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On Bernstein's Theorem for Quasiminimal Surfaces
Part II

E. Hoy

Abstract. A Bernstein-type theorem is proved for surfaces with a quasiconformal Gauss map and with a growth condition for the total curvature.

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1. Introduction

In the present paper, a Bernstein-type theorem which was stimulated by a result in [8] is proved for parametric quasiminimal surfaces. Unlike [8], we assume only a growth condition for the total curvature with respect to geodesic disks. Here, the Bernstein-type theorem is a consequence of an apriori estimate for the mean value of the Gaussian curvature with respect to a geodesic disk which has been derived before. In particular, if the Gauss map of a complete quasiminimal surface S in $\mathbb{R}^3$ omits a neighborhood of the unit sphere and the total curvature with respect to geodesic disks does not increase too fast, then S must be a plane. The considerations of the present paper are based mainly on [4]. For the classification of the present paper and for the notations we refer to [2], which is the precursor of this paper.

2. Assumptions and notations

Let S be an open oriented differential-geometric surface in $\mathbb{R}^3$ with three times continuously differentiable representations in local parameters, where the Gauss map of S is quasiconformal. Due to [7: Section 4] these surfaces are called *quasiminimal*.

Because of the orientability of S and the smoothness of its Gauss map we can regard S as a Riemannian surface. Without loss of generality we may assume that S is simply connected (see [2: Bemerkung 5]). Furthermore, let the Gauss map of S omit the north pole of the unit sphere. Since S is a quasiminimal surface, the Gaussian curvature is non-positive everywhere on S. Due to a theorem of Hadamard (see, e.g., [5: Theorem 6.6.4]) there exists a diffeomorphism of a plane to S, if S is additionally complete. This

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E. Hoy: Peter-Geibel-Str. 10, 61169 Friedberg

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diffeomorphism arises from the global introduction of geodesic polar coordinates $(r, \vartheta) \in [0, \infty) \times [0, 2\pi)$ on S with respect to an origin $P_0 \in S$. Then, the diffeomorphism of the plane onto the complete surface S induces a mapping $h = h(r, \vartheta) : [0, \infty) \times [0, 2\pi) \to S$. Note that $h_\vartheta \cdot h_\vartheta = G^2 = G^2(r, \vartheta)$ holds with a non-negative function $G = G(r, \vartheta)$.

As in [2], the function $g = g(r, \vartheta) : [0, \infty) \times [0, 2\pi) \to \mathbb{C}$ is the composition of h, the Gauss map and the stereographic projection. Denoting $g_1 = \operatorname{Re} g$ and $g_2 = \operatorname{Im} g$ we have the relations

$$T = g_{2r} g_{1\vartheta} - g_{1r} g_{2\vartheta} \geq 0 \tag{1}$$

$$(1 + |g|^2)^2 G_{rr} = 4T \tag{2}$$

$$g_{1\vartheta}^2 + g_{2\vartheta}^2 \leq QTG \tag{3}$$

$$G_{rr} = -KG \tag{4}$$

$$L(r) = \int_0^{2\pi} G(r, \vartheta)\, d\vartheta \tag{5}$$

for all $r \in [0, \infty)$ and all $\vartheta \in [0, 2\pi)$, where (4) is a special case of Gauss' Theorema egregium (see, e.g., [5: Theorem 3.8.7]) and (2) arises after expressing the surface element $d\Sigma$ on the unit sphere by the area element $(g_{2r} g_{1\vartheta} - g_{1r} g_{2\vartheta})\, dr d\vartheta = T\, dr d\vartheta$ in the plane and, otherwise, by the surface element $G\, dr d\vartheta$ on S. So we get $d\Sigma = -4(|g|^2 + 1)^{-2} T\, dr d\vartheta$ and $d\Sigma = KG\, dr d\vartheta$. By means of (4) this yields (2). Since the Gauss map of S is quasiconformal and the first fundamental form of S has the structure $ds^2 = dr^2 + G^2 d\vartheta^2$, the function $g = g(r, \vartheta)$ satisfies

$$g_{1r}^2 + g_{2r}^2 + \frac{g_{1\vartheta}^2 + g_{2\vartheta}^2}{G^2} \leq (Q^2 + 1) \frac{|g_{1r} g_{2\vartheta} - g_{1\vartheta} g_{2r}|}{QG}$$

(see [2: Formula (8)]). Using the inequality

$$2 \frac{|g_{1r} g_{2\vartheta} - g_{1\vartheta} g_{2r}|}{QG} \leq g_{1r}^2 + g_{2r}^2 + \frac{g_{1\vartheta}^2 + g_{2\vartheta}^2}{Q^2 G^2}$$

we obtain (3) by an addition of these inequalities (cf. [1: Formula (7)] in the case $G = r$).

3. A lemma

As in [2] and [4] we show the inequality

$$\int_0^R \Phi^2(r) L''(r)\, dr \leq \sigma \int_0^R \Phi'^2(r) L(r)\, dr$$

where the number $\sigma > 0$ does not depend on the test functions Φ. These test functions $\Phi = \Phi(r) : [0, R] \to \mathbb{R}$ are continuous in $[0, R]$, possess a derivative Φ' (in the sense of distributions) whose square is integrable over $[0, R]$, and fulfill $\Phi(R) = 0$, where $R > 0$ has been fixed. The set of all these functions Φ is denoted by V_R, for $R > 0$.

Starting from [2: Formula (10)] (see also [4: Formula (10)]) we get with the aid of the Cauchy-Schwarz inequality

$$\left[\int_0^R\int_0^{2\pi} \Phi(r)\Phi'(r)(g_1 g_{2\vartheta} - g_{1\vartheta} g_2)\,dr d\vartheta\right]^2 \le \int_0^R\int_0^{2\pi} \Phi^2(r) T\,dr d\vartheta \int_0^R\int_0^{2\pi} \Phi'^2(r)\frac{(g_1 g_{2\vartheta} - g_{1\vartheta} g_2)^2}{T}\,dr d\vartheta$$

(see [1], too). In the case $T = 0$, the latter fraction should be replaced by zero. Actually, (3) yields

$$\frac{(g_1 g_{2\vartheta} - g_{1\vartheta} g_2)^2}{T} \le \frac{(g_1^2 + g_2^2)(g_{1\vartheta}^2 + g_{2\vartheta}^2)}{T} \le |g|^2 QG.$$

Hence, from [2: Formula (10)] we obtain the inequality

$$\int_0^R\int_0^{2\pi} \Phi^2(r) T\,dr d\vartheta \le Q \int_0^R\int_0^{2\pi} \Phi'^2(r) G |g|^2 dr d\vartheta.$$

Thus, we have the following result.

Lemma. *Let* $M = \sup\{|g(r, \vartheta)| : 0 \le r \le R,\ 0 \le \vartheta \le 2\pi\}$ *for an arbitrary* $R > 0$. *Then, the function* $L = L(r)$ *from* (5) *satisfies*

$$\int_0^R \Phi^2(r) L''(r)\,dr \le 4M^2 Q \int_0^R \Phi'^2(r) L(r)\,dr \tag{6}$$

for all test functions $\Phi \in V_R$.

4. An apriori estimate for a mean value of K and a Bernstein-type theorem

By setting

$$\Phi = \Phi(r) = \begin{cases} \displaystyle\int_\rho^R \frac{dt}{L(t)} = \text{const} = c & \text{for } 0 \le r \le \rho \\ \displaystyle\int_r^R \frac{dt}{L(t)} & \text{for } \rho < r \le R \end{cases}$$

property (6) implies

$$\int_0^\rho L''(r)\,dr \le \frac{4M^2 Q}{c^2} \int_\rho^R \frac{1}{L^2(r)} L(r)\,dr = \frac{4M^2 Q}{c} \tag{7}$$

where $\rho \in (0, R)$ denotes a fixed number. Because of $K \leq 0$ and $G \geq 0$ one may derive $L''(r) \geq 0$ from (4) and (5) (see, e.g., [3: Chapter 6]). This yields $\frac{1}{L(r)} \geq \frac{R}{L(R)}\frac{1}{r}$ for all $r \in [\rho, R]$. So, (7) can be written in the form

$$\int_0^\rho L''(r)\,dr \leq \frac{4M^2QL(R)}{R\ln\frac{R}{\rho}}.$$

The quotient $\frac{L(R)}{R}$ is related to the total curvature of the geodesic disk with its center in P_0 and the radius R. Actually, it holds

$$\frac{L(R)}{R} \leq 2\pi + \int_0^R\int_0^{2\pi}(-K)G\,dr d\vartheta$$

due to [10: Corollary 1]. Therefore, we have the apriori estimate

$$\int_0^\rho\int_0^{2\pi}|K|G\,dr d\vartheta = \int_0^\rho L''(r)\,dr \leq \frac{4M^2Q}{\ln\frac{R}{\rho}}\left(2\pi - \int_0^R\int_0^{2\pi}KG\,dr d\vartheta\right) \tag{8}$$

from which we infer the following Bernstein-type theorem.

Theorem. *Let S be a (open oriented differential-geometric) complete quasiminimal surface in $\mathbb{R}^3$ whose Gauss map omits a neighborhood of the north pole of the unit sphere. $P_0 \in S$ denotes a point on S, and $B_R(P_0) \subset S$ denotes a geodesic disk with its center in P_0 and the radius $R > 0$. If for any fixed $\alpha \in (0, \infty)$ the quotient of the total curvature with respect to $B_R(P_0)$ and $\ln^\alpha(1+R)$ tends to zero as $R \to \infty$, then S must be a plane.*

Proof. Since the stereographic projection used for the definition of g maps the north pole to ∞, the function $g = g(r, \vartheta)$ is even bounded in $[0, \infty) \times [0, 2\pi)$. Thus, (8) yields

$$\int_0^\rho\int_0^{2\pi}G_{rr}\,dr d\vartheta = \int_0^\rho L''(r)\,dr = 0$$

for any $\alpha \in (0, 1]$ and any $\rho > 0$. Because of (4), the (non-negative) term G_{rr} vanishes for all $r \in [0, \rho]$ and all $\vartheta \in [0, 2\pi)$. Since we can choose ρ arbitrarily large, the Jacobian of $g = g(r, \vartheta)$ has the value zero for all $r \in [0, \infty)$ and all $\vartheta \in [0, 2\pi)$. Therefore, as in [2] we obtain $g \equiv$ const immediately. Hence, S can only be a plane.

Suppose now $\alpha > 1$ and let

$$C^\beta(t) = \left[\int_0^t L''(r)\,dr\right] \Big/ \ln^\beta(t)$$

for $t > 1$ and $\beta \in \mathbb{R}$. Setting $\rho = \sqrt{R}$ for $R > 1$ we get the inequality

$$C^{\alpha-1}(\sqrt{R}) \leq 8 \cdot 2^{\alpha-1}M^2Q\left[\frac{2\pi}{\ln^\alpha(R)} + C^\alpha(R)\right]$$

from (8). Because of $C^\alpha(R) \to 0$ as $R \to \infty$ the term $C^{\alpha-1}(\sqrt{R})$ tends to zero, too. After a finite number of such steps one can derive $C^1(R) \to 0$ as $R \to \infty$ so that the statement of the theorem follows as in the case $\alpha \in (0, 1]$ ∎

5. Remarks

In the following, the Bernstein-type theorem of this paper will be compared with some other results.

1. The above theorem generalizes [8: Corollary 1] not only with respect to the growth condition of the total curvature, but this theorem also holds for a larger class of quasiminimal surfaces than the result in [8].

2. In [9] Bernstein-type theorems were proved, too. We will show the relation of the growth condition to Assumption III in [8: Theorem 1] and to the corresponding assumption in [9: Section 5]. The first one has the form

$$\int_0^R L(r)\,dr \le d_0 R^2 \tag{9}$$

for any fixed $d_0 \in (0,\infty)$ and all $R > 0$, and the assumption in [9: Section 5] implies even (9) (see [8: Section 3/Remark 2]). Because of the monotonicity of L and $L \ge 0$ we infer

$$L(R)R \le \int_R^{2R} L(r)\,dr \le \int_0^{2R} L(r)\,dr \le 4d_0 R^2$$

for all $R > 0$ from (9). This means $L(R) \le 4d_0 R$. With the aid of $L(r) = \int L'(r)\,dr +$ const the same argument leads to $L'(R) \le 8d_0$. Consequently, we have

$$\lim_{R\to\infty} \left\{ \left[\int_0^R L''(r)\,dr \right] \Big/ \ln^{\alpha}(1+R) \right\} = 0$$

for all $\alpha \in (0,\infty)$. Therefore, the growth condition used in the above theorem is weaker than (9).

3. With regard to the function-theoretical proof of the Bernstein-type theorem for minimal surfaces (see [6]) it is desirable to derive an analogous theorem for quasiminimal surfaces without the growth condition in the above theorem.

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