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On Uryson Operators with Partial Integrals in Lebesgue Spaces with Mixed Norm

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Abstract. In this paper we consider some class of partial Uryson integral operators in spaces with mixed norm. We give some conditions for action, boundedness and continuity of those operators.

Keywords: *Nonlinear Uryson operators, partial integral operators, acting conditions, boundedness conditions, continuity conditions*

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1. Introduction

Let $T \subset \mathbb{R}^n$ and $S \subset \mathbb{R}^m$ be two compact sets with Lebesgue measure, $D = T \times S$, $a_i : D \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3)$ given Carathéodory functions, and

$$A = A_1 + A_2 + A_3,$$

where

$$(A_1 x)(t,s) = \int_T a_1\big(t,s,x(\tau,s)\big)\,d\tau \tag{1}$$

$$(A_2 x)(t,s) = \int_S a_2\big(t,s,x(t,\sigma)\big)\,d\sigma \tag{2}$$

$$(A_3 x)(t,s) = \iint_D a_3\big(t,s,x(\tau,\sigma)\big)\,d\tau d\sigma. \tag{3}$$

The operators A, A_1, and A_2 are so called *partial Uryson integral operators*, which have been studied in $C(D)$, in spaces with mixed quasinorm $L^\alpha[L^\beta]$, and in quasi-Banach ideal spaces (see [1, 4, 8], respectively). The properties of partial Uryson integral operators essentially differ from those of ordinary Uryson integral operators. For example, the operator A_1 with kernel $a_1(t,s,u) \equiv u$ is not completely continuous in $L^p(D)$, but the operator A_3 is completely continuous for $a_3(t,s,u) \equiv u$.

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We remark that linear and nonlinear operators with partial integrals have applications in problems of continuum mechanics, of the theory of transfer, of differential and integro-differential equations and other questions.

In this paper the operator A is studied in *Lebesgue spaces with mixed norm*

$$L^q(T)[L^p(S)] \qquad (1 \le p, q \le \infty).$$

More general classes of partial Uryson integral operators in spaces of summable functions have been studied in [3]. Action, boundedness, and continuity criteria of the operator A_3 in Lebesgue spaces have been obtained by Ojnarov [6].

2. Action, boundedness and continuity

Let $M(D)$ be the space of all real measurable almost everywhere finite functions on D. We denote by $L^q(T)[L^p(S)]$ $(1 \le p, q \le \infty)$ the set of functions $x \in M(D)$ for which the norm

$$\|x\|_{L^q(T)[L^p(S)]} = \left\|t \mapsto \|x(t,\cdot)\|_{L^p(S)}\right\|_{L^q(T)}$$

is defined and finite [2]. These spaces are Banach spaces. Of course, in case $p = q$ we have $L^p(T)[L^p(S)] \cong L^p(T \times S)$. Let $X(x_0, r)$ denote the closed ball of radius r with center x_0 in the space X.

The operators A_1, A_2, and A_3 are defined on functions $x \in M(D)$ for which the functions

$$\begin{aligned}
(t,s,\tau) &\longmapsto a_1\big(t,s,x(\tau,s)\big)\\
(t,s,\sigma) &\longmapsto a_2\big(t,s,x(t,\sigma)\big)\\
(t,s,\tau,\sigma) &\longmapsto a_3\big(t,s,x(\tau,\sigma)\big)
\end{aligned}$$

are summable in the variables τ, σ, and (τ,σ), respectively, for almost all $(t,s) \in D$. Let $D(A)$ be the domain of definition of the operator $A = A_1 + A_2 + A_3$. If $a_1(t,s,0) = a_2(t,s,0) = a_3(t,s,0) = 0$, and $x_1, \ldots, x_n \in D(A)$ are functions with disjoint supports, then $x_1 + \ldots + x_n \in D(A)$ and the operator A is *partially additive*, i.e.,

$$A(x_1 + \ldots + x_n) = Ax_1 + \ldots + Ax_n.$$

In general, the operator $x \mapsto A(x + x_0) - Ax_0$ is partially additive for fixed $x_0 \in D(A)$.

By the partial additivity of A we have the following statement.

Theorem 1. *Let $X = L^q(T)[L^p(S)]$ $(1 \le p, q < \infty)$. Suppose that the operator A acts from $X(x_0, r)$ into $Y = L^\beta(T)[L^\alpha(S)]$ $(1 \le \alpha, \beta \le \infty)$. Then A acts from X into Y and is bounded (i.e., A is bounded on any bounded set). Moreover, A is continuous on X if A is continuous on $X(x_0, r)$.*

By Theorem 1 the boundedness of the operator A follows direct from its action.

The next theorem concerning acting conditions (both sufficient and necessary) for the operator A_3 may be obtained following the idea of [6].

Theorem 2. *The operator A_3 acts from $X = L^q(T)[L^p(S)]$ $(1 \le p, q < \infty)$ into $Y = L^\beta(T)[L^\alpha(S)]$ $(1 \le \alpha, \beta \le \infty)$ if and only if, for any $u \in \mathbb{R}$,*

$$\|a_3(\cdot,\cdot,u)\|_Y \le a\,|u|^{\min\{p,q\}} + b, \tag{4}$$

where a and b are non-negative constants.

Proof. Without loss of generality, we assume that $\text{mes}T = \text{mes}S = 1$. Suppose that condition (4) holds. Then for any $x \in X$ the Hölder and Minkowski inequalities imply that

$$\begin{aligned}
\|A_3 x\|_Y &= \left\| \iint_D a_3(\cdot,\cdot,x(\tau,\sigma))\,d\tau\,d\sigma \right\|_Y \\
&\le \iint_D \|a_3(\cdot,\cdot,x(\tau,\sigma))\|_Y\,d\tau\,d\sigma \\
&\le \iint_D \big(a\,|x(\tau,\sigma)|^{\min\{p,q\}} + b\big)\,d\tau\,d\sigma \\
&\le a\,\|x\|_X^{\min\{p,q\}} + b.
\end{aligned}$$

Hence, A_3 acts from X into Y.

Conversely, suppose that the operator A_3 acts from X into Y. Then, by Theorem 1, there exists a number $b > 0$ such that $\|A_3 x\|_Y \le b$ if $\|x\|_X \le 1$. Let $u \in \mathbb{R}$ and $x_u \equiv u \in X$. If $|u| \le 1$, it is clear that

$$\|a_3(\cdot,\cdot,u)\|_Y = \|A_3 x_u\|_Y \le b \le b\big(|u|^{\min\{p,q\}} + 1\big). \tag{5}$$

If $|u| > 1$, we define a function $\bar{x}_u$ on D by

$$\bar{x}_u(t,s) = \begin{cases} u\chi_{T\times S_u}(t,s) & \text{if } p \le q \\ u\chi_{T_u\times S}(t,s) & \text{if } p > q \end{cases}$$

where S_u is a measurable subset of S with $\text{mes}S_u = |u|^{-p}$ and T_u is a measurable subset of T with $\text{mes}T_u = |u|^{-q}$. Here, $\chi_{T\times S_u}$ and $\chi_{T_u\times S}$ denote the characteristic functions of $T \times S_u$ and $T_u \times S$, respectively. Then $\|\bar{x}_u\|_X = 1$ and

$$\left\| |u|^{-\min\{p,q\}} a_3(\cdot,\cdot,u) + \big(1 - |u|^{-\min\{p,q\}}\big) a_3(\cdot,\cdot,0) \right\|_Y = \|A_3 \bar{x}_u\|_Y \le b.$$

Hence,

$$\|a_3(\cdot,\cdot,u)\|_Y \le b\,|u|^{\min\{p,q\}} + |u|^{\min\{p,q\}} \|a_3(\cdot,\cdot,0)\|_Y \le 2b\,|u|^{\min\{p,q\}}. \tag{6}$$

From (5) and (6) it follows that condition (4) holds ∎

Some acting conditions for the operators A_1 and A_2 in spaces of summable functions have been given in [3]. We will give simple acting conditions (only sufficient) in the next lemma.

Lemma. *Let $1 \le p, q, \alpha, \beta < \infty$, $X = L^q(T)[L^p(S)]$, and $Y = L^\beta(T)[L^\alpha(S)]$. The operators A_1 and A_2 act from X into Y if the kernels a_1 and a_2 satisfy growth conditions of the form*

$$|a_i(t,s,u)| \le c_i\,|u|^{\min\{p,q\}/\rho_i(\alpha,\beta)} + b_i(t,s) \qquad (i=1,2) \tag{7}$$

for some $b_1, b_2 \in Y$ and $c_1, c_2 \ge 0$, where $\rho_1(\alpha,\beta) = \alpha$ and $\rho_2(\alpha,\beta) = \beta$. Moreover, in this case A_1 and A_2 are bounded and continuous.

Proof. It is easy to show the first statement by the Hölder and Minkowski inequalities. The continuity of A_1 and A_2 follows from the principle of majorants [8] ∎

We note that the growth condition (7) is not necessary for the action of A_1 (resp. A_2). Moreover, there exists A_1 acting from X into Y (whence A_1 is even bounded), which is not continuous. In particular, the corresponding kernel a_1 does not satisfy the growth condition (7) (by the previous lemma).

The following example is essentially due to P. P. Zabrejko [5].

Example. Let $D = [0,1]\times[0,1]$, $X = L^q(T)[L^p(S)]$, and $Y = L^\beta(D)$ $(1 \le p, q, \beta < \infty)$. Let $z_n(t,s) = z_n(t) \ge 0$ have disjoint support, and $\|z_n\|_Y = 1$. Define the kernel a_1 on $D \times \mathbb{R}$ by

$$a_1(t,s,u) = \begin{cases} (2^n|u|-1)z_{n-1}(t) + (2-2^n|u|)z_n(t) & \text{if } 2^{-n} \le |u| < 2^{1-n} \\ 0 & \text{if } u = 0 \text{ or } |u| \ge 1. \end{cases}$$

Then the kernel a_1 is a non-negative Carathéodory function, and the operator A_1 acts from X into Y and is bounded (it even has bounded range): Indeed, by Minkowski's inequality we have for any measurable x

$$\begin{aligned}
\|A_1 x\|_Y^\beta &= \int_0^1 \left\| \int_0^1 a_1\big(\cdot, s, x(\tau,s)\big)\,d\tau \right\|_{L^\beta}^\beta ds \\
&\le \int_0^1 \left(\int_0^1 \|a_1\big(\cdot, s, x(\tau,s)\big)\|_{L^\beta}\,d\tau \right)^\beta ds \\
&\le \int_0^1 \left(\int_0^1 1\,d\tau \right)^\beta ds \\
&\le 1.
\end{aligned}$$

However, A_1 is not continuous, since it maps the convergent sequence $(x_n) = (2^{-n})$ into the non-compact sequence $(A_1 x_n) = (z_n)$.

The kernel a_1 not only fails to satisfy the growth condition (7). Even more, a_1 does not satisfy

$$|a_1(t,s,u)| \le c\,|u|^\gamma + b(t,s) \tag{8}$$

for fixed $c, \gamma > 0$ and $b \in Y$. Indeed, for $u_n = 2^{-n}$, (8) would imply $z_n(t,s) = a_1(t,s,u_n) \le c + b(t,s)$, whence $d(t,s) = b(t,s) + c$ satisfies $d \ge z_n$ for all n, which obviously is not possible, since $d \in Y$.

The continuity of the operator A_3 does not follow from its action and boundedness as is shown by the previous example (consider $a_3 = a_1$).

To discuss continuity conditions for the operator A_3, we apply the following theorem. Recall that a set $G \subset X$ is *absolutely bounded* if $\sup\{\|\chi_\Omega x\|_X : x \in G\} \to 0$ as $\mathrm{mes}\Omega \to 0$.

Theorem 3. *Let $1 \leq p, q, \alpha, \beta < \infty$, $X = L^q(T)[L^p(S)]$ and $Y = L^\beta(T)[L^\alpha(S)]$. Suppose that, for each function $x \in X$,*

$$\left\| \iint_D |a_3(\cdot,\cdot,x(\tau,\sigma))|\,d\tau d\sigma \right\|_Y < \infty.$$

Then the operator A_3 acts from X into Y. Moreover, for each absolutely bounded set $G \subset X$ and for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality

$$\sup_{x\in G} \left\| \iint_{D_1} |a_3(\cdot,\cdot,x(\tau,\sigma))|\,d\tau d\sigma \right\|_Y < \varepsilon$$

holds whenever $D_1 \subset D$ satisfies $\mathrm{mes}D_1 < \delta$.

Proof. It is analogous to that of [5: Theorem 18.4] ∎

Theorem 4 (see [7]). *Let $1 \leq p, q, \alpha, \beta < \infty$. The operator A_3 acts from $X = L^q(T)[L^p(S)]$ into $Y = L^\beta(T)[L^\alpha(S)]$ and is continuous if and only if condition* (4) *holds and*

$$\lim_{u\to u_0} \|a_3(\cdot,\cdot,u) - a_3(\cdot,\cdot,u_0)\|_Y = 0 \tag{9}$$

for any $u_0 \in \mathbb{R}$.

Proof. Without loss of generality, assume $a_3(t,s,0) = 0$ and $\mathrm{mes}T = \mathrm{mes}S = 1$. Suppose that the operator A_3 acts from X into Y and is continuous. Then condition (4) holds by Theorem 2. Putting $x \equiv u$ and $x_0 \equiv u_0$, we have $A_3x = a_3(\cdot,\cdot,u)$ and $A_3x_0 = a_3(\cdot,\cdot,u_0)$. Thus the continuity of A_3 implies (9).

Conversely, suppose that conditions (4) and (9) hold. Then the operator A_3 acts from X into Y by Theorem 2. Assume that A_3 is not continuous. This means that there exist a sequence (x_n) converging to a function x_0 in X and a number $\varepsilon_0 > 0$ such that

$$\|A_3x_n - A_3x_0\|_Y \geq \varepsilon_0 \qquad (n \in \mathbb{N}). \tag{10}$$

Since $x_n \to x_0$ in X, the set $\{x_0, x_1, x_2, \ldots\}$ is absolutely bounded. Hence, by Theorem 3 there is a number $\delta > 0$ such that the inequalities

$$\|A_3(\chi_F x_n)\|_Y < \frac{\varepsilon_0}{3} \qquad (n \geq 0) \tag{11}$$

hold whenever $F \subset D$ satisfies $\mathrm{mes}F < \delta$. Let $c = \sup_{n\geq 0}\|x_n\|_X$, $N = c\left(\frac{\delta}{3}\right)^{-1/\min\{p,q\}}$, and $D_n^N = \{(t,s) : |x_n(t,s)| \geq N\}$ $(n \geq 0)$. Then $\mathrm{mes}D_n^N \leq \frac{\delta}{3}$ $(n \geq 0)$. Since $x_n \to x_0$ in X, we can find a subsequence (x_{n_k}) which converges almost everywhere to x_0. Moreover, by Egorov's theorem, there exists a measurable set $D_\delta \subset D$ such that

$\operatorname{mes}(D-D_\delta) < \frac{\delta}{3}$ and (x_{n_k}) converges to x_0 uniformly on D_δ. Let $F_k^\delta = D_\delta - (D_{n_k}^N \cup D_0^N)$ and $\tilde{F}_k^\delta = D - F_k^\delta \; (k \geq 1)$. Then $\operatorname{mes}\tilde{F}_k^\delta < \delta$ for any $k \geq 1$. Now, we estimate

$$\|A_3 x_{n_k} - A_3 x_0\|_Y \leq \|A_3 \chi_{F_k^\delta} x_{n_k} - A_3 \chi_{F_k^\delta} x_0\|_Y + \|A_3 \chi_{\tilde{F}_k^\delta} x_{n_k}\|_Y + \|A_3 \chi_{\tilde{F}_k^\delta} x_0\|_Y.$$

By condition (9) there is a $\delta_0 = \delta_0(N, \varepsilon_0) > 0$ such that

$$\|a_3(\cdot,\cdot,u) - a_3(\cdot,\cdot,u_0)\|_Y < \frac{\varepsilon_0}{3} \tag{12}$$

whenever $|u| < N$, $|u_0| < N$, and $|u - u_0| < \delta_0$. Since (x_{n_k}) converges to x_0 uniformly on D_δ, there exists an integer $m = m(\delta_0)$ such that $|x_{n_m}(t,s) - x_0(t,s)| < \delta_0$ for all $(t,s) \in D_\delta$. Combining inequalities (11) - (13) we get $\|A_3 x_{n_m} - A_3 x_0\|_Y < \varepsilon_0$, which is contradictory to (10). Thus the operator A_3 is continuous ∎

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