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Complete Continuity of Some Nonlinear Volterra Operator in Banach Spaces

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Abstract. We consider continuity and compactness of the particular Volterra operator $Hx(t) = \int_{t_0}^{t} K(\tau)x(\tau)\,d\tau$, where $K(t)$ is a nonlinear continuous and compact or α-Lipschitz operator in some Banach space.

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0. Introduction

The initial value problem in a Banach space

$$\left.\begin{aligned} x'(t) &= K(t)x(t) \\ x(t_0) &= x_0 \end{aligned}\right\} \tag{1}$$

with nonlinear operators $K(t)$ reads in integrated form

$$x(t) - x_0 = \int_{t_0}^{t} K(\tau)x(\tau)\,d\tau. \tag{2}$$

In finite dimensions, the right-hand side usually defines a completely continuous operator H in appropriate function spaces, and thus by Schauder's theorem there exists a local (weak) solution of problem (1) (i.e. a solution of equation (2)). We can proceed similarly in infinite dimensions, if we can say something about continuity and compactness of H.

The existence of solutions of problem (1) has been discussed by many authors, e.g. by Ambrosetti [3], Krasnoselskii [9] and Sadovskii [12] (see also [5: Subsections 2.1 and

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8.1], [1: Subsection 4.1], and the references therein). However it seems that no attempts have been made to consider compactness of H without the assumption that all $K(\tau)$ have their range in the *same* compact set.

We try to fill this gap now. The corresponding general theorems on continuity and compactness of H are established in Sections 1 and 2, while in Section 3 we discuss, how the abstract conditions may be verified. Section 4 is devoted to the case that $K(t)$ are Uryson operators (over arbitrary measure spaces).

1. Continuity

If T is a σ-finite measure space, and U a metric space, we call a function $x : T \to U$ (*strongly Bochner*) *measurable*, if it is a.e. (in the sense of the Lebesgue extension) the limit of a sequence of simple functions $x_n : T \to U$. A function $x : T \to U$ is called *essentially separable-valued*, if $x(T \setminus T_0)$ is separable for some null set $T_0 \subseteq T$.

Proposition 1.1. *Let T be a σ-finite measure space, and U be a metric space. A function $x : T \to U$ is measurable, if and only if it is essentially separable-valued and $x^{-1}(O)$ is measurable (in the sense of the Lebesgue extension of the measure space) for any open set $O \subseteq U$.*

Proof. The proof is standard (cf., e.g., [6: Subsection III.6/Theorem 10] for a Banach space U) ∎

Proposition 1.1 implies the following fact, which we shall frequently use:

Lemma 1.1. *Let T be a σ-finite measure space, U be a metric space, and $M \subseteq U$. If $x : T \to U$ is measurable with $x(T) \subseteq M$, then x is a.e. the limit of a sequence of simple functions $x_n : T \to M$.*

Proof. Proposition 1.1 shows that x is measurable, if it is considered as a function from T into the metric space M ∎

Let I be some compact interval, $t_0 \in I$, U be a normed linear space, V be a Banach space, and $M \subseteq U$. Assume, for almost all $t \in I$ we have a nonlinear operator $K(t) : M \to V$, such that the Carathéodory condition is satisfied:

- **(i)** $t \mapsto K(t)u$ is measurable for all $u \in M$
- **(ii)** $u \mapsto K(t)u$ is continuous on M for almost all t.

Observe that these conditions already ensure that the superposition operator $Fx(t) = K(t)x(t)$ maps measurable functions $x : I \to M$ to measurable functions: Approximate x by simple functions $x_n : I \to M$ (Lemma 1.1). Then each Fx_n is measurable and $Fx_n(t) \to Fx(t)$ a.e.

We consider the special nonlinear Volterra integral operator in Banach spaces

$$Hx(t) = \int_{t_0}^{t} K(\tau)x(\tau)\,d\tau = \int_{t_0}^{t} Fx(\tau)\,d\tau \tag{3}$$

as a mapping from B into $C(I,V)$, where B is some set of measurable functions $x : I \to M$. Let B be equipped with some metric, such that convergence in this metric implies convergence in measure. It is easy to give sufficient conditions for the continuity of H. One of the best is the following consequence of Vitali's convergence theorem:

Theorem 1.1. *Assume that in addition to conditions* (i) *and* (ii) *the following holds:*

(∗) *Whenever $x_n \to x$ in B, and $I \supseteq D_1 \supseteq D_2 \supseteq \ldots$ are measurable with $\cap_k D_k = \emptyset$, we have* $\lim_{k\to\infty} \sup_n \int_{D_k} \|K(\tau)x_n(\tau)\|\, d\tau = 0$.

Then $H : B \to C(I,V)$ is defined and continuous.

Proof. We first observe that condition (∗) is equivalent to the apparently more restrictive condition

$$\lim_{\mathrm{mes}D\to 0} \sup_n \int_D \|K(\tau)x_n(\tau)\|\, d\tau = 0 \tag{4}$$

for any convergent sequence $\{x_n\}$ in B. Indeed, if (4) is violated, there exists a sequence of measurable sets E_k with $\sum_k \mathrm{mes}E_k < \infty$ such that

$$\sup_n \int_{E_k} \|K(\tau)x_n(\tau)\|\, d\tau \not\to 0 \qquad (k \in \mathbb{N}).$$

Then $D_k = \cup_{n\geq k} E_n$ is descending with $\mathrm{mes}D_k \to 0$. Thus, eliminating the null set $\cap_k D_k$ from each D_k, we see that condition (∗) fails since $E_k \subseteq D_k$. Applying (4) for the constant sequence $x_n = x$ we find that Fx is even integrable on I, and thus $Hx(t)$ is defined.

Now, let $x_n \to x$ in B. We have to prove that $\|Hx - Hx_n\| \to 0$. Since it suffices to prove this for some subsequence, we may assume $x_n \to x$ a.e., whence $Fx_n \to Fx$ a.e., and thus Vitali's convergence theorem [6: Subsection III.6/Theorem 15] implies

$$\|Hx - Hx_n\| \leq \int_I \|Fx(t) - Fx_n(t)\|\, dt \to 0 \tag{5}$$

in view of (4) ∎

Observe that condition (∗) is 'almost necessary' for H to be continuous: Since the conditions of Vitali's convergence theorem are also necessary, condition (∗) must be satisfied, if the right-hand side of (5) converges to zero. Surprisingly, it turns out that condition (∗) is not necessary, even for $U = V = \mathbb{R}$ and smooth $t \mapsto K(t)u$. We will construct a counterexample based on the following fact, which shows that the Hahn-Saks theorem [15: Section 45/Theorem 4] fails in a spectacular way for intervals:

Lemma 1.2. *There exists a sequence of smooth functions $\alpha_n : [0,1] \to \mathbb{R}$, such that*

$$\max_{t\in[0,1]} \left| \int_0^t \alpha_n(\tau)\, d\tau \right| \to 0 \qquad \textit{and} \qquad \alpha_n \to 0 \ \textit{a.e.} \tag{6}$$

although

$$\int_0^1 |\alpha_n(\tau)|\, d\tau \not\to 0. \tag{7}$$

Proof. We first approximate Cantor's 'middle-third function' x uniformly by smooth functions x_n in the following way: Let E_n be the union of all intervalls dropped until the n-th step of the construction of the Cantor set. Recall that x is constant on each of these intervals and monotone increasing with $x(0) = 0$ and $x(1) = 1$. Let x_n be any smooth increasing function satisfying $x_n(0) = 0$, $x_n(1) = 1$ and $x_n|_{E_n} = x|_{E_n}$. By construction $x_n \to x$ uniformly, and $x'_n \to 0$ a.e.

Now, let $k = (k_1, k_2) : \mathbb{N} \to \mathbb{N}^2$ be one-to-one and onto, and put $\alpha_n = x'_{k_1(n)} - x'_{k_2(n)}$. Since $\{x_n\}$ is a Cauchy sequence in $C([0,1])$, (6) is satisfied. But since the space $AC([0,1])$ of absolutely continuous functions on $[0,1]$ is complete with respect to the norm

$$\|x\|_1 = \max_{t\in[0,1]} |x(t)| + \int_0^1 |x'(t)|\, dt,$$

and since $x_n \in AC([0,1])$ converge uniformly to $x \notin AC([0,1])$, $\{x_n\}$ is not a Cauchy sequence in this norm, which implies (7) ∎

Now the counterexample is as follows: Put $I = [0,1]$, $U = V = \mathbb{R}$, $M = [0,1]$, and define $K(t)0 = 0$,

$$K(t)(n+\lambda)^{-1} = \lambda\alpha_{n+1}(t) + (1-\lambda)\alpha_n(t) \qquad (n \in \mathbb{N}, 0 \le \lambda < 1).$$

Then each $u \mapsto K(t)u$ is continuous, and $t \mapsto K(t)u$ is smooth, but for $x_n(t) \equiv n^{-1}$ we have $\int_0^1 |K(t)x_n(t)|\, dt \not\to 0$. Thus, Vitali's convergence theorem (necessary part) implies (since a.e. $K(t)x_n(t) \to 0$) that condition $(*)$ fails with this sequence $\{x_n\}$.

2. Compactness

For the moment, let us consider a more general situation: Let T be a σ-finite measure space, U and V as before, and $M \subseteq U$. Assume, for almost all $t \in T$ we have a nonlinear continuous operator $K(t) : M \to V$. Let B consist of measurable functions $x : T \to M$. We are interested in conditions which ensure that the range AB of

$$Ax = \int_T K(t)x(t)\, dt$$

is precompact.

We prepare the result by observing a simple fact for Bochner integrals. Recall that the *essential range* of x is defined as the set of all $y \in Y$, such that $\operatorname{ess\,inf}_{s\in S} \|x(s) - y\| = 0$.

Lemma 2.1. *If S is a measure space with* $\mathrm{mes}S = 1$, *Y is a Banach space, and $x : S \to Y$ is integrable, then the integral $y_0 = \int_S x(s)\,ds$ belongs to the closed convex hull C of the essential range of x.*

Proof. If $y_0 \notin C$, apply the separation theorem [11: Theorem 3.4] to find $l \in Y^*$ with $\mathrm{Re}l(y_0) < \inf \mathrm{Re}l(C)$. Then, by elementary properties of the Bochner integral [6: Subsection III.2/Theorem 19/(c)], we find $\mathrm{Re}l(y_0) = \int_S \mathrm{Re}l(x(s))\,ds > \int_S \mathrm{Re}l(y_0)\,ds = \mathrm{Re}l(y_0)$, which is a contradiction ∎

We emphasize that, although we invoked Hahn-Banach's theorem in the proof, we just needed a countable form of the axiom of choice. Indeed, by the definition of the integral it is clear that it suffices to consider the closed linear hull of the values of the approximating simple functions instead of Y. Thus without loss of generality, we may assume that Y is separable. And for the Hahn-Banach theorem in separable spaces the (uncountable) axiom of choice is not needed [7: p. 183].

Now, let $\mathcal{B}(M,V)$ be the Banach space of all functions $M \to V$ with bounded range, equipped with the sup-norm, $\mathcal{BC}(M,V) \subseteq \mathcal{B}(M,V)$ the subspace of continuous and bounded functions, and $\mathcal{CC}(M,V) \subseteq \mathcal{BC}(M,V)$ the subspace of all continuous functions with precompact range, and assume the following:

(iii) $K(t) \in \mathcal{CC}(M,V)$ for almost all t.

(iv) K is integrable as a function with values in $\mathcal{B}(M,V)$.

Observe that the measurability of $K : T \to \mathcal{B}(M,V)$ implies condition (i), and that condition (ii) is a consequence of $K(t) \in \mathcal{CC}(M,V)$.

Theorem 2.1. *Let conditions* (iii) *and* (iv) *hold. Then AB is precompact.*

Proof. By Lemma 1.1 there exists a sequence of simple functions $K_n : T \to \mathcal{CC}(M,V)$, converging a.e. to K (in the norm of $\mathcal{B}(M,V)$). We may assume that additionally $\|K_n(t)\| \le \|K(t)\|$. Indeed, there exists a sequence of simple non-negative functions α_n, which converges a.e. monotonically increasing to $t \mapsto \|K(t)\|$, whence we may replace K_n by $\widetilde{K}_n(t) = \min\{1, \alpha_n(t)\|K_n(t)\|^{-1}\}K_n(t)$, if necessary. By Lebesgue's dominated convergence theorem,

$$\int_T \|K_n(t) - K(t)\|\,dt \to 0. \tag{8}$$

The range of each $A_n x = \int_T K_n(t)x(t)\,dt$ is precompact. Indeed, for fixed n there exist pairwise disjoint measurable E_k and $C_k \in \mathcal{CC}(M,V)$ with

$$K_n(t) = \sum_{k=1}^{j} \chi_{E_k}(t) C_k.$$

In view of Lemma 2.1 (for the measure space E_k with renormed measure) the range $A_n B$ is contained in $\sum(\mathrm{mes}E_k)\overline{\mathrm{conv}}(C_k M)$, i.e. in a compact set. Now just observe that by (8) we have $A_n \to A$ uniformly on B ∎

Remark 2.1. If B is equipped with some metric, such that convergence in this metric implies convergence in measure on sets of finite measure, then an analogous proof to Theorem 1.1 shows that $A : B \to V$ is also continuous. Indeed, if $x_n \to x$ in B, then for a subsequence and almost all t we have $x_{n_k}(t) \to x(t)$. Thus $K(t)x_{n_k}(t) \to K(t)x(t)$ implies $Ax_{n_k} \to Ax$ by Lebesgue's dominated convergence theorem.

We now turn to the question, under which conditions the operator (3) has precompact range HB in $C(I,V)$. Using the theorem of Arzelà-Ascoli for vector-valued functions (see [8: Theorem 3.10] or [10]) we see that this is the case if and only if the following is true:

(a) HB is equicontinuous.

(b) $\{Hx(t) : x \in B\}$ is precompact in V for each t.

The last assertion is satisfied under the conditions of Theorem 2.1 for $T = [t_0, t]$ (resp. $[t, t_0]$). But condition (iv) evidently implies (a) and condition (*) of Theorem 1.1. Thus we have found the following main result of this paper.

Theorem 2.2. *Let conditions* (iii) *and* (iv) *hold. Then $H : B \to C(I,V)$ is defined, continuous, and has precompact range.*

Considering the initial value problem (1), we now can say:

Corollary 2.1. *Assume, conditions* (iii) *and* (iv) *are satisfied for $M = \{u \in U : \|u - x_0\| \leq \delta\}$ ($\delta > 0$ fixed), $B = C(I,M)$, and $V = U$. Then problem* (1) *has a (local) weak solution in $C(J,M)$ for a small interval $J \subseteq I$ around t_0.*

Proof. Choose J such that $\left|\int_{t_0}^{t} \|K(\tau)\|\, d\tau\right| \leq \delta$ for $t \in J$. Then $x \mapsto x_0 + Hx$ is a compact and continuous mapping from $C(J,M)$ into itself. Since $C(J,M)$ is a closed, bounded and convex subset of the Banach space $C(J,U)$, it remains to apply Schauder's fixed point theorem ∎

We emphasize that $x \mapsto x_0 + Hx$ in the proof even transforms sequences $\{x_n\}$ convergent merely in measure (not uniformly) into uniform convergent sequences: Just equip B with the metric of convergence in measure,

$$d(x,y) = \int_I \frac{\|x(t) - y(t)\|}{1 + \|x(t) - y(t)\|}\, dt.$$

Corollary 2.2. *Let the conditions of Corollary* 2.1 *be satisfied, and assume $C(t) : M \to U$ satisfies a Lipschitz condition*

$$\|C(t)u - C(t)v\| \leq L(t)\|u - v\| \qquad (t \in I;\ u,v \in M)$$

with (in a neighborhood of t_0) integrable L, and $t \mapsto C(t)u$ is measurable for each fixed $u \in M$. Then the initial value problem

$$\left.\begin{aligned} x'(t) &= C(t)x(t) + K(t)x(t) \\ x(t_0) &= x_0 \end{aligned}\right\} \tag{9}$$

has a (local) weak solution in $C(J, M)$ for a small interval $J \subseteq I$ around t_0.

Proof. The operator $Dx(t) = \int_{t_0}^{t} C(\tau)x(\tau)\,d\tau$ is a contraction of $C(J, M)$ for small J. Since weak solutions of problem (9) are the fixed points of $x \mapsto x_0 + Dx + Hx$, which thus is the sum of a contracting and a compact operator (whence a condensing operator [1: Subsection 1.5]), we can argue as in the previous proof, using Darbo's fixed point theorem [4] ∎

Corollary 2.2 might be compared with the classical result in [9]. In this connection, we emphasize that our conditions do not imply that the (essential) range of the mapping $g(t, u) = K(t)u$ on $I \times M$ is precompact, even if $K(t)$ is linear, and K is piecewise constant:

Example 2.1. Let V be an infinite dimensional Banach space, i.e. there exists a bounded sequence $\{v_n\} \subset V$ without convergent subsequence. Let $l \in U^*$ satisfy $l(u) \neq 0$ for some $u \in M$. Choose $I = [0, 1]$, and put $K(t)u = l(u)v_n$ for $(n+1)^{-1} < t \leq n^{-1}$. Then conditions (iii) and (iv) are satisfied, but the (essential) range of $g(t, u) = K(t)u$ is not precompact.

It is a natural question whether analogues of Theorems 2.1 and 2.2 hold also for condensing $K(t)$. This indeed is true in a certain sense. Recall that the Hausdorff measure γ of non-compactness of a set E in a metric space Z is defined as the infimum of all $\varepsilon > 0$ such that E has a finite ε-net in Z. Similarly, the Kuratowski measure α of E is defined as the infimum of all $\delta > 0$ such that E has a finite covering of sets with diameter less than δ.

Let $CC_q(M, V) \subseteq BC(M, V)$ denote the subset of all continuous C with bounded range, which satisfy

$$\alpha(CM_0) \leq q\gamma(M_0) \qquad (M_0 \subseteq M). \tag{10}$$

Condition (10) is appropriate for the following considerations. However, for applications it is more convenient to have a symmetric condition: Observing that $\gamma \leq \alpha \leq 2\gamma$, we may replace (10) by one of the sufficient (and up to the factor 2 necessary) conditions

$$\alpha(CM_0) \leq q\alpha(M_0) \qquad (M_0 \subseteq M) \tag{11}$$

or

$$\gamma(CM_0) \leq \frac{q}{2}\gamma(M_0) \qquad (M_0 \subseteq M). \tag{12}$$

We intend to replace condition (iii) by the condition

(v) $K(t) \in CC_q(M, V)$ for almost all $t \in T$ $(q < \infty$ fixed).

Analogously to Theorems 2.1 and 2.2 we can prove:

Theorem 2.3. *Let conditions* (iv) *and* (v) *hold. Then* $\alpha(AB) \leq (q \operatorname{mes} T)\gamma(M)$. *In case* $I = T$ *we also have* $\alpha(HB) \leq (q \operatorname{mes} I)\gamma(M)$.

Proof. The proof of the first statement is almost the same as that of Theorem 2.1 (with $CC_q(M, V)$ instead of $CC(M, V)$): Observe that $K_n(t) \in CC_q(M, V)$ indeed implies

$\widetilde{K}_n(t) \in CC_q(M,V)$, and that

$$\begin{aligned}\alpha(A_nB) &\le \alpha\Big(\sum(\operatorname{mes} E_k)\overline{conv}(C_kM)\Big)\\ &\le \sum(\operatorname{mes} E_k)\alpha(C_kM) \qquad (13)\\ &\le (\operatorname{mes} T)q\gamma(M).\end{aligned}$$

The second statement follows from the first, since HB is equicontinuous (see [3] or [10]) ∎

The first part of the Theorem 2.3 generalizes [1: Lemma 4.1.3]. For $q = 0$, Theorem 2.3 gives the main statements of Theorems 2.1 and 2.2 again.

We emphasize that Theorem 2.3 holds true for *arbitrary* set functions γ on U, which satisfy estimate (10) ($C = K(t)$). Even more, it suffices to have (10) only for $M_0 = M$ (for the statement of the theorem, it is unimportant, how $\gamma(M_0)$ is defined for $M_0 \neq M$, whence we may assume $\gamma(M_0) = \gamma(M)$ for $M_0 \subseteq M$). Thus, the only requirement on γ for Theorem 2.3 is the estimate $\alpha(K(t)M) \le q\gamma(M)$ (for almost all $t \in T$).

For the first part of Theorem 2.3, also α can be a more general set function: Using the terminology of [1], we see that (13) holds for any measure α of non-compactness which is monotone, semi-homogeneous, and algebraically semi-additive; if additionally α is continuous with respect to the Hausdorff metric, the complete proof of the first part of Theorem 2.3 carries over for such an α.

However, observe that for the second part of Theorem 2.3 we need some connection between α in the spaces $C(I,V)$ and in V.

Whether Theorem 2.3 implies that A resp. H are condensing, depends of course on the metric on B, which has not been used so far. For simplicity, let us restrict ourselves to the most interesting case $B \subseteq C(I,U)$ (i.e. we equip B with the max-norm).

Lemma 2.2. *Let $B_0 \subseteq C(I,U)$ resp. $B_0 \subseteq C(I,M)$, and $M_0 = \{x(I) : x \in B_0\}$. Then $\gamma(M_0) \le \gamma(B_0)$.*

Proof. Given $\varepsilon > 0$, let $x_1, \ldots, x_n$ be a finite $(\gamma(B_0)+\varepsilon)$-net for B_0. Let M_k be a finite ε-net for $x_k(I)$. Then the union of all M_k is a finite $(\gamma(B_0)+2\varepsilon)$-net for M_0, which implies $\gamma(M_0) \le \gamma(B_0)+2\varepsilon$. Indeed, given $x \in B_0$ and $t \in I$, there is some k with $\|x(t) - x_k(t)\| \le \gamma(B_0)+\varepsilon$ and some $u \in M_k$ with $\|u - x_k(t)\| \le \varepsilon$, whence $\|u - x(t)\| \le \gamma(B_0)+2\varepsilon$ ∎

We remark that for Lemma 2.2 it is unimportant whether we consider γ in the spaces U resp. $C(I,U)$, or in the corresponding metric subspaces M resp. $C(I,M)$. The same is true of course for the following combination of Lemma 2.2 with Theorem 2.3.

Theorem 2.4. *Let conditions* (iv) *and* (v) *hold with $B \subseteq C(I,U)$ $(B \subseteq C(I,M))$. Then H is continuous, and*

$$\alpha(HB_0) \le (q \operatorname{mes} I)\gamma(B_0) \qquad (B_0 \subseteq B). \tag{14}$$

In particular,

$$\gamma(HB_0) \le (q \operatorname{mes} I)\gamma(B_0) \qquad (B_0 \subseteq B) \tag{15}$$

and

$$\alpha(HB_0) \leq (2q \operatorname{mes} I)\alpha(B_0) \qquad (B_0 \subseteq B). \tag{16}$$

Thus, if $q \operatorname{mes} I < 1$ $(q \operatorname{mes} I < \frac{1}{2})$, *$H$ is condensing with respect to the Hausdorff (Kuratowski) measure of non-compactness.*

Proof. The continuity of H follows by Theorem 1.1. For (14), apply Theorem 2.3 for $B = B_0$ and $M = M_0$, where M_0 is defined as in Lemma 2.2 ∎

Observe that, if we consider only the Hausdorff measure of non-compactness, we loose a factor 2 in (12) (but not in (15)), and if we consider only the Kuratowski measure, we loose the same factor in (16) (but not in (11)). Thus it is usually the best idea to verify the conditions of Theorem 2.4 for the Kuratowski measure of non-compactness, and to use the conclusion for the Hausdorff measure of non-compactness.

Theorem 2.4 gives the following essential sharpening of Corollary 2.2:

Corollary 2.3. *Corollary* 2.2 *holds even if condition* (iii) *is replaced by condition* (v) *with* $q < \infty$.

Proof. The operator $x \mapsto x_0 + Dx + Hx$ in the proof of Corollary 2.2 is condensing by Theorem 2.4, if the considered interval is sufficiently small ∎

We emphasize that the operators $K(t)$ in the previous corollary need not even be condensing: They must just satisfy *some* '(uniform) α-Lipschitz condition' (this is some analogy to the finite-dimensional case). The price that we have to pay for a larger 'Lipschitz'-constant q is that in general the interval of existence of the solution becomes correspondingly smaller.

The main difference of Corollary 2.3 to the results [3, 12] (see also [5: Subsections 2.1 and 8.1] and [1: Subsection 4.1], and the references therein) is that we have replaced 'uniform continuity' of K by 'uniform (local) integrability'. Note that for uniformly continuous $g(t, u) = K(t)u$ with precompact $K(t)M$ the range $g(I \times M)$ is precompact, contrasting Example 2.1.

3. Checking the conditions

The crucial point in the application of Theorems 2.1 - 2.4 is of course to check condition (iv). This condition is two-fold:

1. $K : T \to \mathcal{B}(M, V)$ is measurable

and

2. the integral of $\|K(\cdot)\|$ over T (which exists by Condition 1) is finite, i.e.

$$\int_T \sup_{u \in M} \|K(t)u\| \, dt < \infty. \tag{17}$$

While, for applications, (17) usually is obvious, the first condition might be very hard to check: If the image space is not separable, measurability of a function K is a very restrictive condition. For example, in $U = L_1([0,1])$, $V = \mathbb{R}$, the harmless looking family of linear integral functionals $K(t)u = \int_0^t u(s)\,ds$ $(t \in [0,1])$ satisfies conditions (i) - (iii) and has uniformly bounded norm, but it is not measurable with respect to the uniform convergence, since it is not essentially separable-valued (see [14: Example 5.1.2].

To check measurability, Proposition 1.1 might be useful. Sometimes it is more convenient to check its conditions not for $K : T \to \mathcal{B}(M,V)$ but just for $K : T \to \mathcal{CC}(M,V)$ or $K : T \to \mathcal{BC}(M,V)$. This is sufficient:

Proposition 3.1. *Let W be a subspace of $\mathcal{B}(M,V)$, and $K : T \to W$. Then K is measurable if and only if $K : T \to \mathcal{B}(M,V)$ is measurable.*

Proof. The proposition is an immediate consequence of Lemma 1.1 ∎

Proposition 1.1 implies that in the typical situation the only problem that might occur is that K is not essentially separable-valued (as in the above example):

Theorem 3.1. *Let M be separable with respect to some topology. Let $K : T \to \mathcal{B}(M,V)$ satisfy condition (i), and assume $u \mapsto K(t)u$ is sequentially continuous with respect to this topology for almost all t. Then K is measurable if and only if it is essentially separable-valued.*

Proof. 1. Let us first show that any open and separable subset O of a metric space X is the union of countable many closed balls. Indeed, let $\{x_n\}$ be dense in O. Let r_n be the supremum of all radii r such that the ball $\{x : d(x,x_n) < r\}$ still is contained in O. Then $B_n = \{x : d(x,x_n) \le \frac{r_n}{2}\}$ are the desired balls: Given $x \in O$, there is some $r > 0$ such that $d(x,y) < 3r$ implies $y \in O$. Choose some n with $d(x,x_n) \le r$. Then $r_n \ge 2r$, whence $x \in B_n$.

2. By Proposition 1.1 we just have to show the sufficient part of the theorem, and it remains to show that $K^{-1}(O)$ is measurable for any open set $O \subseteq \mathcal{B}(M,V)$. Without loss of generality, let KT be separable, and assume $K(t)$ is sequentially continuous for *each* t. Since $O \cap KT$ is separable [2: Lemma 2.6/(2)] and open in KT, it is the union of countable many balls closed in KT. Thus, it remains to show that $K^{-1}(A)$ is measurable for any closed ball A in KT.

Let A be the closed ball in KT with center $C \in KT$ and radius $r > 0$. Then

$$K^{-1}(A) = \Big\{t \in T : \|K(t) - C\| \le r\Big\} = \bigcap_{u \in M} \Big\{t \in T : \|K(t)u - Cu\| \le r\Big\}.$$

Let $\{u_n\}$ be dense in M. Then, by the continuity of $K(t)$,

$$K^{-1}(A) = \bigcap_{n \in \mathbb{N}} \Big\{t \in T : \|K(t)u_n - Cu_n\| \le r\Big\}.$$

By condition (i), each of these sets is measurable ∎

For the application of Theorem 3.1 it is worth noting that the measurability of $K : T \to \mathcal{B}(M,V)$ does not depend on the topology on M. This means the condition that M be separable is not very restrictive: We just have to find *some* separable topology on M such that $K(t)$ is sequentially continuous. Since in our situation $K(t) \in \mathcal{CC}(M,V)$ is a 'good' operator, this might be an essential weakening of the separability assumption:

Example 3.1. Let M be the unit ball of $U = L_\infty([0,1])$, and $K : T \to CC(M,V)$ satisfy condition (i). Assume, moreover, that $u_n \to u$ in L_1 already implies $K(t)u_n \to K(t)u$ (remember, e.g., the remark following Corollary 2.1). Then $K : T \to \mathcal{B}(M,V)$ is measurable if and only if it is essentially separable-valued.

However, we remark the condition that $K : T \to \mathcal{B}(M,V)$ be essentially separable-valued is quite restrictive in most applications, even if $K : T \to CC(M,V)$ and $V = \mathbb{R}$:

Example 3.2 [16]. Let U have infinite dimension, and $M \subseteq U$ contain interior points. Then $CC(M,\mathbb{R})$ is not separable. Indeed, there exist $\delta > 0$ and a sequence $\{u_n\} \subset M$ with $\|u_n - u_k\| \geq 2\delta$ for $n \neq k$. For $N \subseteq \mathbb{N}$ define a continuous map $x_N : U \to [0,\delta]$ by $x_N(u) = \sum_{n\in N} \max\{\delta - \|u - u_n\|, 0\}$, observing that at most one term does not vanish. Since $x_N(u_n) = \delta$ for $n \in N$ and $x_N(u_n) = 0$ for $n \notin N$, we have $\|x_{N_1} - x_{N_2}\| \geq \delta$ for $N_1 \neq N_2$. Since the $x_N \in CC(M,V)$ are uncountable, $CC(M,V)$ is not separable (this conclusion only needs a countable form of the axiom of choice [14: Lemma A.1.1]).

Since we want to have that almost all $K(t)$ belong to a separable subspace of $CC(M,V)$, it might be useful to know 'canonical' examples of such subspaces. One is the closure of the set of polynomials:

Example 3.3 [16]. Let $L \subseteq U^*$ and $\widetilde{V} \subseteq V$ be separable, and $M \subseteq U$ be bounded. Then the closure of the set

$$\left\{ x \,\middle|\, x(u) = \sum_{k=1}^{n} \left(\prod_{j=1}^{n_k} l_{kj}(u) \right) v_k \quad (n, n_k \in \mathbb{N},\ l_{kj} \in L,\ v_k \in \widetilde{V}) \right\}$$

is a separable subset of $CC(M,V)$. Indeed, if $C_1 \subseteq L$ and $C_2 \subseteq V$ are countable and dense, the set of all functions of the form

$$x(u) = \sum_{k=1}^{n} \left(\prod_{j=1}^{n_k} l_{kj}(u) \right) v_k$$

with $l_{kj} \in C_1$ and $v_k \in C_2$ is countable and dense.

4. An example: Uryson operators

The most important examples of compact nonlinear operators are Uryson operators. We consider a family of such operators

$$K(t)u(s) = \int_R k(t,s,\sigma,u(\sigma))\,d\sigma \qquad (s \in S). \tag{18}$$

Here, S and R are arbitrary σ-finite measure spaces, u takes values in a finite-dimensional space W, and k takes values in a Banach space Z.

We consider $K(t)$ as a mapping from $M \subseteq U$ into V, where U and V are normed linear spaces of (classes of) measurable functions, V being even an ideal space, i.e. V is complete, and for any measurable function w satisfying a.e. $\|w(s)\| \leq \|v(s)\|$ for some $v \in V$, we have $w \in V$ and $\|w\| \leq \|v\|$. Assume that convergence in U implies convergence in measure on sets of finite measure, and that $M \subseteq U$ is bounded in norm. Let almost all $k(t,\cdot,\cdot,\cdot)$ satisfy a Carathéodory condition, i.e. $k(t,\cdot,\cdot,w)$ is measurable (on $S \times R$), and for almost all $(s,\sigma) \in S \times R$ the function $k(t,s,\sigma,\cdot)$ is continuous on W. Then, typically (if k is not 'too singular'), $K(t)$ is compact and continuous and even transforms sequences converging in measure on sets of finite measure into sequences converging in norm. Thus, if the measure space R is separable (or, equivalently, $L_1(R)$ is separable), we may equip M with the separable topology generated by 'convergence in measure on sets of finite measure', and apply Theorem 3.1. It thus remains to check that K is essentially separable-valued and that $t \mapsto K(t)u$ is measurable for all $u \in M$. For the latter, it is natural to assume that k even satisfies a joint Carathéodory condition on $(T \times S \times R) \times U$, i.e. additionally $k(\cdot,\cdot,\cdot,u)$ is measurable on $T \times (S \times R)$ for each u.

However, there is another approach: Under the 'typical' conditions guaranteeing the compactness of $K(t)$, the author has shown in [13] by a different method that $K : T \to \mathcal{B}(M,V)$ is measurable, if k satisfies a joint Carathéodory condition (even if R is not separable). Thus, under these assumptions, the conditions of Theorem 2.1 (or Theorem 2.2 in case $T = I$) are equivalent to the finiteness of the integral (17).

We give a special case of this result for L_p spaces, which is sufficient for most applications (cf. the example in [13]):

Example 4.1. Let $U = L_p(R,W)$ and $V = L_q(S,Z)$, with $1 \leq p,q < \infty$. Let $1 \leq p_t \leq \infty$, and $a : T \times R \to \mathbb{R}$ be measurable such that $\operatorname{ess\,sup}_{t \in T} \|a(t,\cdot)\|_{L_{p_t}} < \infty$. Let measurable $z_t : S \times R \to \mathbb{R}$ be given such that the so-called Zaanen norm

$$\|z_t\|_{Z(L_{p_t},L_q)} = \sup_{\|u\|_{p_t} \leq 1} \left\| \int_R |z_t(\cdot,\sigma)| u(\sigma)\, d\sigma \right\|_q$$

is a.e. bounded by an integrable $c(t)$, and

$$\|P_{Q_n} z_t\|_{Z(L_{p_t},L_q)} \to 0 \qquad \text{whenever} \quad \begin{cases} Q_n = S_n \times R, \; S_n \downarrow \emptyset \\ \text{or} \\ Q_n = S \times R_n, \; R_n \downarrow \emptyset. \end{cases}$$

This is satisfied, for example, if $p_t > 1$ and for $\frac{1}{p_t} + \frac{1}{p_t'} = 1$ the function

$$c(t) = \min\left\{ \left(\int_S \left(\int_R |z_t(s,\sigma)|^{p_t'} d\sigma \right)^{\frac{q}{p_t'}} ds \right)^{\frac{1}{q}}, \left(\int_R \left(\int_S |z_t(s,\sigma)|^q ds \right)^{\frac{p_t'}{q}} d\sigma \right)^{\frac{1}{p_t'}} \right\}$$

$$= \begin{cases} \left(\int_S \left(\int_R |z_t(s,\sigma)|^{p_t'} d\sigma \right)^{\frac{q}{p_t'}} ds \right)^{\frac{1}{q}} & \text{if } p_t' \leq q \\ \left(\int_R \left(\int_S |z_t(s,\sigma)|^q ds \right)^{\frac{p_t'}{q}} d\sigma \right)^{\frac{1}{p_t'}} & \text{if } p_t' \geq q \end{cases}$$

is dominated by an integrable function. Then for any Carathéodory function $k : (T \times S \times R) \times W \to Z$ satisfying the growth condition

$$\|k(t,s,\sigma,w)\| \leq |z_t(s,\sigma)| \left(|a(t,\sigma)| + \|w\|^{\frac{p}{p_t}}\right)$$

the corresponding family of Uryson operators (18) satisfies conditions (iii) and (iv) of our main Theorems 2.1 and 2.2 for any bounded $M \subseteq U$. In particular, in case $T = I$, the operator (3) defines a completely continuous mapping $H : C(I,U) \to C(I,V)$.

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