p'_m}(\delta)$ by the definitions of (d_i) and p'_m, we also have $P_m \ge p'_m$.

(c) If $(d_i) = 0^\infty$, then we repeat the proof of (b) with $(d'_i) = 0^\infty$ and $(e_i) = 10^\infty$. $\square$

Example 5.6. Using a computer program we can determine the admissible sequences of Lemma 5.4 for all *integer* values $2 \le m \le 2^{16}$. For all but seven values the corresponding admissible sequence is of finite type with $N = 1$, more precisely $d = (1^{h_1}0)^\infty$ with a suitable value of h_1. (We have $h_1 = [\log_2 m] - 1$ for $m = 4, 8$, 16–17, 32–33, 64–65, 128–129, 256–257, 512–514, 1024–1026, 2048–2050, 4096–4098, 8192–8195, 16384–16387, and $h_1 = [\log_2 m]$ for the remaining values of m.) For the exceptional values $m = 5, 9, 130, 258, 2051, 4099, 32772$ the corresponding admissible sequence is of finite type with $N = 2$ and $h_1 = [\log_2 m]$ as shown in the following table:

m	d	N	h
5	$(1^2 01^2 010)^\infty$	2	$(2, 2)$
9	$(1^3 01^2 0)^\infty$	2	$(3, 1)$
130	$(1^7 01^6 0)^\infty$	2	$(7, 1)$
258	$(1^8 01^7 0)^\infty$	2	$(8, 1)$
2051	$(1^{11} 01^{10} 0)^\infty$	2	$(11, 1)$
4099	$(1^{12} 01^{11} 0)^\infty$	2	$(12, 1)$
32772	$(1^{15} 01^{14} 0)^\infty$	2	$(15, 1)$

Now we need two definitions. The *quasi-greedy* expansion of a real number x in some base q is its lexicographically largest infinite expansion in the alphabet $\{0, 1, m\}$, while the *quasi-lazy* expansion of x is the *conjugate* $(m - c_i)$ of the quasi-greedy expansion (c_i) of $\frac{m}{q-1} - x$ with respect to the conjugate alphabet $\{0, m-1, m\}$. The following lemma follows at once from these definitions.

Lemma 5.7. *Let (c_i) be a sequence on the alphabet $\{0, 1, m\}$ and $q > 1$ a real number.*

(a) *The sequence (c_i) is a quasi-greedy expansion of some x in base q if and only if*

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} \le 1 \qquad \text{whenever } c_n = 0,$$
$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} \le m-1 \qquad \text{whenever } c_n = 1.$$

Hence, if $c = (c_i)$ is a quasi-greedy expansion in base q, then $m^n c$ and (c_{n+i}) are also quasi-greedy expansions in every base $\ge q$, for every positive integer n.

(b) *The sequence (c_i) is a quasi-lazy expansion of some x in base q if and only if*

$$\sum_{i=1}^{\infty} \frac{m - c_{n+i}}{q^i} \le 1 \qquad \text{whenever } c_n = 1,$$

$$\sum_{i=1}^{\infty} \frac{m - c_{n+i}}{q^i} \le m - 1 \quad \text{whenever } c_n = m.$$

Hence, if $c = (c_i)$ is a quasi-lazy expansion in base q, then $0^n c$ and (c_{n+i}) are also quasi-lazy expansions in every base $\ge q$, for every positive integer n.

(c) *If $x \ge y$ and $p \ge q$, then the quasi-greedy (resp. the quasi-lazy) expansion of x in base p is lexicographically larger than or equal to that of y in base q.*

Lemma 5.8. *Given an admissible sequence $d \ne 1^\infty$ and $m \in I_d$ define the sequences d', δ, δ' and the numbers p'_m, p''_m, p_m as at the beginning of Section* 4.

(a) *The sequences δ and $m\delta'$ are quasi-greedy in base p_m.*

(b) *The sequences δ' and (δ_{1+i}) are quasi-lazy in base p_m.*

Proof. (a) Using the admissibility of d and applying Lemma 4.2(b) with $(c_i) := \delta$ and $q := p_m \ge 2$ on the alphabet $\{1, m\}$ we obtain

$$\sum_{i=1}^{\infty} \frac{\delta_{n+i}}{p_m^i} \le \sum_{i=1}^{\infty} \frac{\delta_i}{p_m^i} \quad \text{for all } n.$$

Since $p_m \ge p'_m$ and

$$\sum_{i=1}^{\infty} \frac{\delta_i}{(p'_m)^i} = m - 1,$$

it follows that

$$\sum_{i=1}^{\infty} \frac{\delta_{n+i}}{p_m^i} \le m - 1 \quad \text{for all } n.$$

Applying Lemma 5.7(a) we conclude that δ is a quasi-greedy expansion in base p_m. The same inequalities ensure that $m\delta'$ is also a quasi-greedy expansion in base p_m.

(b) Since $(\delta_{1+i}) = (\delta'_{k+i})$ for some $k \ge 0$, in view of Lemma 5.7(b) it suffices to show that

$$\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \le 1 \qquad \text{whenever } \delta'_n = 1,$$

$$\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \le m - 1 \quad \text{whenever } \delta'_n = m.$$

If $\delta'_n = 1$, then applying Lemma 3.9 and Lemma 4.2(b) with $(c_i) := \delta'$ and $q := p_m \ge 2$ on the alphabet $\{1, m\}$ we obtain

$$\sum_{i=1}^{\infty} \frac{\delta'_{n+i}}{p_m^i} \ge \sum_{i=1}^{\infty} \frac{\delta'_i}{p_m^i}.$$

Using the definition of p''_m and the inequality $p_m \geq p''_m$, the first property follows:

$$\sum_{i=1}^{\infty} \frac{m-\delta'_{n+i}}{p_m^i} \leq \sum_{i=1}^{\infty} \frac{m-\delta'_i}{p_m^i} \leq \sum_{i=1}^{\infty} \frac{m-\delta'_i}{(p''_m)^i} = 1.$$

If $\delta'_n = m$, then let k be the smallest positive integer satisfying $\delta'_{n+k} = 1$. Applying the first property and the inequalities $p_m \geq 2 \geq \frac{m}{m-1}$, the second property follows:

$$\sum_{i=1}^{\infty} \frac{m-\delta'_{n+i}}{p_m^i} \leq \frac{m-1}{p_m^k} + \frac{1}{p_m^k} \cdot 1 = \frac{m}{p_m^k} \leq \frac{m}{2^k} \leq \frac{m}{2} \leq m-1. \qquad \square$$

Remark 5.9. Applying Lemma 4.2(a) instead of (b) we may obtain the stronger result that δ and $m\delta'$ are quasi-greedy expansions in every base $q \geq p'_m$.

Lemma 5.10. *Denoting by $\gamma = (\gamma_i)$ and $\lambda = (\lambda_i)$ the quasi-greedy expansion of $m-1$ in base p_m and the quasi-lazy expansion of $\frac{m}{p_m-1} - 1$ in base p_m, respectively, we have either*

$$(\delta_{1+i}) \leq \lambda \quad \text{and} \quad \gamma = \delta$$

or

$$\delta' = \lambda \quad \text{and} \quad \gamma \leq m\delta'.$$

Proof. If $p'_m \geq p''_m$, then both γ and δ are quasi-greedy expansions of $m-1$ in base $p_m = p'_m$ by Lemma 5.8, so that $\gamma = \delta$. Since furthermore both $\hat{\delta} := (\delta_{1+i})$ and λ are quasi-lazy expansions in base p_m, in view of Lemma 5.7 it remains to show that $\pi_{p_m}(\hat{\delta}) \leq \pi_{p_m}(\lambda)$. Since

$$m-1 = \pi_{p_m}(\delta) = \frac{m}{p_m} + \frac{1}{p_m}\pi_{p_m}(\hat{\delta})$$

and $p_m \leq P_m$, using (4.5) we have

$$\pi_{p_m}(\hat{\delta}) = (m-1)p_m - m \leq \frac{m}{p_m-1} - 1 = \pi_{p_m}(\lambda).$$

If $p''_m \geq p'_m$, then both λ and δ' are quasi-lazy expansions of $\frac{m}{p_m-1} - 1$ in base $p_m = p''_m$ by Lemma 5.8, so that $\lambda = \delta'$. Furthermore $m\delta'$ and γ are quasi-greedy expansions in base p_m. Since $p_m \leq P_m$, using (4.4) we obtain

$$\begin{aligned}\pi_{p_m}(m\delta') &= \frac{m}{p_m} + \frac{1}{p_m}\pi_{p_m}(\delta') = \frac{m}{p_m} + \frac{1}{p_m}\left(\frac{m}{p_m-1} - 1\right) \\ &\geq m-1 = \pi_{p_m}(\gamma).\end{aligned}$$

Applying Lemma 5.7 we conclude that $m\delta' \geq \gamma$. $\square$

Given $m \geq 2$ we choose an admissible sequence $d \neq 1^\infty$ satisfying $m \in I_d$ (see Lemma 5.4) and we define p_m as at the beginning of Section 4 (see Lemma 5.8). The following lemma proves Theorem 1.1(a).

Lemma 5.11. (a) *If $q > p_m$, then δ' is a nontrivial univoque sequence in base q.*
(b) *There are no nontrivial univoque sequences in any base $1 < q < p_m$.*

Proof. (a) Since the sequence δ is quasi-greedy and the sequence δ' is quasi-lazy in base p_m and since δ' is obtained from δ by removing a finite initial block, δ' is both quasi-greedy and quasi-lazy in base p_m. Hence

$$\sum_{i=1}^{\infty} \frac{\delta'_{n+i}}{p_m^i} \le m-1 \qquad \text{whenever } \delta'_n = 1,$$
$$\sum_{i=1}^{\infty} \frac{m-\delta'_{n+i}}{p_m^i} \le 1 \qquad \text{whenever } \delta'_n = 1,$$
$$\sum_{i=1}^{\infty} \frac{m-\delta'_{n+i}}{p_m^i} \le m-1 \qquad \text{whenever } \delta'_n = m.$$

Since $q > p_m$, it follows that

$$\sum_{i=1}^{\infty} \frac{\delta'_{n+i}}{q^i} < m-1 \qquad \text{whenever } \delta'_n = 1,$$
$$\sum_{i=1}^{\infty} \frac{m-\delta'_{n+i}}{q^i} < 1 \qquad \text{whenever } \delta'_n = 1,$$
$$\sum_{i=1}^{\infty} \frac{m-\delta'_{n+i}}{q^i} < m-1 \qquad \text{whenever } \delta'_n = m.$$

Applying Lemma 5.1 we conclude that δ' is a univoque sequence in base q.

(b) Assume first that d is of finite type and assume on the contrary that there exists a nontrivial univoque sequence in some base $1 < q \le p_m$. Since a univoque sequence remains univoque in every greater base and since a univoque sequence remains univoque if we remove an arbitrary finite initial block, by Lemma 5.3 it follows that there exists a univoque sequence (η_i) in base p_m ($\le P_m$) which contains only the digits 1 and m.

It follows from the lexicographic characterization of univoque sequences that

$$\eta_n = 1 \;\Rightarrow\; (\lambda_i) < (\eta_{n+i}) < (\gamma_i)$$

and therefore (using the preceding lemma) that either

$$\eta_n = 1 \;\Rightarrow\; (\delta_{1+i}) < (\eta_{n+i}) < (\delta_i)$$

or

$$\eta_n = 1 \;\Rightarrow\; (\delta'_i) < (\eta_{n+i}) < m(\delta_i).$$

Setting $c_i = 0$ if $\eta_i = 1$ and $c_i = 1$ if $\eta_i = m$ we obtain a sequence (c_i) of zeroes and ones satisfying either

$$(d_{1+i}) < (c_{n+i}) < (d_i) \qquad \text{whenever } c_n = 0 \tag{5.8}$$

or

$$(d_i') < (c_{n+i}) < 1(d_i') \quad \text{whenever } c_n = 0. \tag{5.9}$$

The second inequalities imply that (c_i) has infinitely many zero digits. By removing a finite initial block if necessary we obtain a new sequence (still denoted by (c_i)) which begins with $c_1 = 0$ and which satisfies (5.8) or (5.9).

In the case of (5.8) we claim that

$$0d_2d_3\cdots < (c_{n+i}) < (d_i) \quad \text{for all } n \geq 0. \tag{5.10}$$

Indeed, if $c_n = 1$ for some n then there exist $m < n \leq M$ such that $c_m = c_{M+1} = 0$ and $c_{m+1} = \cdots = c_M = 1$. Using (5.8) it follows that

$$(c_{n+i}) \leq (c_{m+i}) < (d_i),$$
$$(c_{n+i}) \geq (c_{M+i}) = 0(c_{M+1+i}) > 0(d_{1+i}) = 0d_2d_3\cdots.$$

However, (5.10) contradicts Lemma 3.11.

In the case of (5.9) we claim that

$$0(d_i') < (c_{n+i}) < 1(d_i') \quad \text{for all } n \geq 0. \tag{5.11}$$

Indeed, if $c_n = 1$ for some n then choosing again $m < n \leq M$ such that $c_m = c_{M+1} = 0$ and $c_{m+1} = \cdots = c_M = 1$, we have

$$(c_{n+i}) \leq (c_{m+i}) < 1(d_i'),$$
$$(c_{n+i}) \geq (c_{M+i}) = 0(c_{M+1+i}) > 0(d_i').$$

However, (5.11) contradicts Lemma 3.12.

Now assume that d is of infinite type, associated with an infinite sequence $h = (h_1, h_2, \ldots)$, and that there exists a nontrivial univoque sequence (η_i) in some base $1 < q < p_m$. (Note that $m > 2$.) We will then prove the existence of a nontrivial univoque sequence in some base $1 < q' < p_{m'}$ where $m' \in I_{d'}$ with d' of finite type, contradicting what we have already established. (In this part of the proof, d' does not mean the sequence defined in Lemma 3.9.)

We may assume again that $\eta_i \equiv 1 + (m-1)c_i$ for some sequence $(c_i) \subset \{0, 1\}$. By Lemma 5.1 we have

$$\frac{m}{q-1} - 1 < \pi_q((\eta_{n+i})) < m - 1 \quad \text{whenever } \eta_n = 1$$

and

$$\pi_q((\eta_{n+i})) > \frac{m}{q-1} - (m-1) \quad \text{whenever } \eta_n = m.$$

These may be rewritten in the following equivalent form:

$$\pi_q((c_{n+i})) < 1 - \frac{1}{(q-1)(m-1)} \quad \text{whenever } c_n = 0;$$
$$\pi_q((1-c_{n+i})) < \frac{1}{m-1} \quad \text{whenever } c_n = 0;$$
$$\pi_q((1-c_{n+i})) < 1 \quad \text{whenever } c_n = 1.$$

If $2 < m' < m$ and q' is defined by the equation $(q'-1)(m'-1) = (q-1)(m-1)$, then $q' > q$, so that the above three conditions remain valid on changing q to q' and m to m'. (Observe that the left sides decrease and the right sides increase.) Applying Lemma 5.1 again we conclude that the formula $\eta'_i := 1 + (m'-1)c_i$ defines a nontrivial univoque sequence in base q' for the alphabet $\{0, 1, m'\}$. To end the proof it remains to show that we can choose m' such that $1 < q' < p_{m'}$ and $m' \in I_{d'}$ for some d' of finite type. Thanks to the continuity of the maps $m' \mapsto q'$ and $m' \mapsto p_{m'}$ the first condition is satisfied for all m' sufficiently close to m.

If $h = (h_1, h_2, \ldots)$ contains infinitely many elements $h_j \geq 2$, then we may choose d' associated with the finite sequence $h = (h_1, h_2, \ldots, h_{j-1}, h_j - 1)$ for a sufficiently large index j such that $h_j \geq 2$, and an arbitrary element $m' \in I_{d'}$. If $h = (h_1, h_2, \ldots)$ has a last element $h_j \geq 2$, then m is the right endpoint of the interval $I_{d'}$ for d' associated with the finite sequence $h = (h_1, h_2, \ldots, h_{j-1}, h_j - 1)$ (see Example 3.10), and we may choose $m' \in I_{d'}$ sufficiently close to m. The only remaining case $h = (1, 1, \ldots)$ is similar: m is the right endpoint of the interval $I_{d'}$ for $d' = 0^\infty$, and we may choose $m' \in I_{d'}$ sufficiently close to m. (See Example 3.10 again.) □

Remark 5.12. If d is of finite type and $m \in I_d$, then the first part of the proof of Lemma 5.11(b) shows that there are no nontrivial univoque sequences in base $q = p_m$ either. This is also true for $m \in I_{\hat{d}}$ where $\hat{d}$ is the smallest admissible sequence of infinite type, following an admissible sequence d of finite type (see Example 3.10). Indeed, since $(\hat{d}_{1+i}) = (\hat{d})' = d'$ we may apply Lemma 3.12 in the first part of the proof of Lemma 5.11(b).

In the other cases, i.e., when m does not belong to $[m_d, M_d]$ for any d of finite type, δ' is a nontrivial univoque sequence in base $q = p_m = P_m$. Indeed, in this case the first three inequalities in the proof of Lemma 5.11(a) are strict. For otherwise we would have two different quasi-greedy expansions (δ' and (δ'_{n+i})) of $m-1$, $\frac{m}{p_m-1}-1$ or $\frac{m}{p_m-1}-(m-1)$ in base p_m.

The following lemma completes the proof of Theorem 1.1.

Lemma 5.13. (a) *If $d < \tilde{d} < 1^\infty$ are admissible sequences, then $M_d \leq m_{\tilde{d}}$ with equality if and only if $d = S(N, 1)^\infty$ is of finite type and $\tilde{d} = S(N-1, 1)S(N, 1)^\infty$.*

(b) *The sets I_d, where d runs over all admissible sequences $d \neq 1^\infty$, form a partition of the interval $[(1+\sqrt{5})/2, \infty)$.*

(c) *The set C of numbers $m > (1+\sqrt{5})/2$ satisfying $p_m = P_m$ is a Cantor set, i.e., a nonempty closed set having neither interior, nor isolated points. Its smallest element* *is* $1 + x \approx 2.3247$ *where* x *is the first Pisot number, i.e., the positive root of the equation* $x^3 = x + 1$.

Proof. (a) If d and $\tilde{d}$ are of infinite type, then $m_d = M_d$ and $m_{\tilde{d}} = M_{\tilde{d}}$, so that it suffices to prove the inequality $M_d < M_{\tilde{d}}$. For this, it is sufficient to show that $p''_{d,m} > p''_{\tilde{d},m}$ for each $m \in (1, \infty)$ where $p''_{d,m}$ and $p''_{\tilde{d},m}$ denote the expressions p''_m of Section 4 for the admissible sequences d and $\tilde{d}$, respectively. Indeed, then we can conclude that $p''_{d,M_{\tilde{d}}} > p''_{\tilde{d},M_{\tilde{d}}} = P_{M_{\tilde{d}}}$ and therefore, since the function $m \mapsto p''_{d,m} - P_m$ is strictly increasing by Lemma 4.1, $M_d < M_{\tilde{d}}$.

Assuming on the contrary that $p''_{d,m} \le p''_{\tilde{d},m}$ for some m, in base $q := p''_{\tilde{d},m}$ we have

$$\pi_q(m - \tilde{\delta}') = 1 = \pi_{p''_{d,m}}(m - \delta') \ge \pi_q(m - \delta') \ \Rightarrow\ \pi_q(\delta') \ge \pi_q(\tilde{\delta}').$$

Since d and $\tilde{d}$ are of infinite type, we have $\delta = m\delta'$ and $\tilde{\delta} = m\tilde{\delta}'$ by Lemma 3.9, so that the last inequality is equivalent to $\pi_q(\delta) \ge \pi_q(\tilde{\delta})$.

Since quasi-greedy expansions remain quasi-greedy in larger bases, it follows from Lemma 5.8 that both δ and $\tilde{\delta}$ are quasi-greedy expansions in base q. Therefore we deduce from the last inequality that $\delta \ge \tilde{\delta}$, contradicting our assumption.

If $d = S(N, 1)^\infty$ is of finite type and $\tilde{d}$ of infinite type, then we recall from Example 3.10 that $\hat{d} = S(N-1, 1)S(N, 1)^\infty$ is the smallest admissible sequence satisfying $\hat{d} > d$, and that $m_d < M_d = m_{\hat{d}} = M_{\hat{d}}$. Since $\hat{d}$ is of infinite type, we conclude that $M_d = M_{\hat{d}} < M_{\tilde{d}} = m_{\tilde{d}}$. The case of $d = 0^\infty$ is similar with $\hat{d} = 10^\infty$.

If d is arbitrary and $\tilde{d}$ of finite type, then $\tilde{d}$ is associated with a finite sequence $(h_1, \dots, h_N)$ of length $N \ge 1$. If k is a sufficiently large positive integer, then the admissible sequence d^k associated with the infinite sequence $(h_1, \dots, h_N, k, 1, 1, \dots)$ satisfies $d < d^k < \tilde{d}$, so that $M_d \le m_{d^k}$. Letting $k \to \infty$ we conclude that $M_d \le m_{\tilde{d}}$. Indeed, for any fixed $m < m_{\tilde{d}}$ we have $p'_{\tilde{d},m} - P_m > 0$ by Lemma 4.1(b) and therefore $\pi_{P_m}(\tilde{\delta}) > m - 1$. Since the first k digits of $\tilde{\delta}$ and δ^k coincide, for $k \to \infty$ we have

$$\pi_{P_m}(\delta^k) = \sum_{i=1}^{k} \frac{\tilde{\delta}_i}{P_m^i} + \sum_{i=k+1}^{\infty} \frac{\delta_i^k}{P_m^i} = \sum_{i=1}^{k} \frac{\tilde{\delta}_i}{P_m^i} + O\left(\frac{1}{P_m^k}\right) \to \pi_{P_m}(\tilde{\delta}),$$

so that $\pi_{P_m}(\delta^k) > m - 1$ if k is sufficiently large. Hence $p'_{d^k,m} > P_m$ and therefore $m < m_{d^k}$ by Lemma 4.1(b). Similarly, for any fixed $m > m_{\tilde{d}}$ we have $m > m_{d^k}$ for all sufficiently large k.

(b) The sets I_d are disjoint by (a) and they cover the interval $[(1 + \sqrt{5})/2, \infty)$ by Lemma 5.4. In view of (a) the proof will be completed if we show that for the smallest admissible sequence we have

$$I_{0^\infty} = [(1 + \sqrt{5})/2, 1 + P_1) \tag{5.12}$$

where $x > 1$ is the first Pisot number.

The values m_d and M_d are the solutions of the equations

$$\pi_{P_m}(\delta) = m - 1 \quad \text{and} \quad \pi_{P_m}(\delta') = \frac{m}{P_m - 1} - 1.$$

Now we have $\delta = \delta' = 1^\infty$, so that our equations take the form

$$\frac{1}{P_m - 1} = m - 1 \quad \text{and} \quad \frac{1}{P_m - 1} = \frac{m}{P_m - 1} - 1.$$

Using (4.3) we conclude that they are equivalent to $m = (1 + \sqrt{5})/2$ and $m = 1 + P_1$, respectively.

(c) If we denote by D_1 and D_2 the set of admissible sequences $d \neq 1^\infty$ of finite and infinite type, respectively, then

$$C = [2, \infty) \setminus \bigcup_{d \in D_1} (m_d, M_d)$$

so that C is a closed set. The relation (5.12) shows that its smallest element is $1 + P_1$. In order to prove that it is a Cantor set, it suffices to show that

- the intervals $[m_d, M_d]$ $(d \in D_1)$ are disjoint;
- for each $m \in C$ there exist two sequences $(a_N) \subset [2, \infty) \setminus C$ and $(b_N) \subset C \setminus \{m\}$, both converging to m.

The first property follows from (a). To prove the second, consider the infinite sequence $h = (h_j)$ of positive integers defining the admissible sequence d for which $m_d = m$, and set $d_N := S_h(N, 1)^\infty$, $N = 1, 2, \ldots$. This is a decreasing sequence of admissible sequences, converging pointwise to d. Using (a) we conclude that both (m_{d_N}) and (M_{d_N}) converge to $m_d = M_d$. Since $m_{d_N} \in D_1$ and $M_{d_N} \in D_2$ for every N, the proof is complete. □

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References

[1] Daróczy, Z., Kátai, I.: Univoque sequences. Publ. Math. Debrecen **42**, 397–407 (1993) Zbl 0809.11008 MR 1229687

[2] de Vries, M.: A property of algebraic univoque numbers. Acta Math. Hungar. **119**, 57–62 (2008) Zbl 1164.11008 MR 2400796

[3] de Vries, M.: On the number of unique expansions in non-integer bases. Topology Appl. **156**, 652–657 (2009) Zbl 1160.11002 MR 2492313

[4] de Vries, M., Komornik, V.: Unique expansions of real numbers. Adv. Math. **221**, 390–427 (2009) Zbl 1166.11007 MR 2508926

[5] Erdős, P., Horváth, M., Joó, I.: On the uniqueness of the expansions $1 = \sum q^{-n_i}$. Acta Math. Hungar. **58**, 333–342 (1991) Zbl 0747.11005 MR 1153488

[6] Erdős, P., Joó, I., Komornik, V.: Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. Bull. Soc. Math. France **118**, 377–390 (1990) Zbl 0721.11005 MR 1078082

[7] Glendinning, P., Sidorov, N.: Unique representations of real numbers in non-integer bases. Math. Res. Lett. **8**, 535–543 (2001) Zbl 1002.11009 MR 1851269

[8] Komornik, V., Loreti, P.: Unique developments in non-integer bases. Amer. Math. Monthly **105**, 636–639 (1998) Zbl 0918.11006 MR 1633077

[9] Komornik, V., Loreti, P.: Subexpansions, superexpansions and uniqueness properties in non-integer bases. Period. Math. Hungar. **44**, 197–218 (2002) Zbl 1017.11008 MR 1918687

[10] Komornik, V., Loreti, P.: On the topological structure of univoque sets. J. Number Theory **122**, 157–183 (2007) Zbl 1111.11005 MR 2287118

[11] Parry, W.: On the β-expansions of real numbers. Acta Math. Acad. Sci. Hungar. **11**, 401–416 (1960) Zbl 0099.28103 MR 0142719

[12] Pedicini, M.: Greedy expansions and sets with deleted digits. Theoret. Comput. Sci. **332**, 313–336 (2005) Zbl 1080.11009 MR 2122508

[13] Rényi, A.: Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hungar. **8**, 477–493 (1957) Zbl 0079.08901 MR 0097374