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An Inverse Problem for a Viscoelastic Timoshenko Beam Model

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Abstract. We consider the Timoshenko model for a viscoelastic beam. This model consists in a system of two coupled Volterra integrodifferential equations describing the evolution of the mean displacement w and of the mean angle of rotation φ. The damping mechanism is characterized by two time-dependent memory kernels, a and b, which are a priori unknown. Provided that (w, φ) solves a suitable initial and boundary value problem for the evolution system, the inverse problem of determining a and b from supplementary information is analyzed. A result of existence and uniqueness on a given bounded time interval is proved. In addition, Lipschitz continuous dependence of the solution (w, φ, a, b) on the data is shown.

Keywords: *Inverse problems, viscoelasticity of integral type, Timoshenko beam*

AMS subject classification: 35 R 30, 45 K 05, 73 F 05, 73 K 05

1. Introduction

Consider an isotropic homogeneous beam of length $l > 0$ and uniform cross section, which occupies a bounded domain $[0, l] \times \Omega \subset \mathbb{R}^3$, for any time $t \in [0, T]$ $(T > 0)$. Let $\operatorname{diam} \Omega << l$ (i.e., thin beam). Suppose that Ω, lying in the (y, z)-plane, is centered at $(0, 0)$ and is symmetric with respect to the (x, y)-plane. Moreover, assume that the beam (with unit density) is made from a viscoelastic material and the bending takes place only in the (x, z)-plane. Following [15: Section 9.1], we denote by w the *mean displacement* and by φ the *mean angle of rotation* of a cross section (see [15: Equations (9.10) - (9.11)]). Using a linear viscoelastic stress-strain law of integral type, it is possible to deduce a Volterra integrodifferential system governing the evolution of the pair (w, φ). We thus obtain the so-called *viscoelastic Timoshenko beam* model. When the beam lies free of stresses and strains up to $t = 0$, the model reads

$$w'' = k(a(0) + a' *)(w_{xx} + \varphi_x) + f_1 \tag{1.1}$$

$$\varphi'' = (b(0) + b' *)\varphi_{xx} - k\,c(a(0) + a' *)(w_x + \varphi) + f_2 \tag{1.2}$$

in $Q_T = (0, l) \times (0, T)$. Here $'$ denotes the time derivative and $*$ stands for the usual time convolution over the interval $(0, t)$. Besides, the functions f_1 and f_2 are related to components of an external force (like gravity, for instance), c is a known positive

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constant depending on the geometry of Ω, and k is the *shear correction coefficient*. The relaxation kernels $a, b : [0, +\infty) \to \mathbb{R}$ are such that $a(0) > 0$ and $b(0) > 0$ and account for the viscoelastic behavior.

It is worth noting that in the Timoshenko beam model both the rotatory inertia and the transverse shear of deformations are taken into account. In particular, the latter effect ensures that the perturbations propagate at finite speed (cf. [1, 6, 14] and their references).

Let us associate with (1.1) - (1.2) a set of initial and boundary conditions, e.g.,

$$w(x,0) = w_0(x) \qquad w'(x,0) = w_1(x) \qquad (x \in (0,l)) \tag{1.3}$$

$$\varphi(x,0) = \varphi_0(x) \qquad \varphi'(x,0) = \varphi_1(x) \qquad (x \in (0,l)) \tag{1.4}$$

$$w(0,t) = \alpha(t) \qquad w(l,t) = \beta(t) \qquad (t \in (0,T)) \tag{1.5}$$

$$\varphi(0,t) = \gamma(t) \qquad \varphi(l,t) = \delta(t) \qquad (t \in (0,T)). \tag{1.6}$$

Provided that a and b are prescribed smooth functions, suitable assumptions on the data allow to prove that the so-called *direct* problem, i.e., finding (w, φ) satisfying (1.1) - (1.6), is well posed. More precisely, arguing as in [9], existence and uniqueness of a classical solution (w, φ) which depends continuously on the data f_1, f_2, w_0, w_1, φ_0, φ_1, α, β, γ, δ, a, b can be proved.

Nevertheless, in applications the kernels a and b are usually *a priori* unknown. That leads to consider the *inverse* problem of identifying them. As neither w nor φ are prescribed we need some additional information which can be obtained by supplementary measurements. For example, one can measure the *bending moment* and the *shear force* applied at the free end $x = 0$ of the beam. This fact can be expressed by

$$\big(b(0)\varphi_x + b' * \varphi_x\big)(0,t) = g_1(t) \qquad (t \in (0,T)) \tag{1.7}$$

$$k\big(a(0)(w_x + \varphi) + a' * (w_x + \varphi)\big)(0,t) = g_2(t) \qquad (t \in (0,T)) \tag{1.8}$$

where g_1 and g_2 are known functions.

Summing up, the inverse problem can be formulated in the following way:

Problem (P_0). Find (w, φ, a, b) satisfying equations (1.1) - (1.2) and conditions (1.3) - (1.8).

Taking advantage of a technique based on the contraction principle in weighted norm spaces (see, e.g., [2, 5, 11]), we prove existence and uniqueness results, on the whole time interval $[0, T]$, provided that f_1, f_2, w_0, w_1, φ_0, φ_1, α, β, γ, δ, g_1, g_2 are smooth enough. Also, it is shown that the map *data* $\mapsto (w, \varphi, a, b)$ is Lipschitz continuous.

Inverse problems of this kind have been analyzed for viscoelastic strings and some other models of viscoelastic beams (see, e.g., [4, 5, 9] and the references therein). In particular, the present results generalize the ones obtained in [9] for the viscoelastic string case, where the existence of a solution is ensured just locally in time. It is worth noting that the argument used here also allow to treat some kind of non-linearities (cf. [3], in preparation). In this context, other interesting applications of the weighted norm technique, relied on Fourier's method of eigenfunction expansion, can be found in [12, 13].

Here is the plan of the paper. In Section 2 we recall a result for the wave equation which will be useful in the sequel. Section 3 contains the main results. In Section 4 we prove that the inverse problem is equivalent to a system of nonlinear functional equations in a fixed-point form. This system is solved in Section 5. Finally, Section 6 is devoted to discuss the continuous dependence on data.

2. A preliminary result

Here we recall a result concerning the well-posedness of a Cauchy-Dirichlet problem for the wave equation (see [10]).

We introduce first some notation. Let X be a Banach space and $\tau \in (0,T]$. Then, $C^n(0,\tau;X)$ is the space of the n-times continuously differentiable functions from $[0,\tau]$ to X, and $W^{n,1}(0,\tau;X)$ is the usual Sobolev space of order n with values in X. The functional spaces $C^n(0,\tau;X)$ and $W^{n,1}(0,\tau;X)$ are endowed with the norms

$$\|v\|_{C^n(0,\tau;X)} = \sum_{j=0}^{n} \sup_{t\in[0,\tau]} \|v^{(j)}(t)\|_X$$

and

$$\|w\|_{W^{n,1}(0,\tau;X)} = \sum_{j=0}^{n} \int_0^\tau \|w^{(j)}(t)\|_X \, dt.$$

Besides, if $X \equiv \mathbb{R}$, then we simply set $C^n(0,\tau) := C^n(0,\tau;\mathbb{R})$ and $W^{n,1}(0,\tau) := W^{n,1}(0,\tau;\mathbb{R})$.

We indicate by $C^n(0,l)$ the space of n-times continuously differentiable function from $[0,l]$ to $\mathbb{R}$, normed by

$$\|z\|_n = \sum_{j=0}^{n} \sup_{x\in[0,l]} |z^{(j)}(x)|.$$

Further, if Y is a Banach space, we set $Y^n := Y^{n-1} \times Y \quad (n \in \mathbb{N})$, where $Y^0 := Y$, endowing Y^n with the norm

$$\|\mathbf{y}\|_{Y^n} := \sum_{i=1}^{n} \|y_i\|_Y \qquad (\mathbf{y} \in Y^n).$$

Fix now $\varepsilon > 0$ and consider the following

Problem (DP). Find u satisfying

$$u''(x,t) - \varepsilon u_{xx}(x,t) = h(x,t) \qquad ((x,t) \in Q_T) \tag{2.1}$$

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \qquad (x \in (0,l)) \tag{2.2}$$

$$u(0,t) = \alpha(t), \quad w(l,t) = \beta(t) \qquad (t \in (0,T)). \tag{2.3}$$

As far as the data are concerned, we suppose

$$h \in W^{1,1}(0,T;C(0,l)) \tag{2.4}$$

$$u_0 \in C^2(0,l) \tag{2.5}$$

$$u_1 \in C^1(0,l) \tag{2.6}$$

$$\alpha, \beta \in W^{3,1}(0,T) \tag{2.7}$$

and we assume the consistency conditions

$$u_0(0) = \alpha(0) \qquad u_0(l) = \beta(0) \tag{2.8}$$

$$u_1(0) = \alpha'(0) \qquad u_1(l) = \beta'(0) \tag{2.9}$$

$$\varepsilon(u_0)_{xx}(0) + h(0,0) = \alpha''(0) \qquad \varepsilon(u_0)_{xx}(l) + h(l,0) = \beta''(0). \tag{2.10}$$

Then, an immediate consequence of [10: Theorems 2.3 - 2.4] is

Theorem 2.1. *Let assumptions* (2.4) – (2.10) *hold. Then problem* (DP) *admits one and only one (classical) solution*

$$u \in C^2(\overline{Q}_T) := C^2(0,T;C(0,l)) \cap C^1(0,T;C^1(0,l)) \cap C(0,T;C^2(0,l)).$$

Moreover, for any $t \in [0,T]$ *we have*

$$\begin{aligned} &\|u(t)\|_2 + \|u'(t)\|_1 + \|u''(t)\|_0 \\ &\quad \le c_1 \Big\{ \|u_0\|_2 + \|u_1\|_1 + \|(\alpha,\beta)\|_{(C^2(0,t))^2} \\ &\qquad + \|(\alpha,\beta)\|_{(W^{3,1}(0,t))^2} + \|h(0)\|_0 + \int_0^t \|h'(s)\|_0 ds \Big\} \end{aligned} \tag{2.11}$$

where c_1 *is a positive constant depending on* T, l, ε.

Remark 2.1. It is worth observing that, reducing problem (DP) to a Cauchy problem for a non-homogeneous equation in a Banach space, the usual variation of constants formula does not apply (cf. [10]; see also [7]). Nevertheless, thanks to Theorem 2.1, we can introduce the operator

$$S_\varepsilon : W^{1,1}(0,T;C(0,l)) \times C^2(0,l) \times C^1(0,l) \times W^{3,1}(0,T) \times W^{3,1}(0,T) \to C^2(\overline{Q}_T)$$

defined by

$$S_\varepsilon(h, u_0, u_1, \alpha, \beta) := u$$

where u is the unique solution to problem (DP). Moreover, owing to (2.11), we have, for any $t \in [0,T]$,

$$\begin{aligned} &\|S_\varepsilon(h,u_0,u_1,\alpha,\beta)(t)\|_2 \\ &\quad + \|(S_\varepsilon(h,u_0,u_1,\alpha,\beta))'(t)\|_1 + \|(S_\varepsilon(h,u_0,u_1,\alpha,\beta))''(t)\|_0 \\ &\le c_1 \Big\{ \|u_0\|_2 + \|u_1\|_1 + \|(\alpha,\beta)\|_{(C^2(0,t))^2} \\ &\quad + \|(\alpha,\beta)\|_{(W^{3,1}(0,t))^2} + \|h(0)\|_0 + \int_0^t \|h'(s)\|_0 ds \Big\}. \end{aligned} \tag{2.12}$$

3. Main results

Let

$$f_1, f_2 \in W^{2,1}(0,T;C(0,l)) \tag{3.1}$$

$$w_0, w_1, \varphi_0, \varphi_1 \in C^2(0,l) \tag{3.2}$$

$$a_0(w_0)_{xx} + f_1(\cdot,0) \in C^1(0,l) \tag{3.3}$$

$$b_0(\varphi_0)_{xx} + f_2(\cdot,0) \in C^1(0,l) \tag{3.4}$$

$$\alpha,\ \beta,\ \gamma,\ \delta \in W^{4,1}(0,T) \tag{3.5}$$

$$g_1,\ g_2 \in C^2(0,T) \tag{3.6}$$

$$m_1 := (\varphi_0)_x(0) \neq 0 \tag{3.7}$$

$$m_2 := (w_0)_x(0) + \varphi_0(0) \neq 0 \tag{3.8}$$

$$a_0 := m_2^{-1} g_2(0) > 0 \tag{3.9}$$

$$b_0 := m_1^{-1} g_1(0) > 0 \tag{3.10}$$

and set

$$a_1 := m_2^{-1}\left[g_2'(0) - a_0\big((w_1)_x(0) + \varphi_1(0)\big)\right] \tag{3.11}$$

$$b_1 := m_1^{-1}\left[g_1'(0) - b_0(\varphi_1)_x(0)\right] \tag{3.12}$$

where we have assumed $k = 1$, for the sake of simplicity. Moreover, on account of (3.11) - (3.12), let the following consistency conditions hold:

$$w_0(0) = \alpha(0), \quad w_0(l) = \beta(0), \quad \varphi_0(0) = \gamma(0), \quad \varphi_0(l) = \delta(0) \tag{3.13}$$

$$w_1(0) = \alpha'(0), \quad w_1(l) = \beta'(0), \quad \varphi_1(0) = \gamma'(0), \quad \varphi_1(l) = \delta'(0) \tag{3.14}$$

$$\begin{aligned} &a_0[(w_0)_{xx}(j) + (\varphi_0)_x(j)] + f_1(j,0) \\ &\quad = l^{-1}(l-j)\alpha''(0) + l^{-1}j\beta''(0) \quad (j = 0,l) \end{aligned} \tag{3.15}$$

$$\begin{aligned} &b_0(\varphi_0)_{xx}(j) - ca_0[(w_0)_x(j) + \varphi_0(j)] + f_2(j,0) \\ &\quad = l^{-1}(l-j)\gamma''(0) + l^{-1}j\delta''(0) \quad (j = 0,l) \end{aligned} \tag{3.16}$$

$$\begin{aligned} &a_0[(w_1)_{xx}(j) + (\varphi_1)_x(j)] + a_1[(w_0)_{xx}(j) + (\varphi_0)_x(j)] + f_1'(j,0) \\ &\quad = l^{-1}(l-j)\alpha'''(0) + l^{-1}j\beta'''(0) \quad (j = 0,l) \end{aligned} \tag{3.17}$$

$$\begin{aligned} &b_0(\varphi_1)_{xx}(j) + b_1(\varphi_0)_{xx}(j) \\ &\quad - ca_0[(w_1)_x(j) + \varphi_1(j)] - ca_1[(w_0)_x(j) + \varphi_0(j)] + f_2'(j,0) \\ &\quad = l^{-1}(l-j)\gamma'''(0) + l^{-1}j\delta'''(0) \quad (j = 0,l). \end{aligned} \tag{3.18}$$

Then, our existence and uniqueness result reads

Theorem 3.1. *Let* (3.1) – (3.10) *and* (3.13) – (3.18) *hold. Then problem* (P_0) *admits one and only one solution*

$$(w,\varphi,a,b) \in \left(C^3(0,T;C(0,l)) \cap C^2(0,T;C^1(0,l)) \cap C^1(0,T;C^2(0,l))\right)^2 \times \left(C^2(0,T)\right)^2$$

fulfilling

$$a(0) = a_0 \qquad a'(0) = a_1 \tag{3.19}$$

$$b(0) = b_0 \qquad b'(0) = b_1 . \tag{3.20}$$

We can also prove that the solution depends on the data in a Lipschitz way. Indeed we have

Theorem 3.2. *Let*

$$(f_{1i},\, f_{2i},\, w_{0i},\, w_{1i},\, \varphi_{0i},\, \varphi_{1i},\, \alpha_i,\, \beta_i,\, \gamma_i,\, \delta_i,\, g_{1i},\, g_{2i}) \qquad (i = 1,2)$$

be two sets of data satisfying hypotheses (3.1)–(3.10) *and* (3.13)–(3.18). *Assume* $a_{01} = a_{02} =: a_0$ *and* $b_{01} = b_{02} =: b_0$, *where* a_{0i} *and* b_{0i} *are defined by* (3.9) *and* (3.10) *with* g_{1i}, g_{2i}, m_{1i}, m_{2i} *in place of* g_1, g_2, m_1, m_2, *respectively. Denote by* (w_i,φ_i,a_i,b_i) $(i = 1,2)$ *the corresponding solutions to problem* (P_0) *and let* B_1 *be a positive constant such that*

$$\begin{aligned} &\|(f_{1i},f_{2i})\|_{(W^{2,1}(0,T;C(0,l)))^2} + \|(f_{1i},f_{2i})\|_{(C^1(0,T;C(0,l)))^2} \\ &\quad + \|w_{0i}\|_2 + \|w_{1i}\|_2 + \|\varphi_{0i}\|_2 + \|\varphi_{1i}\|_2 \\ &\quad + \|a_0(w_{0i})_{xx} + f_{1i}(0)\|_1 + \|b_0(\varphi_{0i})_{xx} + f_{2i}(0)\|_1 \\ &\quad + \|(\alpha_i,\beta_i,\gamma_i,\delta_i)\|_{(W^{4,1}(0,T))^4} + \|(\alpha_i,\beta_i,\gamma_i,\delta_i)\|_{(C^3(0,T))^4} \\ &\quad + \|(g_{1i},g_{2i})\|_{(C^2(0,T))^2} + |c| + |m_{1i}|^{-1} + |m_{2i}|^{-1} + a_0^{-1} + b_0^{-1} + l^{-1} \\ &\le B_1 \end{aligned} \tag{3.21}$$

for $i = 1,2$. *Then, there exists a positive function* $M_1 \in C((0,+\infty)^2)$ *which is non-decreasing in each of its arguments such that*

$$\begin{aligned} &\|(w_1 - w_2,\varphi_1 - \varphi_2)\|_{(C^3(0,T;C(0,l)))^2} \\ &\quad + \|(w_1 - w_2,\varphi_1 - \varphi_2)\|_{(C^2(0,T;C^1(0,l)))^2} \\ &\quad + \|(w_1 - w_2,\varphi_1 - \varphi_2)\|_{(C^1(0,T;C^2(0,l)))^2} \\ &\quad + \|(a_1 - a_2, b_1 - b_2)\|_{(C^2(0,T))^2} \\ &\le M_1(B_1,T)\Big\{ \|(f_{11} - f_{12}, f_{21} - f_{22})\|_{(W^{2,1}(0,T;C(0,l)))^2} \\ &\quad + \|(f_{11} - f_{12}, f_{21} - f_{22})\|_{(C^1(0,T;C(0,l)))^2} \\ &\quad + \|w_{01} - w_{02}\|_2 + \|w_{11} - w_{12}\|_2 + \|\varphi_{01} - \varphi_{02}\|_2 + \|\varphi_{11} - \varphi_{12}\|_2 \\ &\quad + \|a_0(w_{01} - w_{02})_{xx} + (f_{11} - f_{12})(0)\|_1 \\ &\quad + \|b_0(\varphi_{01} - \varphi_{02})_{xx} + (f_{21} - f_{22})(0)\|_1 \\ &\quad + \|(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \delta_1 - \delta_2)\|_{(W^{4,1}(0,T))^4} \\ &\quad + \|(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \delta_1 - \delta_2)\|_{(C^3(0,T))^4} \\ &\quad + \|(g_{11} - g_{12}, g_{21} - g_{22})\|_{(C^2(0,T))^2} \Big\}. \end{aligned} \tag{3.22}$$

Remark 3.1. As we shall see in Section 4, the non-vanishing conditions (3.7) and (3.8) play a basic role in showing that problem (P_0) is equivalent to a system of functional equations in fixed-point form. Nevertheless, if, e.g., w_0 and φ_0 identically vanish, so that (3.7) and (3.8) fail, Theorem 3.1 still holds, provided that smoother data are considered and w_1 and φ_1 (or f_1 and f_2) substitute w_0 and φ_0 in (3.7) and (3.8), respectively (see, for instance, [4: Remark 7.1]).

Remark 3.2. If we replace assumption (3.6) by the weaker one

$$g_1, g_2 \in W^{2,1}(0,T), \tag{3.23}$$

then it is still possible to prove an analog of Theorem 3.1. In this case, a and b are found in $W^{2,1}(0,T)$.

Remark 3.3. As remarked in [10], the compatibility conditions (3.13) - (3.18) are necessary for the existence of a classical solution to problem (P_0).

4. Equivalent problems

Here we show that problem (P_0) is equivalent to a similar problem for the unknowns w', φ', a', b'. This problem can be further reduced to a fixed-point system of nonlinear functional equations.

Suppose that problem (P_0) admits a solution

$$\begin{aligned}(w,\varphi,a,b) \in &\left(C^3(0,T;C(0,l)) \cap C^2(0,T;C^1(0,l)) \cap C^1(0,T;C^2(0,l))\right)^2 \\ &\times \left(C^2(0,T)\right)^2.\end{aligned}$$

Setting

$$(z,\psi,p,q) := (w',\varphi',a',b') \tag{4.1}$$

and differentiating, with respect to time, both the sides of (1.1) and (1.2), we get, in Q_T,

$$z'' = (a(0)+p*)(z_{xx}+\psi_x) + p[(w_0)_{xx} + (\varphi_0)_x] + f_1' \tag{4.2}$$

$$\begin{aligned}\psi'' = &(b(0)+q*)\psi_{xx} - c(a(0)+p*)(z_x+\psi) \\ &+ q(\varphi_0)_{xx} - cp[(w_0)_x + \varphi_0] + f_2'.\end{aligned} \tag{4.3}$$

Moreover, setting $t=0$ in equations (1.1) - (1.2) and recalling conditions (1.3) - (1.4), one infers, for any $x \in (0,l)$,

$$z(x,0) = w_1(x) \qquad z'(x,0) = a(0)[(w_0)_{xx} + (\varphi_0)_x] + f_1(x,0) \tag{4.4}$$

$$\psi(x,0) = \psi_1(x) \qquad \psi'(x,0) = b(0)(\varphi_0)_{xx} - ca(0)[(w_0)_x + \varphi_0] + f_2(x,0). \tag{4.5}$$

In order to obtain the equations for p and q, let us consider (1.7) - (1.8) and differentiate them with respect to time. We obtain

$$b(0)\psi_x(0,t) + (q*\psi_x)(0,t) + q(t)(\varphi_0)_x(0) = g_1'(t) \tag{4.6}$$

$$a(0)(z_x+\psi)(0,t) + \left(p*(z_x+\psi)\right)(0,t) + p(t)[(w_0)_x(0) + \varphi_0(0)] = g_2'(t) \tag{4.7}$$

for any $t \in (0,T)$. Then, taking (3.7) - (3.8) into account, equations (4.6) - (4.7) become, for $t \in (0,T)$,

$$q(t) = m_1^{-1}\Big\{g_1'(t) - b(0)\psi_x(0,t) - (q * \psi_x)(0,t)\Big\} \tag{4.8}$$

$$p(t) = m_2^{-1}\Big\{g_2'(t) - a(0)(z_x + \psi)(0,t) - \big(p * (z_x + \psi)\big)(0,t)\Big\}. \tag{4.9}$$

Setting now $t = 0$ in (1.7) - (1.8), using again (3.7) - (3.8), and recalling (3.9) - (3.10), we get

$$a(0) = m_2^{-1} g_2(0) = a_0 \qquad \text{and} \qquad b(0) = m_1^{-1} g_1(0) = b_0. \tag{4.10}$$

Similarly, setting $t = 0$ in (4.8) - (4.9) and recalling (3.11) - (3.12), we derive

$$\begin{aligned} q(0) &= m_1^{-1}\left[g_1'(0) - b_0(\varphi_1)_x(0)\right] \\ p(0) &= m_2^{-1}\left[g_2'(0) - a_0\big((w_1)_0(0) + \varphi_1(0)\big)\right]. \end{aligned} \tag{4.11}$$

We have thus shown that the set of functions (z, ψ, p, q) solves the following

Problem (P_1). Find $(z, \psi, p, q) \in \big(C^2(\overline{Q}_T)\big)^2 \times \big(C^1(0,T)\big)^2$ satisfying

$$z'' - a_0 z_{xx} = G_1[(z,\psi,p,q)] + f_1' \qquad (\text{in } Q_T) \tag{4.12}$$

$$\psi'' - b_0 \psi_{xx} = G_2[(z,\psi,p,q)] + f_2' \qquad (\text{in } Q_T) \tag{4.13}$$

$$p(t) = H_1[(z,\psi,p,q)](t) + m_2^{-1} g_2'(t) \qquad (t \in (0,T)) \tag{4.14}$$

$$q(t) = H_2[(z,\psi,p,q)](t) + m_1^{-1} g_1'(t) \qquad (t \in (0,T)) \tag{4.15}$$

$$z(x,0) = z_0(x), \qquad z'(x,0) = z_1(x) \qquad (x \in (0,l)) \tag{4.16}$$

$$\psi(x,0) = \psi_0(x), \qquad \psi'(x,0) = \psi_1(x) \qquad (x \in (0,l)) \tag{4.17}$$

$$z(0,t) = \alpha'(t), \qquad w(l,t) = \beta'(t) \qquad (t \in (0,T)) \tag{4.18}$$

$$\psi(0,t) = \gamma'(t), \qquad \psi(l,t) = \delta'(t) \qquad (t \in (0,T)) \tag{4.19}$$

where

$$\begin{aligned} &G_1[(z,\psi,p,q)] \\ &\quad := a_0\psi_x + p[(w_0)_{xx} + (\varphi_0)_x] + p * (z_{xx} + \psi_x) \end{aligned} \tag{4.20}$$

$$\begin{aligned} &G_2[(z,\psi,p,q)] \\ &\quad := q(\varphi_0)_{xx} + q * \psi_{xx} - ca_0(z_x + \psi) - cp * (z_x + \psi) - cp[(w_0)_x + \varphi_0] \end{aligned} \tag{4.21}$$

$$H_1[(z,\psi,p,q)](t) := -m_2^{-1}\left\{a_0(z_x + \psi)(0,t) + \big(p * (z_x + \psi)\big)(0,t)\right\} \tag{4.22}$$

$$H_2[(z,\psi,p,q)](t) := -m_1^{-1}\left\{b_0\psi_x(0,t) + (q * \psi_x)(0,t)\right\} \tag{4.23}$$

$$z_0(x) := w_1(x), \quad z_1(x) := a_0[(w_0)_{xx} + (\varphi_0)_x] + f_1(x,0) \tag{4.24}$$

$$\psi_0(x) := \psi_1(x), \quad \psi_1(x) := b_0(\varphi_0)_{xx} - ca_0[(w_0)_x + \varphi_0] + f_2(x,0). \tag{4.25}$$

Conversely, if problem (P_1) admits a solution (z, ψ, p, q), then, setting

$$w(x,t) := w_0(x) + \int_0^t z(x,s)ds \qquad ((x,t) \in Q_T) \tag{4.26}$$

$$\varphi(x,t) := \varphi_0(x) + \int_0^t \psi(x,s)ds \qquad ((x,t) \in Q_T) \tag{4.27}$$

$$a(t) := a_0 + \int_0^t p(s)ds \qquad (t \in (0,T)) \tag{4.28}$$

$$b(t) := b_0 + \int_0^t q(s)ds \qquad (t \in (0,T)) \tag{4.29}$$

and recalling (3.13), (3.19) - (3.20) and (4.10) - (4.11), it is not difficult to check that

$$(w, \varphi, a, b) \in \left(C^3(0,T; C(0,l)) \cap C^2(0,T; C^1(0,l)) \cap C^1(0,T; C^2(0,l))\right)^2 \times \left(C^2(0,T)\right)^2$$

and solves problem (P_0).

Hence, we have proved

Proposition 4.1. *Let the assumptions of Theorem* 3.1 *hold. Then problem* (P_0) *has a unique solution*

$$(w, \varphi, a, b) \in \left(C^3(0,T; C(0,l)) \cap C^2(0,T; C^1(0,l)) \cap C^1(0,T; C^2(0,l))\right)^2 \times \left(C^2(0,T)\right)^2$$

if and only if problem (P_1) *has a unique solution*

$$(z, \psi, p, q) \in \left(C^2(\overline{Q}_T)\right)^2 \times \left(C^1(0,T)\right)^2.$$

Moreover, w, φ, a, b *and* z, ψ, p, q *are related by* (4.26) - (4.29).

Remark 4.1. Theorem 2.1 suggests the regularity assumptions we have to make on $z_0, z_1, \psi_0, \psi_1, \alpha', \beta', \gamma', \delta', f_1', f_2'$, and $G_1[(z,\psi,p,q)], G_2[(z,\psi,p,q)]$. Compare, for instance, (2.4) - (2.6) with (3.1) - (3.4).

Suppose now that problem (P_1) admits a solution (z, ψ, p, q). Taking into account Theorem 2.1 and Remark 2.1, we deduce that, in particular, z and ψ solve the system of functional equations (cf. also (3.1) - (3.5), (3.14) - (3.18), (4.12) - (4.13), (4.16) - (4.19) and (4.24) - (4.25))

$$z = S_1(G_1[(z,\psi,p,q)], 0,0,0,0) + S_1(f_1', z_0, z_1, \alpha', \beta') \tag{4.30}$$

$$\psi = S_2(G_2[(z,\psi,p,q)], 0,0,0,0) + S_2(f_2', \psi_0, \psi_1, \gamma', \delta') \tag{4.31}$$

in Q_T. Here, on account of Remark 2.1, S_1 and S_2 indicate the operators S_{a_0} and S_{b_0}, respectively. Consequently, let us set

$$\mathbf{U} = (U_1, U_2, U_3, U_4)^\star := (z, \psi, p, q)^\star \tag{4.32}$$
$$J_i(\mathbf{U}) := S_i(G_i[\mathbf{U}], 0, 0, 0, 0) \qquad (i = 1, 2) \tag{4.33}$$
$$J_i(\mathbf{U}) := H_{i-2}[\mathbf{U}] \qquad (i = 3, 4) \tag{4.34}$$
$$\mathbf{K} := \left[S_1(f_1', z_0, z_1, \alpha', \beta'), S_2(f_2', \psi_0, \psi_1, \gamma', \delta'), m_1^{-1} g_1', m_2^{-1} g_2'\right]^\star \tag{4.35}$$

where $\star$ stands for transposition. Further, introduce the Banach spaces

$$Z_\tau = C^2(0, \tau; C(0, l)) \cap C^1(0, \tau; C^1(0, l)) \cap C(0, \tau; C^2(0, l)) \tag{4.36}$$
$$X_\tau = (Z_\tau)^2 \times C^1(0, \tau) \times C^1(0, \tau) \tag{4.37}$$

for any $\tau \in (0, T]$, endowed with the norms

$$\|v\|_{Z_\tau} = \sup_{t \in [0,\tau]} \|v(t)\|_2 + \sup_{t \in [0,\tau]} \|v'(t)\|_1 + \sup_{t \in [0,\tau]} \|v''(t)\|_0 \tag{4.38}$$
$$\|\mathbf{U}\|_{X_\tau} = \|U_1\|_{Z_\tau} + \|U_2\|_{Z_\tau} + \sup_{t \in [0,\tau]} \left(|U_3(t)| + |U_3'(t)| + |U_4(t)| + |U_4'(t)|\right). \tag{4.39}$$

Then, it can be easily realized that $\mathbf{U}$ is solution to

Problem (P_2). Find $\mathbf{U} \in X_T$ satisfying

$$\mathbf{U} = \mathbf{J}(\mathbf{U}) + \mathbf{K} \tag{4.40}$$

in Q_T.

On the other hand, if $\mathbf{U} \in X_T$ solves equation (4.40), then taking into account (4.12) - (4.15) and (4.33) - (4.37) it can be shown that $(z, \psi, p, q) := (U_1, U_2, U_3, U_4)$ solves problem (P_2).

Summing up, on account of problem (P_1) and Proposition 4.1, we can state the following

Proposition 4.2. *Let the assumptions of Theorem 3.1 hold. Then, problem (P_0) admits a unique solution*

$$(w, \varphi, a, b) \in \left(C^3(0, T; C(0, l)) \cap C^2(0, T; C^1(0, l)) \cap C^1(0, T; C^2(0, l))\right)^2 \times \left(C^2(0, T)\right)^2$$

if and only if there exists a unique function $\mathbf{U} \in X_T$ which solves equation (4.40).

5. Proof of Theorem 3.1

In this section we show that the fixed-point equation (4.40) admits a unique solution $\mathbf{U} \in X_T$. Then, thanks to Proposition 4.2, Theorem 3.1 is proved.

For the sake of brevity, let us set, for $t \in [0,\tau]$,

$$\begin{aligned}\mathcal{N}(\mathbf{V})(t) := & \|V_1(t)\|_2 + \|V_1'(t)\|_1 + \|V_1''(t)\|_0 \\ & + \|V_2(t)\|_2 + \|V_2'(t)\|_1 + \|V_2''(t)\|_0 \\ & + |V_3(t)| + |V_3'(t)| + |V_4(t)| + |V_4'(t)|\end{aligned} \tag{5.1}$$

for any $\mathbf{V} \in X_\tau$ $(\tau \in (0,T])$. Observe that (cf. (4.39))

$$\|\mathbf{V}\|_{X_\tau} = \sup_{t\in[0,\tau]} \mathcal{N}(\mathbf{V})(t). \tag{5.2}$$

First, we are going to estimate $\mathcal{N}(\mathbf{J}(\mathbf{U}) + \mathbf{K})(t)$.

Owing to the assumptions (3.2) - (3.4), from (4.20) - (4.21) we infer (cf. (4.24) - (4.25))

$$G_1[\mathbf{U}](0) = a_0(\psi_0)_x + a_1[(w_0)_{xx} + (\varphi_0)_x] := G_{01} \in C(0,l) \tag{5.3}$$

$$G_2[\mathbf{U}](0) = b_1(\varphi_0)_{xx} - ca_0[(z_0)_x + \psi_0] - ca_1[(w_0)_x + \varphi_0] := G_{02} \in C(0,l). \tag{5.4}$$

Moreover, on account of (3.1) - (3.5) and position (4.35), one easily checks that, for all $t \in [0,T]$ (cf. also (2.12)),

$$\begin{aligned}&\mathcal{N}(\mathbf{K})(t) + \|G_{01}\|_0 + \|G_{02}\|_0 \\ &\quad \le c_2\Big\{\|(f_1,f_2)\|_{(W^{2,1}(0,t;C(0,l)))^2} + \|(f_1,f_2)\|_{(C^1(0,t;C(0,l)))^2} \\ &\qquad + \|w_0\|_2 + \|\varphi_0\|_2 + \|z_0\|_2 + \|z_1\|_1 + \|\psi_0\|_2 + \|\psi_1\|_1 \\ &\qquad + \|(\alpha,\beta,\gamma,\delta)\|_{(W^{4,1}(0,t))^4} + \|(\alpha,\beta,\gamma,\delta)\|_{(C^3(0,t))^4} \\ &\qquad + \|(g_1,g_2)\|_{(C^2(0,t))^2}\Big\}\end{aligned} \tag{5.5}$$

where c_2 is a positive constant depending on $T, l, a_0, b_0, |a_1|, |b_1|, |c|, |m_1|, |m_2|$. Therefore, an application of (2.12) to (4.40), for $i = 1,2$, gives (cf. (4.20) - (4.21) and (4.33) and (4.35))

$$\begin{aligned}&\|(J_1(\mathbf{U}) + K_1)(t)\|_2 + \|(J_1(\mathbf{U}) + K_1)'(t)\|_1 + \|(J_1(\mathbf{U}) + K_1)''(t)\|_0 \\ &\quad \le c_3\Big\{\|K_1(t)\|_2 + \|K_1'(t)\|_1 + \|G_{01}\|_0 \\ &\quad + \int_0^t \Big[\|U_1(s)\|_2 + \|U_2(s)\|_1 + \|U_2'(s)\|_1 + |U_3'(s)|(\|w_0\|_2 + \|\varphi_0\|_1) \\ &\quad + \int_0^s |U_3'(s-\sigma)|(\|U_1(\sigma)\|_2 + \|U_2(\sigma)\|_1)d\sigma\Big] ds\Big\}\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&\|(J_2(\mathbf{U}) + K_2)(t)\|_2 + \|(J_2(\mathbf{U}) + K_2)'(t)\|_1 + \|(J_2(\mathbf{U}) + K_2)''(t)\|_0 \\
&\quad \le c_4 \Big\{ \|K_2(t)\|_2 + \|K_2'(t)\|_1 + \|G_{02}\|_0 \\
&\quad + \int_0^t \Big[\|U_1(s)\|_1 + \|U_1'(s)\|_1 + \|U_2(s)\|_0 + \|U_2'(s)\|_0 \\
&\quad + |U_3'(s)|(\|w_0\|_1 + \|\varphi_0\|_0) + |U_4'(s)|\|\varphi_0\|_2 \\
&\quad + \int_0^s (|U_3'(s-\sigma)|(\|U_1(\sigma)\|_1 + \|U_2(\sigma)\|_0) \\
&\quad + |U_4'(s-\sigma)|\|U_2(\sigma)\|_2) d\sigma \Big] ds \Big\}
\end{aligned} \tag{5.7}$$

for all $t \in [0,T]$, where c_3 and c_4 are two positive constants depending on T, l, a_0, $|a_1|$ and T, l, a_0, b_0, $|a_1|$, $|b_1|$, $|c|$, respectively.

Consider now (4.40), when $i = 3,4$. Recalling (4.22) - (4.23) and (4.34) and setting (cf. (4.16) and (4.32))

$$U_1(x,t) = z_0(x) + \int_0^t U_1'(x,\sigma)\, d\sigma \qquad ((x,t) \in Q_T)$$

$$U_2(x,t) = \psi_0(x) + \int_0^t U_2'(x,\sigma)\, d\sigma \qquad ((x,t) \in Q_T)$$

we get, for $t \in [0,T]$,

$$\begin{aligned}
&|(J_3(\mathbf{U}) + K_3)(t)| + |(J_3(\mathbf{U}) + K_3)'(t)| \\
&\quad \le c_5 \Big\{ |K_3(t)| + |K_3'(t)| + \|(J_1(\mathbf{U}) + K_1)(t)\|_1 + \|(J_1(\mathbf{U}) + K_1)'(t)\|_1 \\
&\quad + \|(J_2(\mathbf{U}) + K_2)(t)\|_0 + \|(J_2(\mathbf{U}) + K_2)'(t)\|_0 \\
&\quad + \int_0^t (|U_3(t-s)| + |U_3'(t-s)|) \\
&\quad \times \Big[\|z_0\|_1 + \|\psi_0\|_0 + \int_0^s (\|U_1'(\sigma)\|_1 + \|U_2'(\sigma)\|_0) d\sigma \Big] ds \Big\}
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
&|(J_4(\mathbf{U}) + K_4)(t)| + |(J_4(\mathbf{U}) + K_4)'(t)| \\
&\quad \le c_6 \Big\{ |K_4(t)| + |K_4'(t)| + \|(J_2(\mathbf{U}) + K_2)(t)\|_1 + \|(J_2(\mathbf{U}) + K_2)'(t)\|_1 \\
&\quad + \int_0^t (|U_4(t-s)| + |U_4'(t-s)|) \Big[\|\psi_0\|_1 + \int_0^s \|U_2'(\sigma)\|_1 d\sigma \Big] ds \Big\}
\end{aligned} \tag{5.9}$$

where c_5 and c_6 are positive constants depending on $|m_2|$, a_0, $|a_1|$ and $|m_1|$, b_0, $|b_1|$, respectively. Recalling (5.1), a combination of (5.6) - (5.9) entails, for all $t \in [0,T]$,

$$\begin{aligned}\mathcal{N}(\mathbf{J}(\mathbf{U})+\mathbf{K})(t) \le c_7\Big\{&\mathcal{N}(\mathbf{K})(t)+\|G_{01}\|_0+\|G_{02}\|_0\\ &+\int_0^t\Big[\|U_1(s)\|_2+\|U_1'(s)\|_1+\|U_2(s)\|_2+\|U_2'(s)\|_1\\ &+|U_3(s)|+|U_3'(s)|+|U_4(s)|+|U_4'(s)|\\ &+\big(|U_3(t-s)|+|U_3'(t-s)|+|U_4(t-s)|+|U_4'(t-s)|\big)\\ &\times\Big(\int_0^s(\|U_1'(\sigma)\|_1+\|U_2'(\sigma)\|_1)d\sigma\Big)\\ &+\int_0^s\big(|U_3'(s-\sigma)|+|U_4'(s-\sigma)|\big)\\ &\times\big(\|U_1(\sigma)\|_2+\|U_2(\sigma)\|_1\big)d\sigma\Big]ds\Big\}\end{aligned} \tag{5.10}$$

where c_7 is a positive constant depending on T, l, a_0, b_0, $|a_1|$, $|b_1|$, $|c|$, $|m_1|$, $|m_2|$ and suitable norms of data.

In the sequel of the proof c_k $(k \in \mathbb{N})$ will stand for a known positive constant similar to c_7. Then, from (5.10) we infer, for all $t \in [0,T]$,

$$\begin{aligned}\mathcal{N}(\mathbf{J}(\mathbf{U})+\mathbf{K})(t) \le c_8\Big\{&\mathcal{N}(\mathbf{K})(t)+\|G_{01}\|_0+\|G_{02}\|_0\\ &+\int_0^t\Big[\mathcal{N}(\mathbf{U})(s)\Big(1+\int_0^{t-s}\mathcal{N}(\mathbf{U})(\sigma)d\sigma\Big)\\ &+\int_0^s\mathcal{N}(\mathbf{U})(s-\sigma)\mathcal{N}(\mathbf{U})(\sigma)d\sigma\Big]ds\Big\}.\end{aligned} \tag{5.11}$$

Let us estimate now the differences

$$(J_i(\mathbf{U})+K_i)-(J_i(\mathbf{V})+K_i)=J_i(\mathbf{U})-J_i(\mathbf{V}) \qquad (i=1,2,3,4) \tag{5.12}$$

for any $\mathbf{U},\mathbf{V} \in X_T$. Recalling positions (4.20) - (4.23), we get, in Q_T,

$$\begin{aligned}G_1[\mathbf{U}]-G_1[\mathbf{V}]=\;&a_0[(U_2-V_2)_x]+(U_3-V_3)[(w_0)_{xx}+(\varphi_0)_x]\\ &+U_3*[(U_1-V_1)_{xx}+(U_2-V_2)_x]\\ &+(U_3-V_3)*[(V_1)_{xx}+(V_2)_x]\end{aligned} \tag{5.13}$$

$$\begin{aligned}G_2[\mathbf{U}]-G_2[\mathbf{V}]=\;&(U_4-V_4)(\varphi_0)_{xx}+U_4*(U_2-V_2)_{xx}\\ &+(U_4-V_4)*(V_2)_{xx}-ca_0[(U_1-V_1)_x+(U_2-V_2)]\\ &-c(U_3-V_3)[(w_0)_x+\varphi_0]-cU_3*[(U_1-V_1)_x+(U_2-V_2)]\\ &-c(U_3-V_3)*[(V_1)_x+V_2]\end{aligned} \tag{5.14}$$

and, for all $t \in (0,T)$,

$$\begin{aligned} H_1[\mathbf{U}](t) - H_1[\mathbf{V}](t) = -m_2^{-1}\{&a_0[(U_1 - V_1)_x + (U_2 - V_2)](0,t) \\ &+ (U_3 * [(U_1 - V_1)_x + (U_2 - V_2)])(0,t) \\ &+ ((U_3 - V_3) * [(V_1)_x + V_2])(0,t)\} \end{aligned} \tag{5.15}$$

$$\begin{aligned} H_2[\mathbf{U}](t) - H_2[\mathbf{V}](t) = -m_1^{-1}\{&b_0[(U_2 - V_2)_x](0,t) \\ &+ (U_4 * [(U_2 - V_2)_x)(0,t) \\ &+ ((U_4 - V_4) * [(V_2)_x])(0,t)\}. \end{aligned} \tag{5.16}$$

Taking (5.13) - (5.14) into account, an application of estimate (2.12) to (5.12), for $i = 1,2$, gives (cf. also (4.33) and (4.35), and Remark 2.1)

$$\begin{aligned} &\|(J_1(\mathbf{U}) - J_1(\mathbf{V}))(t)\|_2 + \|(J_1(\mathbf{U}) - J_1(\mathbf{V}))'(t)\|_1 + \|(J_1(\mathbf{U}) - J_1(\mathbf{V}))''(t)\|_0 \\ &\le c_{10}\Big\{\int_0^t \Big[\|(U_1 - V_1)(s)\|_2 + \|(U_2 - V_2)(s)\|_1 \\ &+ \|(U_2' - V_2')(s)\|_1 + |(U_3' - V_3')(s)| \\ &+ \int_0^s (|U_3'(s-\sigma)|(\|(U_1 - V_1)(\sigma)\|_2 + \|(U_2 - V_2)(\sigma)\|_1) \\ &+ |(U_3' - V_3')(s-\sigma)|(\|V_1(\sigma)\|_2 + \|V_2(\sigma)\|_1))d\sigma\Big]ds\Big\} \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\|(J_2(\mathbf{U}) - J_2(\mathbf{V}))(t)\|_2 + \|(J_2(\mathbf{U}) - J_2(\mathbf{V}))'(t)\|_1 + \|(J_2(\mathbf{U}) - J_2(\mathbf{V}))''(t)\|_0 \\ &\le c_{11}\Big\{\int_0^t \Big[\|(U_1 - V_1)(s)\|_1 + \|(U_1' - V_1')(s)\|_1 + \|(U_2 - V_2)(s)\|_0 \\ &+ \|(U_2' - V_2')(s)\|_0 + |(U_3' - V_3')(s)| + |(U_4' - V_4')(s)| \\ &+ \int_0^s (|U_3'(s-\sigma)|(\|(U_1 - V_1)(\sigma)\|_1 + \|(U_2 - V_2)(\sigma)\|_0) \\ &+ |U_4'(s-\sigma)|\|(U_2 - V_2)(\sigma)\|_2 \\ &+ (|(U_3' - V_3')(s-\sigma)| + |(U_4' - V_4')(s-\sigma)|) \\ &\times (\|V_1(\sigma)\|_1 + \|V_2(\sigma)\|_0))d\sigma\Big]ds\Big\} \end{aligned} \tag{5.18}$$

for all $t \in [0,T]$. Consider now (5.15) - (5.16). Then, on account of (4.34) and (5.12), one easily deduces, for any $t \in [0,T]$,

$$\begin{aligned} &|(J_3(\mathbf{U}) - J_3(\mathbf{V}))(t)| + |(J_3(\mathbf{U}) - J_3(\mathbf{V}))'(t)| \\ &\le c_{12}\Big\{\|(J_1(\mathbf{U}) - J_1(\mathbf{V}))(t)\|_1 + \|(J_1(\mathbf{U}) - J_1(\mathbf{V}))'(t)\|_1 \\ &+ \|(J_2(\mathbf{U}) - J_2(\mathbf{V}))(t)\|_0 + \|(J_2(\mathbf{U}) - J_2(\mathbf{V}))'(t)\|_0 \end{aligned}$$

$$+\int_0^t \Big[(|U_3(t-s)| + |U_3'(t-s)|) \tag{5.19}$$
$$\times \int_0^s \Big(\|(U_1 - V_1)'(\sigma)\|_1 + \|(U_2 - V_2)'(\sigma)\|_0\Big)d\sigma$$
$$+ \Big(|(U_3 - V_3)(t-s)| + |(U_3' - V_3')(t-s)|\Big)$$
$$\times \Big(1 + \int_0^s (\|V_1'(\sigma)\|_1 + \|V_2'(\sigma)\|_0)d\sigma\Big)\Big]ds\Big\}$$

and

$$|(J_4(\mathbf{U}) - J_4(\mathbf{V}))(t)| + |(J_4(\mathbf{U}) - J_4(\mathbf{V}))'(t)|$$
$$\leq c_{13}\Big\{\|(J_2(\mathbf{U}) - J_2(\mathbf{V}))(t)\|_1 + \|(J_2(\mathbf{U}) - J_2(\mathbf{V}))'(t)\|_1$$
$$+ \int_0^t \Big[(|U_4(t-s)| + |U_4'(t-s)|) \int_0^s \|(U_2' - V_2')(\sigma)\|_1 d\sigma \tag{5.20}$$
$$+ \Big(|(U_4 - V_4)(t-s)| + |(U_4' - V_4')(t-s)|\Big)$$
$$\times \Big(1 + \int_0^s \|V_2'(\sigma)\|_1 d\sigma\Big)\Big]ds\Big\}.$$

Combining (5.17) - (5.20) and recalling (5.1), we deduce, for all $t \in (0,T)$,

$$\mathcal{N}(\mathbf{J}(\mathbf{U}) - \mathbf{J}(\mathbf{V}))(t) \leq c_{14} \int_0^t \Big\{\mathcal{N}(\mathbf{U} - \mathbf{V})(s)\Big(1 + \int_0^{t-s} \mathcal{N}(\mathbf{V})(\sigma)d\sigma\Big)$$
$$+ \int_0^s \Big(\mathcal{N}(\mathbf{U})(s-\sigma) + \mathcal{N}(\mathbf{V})(s-\sigma)\Big)\mathcal{N}(\mathbf{U} - \mathbf{V})(\sigma)\, d\sigma \tag{5.21}$$
$$+ \mathcal{N}(\mathbf{U})(t-s) \int_0^s \mathcal{N}(\mathbf{U} - \mathbf{V})(\sigma)\, d\sigma\Big\}ds$$

for any $\mathbf{U}, \mathbf{V} \in X_T$.

Applying the properties of the convolution product, from inequalities (5.11) and (5.21) we get, for any $\mathbf{U}, \mathbf{V} \in X_T$ and any $t \in [0,T]$,

$$\mathcal{N}(\mathbf{J}(\mathbf{U}) + \mathbf{K})(t)$$
$$\leq c_{15}\Big\{\mathcal{N}(\mathbf{K})(t) + \|G_{01}\|_0 + \|G_{02}\|_0 + \int_0^t \lambda(\mathbf{U})(t-s)\mathcal{N}(\mathbf{U})(s)\, ds\Big\} \tag{5.22}$$

$$\mathcal{N}(\mathbf{J}(\mathbf{U}) - \mathbf{J}(\mathbf{V}))(t) \leq c_{16} \int_0^t \Lambda(\mathbf{U}, \mathbf{V})(t-s)\mathcal{N}(\mathbf{U} - \mathbf{V})(s)\, ds \tag{5.23}$$

where λ and Λ are defined by

$$\lambda(\mathbf{U})(t) := 1 + \int_0^t \mathcal{N}(\mathbf{U})(\sigma)d\sigma \tag{5.24}$$

$$\Lambda(\mathbf{U}, \mathbf{V})(t) := 1 + \int_0^t (\mathcal{N}(\mathbf{U})(\sigma) + \mathcal{N}(\mathbf{V})(\sigma))d\sigma. \tag{5.25}$$

Let us set now

$$\mathbf{L}(\mathbf{U}) := \mathbf{J}(\mathbf{U}) + \mathbf{K}. \tag{5.26}$$

Then, on account of (4.40), our proof reduces to showing that the mapping $\mathbf{L}$ has a unique fixed-point in X_T.

First of all, observe that, owing to (5.2) and (5.22), $\mathbf{L} : X_T \to X_T$ is well defined. Following [2] and [5] (see also [11]), introduce in X_τ ($\tau \in (0,T]$) the weighted norm (cf. (5.1))

$$\|\mathbf{U}\|_\tau^\sigma = \sup_{t\in[0,\tau]} e^{-\sigma t}\mathcal{N}(\mathbf{U})(t) \qquad (\mathbf{U} \in X_\tau) \tag{5.27}$$

where $\sigma \in [0,+\infty)$. One can easily realize that the norm defined by (5.27) is equivalent to the norm (4.39). Indeed, we have, for any $\sigma \in [0,+\infty)$,

$$\|\mathbf{U}\|_\tau^\sigma \le \|\mathbf{U}\|_{X_\tau} \le e^{\sigma\tau}\|\mathbf{U}\|_\tau^\sigma. \tag{5.28}$$

Moreover, if $\tau = T$ and $\sigma = 0$, then $\|\mathbf{U}\|_T^0 = \|\mathbf{U}\|_{X_T}$ follows. Consider now the closed ball of X_T

$$E_{\rho,\sigma} = \left\{\mathbf{U} \in X_T : \|\mathbf{U}\|_T^\sigma \le \rho\right\} \tag{5.29}$$

for some $(\rho,\sigma) \in (0,+\infty) \times [0,+\infty)$ and let $\mathbf{U} \in E_{\rho,\sigma}$. Taking advantage of (5.22), and recalling (5.26), we derive the chain of inequalities (cf. also (5.3) - (5.5))

$$\begin{aligned} e^{-\sigma t}\mathcal{N}(\mathbf{L}(\mathbf{U}))(t) &\le c_{15}\Big\{e^{-\sigma t}\mathcal{N}(\mathbf{K})(t) + e^{-\sigma t}(\|G_{01}\|_0 + \|G_{02}\|_0) \\ &\quad + \int_0^t e^{-\sigma(t-s)}\lambda(\mathbf{U})(t-s)e^{-\sigma s}\mathcal{N}(\mathbf{U})(s)\,ds\Big\} \\ &\le c_{15}\Big\{\|\mathbf{K}\|_T^\sigma + \|G_{01}\|_0 + \|G_{02}\|_0) + \|\mathbf{U}\|_T^\sigma \int_0^t e^{-\sigma s}\lambda(\mathbf{U})(s)\,ds\Big\} \end{aligned} \tag{5.30}$$

for all $t \in [0,T]$. On account of (5.24), one can easily show that

$$\int_0^T e^{-\sigma s}\lambda(\mathbf{U})(s)\,ds \le \sigma^{-1}\Big\{1 - e^{-\sigma T} + \|\mathbf{U}\|_T^\sigma[T + (e^{-\sigma T} - 1)\sigma^{-1}]\Big\}. \tag{5.31}$$

Hence, a combination of (5.30) and (5.31) yields (cf. also (5.27) - (5.28))

$$\begin{aligned} \|\mathbf{L}(\mathbf{U})\|_T^\sigma \le c_{15}\Big\{&\|\mathbf{K}\|_T^\sigma + \|G_{01}\|_0 + \|G_{02}\|_0 \\ &+ \rho\sigma^{-1}\left[1 - e^{-\sigma T} + \rho(T + (e^{-\sigma T} - 1)\sigma^{-1})\right]\Big\} \end{aligned} \tag{5.32}$$

for any $\mathbf{U} \in E_{\rho,\sigma}$. Choosing $(\rho,\sigma) \in (0,+\infty) \times [0,+\infty)$ such that (cf. Remark 5.1 below)

$$\begin{aligned} c_{15}\Big\{&\|\mathbf{K}\|_T^{\tilde\sigma} + \|G_{01}\|_0 + \|G_{02}\|_0 \\ &+ \tilde\rho\tilde\sigma^{-1}\left[1 - e^{-\tilde\sigma T} + \tilde\rho(T + (e^{-\tilde\sigma T} - 1)\tilde\sigma^{-1})\right]\Big\} \le \tilde\rho \end{aligned} \tag{5.33}$$

inequality (5.32) entails

$$\mathbf{L}(E_{\tilde{\rho},\tilde{\sigma}}) \subseteq E_{\tilde{\rho},\tilde{\sigma}}. \tag{5.34}$$

Consider now (5.23). Taking (5.27) into account, we have

$$\begin{aligned} &e^{-\tilde{\sigma}t}\mathcal{N}(\mathbf{J}(\mathbf{U}) - \mathbf{J}(\mathbf{V}))(t) \\ &\quad \leq c_{16} \int_0^t e^{-\tilde{\sigma}(t-s)}\Lambda(\mathbf{U},\mathbf{V})(t-s)e^{-\tilde{\sigma}s}\mathcal{N}(\mathbf{U}-\mathbf{V})(s)\,ds \\ &\quad \leq c_{16}\|\Lambda(\mathbf{U},\mathbf{V})\|_T^{\tilde{\sigma}} \int_0^t \|\mathbf{U}-\mathbf{V}\|_s^{\tilde{\sigma}} ds \end{aligned} \tag{5.35}$$

for any $\mathbf{U}, \mathbf{V} \in E_{\tilde{\rho},\tilde{\sigma}}$ and any $t \in [0, T]$.

Recalling (5.25), simple computations show that

$$c_{16}\|\Lambda(\mathbf{U},\mathbf{V})\|_T^{\tilde{\sigma}} \leq c_{17} := c_{16}\left[1 + 2\tilde{\rho}(1 - e^{-\tilde{\sigma}T})\tilde{\sigma}^{-1}\right] \tag{5.36}$$

for any $\mathbf{U}, \mathbf{V} \in E_{\tilde{\rho},\tilde{\sigma}}$. Hence, combining (5.35) and (5.36), we infer (cf. (5.26) and (5.27))

$$\|\mathbf{L}(\mathbf{U}) - \mathbf{L}(\mathbf{V})\|_t^{\tilde{\sigma}} = \|\mathbf{J}(\mathbf{U}) - \mathbf{J}(\mathbf{V})\|_t^{\tilde{\sigma}} \leq c_{17} \int_0^t \|\mathbf{U}-\mathbf{V}\|_s^{\tilde{\sigma}} ds \tag{5.37}$$

for all $\mathbf{U}, \mathbf{V} \in E_{\tilde{\rho},\tilde{\sigma}}$. Taking (5.34) into account, from inequality (5.37) one deduces that there exists $n \in \mathbf{N}$ such that the iterated operator $\mathbf{L}^n$ is a contraction from $E_{\tilde{\rho},\tilde{\sigma}}$ into itself. Then, an application of the Picard-Banach fixed-point theorem (see, e.g., [8: Chapter 2/Theorem 2.2]) proves that $\mathbf{L}$ has a unique fixed-point in $E_{\tilde{\rho},\tilde{\sigma}}$, i.e., in X_T.

Remark 5.1. In order to find $(\tilde{\rho},\tilde{\sigma}) \in (0,+\infty)\times[0,+\infty)$ satisfying (5.33), consider a positive constant B_2 such that

$$\begin{aligned} &\|(f_1, f_2)\|_{(W^{2,1}(0,T;C(0,l)))^2} + \|(f_1, f_2)\|_{(C^1(0,T;C(0,l)))^2} \\ &\quad + \|w_0\|_2 + \|w_1\|_2 + \|\varphi_0\|_2 + \|\varphi_1\|_2 \\ &\quad + \|a_0(w_0)_{xx} + f_1(0)\|_1 + \|b_0(\varphi_0)_{xx} + f_2(0)\|_1 \\ &\quad + \|(\alpha,\beta,\gamma,\delta)\|_{(W^{4,1}(0,T))^4} + \|(\alpha,\beta,\gamma,\delta)\|_{(C^3(0,T))^4} \\ &\quad + \|(g_1, g_2)\|_{(C^2(0,T))^2} + |c| + |m_1|^{-1} + |m_2|^{-1} + a_0^{-1} + b_0^{-1} + l^{-1} \\ &\leq B_2. \end{aligned} \tag{5.38}$$

Then, there exist two positive functions M_2 and M_3, which are continuous and non-decreasing in each of their arguments, satisfying (cf. (5.5))

$$c_{14} \leq M_2(B_2, T) \tag{5.39}$$

and, for all $\sigma \in [0, +\infty)$,

$$\|\mathbf{K}\|_T^{\sigma} + \|G_{01}\|_0 + \|G_{02}\|_0 \leq \|\mathbf{K}\|_{X_T} + \|G_{01}\|_0 + \|G_{02}\|_0 \leq M_3(B_2, T). \tag{5.40}$$

Owing to (5.39) - (5.40), we easily obtain

$$c_{15}\Big\{\|\mathbf{K}\|_T^\sigma + \|G_{01}\|_0 + \|G_{02}\|_0 + \rho\sigma^{-1}\big[1 - e^{-\sigma T} + \rho(T + (e^{-\sigma T} - 1)\sigma^{-1})\big]\Big\} \\ \leq M_2(B_2, T)\Big\{M_3(B_2, T) + \sigma^{-1}\rho(1 + \rho T)\Big\} \tag{5.41}$$

and choosing, for instance,

$$\tilde{\rho} = 2M_2(B_2, T)M_3(B_2, T) \tag{5.42}$$

$$\tilde{\sigma} = \big[1 + 2M_2(B_2, T)M_3(B_2, T)T\big]2M_2(B_2, T) \tag{5.43}$$

from (5.41), we deduce (5.33).

6. Proof of Theorem 3.2

Observe that, thanks to Proposition 4.2, there exists a unique function $\mathbf{U}^i \in X_T$ solution to

$$\mathbf{U}^i = \mathbf{J}^i(\mathbf{U}^i) + \mathbf{K}^i \qquad (i = 1, 2) \tag{6.1}$$

which is related to the solution $(w_i, \varphi_i, a_i, b_i)$ to problem (P_0) by (4.26) - (4.29) and (4.32). Here $\mathbf{J}^i$ and $\mathbf{K}^i$ are defined by (4.33) - (4.35) in correspondence of the set of data

$$\big(f_{1i}, f_{2i}, w_{0i}, w_{1i}, \varphi_{0i}, \varphi_{1i}, \alpha_i, \beta_i, \gamma_i, \delta_i\big).$$

Consequently, from (6.1) we derive the identity

$$\mathbf{U}^1 - \mathbf{U}^2 = \mathbf{J}^1(\mathbf{U}^1) - \mathbf{J}^1(\mathbf{U}^2) + \mathbf{J}^1(\mathbf{U}^2) - \mathbf{J}^2(\mathbf{U}^2) + \mathbf{K}^1 - \mathbf{K}^2. \tag{6.2}$$

Reasoning as in Remark 5.1 and recalling (3.21), one can find two positive and continuous functions M_4 and M_5, non-decreasing in each of their arguments, such that, for $i = 1, 2$,

$$c_{15}^i \leq M_4(B_1, T) \tag{6.3}$$

$$\|\mathbf{K}^i\|_T^\sigma + \|G_{01}^i\|_0 + \|G_{02}^i\|_0 \leq M_5(B_1, T) \qquad (\sigma \in [0, +\infty)). \tag{6.4}$$

Here c_{15}^i is the constant appearing in (5.22) and associated with $\mathbf{J}^i$ and $\mathbf{K}^i$, while G_{01}^i and G_{02}^i are defined as in (5.3) - (5.4), accordingly to the correspondent set of data. From now on, M_n $(n \in \mathbb{N})$ denotes a function quite similar to M_5. Then, since $\tilde{\rho}_i$ and $\tilde{\sigma}_i$ can be chosen only in dependence of M_4 and M_5 (see (5.39) - (5.40) and (5.42) - (5.43)), we infer (cf. (5.27))

$$\|\mathbf{U}^i\|_{X_T} \leq \tilde{\rho}_i e^{\tilde{\sigma}_i T} \leq M_6(B_1, T) \qquad (i = 1, 2). \tag{6.5}$$

Thanks to (3.2) and (6.5), computations similar to the ones done to obtain (5.23) lead to

$$\mathcal{N}(\mathbf{J}^1(\mathbf{U}^1) - \mathbf{J}^1(\mathbf{U}^2))(t) \\ \leq M_7(B_1, T)\Big\{\|G_{01}^1 - G_{01}^2\|_0 + \|G_{02}^1 - G_{02}^2\|_0 + \int_0^t \mathcal{N}(\mathbf{U}^1 - \mathbf{U}^2)(s)\,ds\Big\} \tag{6.6}$$

for any $t \in [0,T]$. Recalling now (4.33) - (4.34), we have

$$\mathbf{J}_1^1(\mathbf{U}^2) - \mathbf{J}_1^2(\mathbf{U}^2) = S_1(G_1^1[\mathbf{U}^2] - G_1^2[\mathbf{U}^2], 0, 0, 0, 0) \tag{6.7}$$
$$\mathbf{J}_2^1(\mathbf{U}^2) - \mathbf{J}_2^2(\mathbf{U}^2) = S_2(G_2^1[\mathbf{U}^2] - G_2^2[\mathbf{U}^2], 0, 0, 0, 0) \tag{6.8}$$
$$\mathbf{J}_3^1(\mathbf{U}^2) - \mathbf{J}_3^2(\mathbf{U}^2) = H_1^1[\mathbf{U}^2] - H_1^2[\mathbf{U}^2] \tag{6.9}$$
$$\mathbf{J}_4^1(\mathbf{U}^2) - \mathbf{J}_4^2(\mathbf{U}^2) = H_2^1[\mathbf{U}^2] - H_2^2[\mathbf{U}^2] \tag{6.10}$$

where G_1^i, G_2^i, H_1^i, H_2^i $(i = 1,2)$ are defined by (4.20) - (4.23) according to the correspondent set of data. Analogously, from (3.7) - (3.8) and (3.11) - (3.12), one deduces

$$m_{11}^{-1} - m_{12}^{-1} = m_{11}^{-1} m_{12}^{-1} \left[(\varphi_{02})_x(0) - (\varphi_{01})_x(0)\right] \tag{6.11}$$

$$m_{21}^{-1} - m_{22}^{-1} = m_{21}^{-1} m_{22}^{-1} \left[\varphi_{02}(0) - \varphi_{01}(0) - ((w_{02})_x(0) - (w_{01})_x(0))\right] \tag{6.12}$$

and

$$\begin{aligned} a_{11} - a_{12} = m_{21}^{-1} \Big[& g_{21}'(0) - g_{22}'(0) \\ & - a_0 \Big((w_{11})_x(0) - (w_{12})_x(0) + \varphi_{11}(0) - \varphi_{12}(0)\Big)\Big] \\ & + (m_{21}^{-1} - m_{22}^{-1}) \Big[g_{22}'(0) - a_0\big((w_{12})_x(0) + \varphi_{12}(0)\big)\Big] \end{aligned} \tag{6.13}$$

$$\begin{aligned} b_{11} - b_{12} = {} & m_{11}^{-1} \Big[g_{11}'(0) - g_{12}'(0) - b_0((\varphi_{11})_x - (\varphi_{12})_x)(0)\Big] \\ & + (m_{11}^{-1} - m_{12}^{-1}) \big[g_{12}'(0) - b_0(\varphi_{12})_x(0)\big]. \end{aligned} \tag{6.14}$$

Taking advantage of (3.2), (6.4) - (6.5) and (6.7) - (6.14), one can derive, for all $t \in [0,T]$,

$$\begin{aligned} & \mathcal{N}(\mathbf{J}^1(\mathbf{U}^2) - \mathbf{J}^2(\mathbf{U}^2))(t) + \mathcal{N}(\mathbf{K}^1 - \mathbf{K}^2)(t) \\ & \le M_8(B_1, T) \Big\{ \|(f_{11} - f_{12}, f_{21} - f_{22})\|_{(W^{2,1}(0,t;C(0,l)))^2} \\ & + \|(f_{11} - f_{12}, f_{21} - f_{22})\|_{(C^1(0,t;C(0,l)))^2} \\ & + \|w_{01} - w_{02}\|_2 + \|w_{11} - w_{12}\|_1 + \|\varphi_{01} - \varphi_{02}\|_2 + \|\varphi_{11} - \varphi_{12}\|_1 \\ & + \|\psi_{01} - \psi_{02}\|_2 + \|\psi_{11} - \psi_{12}\|_1 + \|z_{01} - z_{02}\|_2 + \|z_{11} - z_{12}\|_1 \\ & + \|(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \delta_1 - \delta_2)\|_{(W^{4,1}(0,t))^4} \\ & + \|(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \delta_1 - \delta_2)\|_{(C^3(0,t))^4} \\ & + \|(g_{11} - g_{12}, g_{21} - g_{22})\|_{(C^2(0,t))^2} \Big\}. \end{aligned} \tag{6.15}$$

Hence, combining (6.6) and (6.15), from (6.2) we get, for all $t \in [0,T]$ (cf. also (5.2)),

$$
\begin{aligned}
&\|\mathbf{U}^1 - \mathbf{U}^2\|_{X_t} \\
&\quad \le M_9(B_1,T)\Big\{ \|(f_{11} - f_{12}, f_{21} - f_{22})\|_{(W^{2,1}(0,t;C(0,l)))^2} \\
&\qquad + \|(f_{11} - f_{12}, f_{21} - f_{22})\|_{(C^1(0,t;C(0,l)))^2} \\
&\qquad + \|w_{01} - w_{02}\|_2 + \|w_{11} - w_{12}\|_1 + \|\varphi_{01} - \varphi_{02}\|_2 + \|\varphi_{11} - \varphi_{12}\|_1 \\
&\qquad + \|\psi_{01} - \psi_{02}\|_2 + \|\psi_{11} - \psi_{12}\|_1 + \|z_{01} - z_{02}\|_2 + \|z_{11} - z_{12}\|_1 \\
&\qquad + \|(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \delta_1 - \delta_2)\|_{(W^{4,1}(0,t))^4} \\
&\qquad + \|(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \delta_1 - \delta_2)\|_{(C^3(0,t))^4} \\
&\qquad + \|(g_{11} - g_{12}, g_{21} - g_{22})\|_{(C^2(0,t))^2} + \int_0^t \|\mathbf{U}^1 - \mathbf{U}^2\|_{X_s} ds \Big\}.
\end{aligned}
\tag{6.16}
$$

Finally, recalling positions (4.24) - (4.25), an application of Gronwall lemma to (6.16) implies (3.22).

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