

Schrödinger Operators with Moving Point Perturbations and Related Solvable Models of Quantum Mechanical Systems

V. A. Geyler and I. V. Chudaev

Abstract. A class of explicitly solvable models including those of low-dimensional systems in a magnetic field is considered. The spectral analysis of these models is reduced to the investigation of the spectrum of an ordinary differential operator perturbed by a point potential with varying support. The complete analysis of the variation of the eigenvalues and eigenfunctions with position of the potential support is given.

Keywords: *Schrödinger operators, point perturbation, low-dimensional systems in a magnetic field, impurity in a well*

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0. Introduction

We start with an example of special interest to low-dimensional system physics. Consider a two-dimensional electron system [5] in the presence of a uniform magnetic field parallel to the system plane. The problem to describe the spectrum of the system arises in studies of the optical and transport properties of two-dimensional charge carriers subjected to a uniform in-plane magnetic field (see, e.c., [3 - 5, 8, 14, 20, 23, 33]). The most frequently used picture of the spectrum is that obtained by means of a perturbation with respect to the field strength. Unfortunately, such perturbation methods are satisfactory in a rather restricted region of weak fields only [5]. The reason is that an arbitrary small variation of the field strength leads to an arbitrarily large variation of the vector potential of the field. In fact, the Hamiltonian H_s of a charged (charge e) particle of mass m in the considered system can be chosen in the form

$$H_s = -\frac{\hbar^2}{2m} \left[\left(\frac{\partial}{\partial x} + \frac{ieB}{c\hbar} y \right)^2 + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] + \lambda \delta(y). \quad (1)$$

Here B is the strength of the uniform magnetic field which is parallel to the z -axis, δ is the Dirac delta function, and λ is a coupling constant of the system confinement

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potential $U(x, y, z) = \lambda \delta(y)$, $\lambda < 0$. In the momentum representation with respect to the momenta p_x and p_z , the operator (1) has the form 2

$$H_s = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{m\omega^2}{2}(y+q)^2 + \lambda \delta(y) + \frac{1}{2m} p_z^2 \quad (2)$$

where $\omega = \frac{|eB|}{cm}$ is the cyclotron frequency and $q = \frac{p_x}{\omega m}$ is the y -coordinate of the classical cyclotron orbit centre [18]. From (2) we obtain the following dispersion laws for the energy spectrum of H_s :

$$E_n(q, p_z) = E_n(q) + \frac{1}{2m} p_z^2,$$

where $\{E_n(q) : n \in \mathbb{N}\}$ is the set of all the eigenvalues of the perturbed harmonic oscillator

$$H_{os} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{m\omega^2}{2} y^2 + \lambda \delta(y-q).$$

It is important to note that choosing $\lambda > 0$ in (1) we get a Hamiltonian of magnetosurface states in the presence of an in-plane magnetic field [7]. So in both cases ($\lambda > 0$ and $\lambda < 0$) we come to the spectral problem for an operator of the form

$$H = -\frac{d^2}{dx^2} + V(x) + \lambda \delta(x-q) \quad (3)$$

where the unperturbed operator $-\frac{d^2}{dx^2} + V(x)$ has a discrete spectrum. The principal interest in studying H is investigating the q -dependence of the eigenvalues of the operator H .

Here are more examples of quantum mechanical systems for which the spectral analysis reduces to the study of an operators having the form (3).

1. Narrow conducting channel in an inversion layer or heterostructure subjected to a quantizing magnetic field perpendicular to the layer plane. The Hamiltonian of a charged particle in the system has the form

$$H_s = -\frac{\hbar^2}{2m} \left[\left(\frac{\partial}{\partial x} - \frac{ieB}{c\hbar} y \right)^2 + \frac{\partial^2}{\partial y^2} \right] + \lambda \delta(y). \quad (4)$$

The recent progress in fabrication of two-dimensional electron systems with additional one-dimensional structures makes the possibility of experimental studying quantum transport in systems with Hamiltonian (4) (see, e.g., [25, 30, 31]).

2. Taking $\lambda > 0$ in formula (4), we obtain a model of edge states in a two-dimensional system at the presence of a magnetic field [21, 27], and the model of tunneling between quantum Hall systems [16].

3. Equation (3) describes a short-range impurity in a potential well. In this case $V(x)$ is the confining potential of the well and λ is the strength of the impurity potential ($\lambda < 0$ for attractive impurities and $\lambda > 0$ for repulsive ones). The considered model

was employed, e.g., in [10, 19, 22, 24]. Besides, taking $\lambda > 0$ we obtain the Hamiltonian of a charged particle tunneling through a moving barrier [6, 29].

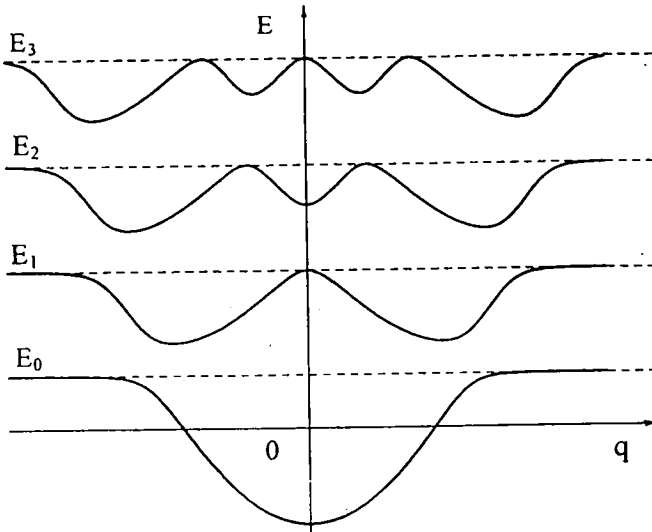


Figure 1: The plot of the eigenvalues $E_n(q)$ as functions of q in the case of $\lambda < 0$

In the papers cited, the q -dependence of the spectrum of H has been studied by numerical methods. In the case of a harmonic oscillator potential V ($V(x) = \frac{x^2}{4}$) the typical plot of the eigenvalues $E_n(q)$ as functions of q is presented on Figures 1 - 3.

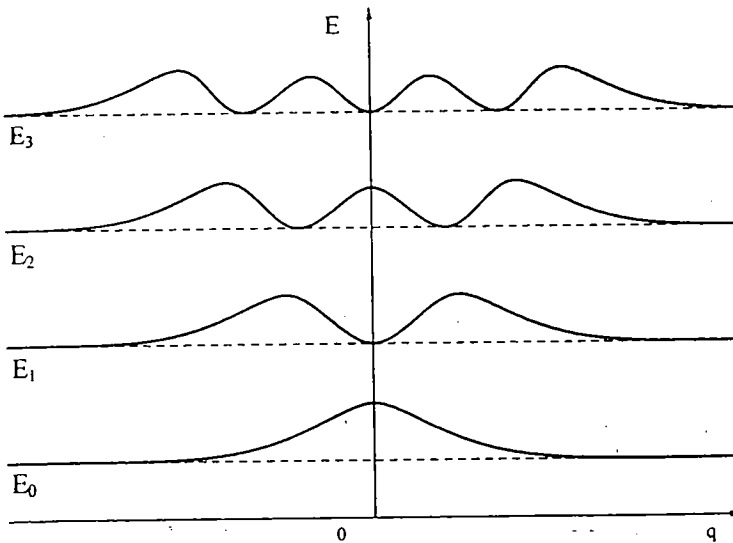


Figure 2: The plot of the eigenvalues $E_n(q)$ as functions of q in the case of $\lambda > 0$

They present the case of $\lambda < 0$, $\lambda > 0$ and $\lambda = 0$, respectively. On these figures, ϵ_n are the levels of the harmonic oscillator: $\epsilon_n = \frac{n+1}{2}$. The peculiarities of the plots are used for explanation of series of experimental data (see, e.c., [8, 14, 33]). As far as we know, there is no strict mathematical analysis of these peculiarities in the general case. For the case of a harmonic oscillator potential $V(x)$ such an analysis is presented in [13]; detailed proofs are contained in [11]. The proofs of main results of the mentioned paper are based on specific properties of the Green function of the harmonic oscillator and are not extended to more general potentials $V(x)$. The aim of the present paper is to present strict mathematical results explaining the above-mentioned peculiarities in the case of a sufficiently general form of the potential $V(x)$ in equation (3). We use an approach based on operator extension theory [2, 28]. In this approach, the Krein resolvent formula for selfadjoint extensions of symmetric operator plays the crucial role [17]. The possibility to use the Green function version of this formula for operators of the form (3) follows from results of [1, 12].

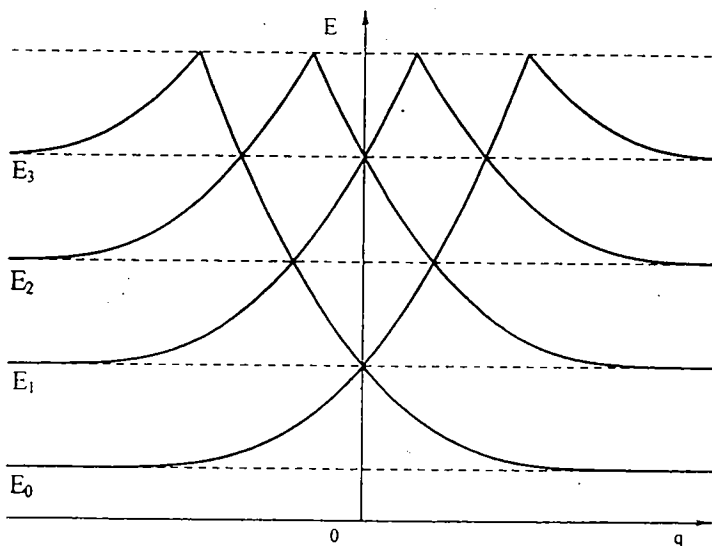


Figure 3: The plot of the eigenvalues $E_n(q)$ as functions of q in the case of $\lambda = 0$

1. Preliminaries

In this section we introduce basic notations and state some auxiliary results (we refer to [9, 26] for details). During the paper, H_0 denotes a self-adjoint operator in the space $L^2(a, b)$ ($-\infty \leq a < b \leq +\infty$) generated by a formally self-adjoint differential expression τ of order 2ν :

$$\tau(f) = (-1)^\nu (p_0 f^{(\nu)})^{(\nu)} + (-1)^{(\nu-1)} (p_1 f^{(\nu-1)})^{(\nu-1)} + \dots + p_\nu f.$$

To avoid some pathologies we suppose that the following conditions are valid:

(L1) The derivatives (in sense of distributions) of the functions p_k ($0 \leq k \leq \nu - 1$) up to order $\nu - k - 1$ are locally absolutely continuous functions on (a, b) .

(L2) $p_0(x) \neq 0$ for all $x \in (a, b)$.

(L3) $p_\nu \in L^2_{loc}(a, b)$.

In particular, under these conditions we have

$$C^\infty_0(a, b) \subset \mathcal{D}(H_0) \subset C(a, b), \tag{5}$$

where $\mathcal{D}(H_0)$ is the domain of H_0 .

As usually, $\sigma(H_0)$ denotes the spectrum of the operator H_0 , $\rho(H_0) = \mathbb{C} \setminus \sigma(H_0)$, and $R_0(\zeta)$ denotes the resolvent of the operator H_0 : $R_0(\zeta) = (H_0 - \zeta)^{-1}$. By I we denote the interval (a, b) with added regular endpoints. The operator $R_0(\zeta)$ has an integral kernel $G_0(x, y; \zeta)$ (the Green function of H_0), this function satisfies the following conditions:

(G1) $\int_a^b |G_0(x, y; \zeta)|^2 dy < \infty$ for all $x \in I$ and $\zeta \in \rho(H_0)$.

(G2) If $m, n \in \mathbb{N}$ and $m + n \leq 2\nu - 1$, then all the functions $\partial_x^m \partial_y^n G_0(x, y; \zeta)$ are continuous in the region $I \times I \times \rho(H_0)$ and holomorphic in $\zeta \in \rho(H_0)$ for any fixed x and y .

Finally, the following supposition will be always assumed:

(S) The spectrum of the operator H_0 is discrete and bounded from below: $\sigma(H_0) = \{\lambda_n : n \in \mathbb{N}\}$ where $\lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$.

We shall denote by $\kappa(n)$ the multiplicity of the eigenvalue λ_n ; it is well-known that $\kappa(n) \leq 2\nu$. For every n we fix an orthonormal system $\Phi_{n,k}$ ($k = 1, \dots, \kappa(n)$) of eigenfunctions of the operator H_0 corresponding to the eigenvalue λ_n . Then

$$G_0(x, y; \zeta) = \sum_{n=0}^{\infty} \sum_{k=1}^{\kappa(n)} (\lambda_n - \zeta)^{-1} \Phi_{n,k}(x) \overline{\Phi_{n,k}(y)}, \tag{6}$$

and by virtue of the Mercer theorem the series

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\kappa(n)} |(\lambda_n - \zeta)^{-1} \Phi_{n,k}(x) \overline{\Phi_{n,k}(y)}|$$

converges locally uniformly on $I \times I \times \rho(H_0)$.

Let $q \in (a, b)$ and $-\infty < \mu < +\infty$, $\mu \neq 0$. We shall consider the self-adjoint operator $H \equiv H(q, \mu)$ in the space $L^2(a, b)$ formally defined by the expression

$$H(q, \mu) = H_0 + \mu \delta(x - q). \tag{7}$$

To attach a rigorous mathematical meaning to the expression (7) we shall use the Krein resolvent formula [17]. According to that formula H is defined by its resolvent $R(\zeta)$ which has the integral kernel

$$G(x, y; \zeta) = G_0(x, y; \zeta) - [Q(\zeta) + \mu^{-1}]^{-1} G_0(x, q; \zeta) G_0(q, y; \zeta) \quad (8)$$

where $Q(\zeta)$ is the so-called Krein function. Using equation (2) and property (G1) we can show that $Q(\zeta)$ has the form $Q(\zeta) \equiv Q(\zeta, q) = G_0(q, q; \zeta)$ (see [12] for details). Hence

$$Q(\zeta) = \sum_{n=0}^{\infty} \sum_{k=1}^{\kappa(n)} (\lambda_n - \zeta)^{-1} |\Phi_{n,k}(q)|^2. \quad (9)$$

Formula (8) allows one to consider the operator $H(q, \mu)$ at $\mu = +\infty$; it can be shown that in this case $H(q, \mu)$ is the direct sum of two operators on the intervals (a, q) and (q, b) , respectively, with Dirichlet boundary conditions at the point q .

We start with some auxiliary assertions.

Lemma 1. *Let P_0 and P_1 are orthoprojections in a Hilbert space, and let $\alpha \in \mathbb{C}$. If $P_0 + \alpha P_1$ is an orthoprojection, then α equals one of the numbers 0, 1, -1.*

Lemma 2. *For every $x_0 \in (a, b)$ the relation $\Phi_{n,k}(x_0) \neq 0$ is valid for infinitely many values of the index n .*

Proof. Otherwise, we obtain from (5) that there exists a function $h \in C(a, b)$ which is a finite linear combination of functions $\Phi_{n,k}$ and obeys $\varphi(x_0) = \langle h | \varphi \rangle$ for all $\varphi \in C_0^\infty(a, b)$. Obviously, this is impossible ■

Lemma 3. *For a fixed $q \in (a, b)$ the following statements are true:*

(i) $\zeta \mapsto Q(\zeta, q)$ is a meromorphic function of $\zeta \in \mathbb{C}$ with infinitely many simple poles. The poles of this function are exactly those points λ_n for which there exists $k \in \{1, \dots, \kappa(n)\}$ such that $\Phi_{n,k}(q) \neq 0$.

(ii) $\frac{\partial Q(\zeta, q)}{\partial \zeta} > 0$ if $\zeta \in \rho(H_0) \cap \mathbb{R}$.

(iii) For real E the function $E \mapsto Q(E, q)$ increases from $-\infty$ to $+\infty$ as E varies between any two neighbouring poles.

(iv) For every $\zeta \in \rho(H_0)$ the function $x \mapsto G_0(x, q; \zeta)$ do not vanish identically on the interval (a, b) .

(v) $Q(\zeta, q) \rightarrow 0$ as $\Re \zeta \rightarrow -\infty$ locally uniformly with respect to q on the interval (a, b) .

Proof. It follows immediately from Lemma 2 and formulae (6) and (9) ■

The following theorem establishes a relation between formulae (7) and (8); the proof of the theorem can be found in [1]. Before to formulate the theorem, we remind that the symbol $\mathcal{Q}(H)$ denotes as usually the form domain of H .

Theorem 1. *Let $\mu \neq +\infty$. Then the following statements are true:*

- (i) $\mathcal{Q}(H_0) = \mathcal{Q}(H) \subset C(a, b)$.
- (ii) *If $\psi \in \mathcal{Q}(H)$, then $h(\psi) = h_0(\psi) + \mu|\psi(q)|^2$ where h and h_0 are the quadratic forms associated with the operators H and H_0 , respectively.*
- (iii) *The quadratic form $|\psi(q)|^2$ ($\psi \in \mathcal{Q}(H_0)$) is infinitesimal small with respect to the form h_0 .*

2. Spectral properties of the operator $H(q, \mu)$ for fixed q

Fix a point $q \in \mathbb{R}$, and let $\lambda_{n_0} < \lambda_{n_1} < \dots < \lambda_{n_k} < \dots$ be the increasing sequence of all the poles of the function $\zeta \mapsto Q(\zeta, q)$ (by virtue of Lemma 3 this sequence is infinite). We shall assume further that if a point $\zeta = \lambda_n$ does not belong to the sequence, then the function $Q(\zeta, q)$ is equal to its continuous extension to the point. In the spectral analysis of the operator $H(q, \mu)$, the key part is played by the following equation with respect to the spectral parameter ζ (known as *dispersion equation*):

$$Q(\zeta, q) + \mu^{-1} = 0. \tag{10}$$

For any $\zeta \neq \lambda_{n_k}$ the function $G_0(\cdot, q; \zeta)$ does not vanish identically (Lemmas 2 and 3). Hence by virtue of formula (8) every ζ satisfying equation (10) is a pole of the resolvent $R(\zeta)$ and therefore belongs to the spectrum $\sigma(H)$. Thus equation (10) has only real solutions. It follows from Lemma 3 that for every μ this equation has exactly one solution in each interval $(\lambda_{n_0}, \lambda_{n_1}), (\lambda_{n_1}, \lambda_{n_2}), \dots$. We arrange these solutions in an increasing sequence $\mathcal{E}_1(q) < \mathcal{E}_2(q) < \dots$. If $\mu > 0$, equation (10) has no other solutions; otherwise it has an additional solution $\mathcal{E}_0(q)$ lying in the interval $(-\infty, \lambda_{n_0})$.

Theorem 2. *The spectrum of the operator $H(q, \mu)$ is a discrete one and consists of four non-intersecting parts σ_k ($k = 1, \dots, 4$) which are described as follows.*

(i) *The part σ_1 consists of all the solutions $\mathcal{E}_j(q)$ of equation (10) which are different from the eigenvalues λ_n of the operator H_0 . Each solution $\mathcal{E}_j(q)$ is a simple eigenvalue of the operator $H(q, \mu)$, the corresponding normalized eigenfunction has the form*

$$\hat{\Phi}_j(x) = G_0(x, q; \mathcal{E}_j(q)) \left[\frac{\partial Q}{\partial E}(\mathcal{E}_j(q), q) \right]^{-\frac{1}{2}}.$$

(ii) *The part σ_2 consists of all the eigenvalues λ_m of the operator H_0 satisfying equation (10) and such that $\Phi_{m,k}(q) = 0$ for all $k = 1, \dots, \kappa(m)$. Each point of σ_2 is an eigenvalue of the operator $H(q, \mu)$ of multiplicity $\kappa(m) + 1$; an orthonormal bases of the corresponding eigensubspace is spanned by the functions $\Phi_{m,k}$ and the function*

$$\Psi_m(x) = G_0(x, q; \lambda_m) \left[\frac{\partial Q}{\partial E}(\lambda_m, q) \right]^{-\frac{1}{2}}.$$

(iii) *The part σ_3 consists of all the eigenvalues λ_m of the operator H_0 which do not satisfy equation (10) and such that $\Phi_{m,k}(q) = 0$ for all $k = 1, \dots, \kappa(m)$. Each point of*

σ_3 is an eigenvalue of the operator $H(q, \mu)$ of multiplicity $\kappa(m)$; an orthonormal bases of the corresponding eigensubspace is spanned by the functions $\Phi_{m,k}$.

(iv) The part σ_4 consists of all the eigenvalues λ_m of the operator H_0 for which $\kappa(m) \geq 2$ and which are poles of the function $\zeta \mapsto Q(\zeta, q)$ (i.e., for which $\Phi_{m,k}(q) \neq 0$ at least for one k). Each point of σ_4 is an eigenvalue of the operator $H(q, \mu)$ of multiplicity $\kappa(m) - 1$; the corresponding eigensubspace is the orthogonal complement of the function

$$\widehat{F}_m(x) = \overline{\Phi_{m,1}(q)} \Phi_{m,1}(x) + \dots + \overline{\Phi_{m,\kappa(m)}(q)} \Phi_{m,\kappa(m)}(x)$$

in the eigensubspace of the operator H_0 associated with λ_m .

Proof. It is convenient now to rewrite the Krein formula (8) in the operator form

$$R(\zeta) = R_0(\zeta) - [Q(\zeta, q) + \mu^{-1}]^{-1} g_q(\zeta) \langle g_q(\zeta) | \cdot \rangle \quad (11)$$

where $g_q(\zeta) = G_0(\cdot, q; \zeta)$. Let us introduce the notation

$$A(\zeta) = [Q(\zeta, q) + \mu^{-1}]^{-1} g_q(\zeta) \langle g_q(\zeta) | \cdot \rangle.$$

Further, we shall use the following well-known assertion: the orthoprojector P on the eigensubspace of a selfjoint operator T corresponding to an isolated point E_0 of its spectrum has the form

$$P = -\text{Res}[(T - \zeta)^{-1}; E_0]. \quad (12)$$

If $\mathcal{E}_j(q) \neq \lambda_n$ for all n , then from (11) it follows immediately that $\mathcal{E}_j(q)$ is a pole of the resolvent $R(\zeta)$. In this case equation (12) determines the orthoprojector P on the subspace spanned by the vector $g_q(\mathcal{E}_j(q))$. An elementary calculation of the residue leads to the expression $P = \hat{g}_q(\mathcal{E}_j(q)) \langle \hat{g}_q(\mathcal{E}_j(q)) | \cdot \rangle$ where

$$\hat{g}_q(\mathcal{E}_j(q)) = g_q(\mathcal{E}_j(q)) \left[\frac{\partial Q}{\partial E}(\mathcal{E}_j(q), q) \right]^{-\frac{1}{2}}.$$

Let now $\mathcal{E}_j(q) = \lambda_m$ where λ_m is not a pole of the function $\zeta \mapsto Q(\zeta, q)$, i.e. let $\Phi_{m,k}(q) = 0$ for all $k = 1, \dots, \kappa(m)$. Then in a neighbourhood of the point λ_m we have

$$Q(\zeta, q) + \mu^{-1} = \alpha_1(\zeta - \lambda_m) + \alpha_2(\zeta - \lambda_m)^2 + \dots$$

where $\alpha_1 = \frac{\partial Q}{\partial \zeta}(\lambda_m, q) > 0$. Thus $\text{Res}[A(\zeta); \lambda_m] = \alpha P_1$ where $\alpha > 0$ and P_1 is the orthoprojector on the subspace spanned by the vector $g_q(\lambda_m)$. Hence $-\text{Res}[R(\zeta); \lambda_m] = P_0 + \alpha P_1$, where $P_0 = -\text{Res}[R_0(\zeta); \lambda_m]$; P_0 is the orthoprojector on the subspace spanned by the functions $\Phi_{m,k}$ ($k = 1, \dots, \kappa(m)$). Using Lemma 1, we obtain that $\alpha = 1$. Therefore

$$P \equiv -\text{Res}[R(\zeta); \lambda_m] = P_0 + P_1. \quad (13)$$

It follows from equation (13) that P is the orthoprojector on the subspace spanned by the functions $\Phi_{m,k}$ and the function Ψ_m from item (ii) of the theorem.

Let us suppose now that λ_m is not a pole of the function $\zeta \mapsto Q(\zeta, q)$ and is not a solution of equation (10), i.e. it is not equal to any $\mathcal{E}_j(q)$. Then $\text{Res}[A(\zeta); \lambda_m] = 0$,

therefore $-\text{Res}[R(\zeta); \lambda_m] = -\text{Res}[R_0(\zeta); \lambda_m]$. Thus λ_m is an eigenvalue of the operator H associated with the eigensubspace spanned by the vectors $\Phi_{m,k}$.

Finally, let λ_m be a pole of the function $\zeta \mapsto Q(\zeta, q)$, i.e. $\Phi_{m,k}(q) \neq 0$ at least for one $k \in \{1, \dots, \kappa(m)\}$. Then λ_m is a pole of the function $\zeta \mapsto g_q(\zeta)$; the corresponding residue we denote by \widehat{F}_m . Hence $\text{Res}[A(\zeta); \lambda_m] = \beta P_1$ where $\beta \neq 0$ and P_1 is the orthoprojector on the subspace spanned by the function $\widehat{F}_m(x)$. Therefore $P \equiv -\text{Res}[R(\zeta); \lambda_m] = P_0 + \beta P_1$. Because P is an orthoprojector and $P \neq P_0$, then by virtue of Lemma 1, $\beta = \pm 1$. Since the range of P_1 is contained in the range of P_0 , then $\beta = -1$. Since P_1 is a one-dimensional projector, $\dim \text{Rang} P = \dim \text{Rang} P_0 - 1$. The theorem is proved ■

3. The dispersion laws $E_n(q)$

Our next aim is to analyse the q -dependence of the eigenvalues $\mathcal{E}_j(q)$. Since the set $\{\lambda_{n_k}\}$ of the poles of the function $\zeta \mapsto Q(\zeta, q)$ depends on q , the use of the eigenvalues $\mathcal{E}_j(q)$ is not relevant to the study of their q -dependence properties. In this connection we give another parametrization of the eigenvalues of H . As a preliminary step we introduce the notations

$$\lambda_{-1} = -\infty$$

$$X_{-1} = \mathbb{R}$$

$$X_n = \{q \in \mathbb{R} : \Phi_{n,j}(q) \neq 0 \text{ for all } j = 1, \dots, \kappa(n)\} \quad (n \geq 0).$$

Since each function $\Phi_{n,j}$ is a solution of a linear differential equation, the sets $\mathbb{R} \setminus X_n$ are discrete. The function $E \mapsto Q(E, q)$ is continuously differentiable on each set $(\lambda_{k-1}, \lambda_k) \times (X_{k-1} \cap X_k)$ and real-analytic on the interval $(\lambda_{k-1}, \lambda_k)$ as a function of E for any fixed $q \in X_{k-1} \cap X_k$. As is shown above, equation (10) for any $q \in X_{k-1} \cap X_k$ has just one solution belonging to the interval $(\lambda_{k-1}, \lambda_k)$; we denote this solution by $E_k(q)$. Recall that the index k runs from 1 if $\mu > 0$ and from 0 if $\mu < 0$. Because $\frac{\partial Q}{\partial E} \neq 0$ on the set $(\lambda_{k-1}, \lambda_k) \times (X_{k-1} \cap X_k)$, the implicit function theorem implies that $E_k(q)$ is a continuously differentiable function on the set $X_{k-1} \cap X_k$.

Proposition 1. *Each function $E_k(q)$ has a continuous extension on the whole real line \mathbb{R} .*

Proof. We fix $s \in \mathbb{R} \setminus X_{k-1} \cap X_k$ and choose a sequence $(q_n) \subset X_{k-1} \cap X_k$, with $q_n \rightarrow s$. From Lemma 3 it is easy to show that the sequence $E_k(q_n)$ is bounded, hence, we can extract a subsequence $E_k(q_{n_m})$ converging to a limit which we denote by \widehat{E} . To complete the proof of the statement we need two lemmas.

Lemma 4. *The following assertions are true:*

- (i) *The number \widehat{E} is not a pole of the function $\zeta \mapsto Q(\zeta, s)$.*
- (ii) *If $\lambda_{k-1} < \widehat{E} < \lambda_k$, then $\zeta = \widehat{E}$ is a solution of the equation $Q(\zeta, s) + \mu^{-1} = 0$ (see (10)), and there exists no other solutions of the equation on the interval $(\lambda_{k-1}, \lambda_k)$.*

(iii) If $\widehat{E} = \lambda_{k-1}$, then $\lim_{E \rightarrow \widehat{E}} [Q(E, s) + \mu^{-1}] \geq 0$.

(iv) If $\widehat{E} = \lambda_k$, then $\lim_{E \rightarrow \widehat{E}} [Q(E, s) + \mu^{-1}] \leq 0$.

Proof. Assertion (i): Let $f(\zeta) = (\zeta - \lambda_{k-1})(\zeta - \lambda_k)$; we consider the function

$$\widehat{Q}_\mu(\zeta, q) = [Q(\zeta, q) + \mu^{-1}]f(\zeta), \tag{14}$$

which is continuous on the set $(\lambda_{k-2}, \lambda_{k+1}) \times \mathbb{R}$. Since $\widehat{Q}_\mu(E_k(q_{n_m}), q_{n_m}) = 0$, then by passing to the limit $m \rightarrow \infty$ we obtain

$$\widehat{Q}_\mu(\widehat{E}, s) = 0. \tag{15}$$

If \widehat{E} is a pole of the function $\zeta \mapsto Q(\zeta, s)$, then $\widehat{Q}_\mu(\widehat{E}, s) \neq 0$ by virtue of Formula (14), what contradicts to the equation (15).

Assertion (ii): Taking into account Lemma 3/(ii), we obtain the statement of the item by passing to the limit in the equality $Q(E_k(q_{n_m}), q_{n_m}) + \mu^{-1} = 0$.

Assertion (iii): By virtue of item (i) the function $\zeta \mapsto Q(\zeta, s)$ is continuous in a neighbourhood of λ_{k-1} , therefore, the limit $\lim_{E \rightarrow \lambda_{k-1}} (Q(E, s) + \mu^{-1}) \equiv L$ exists. Let us assume that $L < 0$, then we have $Q(E_0, s) + \mu^{-1} < 0$ for some $E_0 \in (\lambda_{k-1}, \lambda_k)$. Hence for some m the inequalities $Q(E_0, s) + \mu^{-1} < 0$ and $E_k(q_{n_m}) < E_0$ are satisfied simultaneously. Since $Q(E, q)$ increases on the interval $(\lambda_{k-1}, \lambda_k)$ as function of E , we obtain the contradiction $0 < Q(E_k(q_{n_m}), q_{n_m}) + \mu^{-1} < Q(E_0, q_{n_m}) + \mu^{-1} < 0$.

Assertion (iv): The proof of this item is similar to that of item (iii) ■

Lemma 5. For any sequence $(s_n) \subset X_{k-1} \cap X_k$ converging to s the relation $E_k(s_n) \rightarrow \widehat{E}$ takes place.

Proof. Suppose that for some sequence $(s_n) \subset X_{k-1} \cap X_k$ we have $s_n \rightarrow s$, but $E_k(s_n) \not\rightarrow \widehat{E}$. Then we can choose a subsequence (s_{n_k}) such that $E_k(s_{n_k}) \rightarrow E^*$, and $E^* \neq \widehat{E}$. Suppose that $E^* > \widehat{E}$.

Taking into account Lemma 4/(ii) and the fact that Q is strictly increasing of E , we show that $\widehat{E} = \lambda_{k-1}$ or $E^* = \lambda_k$. In both cases we have

$$\lim_{E \rightarrow \widehat{E}} [Q(E, s) + \mu^{-1}] \geq 0 \quad \text{and} \quad \lim_{E \rightarrow E^*} [Q(E, s) + \mu^{-1}] \leq 0. \tag{16}$$

Let us take some numbers E_1 and E_2 so that $\widehat{E} < E_1 < E_2 < E^*$. Then from (16) with regard to the strict monotonicity of Q as function of E we have

$$0 \leq Q(E_1, s) + \mu^{-1} < Q(E_2, s) + \mu^{-1} \leq 0.$$

This is a contradiction. Taking into account Lemmas 4 and 5, we complete the proof of the Proposition 1 ■

The theorem below is the main result of this section.

Theorem 3. For each fixed $0 \neq \mu \in (-\infty, +\infty]$ there is a sequence $(E_n(q))$ of continuous functions of $q \in \mathbb{R}$ such that the following properties are satisfied:

- (i) $\lambda_{n-1} \leq E_n(q) \leq \lambda_n$ for all n .
- (ii) $E_n(q) \leq E_{n+1}(q)$ for all admissible values of n .
- (iii) For each q the set consisting of all $E_n(q)$ and all the numbers λ_m with $\kappa(m) \geq 2$ is the complete collection of all the eigenvalues of the operator $H(q, \mu)$.
- (iv) If $\lambda_{n-1} < E_n(q) < \lambda_n$, then $E_n(q)$ is the unique solution of the dispersion equation (10).
- (v) If λ_n is a pole of the function $\zeta \mapsto Q(\zeta, q)$ (i.e. if $\Phi_{n,k}(q) \neq 0$ at least for one k), then $\lambda_{n-1} < E_n(q) < \lambda_n$.
- (vi) Provided that λ_n is not a pole of the function $\zeta \mapsto Q(\zeta, q)$ (i.e. $\Phi_{n,k}(q) = 0$ for all k), we have the following:
 - (a) If $Q(\lambda_n, q) + \mu^{-1} < 0$, then $E_n(q) = \lambda_n < E_{n+1}(q)$.
 - (b) If $Q(\lambda_n, q) + \mu^{-1} > 0$, then $E_n(q) < \lambda_n = E_{n+1}(q)$.
 - (c) If $Q(\lambda_n, q) + \mu^{-1} = 0$, then $E_n(q) = \lambda_n = E_{n+1}(q)$.

Definition. The points q for which condition (iv) of Theorem 3 is satisfied we shall call the *non-singular points* of the function $E_n(q)$. If condition (vi)/(a), (vi)/(b) or (vi)/(c) is satisfied, then q will be called *singular point of kind* (a), (b) or (c), respectively.

Proof of Theorem 3. By virtue of Proposition 1 each function $E_n(q)$ can be extended by continuity on the whole real line \mathbb{R} ; the extension we denote by $E_n(q)$ again. Properties (i) and (ii) of the function $E_n(q)$ are obvious, property (iii) follows from Theorem 2 and properties (iv) and (vi). Properties (iv) and (v) follow immediately from the definition of the functions $E_n(q)$ and Lemma 4. It remains to prove property (vi).

Let λ_n be not a pole of the function $\zeta \mapsto Q(\zeta, q)$, and let $Q(\lambda_n, q) + \mu^{-1} < 0$. For any positive integer m we choose a number E_m such that $\lambda_n - \frac{1}{m} < E_m < \lambda_n$; then $Q(E_m, q) + \mu^{-1} < 0$. Further, we choose points $q_m \in \mathbb{R}$ for which λ_{n-1} and λ_n are not the poles of the function $\zeta \mapsto Q(\zeta, q_m)$ (that is $q_m \in X_{m-1} \cap X_m$), and such that $|q - q_m| < \frac{1}{m}$ and $Q(E_m, q_m) + \mu^{-1} < 0$. Then $E_n(q_m)$ is a solution of the equation $Q(E, q_m) + \mu^{-1} = 0$ laying in the interval $(\lambda_{n-1}, \lambda_n)$. Since Q is a strictly monotone function of E in this interval, the inequalities $E_m < E_n(q_m) < \lambda_n$ take place for all m . Thus $E_n(q_m) \rightarrow \lambda_n$ and $q_m \rightarrow q$ as $m \rightarrow \infty$; therefore $\lambda_n = E_n(q)$ by the definition of the function $E_n(q)$. According to Lemma 4/(iii), $Q(\lambda_n, q) + \mu^{-1} \geq 0$ if $\lambda_n = E_{n+1}(q)$; therefore $\lambda_n < E_{n+1}(q)$. Thus item (vi)/(a) is proved. The proofs of items (vi)/(b) and (vi)/(c) are similar ■

4. Properties of the dispersion laws $E_n(q)$ for Schrödinger operators

Here we consider the most important class of operators H_0 , namely Schrödinger operators on the real line \mathbb{R} . Throughout the section, H_0 denotes the operator in $L^2(\mathbb{R})$ given by the differential expression

$$\tau = -\frac{d^2}{dx^2} + V(x) \quad (17)$$

where the potential V belongs to the space $L^2_{loc}(\mathbb{R})$ and is bounded from below. In the considered case the expression (17) defines an essentially selfadjoint operator on $C_0^\infty(\mathbb{R})$.

We need some additional restrictions on the potential V . First of all we shall assume that the condition

$$(V1) \lim_{x \rightarrow \pm\infty} V(x) = +\infty$$

is fulfilled. Under this condition the operator H_0 has a simple discrete spectrum, and each eigenfunction $\Phi_m \equiv \Phi_{m,1}$ has strictly m zeros on the real line. In addition, we can suppose that all the functions Φ_m are real-valued ones.

In the case of the Schrödinger operator, the following representation of the Green function G_0 is convenient. Let $\zeta_0 \in \rho(H_0)$. Then the equation $H_0\Psi = \zeta_0\Psi$ has up to a multiplicative constant only one non-zero square integrable solution $\Psi_+(x; \zeta_0)$ and $\Psi_-(x; \zeta_0)$ on the half-lines $(0, +\infty)$ and $(-\infty, 0)$, respectively. Let

$$\omega(\zeta_0) = W(\Psi_+(x; \zeta_0), \Psi_-(x; \zeta_0))$$

be the Wronskian of these solutions. Then

$$G_0(x, y; \zeta_0) = \frac{\Psi_+(\max(x, y); \zeta_0) \Psi_-(\min(x, y); \zeta_0)}{\omega(\zeta_0)} \quad (18)$$

(see [32]). In a neighbourhood of the point ζ_0 the functions $\Psi_+(x; \zeta)$ and $\Psi_-(x; \zeta)$ can be chosen to be analytical ones of ζ . In addition, the function $\omega(\zeta)$ has only simple zeros, and these zeros coincide with the eigenvalues λ_n of the operator H_0 . From (18) we obtain

$$Q(\zeta, q) = \frac{\Psi_+(q; \zeta) \Psi_-(q; \zeta)}{\omega(\zeta)}. \quad (19)$$

Let $\zeta \in \mathbb{R}$ and $\zeta < \lambda_0$. Then the functions $\Psi_+(x; \zeta)$ and $\Psi_-(x; \zeta)$ can be chosen to be strictly positive for all $x \in \mathbb{R}$. For the same values of ζ , the Green function G_0 is the integral kernel of a positive operator, therefore

$$Q(\zeta, q) = G_0(q, q; \zeta) \geq 0 \quad (20)$$

for all $q \in \mathbb{R}$. On the other hand, by virtue of Lemma 3, $\frac{\partial Q}{\partial \zeta} > 0$ and $\lim_{\zeta \rightarrow -\infty} Q(\zeta, q) = 0$, hence for all $q \in \mathbb{R}$ and for the same values of ζ , we have

$$Q(\zeta, q) = G_0(q, q; \zeta) > 0. \quad (21)$$

In particular, we have

$$\omega(\zeta) > 0. \quad (22)$$

The case of even potentials $V(x)$ is of principal interest. In this case the functions Φ_{2m} are even and Φ_{2m+1} are odd. Besides, we can assume $\Psi_-(q; \zeta) = \Psi_+(-q; \zeta)$, therefore

$$Q(\zeta, q) = \frac{\Psi_+(q; \zeta) \Psi_+(-q; \zeta)}{\omega(\zeta)}. \quad (23)$$

Lemma 6. *Let the potential $V(x)$ be an even function. If $\Phi_m(q) = 0$, then $Q(\lambda_m, q) = 0$.*

Proof. Let $\Phi_m(q) = 0$. Then $\Phi_m(-q) = 0$ because $\Phi_m(x)$ is either odd or even. It is known (see, e.g., [31]) that for some neighbourhood U of λ_m there exists a family $\Psi(q; \zeta)$ which is analytic with respect to $\zeta \in U$, and such that $\Psi(q; \lambda_m) = \Phi_m(q)$ and $\Psi(q; \zeta) = \Psi_+(q; \zeta)$ for $\zeta \neq \lambda_m$. Since the function $\omega(\zeta)$ has a simple zero at the point λ_m , then the statement follows from equality (23) ■

Lemma 6 implies the following refinement of Theorem 2.

Proposition 2. *The following assertions hold:*

(i) *If H_0 is the Schrödinger operator, then the set σ_4 from Theorem 2 is empty.*

(ii) *Let the potential $V(x)$ be an even function. Then the following assertions take place:*

(a) *For $\mu \neq +\infty$ all the eigenvalues of the operator $H(q, \mu)$ are simple and belong either to the set σ_1 or to the set σ_3 from Theorem 2.*

(b) *For $\mu = +\infty$ the set σ_3 is empty, and all the eigenvalues λ_m from the spectrum of $H(q, \mu)$ are doubly degenerate.*

(c) *λ_0 does not belong to the spectrum of all the operators $H(q, \mu)$, while every eigenvalue of the form λ_{2m+1} belongs to the spectrum of any operator $H(0, \mu)$.*

Note that in the case of the Schrödinger operator, the number of the singular points of each function $E_n(q)$ is finite and does not exceed n . Moreover, in this case it is easy to establish the following statement.

Proposition 3. *If the potential $V(x)$ is a function of the class C^k ($0 \leq k \leq \infty$) on some interval (c, d) , then the functions $E_n(q)$ belong to the class C^{k+2} on the set of all non-singular points of this interval. If $V(x)$ is a real-analytic function on an interval (c, d) , then the functions $E_n(q)$ are real-analytic at all non-singular points of the interval as well as at all singular points of the kind (a) or (b). In particular, for $\mu \neq +\infty$ the functions $E_k(q)$ are real-analytic if the potential $V(x)$ is a real-analytic function.*

Proof. For non-singular points the statement of the proposition immediately follows from formula (19) and the implicit function theorem. Let now q_0 be a singular point of the kind (a) for the function $E_k(q)$. We consider the function $\hat{Q}_\mu(\zeta, q)$ defined by the equality

$$\hat{Q}_\mu(\zeta, q) = \omega(\zeta)(Q(\zeta, q) + \mu^{-1}). \quad (24)$$

In some neighbourhood of the point (λ_k, q_0) the function \widehat{Q}_μ is analytical, and in some neighbourhood of the point q_0 the function $E_k(q)$ is defined as an implicit function by the equation

$$\widehat{Q}_\mu(E, q) = 0. \tag{25}$$

It remains to verify that

$$\frac{\partial \widehat{Q}_\mu}{\partial E}(\lambda_m, q_0) \neq 0. \tag{26}$$

We have

$$\frac{\partial \widehat{Q}_\mu}{\partial E} = \left(\frac{\partial Q}{\partial E} \right) \omega(E) + (Q + \mu^{-1}) \omega'(E).$$

Because $\left| \frac{\partial Q}{\partial E}(\lambda_m, q_0) \right| < \infty$, we have $\frac{\partial Q}{\partial E}(\lambda_m, q_0) \omega(\lambda_m) = 0$. Since $\omega(\zeta)$ has a single zero in the point λ_m , then $\omega'(\lambda_m) \neq 0$. Besides, $Q + \mu^{-1}$ does not vanish in the point (λ_m, q_0) . Thus relation (26) is proved. In the case of points of the kind (b) the proof is similar ■

As a corollary of Theorem 1 we obtain the following statement.

Proposition 4. *For all $q \in \mathbb{R}$ the inequality*

$$-\frac{\mu^2}{4} \leq E_0(q) \tag{27}$$

holds.

Proof. We denote by B the selfadjoint operator in $L^2(\mathbb{R})$ associated with the quadratic form $\widehat{B}(\psi) = \|\psi'\|^2 + \mu |\psi(q)|^2$. It follows from Theorem 1 that $B \leq H$. The ground state of the operator B is equal to $-\frac{\mu^2}{4}$ (see [2]). Hence we obtain (27) ■

Further we shall use the following hypothesis about the potential V :

- (V2) The set of all discontinuity points of the function $V(x)$ is finite (possibly, empty); $V(x)$ decreases on the half-line $x \leq 0$ and increases on the half-line $x \geq 0$. This function is differentiable at continuity points.
- (V3) There exists a point $x_0 > 0$ such that the derivative $V'(x)$ is a locally absolutely continuous function for $|x| > x_0$, and $\int_{|x|>x_0} \left| \frac{5V'^2(x)}{4V^3(x)} - \frac{V''(x)}{V^2(x)} \right| \sqrt{V(x)} dx < \infty$.

For convenience, the partial derivatives of the function $Q(\zeta, q)$ with respect to q are denoted by a prime: $\frac{\partial Q(\zeta, q)}{\partial q} = Q'(\zeta, q)$, etc.

Lemma 7. *Suppose that condition (V2) is fulfilled. Then the following statements are true:*

(i) *For all $\zeta \in \rho(H_0)$ the function $q \mapsto Q(\zeta, q)$ has the continuous second derivative $Q''(\zeta, q)$ at all the continuity points of the function $V(q)$; at the discontinuity points there exist the one-sided limits $Q''(\zeta, s - 0)$ and $Q''(\zeta, s + 0)$.*

(ii) *Let $E \in \mathbb{R}$ and $E \neq \lambda_n$ for every n . If $\Psi_+(s; E) \neq 0$ and $Q'(E, s) = 0$ ($s \in \mathbb{R}$), then $Q''(E, s \pm 0) < 0$.*

(iii) Let $E \in \mathbb{R}$ and $E < \lambda_0$. There exists at most one point $q_0 \in \mathbb{R}$ such that $Q'(E, q_0) = 0$. If $Q'(E, q_0) = 0$, then $Q'(E, q) > 0$ for $q > q_0$, and $Q'(E, q_0) < 0$ for $q < q_0$.

Proof. Assertion (i): If condition (V2) is fulfilled, then each solution of the equation $H_0 F = \zeta F$ has piecewise continuous second derivative F'' , and in each point $s \in \mathbb{R}$ there exist the one-sided limits $F''(s-0)$ and $F''(s+0)$. Therefore assertion (i) follows from equality (19).

Assertion (ii): Using equality (19), we obtain in a neighbourhood of s

$$Q'(E, q) = \frac{[\Psi'_+(q; E)\Psi_-(q; E) + \Psi_+(q; E)\Psi'_-(q; E)]}{\omega(E)}, \quad (28)$$

and

$$\begin{aligned} Q''(E, q) &= \frac{\Psi''_+(q; E)\Psi_-(q; E) + \Psi_+(q; E)\Psi''_-(q; E) + 2\Psi'_+(q; E)\Psi'_-(q; E)}{\omega(E)} \\ &= 2 \frac{(V(q) - E)\Psi_+(q; E)\Psi_-(q; E) + \Psi'_+(q; E)\Psi'_-(q; E)}{\omega(E)} \end{aligned} \quad (29)$$

if q is a continuity point of the function V . From the equality $Q'(E, s) = 0$ and equality (19) we obtain

$$\Psi'_-(q; E) = -\frac{\Psi_-(q; E)\Psi'_+(q; E)}{\Psi_+(q; E)} + o(1) \quad \text{as } q \rightarrow s.$$

Substituting this expression into (29), we obtain

$$Q''(E, q) = 2\Psi_-(q; E) \frac{(V(q) - E)\Psi_+^2(q; E) - \Psi_+^{\prime 2}(q; E)}{\omega(E)\Psi_+(q; E)} + o(1) \quad (30)$$

as $q \rightarrow s$. Denote

$$f(q) = (V(q) - E)\Psi_+^2(q; E) - \Psi_+^{\prime 2}(q; E).$$

We have, at the continuity points q of the function V ,

$$\begin{aligned} f'(q) &= V'(q)\Psi_+^2(q; E) + 2(V(q) - E)\Psi_+(q; E)\Psi'_+(q; E) - 2\Psi_+(q; E)\Psi''_+(q; E) \\ &= V'(q)\Psi_+^2(q; E) + 2\Psi'_+(q; E)[(V(q) - E)\Psi_+(q; E) - \Psi_+^{\prime 2}(q; E)] \\ &= V'(q)\Psi_+^2(q; E). \end{aligned} \quad (31)$$

Thus $f'(q) \geq 0$ at the continuity points q of the function V obeying $q \geq 0$. Since $V(q)$ has positive jumps provided $q \geq 0$, the function $f(q)$ increases for $q \geq 0$. Now we shall prove that

$$\liminf_{q \rightarrow +\infty} f(q) \leq 0. \quad (32)$$

Let us rewrite $f(q)$ in the form

$$f(q) = g(q) - 2\Psi_+^{\prime 2}(q; E) \quad \text{where } g(q) = (V(q) - E)\Psi_+^2(q; E) + \Psi_+^{\prime 2}(q; E).$$

To prove (32) it is sufficient to show that

$$\liminf_{q \rightarrow +\infty} g(q) \leq 0. \tag{33}$$

In order to prove this, we show that $\int_a^\infty g(x) dx < +\infty$ for some $a > 0$. Since the function $\Psi_+(\cdot; E)$ is square integrable on the half-line $(0, +\infty)$ and $\Psi_-(\cdot; E) \notin L^2(0, +\infty)$, then $\Psi_+(x_1; E) < \Psi_-(x_1; E)$ for some $x_1 > 0$. Similarly $\Psi_+(x_2; E) > \Psi_-(x_2; E)$ for some $x_2 < 0$. Therefore there exists $x_0 \in [x_2, x_1]$ such that $\Psi_+(x_0; E) = \Psi_-(x_0; E)$. We define now an auxiliary function h as

$$h(x) = \begin{cases} \Psi_+(x; E) & \text{for } x \geq x_0 \\ \Psi_-(x; E) & \text{for } x \leq x_0. \end{cases}$$

It is easy to show that for any function $\varphi \in \mathcal{D}(H_0)$

$$\langle h | (H_0 - E)\varphi \rangle = -\gamma\varphi(x_0), \tag{34}$$

where γ is the jump of the function h' at the point x_0 . From (34) and Theorem 1 we obtain that $h \in \mathcal{Q}(H_0)$, therefore the function $(V(x) - E)h^2(x) + h'^2(x)$ belongs to the space $L^1(\mathbb{R})$. Hence $g \in L^1(x_0, \infty)$ and the relation (32) is proved. Since f increases for $q \geq 0$, then $f(q) \leq 0$ for all $q \geq 0$. Let us suppose that $f(q) = 0$ for all $q \geq q_1$. Without loss of generality we can assume that $V(q)$ is continuous for $q \geq q_1$. Then $V'(q) = 0$ for the same q (see equality (31)), i.e. the function $V(q)$ is constant for $q \geq q_1$, but this contradicts to condition (V1). Thus $f(q) < 0$ for all $q \geq 0$. Remember that for $E < \lambda_0$ the values of the functions $\Psi_+(q; E)$, $\Psi_-(q; E)$ and $\omega(E)$ are strictly positive. Therefore for all points $s \geq 0$ statement (ii) follows from (30). For points $s < 0$, we argue similarly.

(iii): It is sufficient to prove the uniqueness of the point q_0 for which $Q'(E, q_0) = 0$. Note that q_0 is an isolated zero of the function $Q'(E, \cdot)$; otherwise, we have $Q''(E, q_0 - 0) = 0$ or $Q''(E, q_0 + 0) = 0$ with contradictory to statement (ii). Let $Q'(E, q_1) = 0$ where q_1 is such a point that $q_1 > q_0$ and $Q'(E, q) \neq 0$ if $q \in (q_0, q_1)$. Then $Q''(E, q_0 + 0) < 0$ and $Q''(E, q_1 - 0) < 0$ simultaneously, hence for sufficiently small $\delta > 0$ we obtain that $Q'(E, q_0 + \delta) < 0$ and $Q'(E, q_1 - \delta) > 0$. Therefore $Q'(E, q) = 0$ for some $q \in (q_0 + \delta, q_1 - \delta)$. This contradiction completes the proof ■

Lemma 8. *Suppose that the conditions (V1) and (V3) are fulfilled. If $E \in \mathbb{R} \setminus \sigma(H_0)$, then $Q(q, E) \rightarrow 0$ as $q \rightarrow \pm\infty$.*

Proof. It is known that under condition (V3) the following asymptotics take place as $x \rightarrow +\infty$ (see [15,32]):

$$\begin{aligned} \Psi_\pm(x; \zeta) &= (V(x) - \zeta)^{-\frac{1}{4}} \exp \left[\mp \int_{x_0}^x \sqrt{V(x) - \zeta} dx \right] (1 + o(x)) \\ \Psi_\pm(x; \zeta) &= (V(x) - \zeta)^{-\frac{1}{4}} \exp \left[\pm \int_{x_0}^x \sqrt{V(x) - \zeta} dx \right] (1 + o(x)). \end{aligned}$$

From this asymptotics we obtain that, for a fixed $E \in \mathbb{R}$, $E \neq \lambda_n$,

$$Q(E, q) \sim V(q)^{-\frac{1}{2}} \quad \text{for } q \rightarrow \pm\infty. \tag{35}$$

From (35) and assumption (V1) the lemma follows ■

The following theorem is the main result of the paper.

Theorem 4. For the Schrödinger operator H_0 the functions $E_k(q)$ have the following properties:

(i) Let $\mu < 0$ and $k \geq 1$. Then the function $E_k(q)$ attains the maximum λ_k at those points $q \in \mathbb{R}$ for which $\Phi_k(q) = 0$ and has no other points of local maximum. Besides, $E_k(q) > \lambda_{k-1}$ for all $q \in \mathbb{R}$.

(ii) Let $0 < \mu < \infty$ and $k \geq 1$. Then the function $E_k(q)$ attains its minimum λ_{k-1} at those points $q \in \mathbb{R}$ for which $\Phi_{k-1}(q) = 0$ and has no other points of local minimum. Besides, $E_k(q) < \lambda_k$ for all $q \in \mathbb{R}$.

(iii) Let $\mu = \infty$ and $k \geq 1$. Then:

(a) $\lambda_0 < E_1(q)$ for all $q \in \mathbb{R}$.

(b) The function $E_1(q)$ attains its maximum which is equal to λ_1 at that point q for which $\Phi_1(q) = 0$ and has no other points of local maximum.

(c) For $k \geq 2$ the function $E_k(q)$ attains its minimum λ_{k-1} at those points $q \in \mathbb{R}$ for which $\Phi_{k-1}(q) = 0$, and attains its maximum λ_k at those points $q \in \mathbb{R}$ for which $\Phi_k(q) = 0$; this function has no other points of local maximum or local minimum.

(iv) Let the conditions (V2) and (V3) are fulfilled. Then:

(a) If $0 < \mu \leq \infty$, then $\lim_{q \rightarrow \pm\infty} E_k(q) = \lambda_{k-1}$ for $k \geq 1$.

(b) If $\mu < 0$, then $\lim_{q \rightarrow \pm\infty} E_k(q) = \lambda_k$ for $k \geq 0$. For the same values of μ , there exists a unique point q_0 such that $E'_0(q_0) = 0$, moreover, $E'_0(q) > 0$ if $q > q_0$, and $E'_0(q) < 0$ if $q < q_0$.

Proof. (i) - (iii): These assertions immediately follow from Theorem 3 and Lemma 7/(ii).

(iv): Let $\mu < 0$; we shall consider the function $E_k(q)$. Take $E^* \in (\lambda_{k-1}, \lambda_k)$. By virtue of Lemma 8 there exists $q_0 \in \mathbb{R}$ such that all points $q > q_0$ are non-singular ones for the function $E_k(q)$, and at these points

$$Q(E^*, q) + \mu^{-1} < 0. \tag{36}$$

Since Q increases as a function of E , we obtain from (32) that $E^* < E_k(q) < \lambda_k$ for all $q > q_0$. Thus for $\mu < 0$ we have $\lim_{q \rightarrow \pm\infty} E_k(q) = \lambda_k$. If $\mu > 0$, we argue in a similar manner. Let now $\mu < 0$. Then $-\frac{\mu^2}{4} \leq E_0(q) < \lambda_0$ for all q , therefore the function $E_0(q)$ has at least one minimum. From (10) we obtain

$$E'_0(q) = -Q'(E_0(q), q) \left[\frac{\partial Q}{\partial E}(E_0(q), q) \right]^{-1},$$

hence the reference to Lemma 7 completes the proof ■

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