An Example of Blowup for a Degenerate Parabolic Equation with a Nonlinear Boundary Condition

M. Chipot **and J. Fib**

Abstract. In this paper, a nonlinear parabolic equation of the form $u_t = (a(u_x))_x$ for $x \in (0,1)$, *I* > 0, $a(u_x) = |u_x|^{p-2}u_x$ if $u_x \ge \eta > 0$, $1 < p < 2$, with nonlinear boundary condition $a(u_x(1,t)) = |u|^{q-2}u(1,t)$ is considered. It is proved that if $qp - 3p + 2 > 0$, then the solutions blow up in finite time. Moreover, estimates on the blowup profile (in x) and the blowup rate (in *t*) for $x = 1$ are derived.

Keywords: *Degenerate parabolic equations, nonlinear boundary conditions, blowup* AMS subject classification: *35 K 65, 35K 55*

1. Introduction

In the recent years questions like blowup, global solvability and qualitative behaviour of solutions near blowup time for semilinear parabolic problems have attracted considerable interest. Should we restrict ourselves to problems with nonlinear boundary conditions, the heat equation has been discussed to a fair degree, however, nonlinear diffusion problems have not been studied in that extent (see [3] and references therein).

The purpose of this paper is to choose an interesting model problem to demonstrate how the diffusion coefficient, nonlinearly depending on the gradient of a solution, affects the value of critical exponent for global solvability and to describe the behaviour of solutions that blow up in a finite time as detailed as possible. Thus, consider the problem ussed to a fair degree, however, nonlinearly that extent (see [3] and references the policies of the problem to choose an interesting model problem to meanly depending on the gradient of a so global solvability and to des

years questions like blowup, global solvability and qualitative behaviour of
blowup time for semilinear parabolic problems have attracted considerable
ld we restrict ourselves to problems with nonlinear boundary conditions,
tion has been discussed to a fair degree, however, nonlinear diffusion
inote the total number of the same number of terms of the same number of terms of this paper is to choose an interesting model problem to demonstrate
non coefficient, nonlinearly depending on the gradient of a solution, affects
critical exponent for global solvability and to describe the behaviour of
below up in a finite time as detailed as possible. Thus, consider the

$$
u_t = (|u_x|^{p-2}u_x)_x \text{ in } \Omega_T \equiv (0,1) \times (0,T)
$$

$$
u_x(0,t) = 0
$$

$$
|u_x(1,t)|^{p-2}u_x(1,t) = |u(1,t)|^{q-2}u(1,t) \text{ for } t \in (0,T)
$$

$$
u(\cdot,0) = \varphi \in W_p^1(0,1)
$$

M. Chipot: Inst. Angcw. Math., Univ. Zurich, Winterthurerstr. *190, CH - 8057* Zurich, Switzerland

J. Filo: Inst. Appl. Math., Comenius Univ., Mlynská dolina, 842 15 Bratislava, Slovakia; the author has been *partially supported by VEGA grant 1/4195/97.*
ISSN 0232-2064 / S 2.50 © Heldermann Verlag Berlin

where $1 < p < 2$ and $q > 1$ are given. The elementary inequality

M. Chipot and J. Filo
\n
$$
1 < p < 2 \text{ and } q > 1 \text{ are given. The elementary inequality}
$$
\n
$$
\max_{0 \le x \le 1} |v(x)|^{(3p-2)/p} \le \frac{3p-2}{p} \left(\int_0^1 |v_x(x)|^p dx \right)^{1/p} \left(\int_0^1 |v(x)|^2 dx \right)^{(p-1)/p}
$$
\n
$$
+ \int_0^1 |v(x)|^{(3p-2)/p} dx \qquad \forall v \in W_p^1(0,1)
$$
\nwe help of Young's inequality easily yields

\n
$$
|v(1)|^{(3p-2)/p} \le \varepsilon \int_0^1 |v_x(x)|^p dx + C_{\varepsilon} \int_0^1 |v(x)|^2 dx + C
$$
\n
$$
0 < \varepsilon < \infty. \text{ Using (1.2) it is not difficult to show (see Remark 1) that Problem globally solvable, i.e., it has a weak solution for arbitrarily large *T* whenever
$$

with the help of Young's inequality easily yields

$$
\int_{0}^{1} |v(x)|^{p(x)} dx \quad \forall \ v \in W_{p}(0, 1)
$$
\n
$$
[V(1)]^{(3p-2)/p} \leq \varepsilon \int_{0}^{1} |v_{x}(x)|^{p} dx + C_{\varepsilon} \int_{0}^{1} |v(x)|^{2} dx + C \qquad (1.2)
$$
\n
$$
\infty. Using (1.2) it is not difficult to show (see Remark 1) that Problem solvable, i.e. it has a weak solution for arbitrarily large T whenever\n
$$
q \leq \frac{3p-2}{p}.
$$
\n
$$
\text{this note to show that the growth condition (1.3) for global solvability (1.1) is optimal.}
$$
\n
$$
\text{precise, let us take a function } a \in C^{4}(\mathbb{R}),
$$
\n
$$
a(\xi) = |\xi|^{p-2}\xi \quad \text{if } |\xi| \geq \eta > 0 \qquad (1.4)
$$
\n
$$
a'(\xi) > 0 \quad \text{and} \quad |a(\xi)| \leq |\xi|^{p-1} \qquad \text{for all } \xi \in \mathbb{R}, \qquad (1.5)
$$
\n
$$
\text{as arbitrary but from now on fixed and instead of (1.1) consider the}
$$
$$

for any $0 < \varepsilon < \infty$. Using (1.2) it is not difficult to show (see Remark 1) that Problem (1.1) is globally solvable, i.e. it has a weak solution for arbitrarily large *T* whenever 9. Using (1.2) it is not difficult to show
that, i.e. it has a weak solution for
 $q \leq \frac{3p-2}{p}$.
s note to show that the growth cone
(1) is optimal.
ccise, let us take a function $a \in C^4(\mathbb{I})$
 $a(\xi) = |\xi|^{p-2}\xi$ if $|\xi| \$

not difficult to show (see Remark 1) that Problem
weak solution for arbitrarily large *T* whenever

$$
q \leq \frac{3p-2}{p}
$$
. (1.3)
it the growth condition (1.3) for global solvability
function $a \in C^4(\mathbb{R})$,
 $q = 2\zeta$ if $|\xi| \geq \eta > 0$ (1.4)
 $(\xi)| \leq |\xi|^{p-1}$ for all $\xi \in \mathbb{R}$, (1.5)
in 0.

It is the aim of this note to show that the growth condition (1.3) for global solvability of problems like (1.1) is optimal. arrow that the group
of show that the group
imal.
 $\log \frac{1}{\xi}$ is a function
 $a(\xi) = |\xi|^{p-2} \xi$

To be more precise, let us take a function $a \in C^4(\mathbb{R})$,

$$
a(\xi) = |\xi|^{p-2}\xi \quad \text{if } |\xi| \ge \eta > 0 \tag{1.4}
$$

$$
a'(\xi) > 0 \quad \text{and} \quad |a(\xi)| \le |\xi|^{p-1} \qquad \text{for all} \quad \xi \in \mathbb{R}, \tag{1.5}
$$

problem

$$
q \leq \frac{3p-2}{p}.
$$
 (1.3)
It is the aim of this note to show that the growth condition (1.3) for global solvability
of problems like (1.1) is optimal.
To be more precise, let us take a function $a \in C^4(\mathbb{R})$,
 $a(\xi) = |\xi|^{p-2}\xi$ if $|\xi| \geq \eta > 0$ (1.4)
 $a'(\xi) > 0$ and $|a(\xi)| \leq |\xi|^{p-1}$ for all $\xi \in \mathbb{R}$, (1.5)
 $0 < \eta \ll 1$ being arbitrary, but from now on fixed, and instead of (1.1) consider the
problem
 $u_t = (a(u_x))_x$ in Ω_T
 $u_x(0,t) = 0$
 $a(u_x(1,t)) = |u(1,t)|^{q-2}u(1,t)$ for $t \in [0,T)$ (1.6)
 $u(x,0) = \varphi(x) > 0$ for $x \in [0,1]$.
We would like to show that Problem (1.6) has for smooth φ a classical solution which
becomes unbounded in a finite time if
 $q > \frac{3p-2}{p}$. (1.7)
Moreover, we will derive estimates on the profile (in x) and the rate (in t) for $x = 1$

We would like to show that Problem (1.6) has for smooth φ a classical solution which becomes unbounded in a finite time if

$$
q > \frac{3p-2}{p}.\tag{1.7}
$$

Moreover, we will derive estimates on the profile (in x) and the rate (in t) for $x = 1$ of the solutions to problem (1.6) when they approach the blowup time. The methods, that we apply, borrow material from [3] and the references therein, nevertheless, their application to our problem seems to he not straightforward.

Before stating our main result, let us make precise some notion we use throughout the paper. By a *classical solution* or shortly by a *solution* of Problem (1.6) on Ω_T we mean a smooth function *u*, say $u \in C^{2,1}(\overline{\Omega}_T)$, such that all identities in (1.6) are satisfied. Put

$$
T_{\max} \equiv \sup \Big\{ T \Big| \text{ there exists a classical solution of Problem (1.6) on } \Omega_T \Big\}
$$

By a *maximal solution* of Problem (1.6) we shall understand a function $u : \overline{\Omega} \times$ $[0, T_{\text{max}}) \longrightarrow \mathbb{R}$ such that *u* is a solution of (1.6) on Ω_T for any T , $T < T_{\text{max}}$. If $T_{\text{max}} = \infty$, we say that Problem (1.6) is globally solvable. On the other hand, if $T_{\text{max}} < \infty$ and $\max_{x \in \overline{\Omega}} |u(x,t)| \longrightarrow \infty$ as $t \to T_{\text{max}}$, we say that the solution *u blows up in a finite time.* Unit Problem (1.6) is globally solvable. On the other hand, if $\frac{1}{\sqrt{2}}|u(x,t)| \rightarrow \infty$ as $t \rightarrow T_{\text{max}}$, we say that the solution *u* blows

In summarizes our knowledge of the behaviour of the solutions to
 $t \leq t \leq 2$, pq

2. Main results

The following theorem summarizes our knowledge of the behaviour of the solutions to Problem (1.6).

Theorem 1. Let $1 < p < 2$, $pq - 3p + 2 > 0$, $\varphi'(1) > p$ and assume that u is a *maximal positive solution of Problem (1.6) such that*

$$
u_t, u_x, u_{xt} \quad \text{are continuous and non-negative} \tag{2.1}
$$

on $[0,1] \times [0,T_{\text{max}})$. *Then:*

(i) The solution u blows up in a finite time and the maximal time of existence,

can be estimated from above as follows:
 $0 < T_{\text{max}} \le \tau(\varphi(1))^{-(pq-3p+2)/(p-1)},$ (2.2) *Trnax, can be estimated from above as follows: T((1))321), (2.2)*

$$
0 < T_{\max} \le \tau(\varphi(1))^{-(pq-3p+2)/(p-1)},\tag{2.2}
$$

where $\tau \equiv p/(pq - 3p + 2)$.

(ii) *Moreover, if* $q < 2$, then there exist positive constants m, C and for each $\varepsilon \in (0, 1)$ a constant c_{ε} such that

$$
[0, T_{\max})\text{ Then:}
$$
\nthe solution u blows up in a finite time and the maximal time of existence,
\nbe estimated from above as follows:
\n
$$
0 < T_{\max} \le \tau(\varphi(1))^{-(pq-3p+2)/(p-1)}, \qquad (2.2)
$$
\n
$$
p/(pq-3p+2).
$$
\n
$$
[0, pq-3p+2].
$$
\n
$$
[1, pq-3p+2] \le \tau(\varphi(1))^{-(pq-3p+2)/(p-1)}, \qquad (2.2)
$$
\n
$$
[0, T_{\max} - t)^{(p-1)/(pq-3p+2+\epsilon)} \le u(1, t) \le \frac{C}{(T_{\max} - t)^{(p-1)/(pq-3p+2)}} \qquad (2.3)
$$
\n
$$
[0, T_{\max}), and \qquad \frac{m}{(1-x)^{(p-1)/(q-p)}} \le u(x, t) \qquad (2.4)
$$
\n
$$
[0, 1] \times [0, T_{\max}) \text{ such that}
$$
\n
$$
\zeta \le x \le 1 - \Lambda^{\epsilon}(T_{\max} - t)^{(q-p)/(pq-3p+2+\epsilon)}, \qquad (2.5)
$$
\n
$$
(1 - (\sigma - 1)^{-1}(\varphi(1))^{2-\sigma})_{+} = \max\{0, 1 - (\sigma - 1)^{-1}(\varphi(1))^{2-\sigma}\}, \Lambda^{\epsilon} \equiv (\sigma - 1)^{-1}(\varphi(1))^{2-\epsilon} \text{ and } \Lambda^{\epsilon} \equiv \Lambda^{\epsilon}(\sigma - 1)^{-1}(\varphi(1))^{
$$

for all $t \in [0, T_{\text{max}})$, and

$$
(-3p+2+\epsilon) \stackrel{(1,0)}{=} (T_{\max} - t)^{(p-1)/(pq-3p+2)}
$$

for all $(x, t) \in [0, 1] \times [0, T_{\max})$ *such that*

$$
\zeta \leq x \leq 1 - \Lambda^{\epsilon} (T_{\max} - t)^{(q-p)/(pq-3p+2+\epsilon)}, \qquad (2.5)
$$

for all $(x, t) \in [0, 1] \times [0, T_{\max})$ such that
 $\zeta \leq x \leq 1 - \Lambda^{\epsilon} (T_{\max} - t)^{(q-p)/(pq-3p+2+\epsilon)},$

where $\zeta \equiv (1 - (\sigma - 1)^{-1} (\varphi(1))^{2-\sigma})_{+} = \max \{0, 1 - (\sigma - 1)^{-1} (\varphi(1))^{2-\sigma}\}, \Lambda^{\epsilon} \equiv 1$
 $1)^{-1} c_{\epsilon}^{2-\sigma}$ and $(\sigma$ *tor all* $(x, t) \in$
 uhere $\zeta \equiv (1$
 $(1)^{-1} c_{\epsilon}^{2-\sigma}$ and

$$
\sigma - 1 = \frac{q-1}{p-1}.\tag{2.6}
$$

that

M. Chipot and J. Filo
\n(iii) On the other hand, if
$$
q \ge 2
$$
, then there exist positive constants c, C, m, M such
\n
$$
\frac{c}{(T_{\max} - t)^{(p-1)/(pq-3p+2)}} \le u(1,t) \le \frac{C}{(T_{\max} - t)^{(p-1)/(pq-3p+2)}}
$$
\n(2.7)

for all $t \in [0, T_{\text{max}})$ *and*

\n (a) Find the other hand, if
$$
q \geq 2
$$
, then there exist positive constants c, C, m, M such that\n $\frac{c}{(T_{\max} - t)^{(p-1)/(pq-3p+2)}} \leq u(1, t) \leq \frac{C}{(T_{\max} - t)^{(p-1)/(pq-3p+2)}} \qquad (2.7)$ \n

\n\n (2.7) T_{\max} and\n

\n\n (2.8) $\frac{m}{(1-x)^{(p-1)/(q-p)}} \leq u(x, t) \leq \frac{M}{(\min\{1-\rho, 1-x\})^{(p-1)/(q-p)}} \qquad (2.8)$ \n

\n\n (2.9) $\frac{m}{(1-x)^{(p-1)/(q-p)}} \leq u(x, t) \leq \frac{M}{(\min\{1-\rho, 1-x\})^{(p-1)/(q-p)}} \qquad (2.9)$ \n

\n\n (2.9) $\frac{m}{(1-x)^{(p-1)/(q-p)}} \leq x \leq 1 - \Lambda(T_{\max} - t)^{(q-p)/(pq-3p+2)} \qquad (2.9)$ \n

for some $\rho, 0 < \rho < 1$ *, where the right-hand side inequality holds for all* $(x, t) \in [0, 1] \times$ $(0, T_{\text{max}})$ and the left-hand side inequality only for those (x, t) , that satisfy $\frac{1}{\sqrt{2}}$
 $\frac{1}{\sqrt{2}}$

$$
\zeta \leq x \leq 1 - \Lambda (T_{\max} - t)^{(q-p)/(pq-3p+2)} \tag{2.9}
$$

with $\Lambda \equiv (\sigma - 1)^{-1}c^{2-\sigma}$ and ζ as above.

The constants c_{ϵ} , c , C , m , M , ρ are specified in the proof below.

The next theorem says that it is possible to find initial functions such that (2.1) *is* satisfied for corresponding solutions of Problem (1.6).

some
$$
\rho, 0 < \rho < 1
$$
, where the right-hand side inequality holds for all $(x, t) \in [0, 1] \times$
\n \lim_{\max} and the left-hand side inequality only for those (x, t) , that satisfy
\n
$$
\zeta \leq x \leq 1 - \Lambda(T_{\max} - t)^{(q-p)/(pq-3p+2)}
$$
\n $\Lambda \equiv (\sigma - 1)^{-1}c^{2-\sigma}$ and ζ as above.
\nThe constants $c_{\epsilon}, c, C, m, M, \rho$ are specified in the proof below.
\nThe next theorem says that it is possible to find initial functions such that (2.1) is
\nsfied for corresponding solutions of Problem (1.6).
\nTheorem 2. Assume that φ is a positive smooth solution of the problem
\n $(a(v_x))_x = \lambda v^{(q-1)/(p-1)}(x)$ $(0 < x < 1)$
\n $v_x(0) = 0$
\n $a(v_x(1)) = v^{q-1}(1)$
\n $g(x, t) = \lambda v^{(q-1)/(p-1)}(x)$ $(0 < x < 1)$
\n (2.10)
\n $a(v_x(1)) = v^{q-1}(1)$
\n $g(x, t) = \lambda v^{(q-1)/(p-1)}(x)$ $(0 < x < 1)$
\n (2.10)
\n $a(v_x(1)) = v^{q-1}(1)$
\n $a(v_x(1)) = (2.10)$
\n $a(v_x(1)) = (2.10$

for some $\lambda > 0$ and such that $\varphi'(1) > \eta$. Then there exists a unique maximal classical *solution of Problem (1.6) such that* $u_t, u_x \in C^{2,1}(\overline{\Omega}_T)$ *for all* $T \in [0, T_{\max})$ *and* u_t, u_x, u_{xt} *are positive on* $(0, 1] \times [0, T_{\text{max}})$. *(i)* \int *(2.10) (i) f froblem* (2.10) *with* (2.11)

Finally, the following theorem guarantees the classical solvability of Problem (2.10).

Theorem 3. Let $\mathcal{L}^{q-1} > \eta^{p-1}$ be given. Then there exists a positive function $v \in C^{\infty}([0,1])$, $v(1) = \mathcal{L}$ such that v is a solution of Problem (2.10) with

$$
\lambda = \frac{\sigma A (\mathcal{L}^{\sigma - 1})}{\mathcal{L}^{\sigma} - \mu^{\sigma}},
$$
\n(2.11)

where $A(\xi) = \int_0^{\xi} a'(\tau) \tau \, d\tau$ and $\mu = v(0)$.

3. Proof of Theorem 1

1. First of all note that our assumption $\varphi'(1) > \eta$ yields $a(u_x(1,t)) = u_x^{p-1}(1,t)$ for all $t \in [0, T_{\text{max}})$ and one can compute

$$
u^{p(\sigma-1)}(1,t)=u_x^p(1,t)=a(u_x(1,t))u_x(1,t)=\int_0^1((a(u_x))_xu_x+a(u_x)u_{xx})dx.
$$

As $(a(u_x))_x = u_t$, integrating the first term on the right-hand side of the above identity by parts we arrive at $((a(u_x))_x u_x + b$
ht-hand side of
 $\int u_{x} u dx + \int u$

$$
u^{p(\sigma-1)}(1,t) = u_x^p(1,t) = a(u_x(1,t))u_x(1,t) = \int_0^1 \left((a(u_x))_x u_x + a(u_x) u_{xx} \right) dx
$$

As $(a(u_x))_x = u_t$, integrating the first term on the right-hand side of the above ide
by parts we arrive at

$$
u^{p(\sigma-1)}(1,t) = u_t(1,t)u(1,t) - u_t(0,t)u(0,t) - \int_0^1 u_{xt}u \, dx + \int_0^1 a(u_x)u_{xx} \, dx.
$$

In view of (1.5) and (2.1) we have

$$
\int_0^1 a(u_x)u_{xx} \, dx \le \int_0^1 u_x^{p-1}u_{xx} \, dx
$$

that with the previous inequality yield

$$
u^{p(\sigma-1)}(1,t) \le u_t(1,t)u(1,t) + p^{-1} \int_0^1 (u_x^p)_x \, dx,
$$

i.e.

$$
u_t(1,t) \ge \alpha u^{1+\theta}(1,t),
$$

where $\alpha \equiv \frac{p-1}{p}$ and $\theta \equiv \frac{pq-3p+2}{p-1}$. Solving this differential inequality one can see t

In view of (1.5) and (2.1) we have

$$
= a(u_x(1,t)) u_x(1,t) = \int_0^1 \left(\left(a(u_x \right) \right) du_x(1,t) dt + \int_0^1 \left(a(u_x \right) du_x(1,t) - u_t(0,t) u(0,t) - \int_0^1 u_{xt} u \right)
$$
\nwe have\n
$$
\int_0^1 a(u_x) u_{xx} dx \leq \int_0^1 u_x^{p-1} u_{xx} dx
$$
\n
$$
\text{quality yield}
$$

that with the previous inequality yield

$$
u_t(1, t)u(1, t) - u_t(0, t)u(0, t) - \int_0^t u_{xt}u \, dx + \int_0^t a(u_x)u_{xx} \, dx.
$$

\n(2.1) we have
\n
$$
\int_0^1 a(u_x)u_{xx} \, dx \le \int_0^1 u_x^{p-1}u_{xx} \, dx
$$

\nous inequality yield
\n
$$
u^{p(\sigma-1)}(1, t) \le u_t(1, t)u(1, t) + p^{-1} \int_0^1 (u_x^p)_x \, dx,
$$

\n
$$
u_t(1, t) \ge \alpha u^{1+\theta}(1, t),
$$

\n
$$
\theta \equiv \frac{pq-3p+2}{p-1}.
$$
 Solving this differential inequality one can see that
\n
$$
T_* \equiv \frac{1}{\alpha \theta(\varphi(1))^{\theta}}
$$
 and
$$
\frac{\varphi(1)}{(1 - T_*^{-1}t)^{1/\theta}} \le u(1, t)
$$

\nNow it is not difficult to show that $u(1, t) \to \infty$ if $t \to T_{\text{max}}$ (see

$$
u_t(1,t) \ge \alpha u^{1+\theta}(1,t) , \qquad (3.1)
$$

that with the previous in $u^{p(\sigma-1)}$.
i.e.
where $\alpha \equiv \frac{p-1}{p}$ and $\theta \equiv$ $\frac{pq-3p+2}{p^2-3p+2}$. Solving this differential inequality one can see that

$$
u_t(1,t) \ge \alpha u^{1+\theta}(1,t) ,
$$

\n
$$
\frac{u_{\theta}}{p}
$$
 and $\theta \equiv \frac{pq-3p+2}{p-1}$. Solving this differential inequality one can
\n
$$
T_{\max} \le T_{\star} \equiv \frac{1}{\alpha \theta(\varphi(1))^{\theta}}
$$
 and
$$
\frac{\varphi(1)}{(1-T_{\star}^{-1}t)^{1/\theta}} \le u(1,t)
$$

for all $t \in [0, T_{\text{max}})$. Now it is not difficult to show that $u(1,t) \to \infty$ if $t \to T_{\text{max}}$ (see the proof of Theorem 2 below). Next, integrate (3.1) over (s, t) , $s < t$:
 $-(u(1,s))^{-\theta} \le (u(1,t))^{-\theta} - (u(1,s))^{-\theta} \le -\alpha \theta(t-s).$ (3.2) the proof of Theorem 2 below). Next, integrate (3.1) over (s, t) , $s < t$: *(u(1,* t))° — (u(1, *s))_8 —a* 8(t — *s). (3.2) J J J (yet)* \int *J (x, t) (2.3) (2.3) (2.3) (2.7) 2 with C* = $(\alpha \theta)^{-1/\theta}$. Note that understand th

$$
-(u(1,s))^{-\theta} \le (u(1,t))^{-\theta} - (u(1,s))^{-\theta} \le -\alpha \theta(t-s).
$$
 (3.2)

For t going to T_{max} (3.2) easily yields $(2.3)_2$ and $(2.7)_2$ with $C \equiv (\alpha \theta)^{-1/\theta}$. Note that by the notation $(2.3)_2$ we understand the second inequality in (2.3) , i.e. by the notation $(2.3)_2$ we understand the second inequality in (2.3) , i.e.

$$
(-1)^{-\circ} \le (u(1, t))^{-\circ} - (u(1, s))^{-\circ} \le -\circ
$$

easily yields (2.3)₂ and (2.7)₂ with
understand the second inequality is

$$
u(1, t) \le \frac{C}{(T_{\text{max}} - t)^{(p-1)/(pq-3p+2)}}
$$

2. Now choose $0 < \rho < 1$, $n \ge 2$, define $\chi(x) \equiv \left(\frac{(x-\rho)_+}{1-\rho}\right)^n$ and write

$$
J(x,t) \equiv u_x(x,t) - \chi(x) u^{\sigma-1}(x,t).
$$
 (3.3)

We now claim that if ρ is close enough to 1 and n is sufficiently large, then

$$
J(x,t) \ge 0 \qquad \text{on} \quad [0,1] \times [0,T_{\text{max}}) \tag{3.4}
$$

close enough to 1 and *n* is sufficiently large, then
 $J(x, t) \ge 0$ on $[0, 1] \times [0, T_{\text{max}})$ (3.4)

rst that due to our assumptions $u_x(1, 0) = \varphi'(1) > \eta$, $u_{xt} \ge 0$,
 $|u, 1| \times [0, T_{\text{max}})$ for some $0 < \mu < 1$. Taking $\rho > \mu$ if $q \ge 2$. In fact, observe first that due to our assumptions $u_z(1, 0) = \varphi'(1) > \eta$, $u_{zt} \ge 0$, difficult to verify that *J* satisfies the following set of equations:

$$
H_{\mathcal{H}} \leq \sum \text{ in fact, observe first that due to our assumptions } a_{\mathcal{I}}(1,0) = \varphi(1) > \eta, a_{\mathcal{I}}(1,0)
$$
\nwe have $a_{\mathcal{I}}(x,t) \geq \eta$ on $[\mu,1] \times [0,T_{\text{max}})$ for some $0 < \mu < 1$. Taking $\rho \geq \mu$ it is not difficult to verify that J satisfies the following set of equations:\n
$$
\mathcal{P}J \equiv J_t - a'(a_x)J_{xx} + A(x,t)J_x + B(x,t)J = (p-1)u_x^{p-3}u^{2(\sigma-1)}\left(\chi\chi'' - (2-p)(\chi')^2 + 2(q-1)u^{\sigma-2}\chi'\chi^2 + (\sigma-1)(q-2)u^{2\sigma-4}\chi^4\right) \quad \text{on } \Omega_T \qquad (3.5)
$$
\n
$$
J(0,t) = J(1,t) = 0 \quad \text{for } t \in [0,T]
$$
\n
$$
J(x,0) = \varphi'(x) - \chi(x)\varphi^{\sigma-1}(x) \quad \text{for } x \in [0,1]
$$
\nfor some bounded and continuous functions A and B , and for any $T < T_{\text{max}}$. Now, if $q \geq 2$ and $n \geq \max\{2, (p-1)^{-1}\},$ \n(3.6)\nthen it is easy to verify that $\mathcal{P}J \geq 0$ on $\overline{\Omega}_T$. We next claim if ρ is close enough to 1, $J(x,0) \geq 0 \quad \text{for } x \in [0,1]$.\n\nTo see this, we note that due to (2.1), $\varphi', \varphi'', \varphi''' \geq 0$ on [0,1]. Then

for some bounded and continuous functions A and B, and for any $T < T_{\text{max}}$. Now, if

$$
q \ge 2 \qquad \text{and} \qquad n \ge \max\{2, (p-1)^{-1}\}, \tag{3.6}
$$

then it is easy to verify that $\mathcal{P}J \geq 0$ on $\overline{\Omega}_T$. We next claim if ρ is close enough to 1,

$$
J(x,0) \ge 0 \quad \text{for } x \in [0,1]. \tag{3.7}
$$

To see this, we note that due to (2.1) , $\varphi', \varphi'', \varphi''' \ge 0$ on $[0, 1]$. Then

$$
J(x,0) \ge 0 \quad \text{for } x \in [0,1].
$$

at the $\text{the to } (2.1), \varphi', \varphi'', \varphi''' \ge 0 \text{ on } [0,1].$ T

$$
\varphi'(x) \ge \ell(x) \equiv (\varphi''(1)(x-1) + (\varphi(1))^{\sigma-1})_+.
$$

This for $x = 0$ yields $\varphi''(1) \geq (\varphi(1))^{n-1}$ and one can conclude that $\ell(x) \equiv 0$ for all $x \in [0, x_0], x_0 = 1 - (\varphi(1))^{n-1} / \varphi''(1)$. Taking $\rho \ge x_0$ and making use the convexity of the function $\chi(x)\varphi^{\sigma-1}(x)$ together with the fact that $\varphi^{\sigma-1}(1) = \ell(1)$ and $\chi(x_0)\varphi^{\sigma-1}(x_0) =$ $\ell(x_0)=0$ we arrive at $\varphi''(1) \ge (\varphi(1))^{\sigma-1}$ and one can conclude $f(x) = \varphi(x)$
 $f(x) = \varphi(x) + \varphi''(1)$. Taking $\rho \ge x_0$ and making ogether with the fact that $\varphi^{\sigma-1}(1) = \ell(1)$
 $\chi(x)\varphi^{\sigma-1}(x) \le \ell(x)$ for all $x \in [0,1]$

$$
\chi(x)\varphi^{\sigma-1}(x)\leq \ell(x)\qquad\text{for all}\;\;x\in[0,1]
$$

and (3.7) follows. Thus, the maximum principle yields (3.4), and so the estimate $(2.8)_2$ follows by integration. Indeed, integrating (3.4) over $(x, 1)$ we find

$$
\chi(x)\varphi^{\sigma-1}(x) \le \ell(x) \qquad \text{for all } x \in [0,1]
$$

Thus, the maximum principle yields (3.4), and so t
tion. Indeed, integrating (3.4) over $(x, 1)$ we find

$$
u^{\sigma-2}(x,t) \le 1/(\sigma-2)g(x), \qquad g(x) \equiv \int_x^1 \chi(\tau) d\tau.
$$

Note $g \in C^3$, $g(x) = \int_{\rho}^{1} \chi(\tau) d\tau$ for all $x \in [0, \rho], g(1) = 0$ and $g''(x) \le 0$. Thus

$$
g(x) \ge \int_{\rho}^{1} \chi(\tau) d\tau \min\left\{1, \frac{x-1}{\rho-1}\right\}
$$

and

$$
u(x,t) \le \left[\frac{\sigma-2}{1-\rho} \int_{\rho}^{1} \chi(\tau) d\tau\right]^{1/(2-\sigma)} \frac{1}{(\min\{1-\rho, 1-x\})^{(p-1)/(q-p)}}
$$

[0,1], i.e. (2.8)₂.

$$
w \text{ of } (3.4), i.e. \quad J(x,t) \ge 0 \text{ and due to the fact that } J(1,t) = 0 \text{ we find}
$$

0, i.e.

$$
u_t(1,t) \le (p-1)\chi'(1)u^{(p-1)(\sigma-1)}(1,t) + (q-1)u^{p(\sigma-1)-1}(1,t), \qquad (3.8)
$$

$$
= n/(1-\rho). \text{ Note that } p(\sigma-1) - 1 - (p-1)(\sigma-1) = \sigma - 2 > 0. \text{ Therefore}
$$

for all $x \in [0, 1]$, i.e. $(2.8)_2$.

3. In view of (3.4), i.e. $J(x,t) \geq 0$ and due to the fact that $J(1,t) = 0$ we find for all $x \in [0,1]$
3. In view of
 $J_z(1,t) \le 0$, i.e. $\begin{align*}\n u^{(p-1)(\sigma-1)}(1,t) + (q-1)u^{p(\sigma-1)-1}(1,t), \quad & (3.8) \\
 u^{(p-1)(\sigma-1)}(1,t) + (q-1)u^{p(\sigma-1)-1}(1,t), \quad & (3.8) \\
 u^{p(\sigma-1)-1} - (p-1)(\sigma-1) &= \sigma - 2 > 0. \quad & \text{Therefore} \\
 u^{p(\sigma-1)-1}(1,t) \quad & \forall \ t \in [\iota, T_{\text{max}}), \quad & (3.9) \\
 u^{p(\sigma-1)-1}(1,t) \quad & \forall \ t \in [\iota$

$$
u_t(1,t) \le (p-1)\chi'(1) u^{(p-1)(\sigma-1)}(1,t) + (q-1) u^{p(\sigma-1)-1}(1,t), \tag{3.8}
$$

where $\chi'(1)=n/(1-\rho)$. Note that $p(\sigma-1)-1-(p-1)(\sigma-1)=\sigma-2>0$. Therefore

where
$$
\chi'(1) = n/(1 - \rho)
$$
. Note that $p(\sigma - 1) - 1 - (p - 1)(\sigma - 1) = \sigma - 2 > 0$. Therefore\n
$$
u_t(1, t) \leq \left((p - 1)\chi'(1) + q - 1 \right) u^{p(\sigma - 1) - 1}(1, t) \quad \forall \ t \in [\iota, T_{\text{max}}), \tag{3.9}
$$
\nwhere $\iota \equiv \min\{s \mid s \in [0, T_{\text{max}}), u(1, s) \geq 1\}$. Consequently,

$$
u_t(1,t) \le ((p-1)\chi'(1) + q - 1)u^{p(\sigma-1)-1}(1,t) \quad \forall t \in [\iota, T_{\max}),
$$

\n
$$
\text{where } \iota \equiv \min\{s | s \in [0, T_{\max}), u(1, s) \ge 1\}. \text{ Consequently,}
$$

\n
$$
(u(1,t))^{-\theta} - (u(1,s))^{-\theta} \ge -(q-1+(p-1)\chi'(1))\theta(t-s) \quad \forall t, s \in [\iota, T_{\max}).
$$

Thus, after $t \to T_{\text{max}}$ (lim_{$t \to T_{\text{max}}$} $u(1, t) = \infty$), we arrive at

$$
\leq (p-1)\chi'(1)u^{(p-1)(\sigma-1)}(1,t) + (q-1)u^{p(\sigma-1)-}
$$

\n- ρ). Note that $p(\sigma-1) - 1 - (p-1)(\sigma-1) = \sigma$
\n
$$
\leq ((p-1)\chi'(1) + q - 1)u^{p(\sigma-1)-1}(1,t) \quad \forall t \in [t]
$$

\n $\in [0, T_{\max}), u(1, s) \geq 1$]. Consequently,
\n $u(1, s))^{-\theta} \geq -(q-1 + (p-1)\chi'(1))\theta(t - s) \quad \forall$
\n $ax (\lim_{t \to T_{\max}} u(1, t) = \infty)$, we arrive at
\n $u(1, t) \geq \left(\frac{1}{(q-1 + (p-1)\chi'(1))\theta(T_{\max} - t)}\right)^{1/\theta}$
\n, and (2.7)₁ follows.
\n $atimate$ for $q < 2$ we have to proceed differently. I
\n
$$
\int_{1-\xi(t)}^{1} u(x, t) dx, \xi(t) = u^{-\omega}(1, t), \omega = \frac{q-p}{p-1} + \varepsilon,
$$

\n $= -u^{\nu-1}(1, t)u_t(1, t)I(t) + u^{\nu}(1, t) \int_{1}^{1} (a(u_x))$

4. To obtain the estimate for $q < 2$ we have to proceed differently. Put

for all
$$
t \in [i, T_{\text{max}})
$$
, and (2.7)₁ follows.
\n4. To obtain the estimate for $q < 2$ we have to proceed differently. Put
\n
$$
y(t) = u^{\nu}(1, t) \int_{1-\xi(t)}^{1} u(x, t) dx, \ \xi(t) = u^{-\omega}(1, t), \ \omega = \frac{q-p}{p-1} + \varepsilon, \ \nu = \omega - 1 + \varepsilon.
$$
\nThen
\n
$$
y'(t) = -u^{\nu-1}(1, t)u_t(1, t)I(t) + u^{\nu}(1, t) \int_{1-\xi(t)}^{1} (a(u_x))_x dx
$$
\n(3)

Then

$$
\int_{\max}^{T} f(t) \, d\mu(t) \, d\mu(t) \, d\mu(t) \, d\mu(t)
$$
\nThus, the estimate for $q < 2$ we have to proceed differently. Put

\n
$$
\int_{1-\xi(t)}^{1} u(x,t) \, dx, \, \xi(t) = u^{-\omega}(1,t), \, \omega = \frac{q-p}{p-1} + \varepsilon, \, \nu = \omega - 1 + \varepsilon.
$$
\n
$$
y'(t) = -u^{\nu-1}(1,t)u_t(1,t)I(t) + u^{\nu}(1,t) \int_{1-\xi(t)}^{1} (a(u_x))_x dx \,, \tag{3.10}
$$
\n
$$
I(t) = \omega u(1 - \xi(t), t) \xi(t) - \nu \int_{1-\xi(t)}^{1} u(x,t) dx.
$$

where

$$
I(t) = \omega u(1 - \xi(t), t) \xi(t) - \nu \int_{1 - \xi(t)}^{1} u(x, t) dx
$$

Let us first consider the more difficult case $\nu > 0$. Below we shall use the following estimate, which follows from the non-negativeness of u_x, u_{xz} and from the form of our boundary condition at $x = 1$: *U*₁ = $-u^{\nu-1}(1, t)u_t(1, t)I(t) + u^{\nu}(1, t)\int_{1-\xi(t)}^{1}(a(u_x))_x dx$, (3.10)
 I(*t*) = $\omega u(1 - \xi(t), t)\xi(t) - \nu \int_{1-\xi(t)}^{1} u(x, t) dx$.

Sider the more difficult case $\nu > 0$. Below we shall use the following

follows from the non-negativ $\int_{1-\xi(t)}^{t} u(x,t) dx$.

0. Below we shall use the following

of u_x, u_{xx} and from the form of our
 $t) \le u(x,t) \le u(1,t)$ (3.11)

ration yields
 $u^{\epsilon}(1,t)$. (3.12)

$$
u(1,t)(1-u^{-\epsilon}(1,t)) \le u(1-\xi(t),t) \le u(x,t) \le u(1,t) \tag{3.11}
$$

for all $x \in [1 - \xi(t), 1], t \in [0, T_{\text{max}})$, that by integration yields T_{max}), that by integring
 $u^{\epsilon}(1, t) - 1 \leq y(t) \leq$

$$
u^{\epsilon}(1,t) - 1 \le y(t) \le u^{\epsilon}(1,t). \tag{3.12}
$$

Now we show that there exists $\tau \geq 0$ such that $I(t) \geq 0$ for $t \geq \tau$. As

Chipot and J. Filo
\nhow that there exists
$$
\tau \ge 0
$$
 such that $I(t) \ge 0$ for $t \ge \tau$. As
\n
$$
I(t) = (1 - \varepsilon) \int_{1 - \xi(t)}^{1} u(x, t) dx - \omega \int_{1 - \xi(t)}^{1} (u(x, t) - u(1 - \xi(t), t)) dx
$$
\n
$$
\ge (1 - \varepsilon) \int_{1 - \xi(t)}^{1} u(x, t) dx - \omega(u(1, t) - u(1 - \xi(t), t)) \xi(t),
$$

applying (3.11) we arrive at

$$
f_{1-\xi(t)}
$$

\n
$$
I(t) \geq u^{1-\epsilon}(1,t)\xi(t)(1-\epsilon)\left[u^{\epsilon}(1,t) - \frac{\sigma-1}{1-\epsilon}\right],
$$

\n
$$
t \text{ is sufficiently large. Note that } \sigma - 1 > 1 - \epsilon. \text{ Thus, (3.10) yields}
$$

\n
$$
y'(t) \leq u^{\theta+2\epsilon}(1,t), \qquad \theta = (qp - 3p + 2)/(p - 1)
$$

\n
$$
\text{stance of (3.12) gives}
$$

\n
$$
z'(t) \leq z^{1+(\theta+\epsilon)/\epsilon}(t), \qquad z(t) = y(t) + 1.
$$

\n
$$
\in [\tau, T) \text{ such that } s < t, \text{ by integrating of (3.13) we get}
$$

which is positive if t is sufficiently large. Note that $\sigma - 1 > 1 - \varepsilon$. Thus, (3.10) yields

$$
y'(t) \le u^{\theta+2\epsilon}(1,t), \qquad \theta = (qp-3p+2)/(p-1)
$$

which with the assistance of *(3.12)* gives

$$
z'(t) \le z^{1+(\theta+\epsilon)/\epsilon}(t), \qquad z(t) = y(t) + 1. \tag{3.13}
$$

Now, if we take $t, s \in [\tau, T)$ such that $s < t$, by integrating of (3.13) we get

which with the assistance of (3.12) gives
\n
$$
z'(t) \leq x^{1+(\theta+\varepsilon)/\varepsilon}(t), \qquad z(t) = y(t) + 1.
$$
\nNow, if we take $t, s \in [\tau, T)$ such that $s < t$, by integrating of (3.13) we get
\n
$$
z^{-\delta}(s) - z^{-\delta}(t) \leq \delta(t - s), \qquad \delta = \frac{\theta + \varepsilon}{\varepsilon} = \frac{pq - 3p + 2 + \varepsilon(p - 1)}{\varepsilon(p - 1)}.
$$
\nNext, (3.13) and (3.12) yield
\n
$$
\frac{1}{2^{\delta}u^{\varepsilon\delta}(1,s)} \leq \frac{1}{u^{\varepsilon\delta}(1,t)} + \delta(t - s),
$$
\nthat by letting $t \to T_{\text{max}}$ gives

Next, *(3.13)* and *(3.12)* yield

$$
u^{\theta+2\epsilon}(1,t), \qquad \theta = (qp - 3p + 2),
$$

of (3.12) gives

$$
0 \le z^{1+(\theta+\epsilon)/\epsilon}(t), \qquad z(t) = y(t).
$$

such that $s < t$, by integrating c

$$
\le \delta(t - s), \qquad \delta = \frac{\theta + \epsilon}{\epsilon} = \frac{pq - 3}{\epsilon}
$$

and

$$
\frac{1}{2^{\delta}u^{\epsilon\delta}(1,s)} \le \frac{1}{u^{\epsilon\delta}(1,t)} + \delta(t - s),
$$

ives

$$
\frac{1}{(2^{\delta} \delta (T_{\max} - s))^{(p-1)/(pq-3p+\epsilon(p-1))}} \leq u(1,s).
$$

Hence, $(2.3)_1$ follows. In the case of $\nu \leq 0$, $I(t) \geq 0$ and the rest of the proof is the same as above. *u*: $u(x,t) \geq u(x,s) \geq u_x(1,s)(x-1) + u(1,s) = u^{2-1}(1,s)(x-1) + u(1,s)$
 u $(u(x,t) \geq u(x,s) \geq u_x(1,s)(x-1) + u(1,s) = u^{2-1}(1,s)(x-1) + u(1,s)$ (3.14) follows. In the case
 e.
 p(*x*, *s*) $\geq u_x(1, s)(x -$
 p(*x*, *s*) $\geq u_x(1, s)(x -$
 p(*x*), *t*). Consequently,
 $\max_{\varphi(1) \leq \alpha \leq u(1, t)} (\alpha^{\sigma - 1}(x -)$

5. Now, at last, we show the validity of (2.4) and $(2.8)_1$. We know from (2.1) that

$$
u(x,t) \ge u(x,s) \ge u_x(1,s)(x-1) + u(1,s) = u^{\sigma-1}(1,s)(x-1) + u(1,s) \tag{3.14}
$$

for any $s \in [0,t]$. Consequently,

$$
u(x,t) \ge u(x,s) \ge u_x(1,s)(x-1) + u(1,s) = u^{\sigma-1}(1,s)(x-1) + u(1,s)
$$
 (3.14)
any $s \in [0,t]$. Consequently,

$$
u(x,t) \ge \max_{\varphi(1) \le \alpha \le u(1,t)} (\alpha^{\sigma-1}(x-1) + \alpha) \ge \max_{\varphi(1) \le \alpha \le \vartheta(t)} (\alpha^{\sigma-1}(x-1) + \alpha),
$$
 (3.15)

where $\vartheta(t) \equiv d(T_{\text{max}} - t)^{-\beta}$ and

$$
\begin{aligned}\n&\equiv d(T_{\max} - t)^{-\beta} \text{ and} \\
d &\equiv \begin{cases}\nc_{\epsilon} & \text{if } q < 2 \\
c & \text{if } q \ge 2\n\end{cases} \text{ and } \beta \equiv \begin{cases}\n\frac{p-1}{pq-3p+2+\epsilon} & \text{if } q < 2 \\
\frac{p-1}{pq-3p+2} & \text{if } q \ge 2.\n\end{cases} \\
&\text{or fixed } 0 < x < 1
$$
\n
$$
\int_{-\infty}^{\infty} (\alpha^{\sigma-1}(x-1) + \alpha) = (\alpha^{\sigma-1}(x-1) + \alpha) \Big|_{\alpha = \alpha_0} = \frac{m}{(1-x)^{(p-1)}}
$$
\n
$$
= \left[(\sigma - 1)(1-x) \right]^{1/(2-\sigma)} \quad \text{and} \quad m = (\sigma - 2)(\sigma - 1)^{(1-\sigma)},
$$
\n
$$
\text{at that } \varphi(1) \le \alpha_0 \le \vartheta(t) \text{ whenever}
$$
\n
$$
1 - \frac{p-1}{q-1}(\varphi(1))^{-(q-p)/(p-1)} \le x \le 1 - \frac{p-1}{q-1}(\vartheta(t))^{-(q-p)/(p-1)}
$$
\n
$$
t) \ge m(1-x)^{(1-p)/(q-p)} \text{ if (3.16) holds.}\n\end{aligned}
$$

Note that for fixed $0 < x < 1$

that for fixed
$$
0 < x < 1
$$

\n
$$
\max_{0 \leq \alpha < \infty} \left(\alpha^{\sigma - 1}(x - 1) + \alpha \right) = \left(\alpha^{\sigma - 1}(x - 1) + \alpha \right) \Big|_{\alpha = \alpha_0} = \frac{m}{(1 - x)^{(p - 1)/(q - p)}},
$$
\nand

\n
$$
\alpha_0 = \left[(\sigma - 1)(1 - x) \right]^{1/(2 - \sigma)} \quad \text{and} \quad m = (\sigma - 2)(\sigma - 1)^{(1 - \sigma)/(\sigma - 2)}.
$$

where \rightarrow

$$
\alpha_0 = [(\sigma - 1)(1 - x)]^{1/(2 - \sigma)} \quad \text{and} \quad m = (\sigma - 2)(\sigma - 1)^{(1 - \sigma)/(\sigma - 2)}.
$$

Observe next that $\varphi(1) \leq \alpha_0 \leq \vartheta(t)$ whenever

$$
1 - \frac{p-1}{q-1}(\varphi(1))^{-(q-p)/(p-1)} \le x \le 1 - \frac{p-1}{q-1}(\vartheta(t))^{-(q-p)/(p-1)}.\tag{3.16}
$$

Hence, $u(x,t) \geq m(1-x)^{(1-p)/(q-p)}$ if (3.16) holds.

4. Proof of Theorem 2

Let u_0 be a positive smooth solution of Problem (2.10) and denote $\mu = u_0(0), \mathcal{L} = u_0(1)$. Let $\phi \in C^3(\mathbb{R})$, $\phi(0) = 0$, $\phi_u \leq 0$, $|\phi, \phi_u, \phi_{uu}| \leq C$ on R for some positive constant *C* and

$$
\phi(u) = -|u|^{q-2}u
$$
 if $0 < \mu \le u \le M$ (= 2L).

Moreover, let $a_1 \in C^4(\mathbb{R})$, $0 < \varrho \le a'_1(\xi) \le \varrho^{-1}$ on \mathbb{R} for some $0 < \varrho < 1$ and

$$
0 = -|u|^{q-2}u \quad \text{if } 0 < \mu \le u \le M \text{ (}
$$

$$
R), 0 < \varrho \le a'_1(\xi) \le \varrho^{-1} \text{ on } \mathbb{R} \text{ for some } 0
$$

$$
a_1(\xi) = a(\xi) \quad \text{if } |\xi| \le K \text{ (} = M^{\sigma-1}),
$$

and consider the following problem:

$$
u_t = (a_1(u_x))_x = 0 \text{ and } a_1(u_x) = 0 \text{ and } a_2(u_x) = 0
$$
\n
$$
u_t = (a_1(u_x))_x = 0 \text{ and } a_1(v_x) = 0 \text{ and } a_1(v_x) = 0
$$
\n
$$
u_t = (a_1(u_x))_x = 0 \text{ and } a_1(u_x(1, t)) + \phi(u(1, t)) = 0, u_x(0, t) = 0 \text{ and } a_1(u_x) = 0
$$
\n
$$
u_t = (a_1(u_x))_x = 0 \text{ and } a_1(u_x(1, t)) + \phi(u(1, t)) = 0, u_x(0, t) = 0 \text{ on } [0, T] \text{ (4.1)}
$$
\n
$$
u(x, 0) = u_0(x) \text{ on } [0, 1].
$$
\nIt is of [10: Chapter V/p. 429] guarantees the existence of a unique solution

\n1.1) in the class $H^{2+\beta, 1+\beta/2}(\overline{\Omega_T})$ for any $0 < \beta < 1$. However, to show

\n
$$
u_t = \alpha(x, t)u_{xx} = 0 \text{ in } \Omega_T
$$
\n
$$
u_t = \alpha(x, t)u_{xx} = 0 \text{ in } \Omega_T
$$
\n
$$
u_t = \alpha(x, t)u_{xx} = 0 \text{ in } \Omega_T
$$
\n
$$
u_t = \alpha(x, t)u_{xx} = 0 \text{ in } \Omega_T
$$
\n
$$
u_t = \alpha(x, t)u_{xx} = 0 \text{ on } [0, T]
$$
\n
$$
u(x, 0) = u_0(x) \text{ on } [0, 1].
$$
\n(4.2)

Now, the results of [10: Chapter V/p . 429] guarantees the existence of a unique solution to Problem (4.1) in the class $H^{2+\beta,1+\beta/2}(\overline{\Omega}_T)$ for any $0 < \beta < 1$. However, to show that u_t, u_x, u_{xx}, u_{xt} are non-negative we first show that u is indeed more regular. To do this we rewrite Problem (4.1) into the following form: *u*_t - (a₁(u_z))_x = 0 in Ω _T

a₁(u_z(1, t)) + ϕ (u(1, t)) = 0, u_z(0, t) = 0 on [0, T] (4.1)
 u(x, 0) = u₀(x) on [0, 1].
 u, the results of [10: Chapter V/p. 429] guarantees the existence of a uniqu

Problem (4.1) into the following form:
\n
$$
u_t - \alpha(x, t)u_{xx} = 0 \qquad \text{in} \ \Omega_T
$$
\n
$$
u_x(1, t) + b(t)u(1, t) = 0, \ u_x(0, t) = 0 \qquad \text{on} \ [0, T]
$$
\n
$$
u(x, 0) = u_0(x) \qquad \text{on} \ [0, 1].
$$
\n(4.2)

where

hipot and J. Filo

\n
$$
\alpha(x,t) = a_1'(u_x(x,t)) \qquad \text{and} \qquad b(t) = -\frac{a_1^{-1}(-\phi(u(1,t)))}{u(1,t)}
$$

As $D_x^m \alpha$ ($m = 0, 1$) are Hölder continuous in $\overline{\Omega}_T$, by [7: Chapter 3, Section 5/Theorem 10] we obtain that $D_x^m u$, $D_t D_x^k u$ ($0 \leq m \leq 3, 0 \leq k \leq 1$) exist and are Hölder continuous in Ω_T . Hence, the equation in (4.1) can be differentiated with respect to x and for $w = u_x$ we obtain *t*)) and $b(t) = -\frac{a_1^{-1}(-\phi(u(1)))}{u(1,t)}$

continuous in $\overline{\Omega}_T$, by [7: Chapter 3, S
 D_x^ku ($0 \le m \le 3, 0 \le k \le 1$) exis

equation in (4.1) can be differentiated
 $\tau_x - a_1''(u_x)u_{xx}w_x = 0$ in Ω_T
 $\phi(u(1,t)), w(0,t) = 0$ on [0

t and J. Filo
\n
$$
x, t) = a'_1(u_x(x, t)) \quad \text{and} \quad b(t) = -\frac{a_1^{-1}(-\phi(u(1, t)))}{u(1, t)}.
$$
\n0,1) are Hölder continuous in $\overline{\Omega}_T$, by [7: Chapter 3, Section 5/Theorem
\nthat $D_T^m u$, $D_t D_x^k u$ ($0 \le m \le 3, 0 \le k \le 1$) exist and are Hölder
\n T_T . Hence, the equation in (4.1) can be differentiated with respect to x
\nwe obtain
\n
$$
w_t - a'_1(u_x)w_{xx} - a''_1(u_x)u_{xx}w_x = 0 \quad \text{in} \quad \Omega_T
$$
\n
$$
w(1, t) = a_1^{-1}(-\phi(u(1, t)), w(0, t) = 0 \quad \text{on} \quad [0, T] \quad (4.3)
$$
\n
$$
w(x, 0) = u'_0(x) \quad \text{on} \quad [0, 1].
$$
\n
$$
r \cdot W / \text{Theorem 5-21 it has a unique solution we } \subset H^{2+\beta, 1+\beta/2}(\overline{\Omega}) \quad \text{Thus,}
$$

By [10: Chapter IV/Theorem 5.2] it has a unique solution $w \in H^{2+\beta,1+\beta/2}(\overline{\Omega}_T)$. Thus, $D_x^m D_t^k \alpha$ ($0 \leq m + 2k \leq 2, k \leq 1$) exist and are Hölder continuous in $\overline{\Omega}_T$ and by [7: Chapter 3, Section 5/Theorem 11] for (4.2) we arrive at $D_x^m D_t^k u$ ($0 \le m+2k \le 4, k \le 2$) to be Hölder continuous in Ω_T . As, moreover, $u_{xt} \in C(\overline{\Omega}_T)$, we can differentiate the equation in (4.1) with respect to t, and the boundary condition too. Putting $U = u_t$ one can check that *U1* $a_1 = a_1'(u_x)w_{xx} - a_1''(u_x)u_{xx}w_x = 0$ in Ω_T
 $D = a_1^{-1}(-\phi(u(1, t)), w(0, t) = 0$ on $[0, T]$
 $w(x, 0) = u'_0(x)$ on $[0, 1]$.

Theorem 5.2] it has a unique solution $w \in H^{2+\beta, 1+\beta/2}$
 $2k \le 2, k \le 1$ exist and are Hölder contin $w_t - a'_1(u_x)w_{xx} - a''_1(u_x)u_{xx}w_x = 0 \qquad \text{in} \ \Omega_T$ $w(1,t) = a_1^{-1}(-\phi(u(1,t)), \ w(0,t) = 0 \qquad \text{on} \ [0,T] \qquad (4.3)$ $w(x,0) = u'_0(x) \qquad \text{on} \ [0,1].$ $[10: Chapter IV/Theorem 5.2] it has a unique solution $w \in H^{2+\beta,1+\beta/2}(\overline{\Omega}_T)$. Thus,
 $D_t^k \alpha \ (0 \le m + 2k \le 2, k \le 1)$ exist and$

e can check that
\ne
$$
u_t = a_1'(u_x)U_{xx} - a_1''(u_x)u_{xx}U_x = 0
$$
 in Ω_T
\n $a'_1(u_x)U_x(1,t) + \phi'(u)U(1,t) = 0, a'_1(u_x)U_x(0,t) = 0$ on [0, T] (4.4)
\n $U(x,0) = (a_1(u'_0))'(x)$ on [0,1].
\n
\nre, the hypotheses of [10: Chapter IV/Theorem 5.2] are fulfilled and we obtain $U \in$
\n $(x,0) = (a_1(u'_0))'(x)$ on [0,1].
\n
\nre, the hypotheses of [10: Chapter IV/Theorem 5.2] are fulfilled and we obtain $U \in$
\n $(x,0) = (a_1(u'_0))'(x)$ on [0,1].
\n
\n $U(x,0) = (a_1(u'_0))'(x)$ Thus, similarly as above, one can conclude that $D_x^m D_t^ku$ exist and
\n $U_t = a'_1(u_x)W_{xx} + (a''_1(u_x)u_{xxx} + a'''_1(u_x)u_{xx}^2)W + 2a''_1(u_x)u_{xx}W_x$ in Ω_T
\n $a'_1(u_x)W(1,t) + \phi'(u)U(1,t) = 0, a'_1(u_x)W(0,t) = 0$ on [0, T] (4.5)
\n $W(x,0) = (a_1(u'_0))''(x)$ on [0,1].

Here, the hypotheses of [10: Chapter IV/Theorem 5.2] are fulfilled and we obtain $U \in$ $H^{2+\beta,1+\beta/2}(\overline{\Omega}_T)$. Thus, similarly as above, one can conclude that $D_x^m D_t^k u$ exist and are Hölder continuous in Ω_T for $0 \le m + 2k \le 6, k \le 3$. Hence, $W = u_{xt}$ satisfies

Hölder continuous in
$$
\Omega_T
$$
 for $0 \le m + 2k \le 6$, $k \le 3$. Hence, $W = u_{xt}$ satisfies $W_t = a'_1(u_x)W_{xx} + (a''_1(u_x)u_{xxx} + a'''_1(u_x)u_{xx}^2)W + 2a''(u_x)u_{xx}W_x$ in Ω_T $a'_1(u_x)W(1,t) + \phi'(u)U(1,t) = 0$, $a'_1(u_x)W(0,t) = 0$ on $[0,T]$ (4.5) $W(x,0) = (a_1(u'_0))''(x)$ on $[0,1]$.

After these preliminaries it is not difficult to prove Theorem 2. Indeed, as $u_t(x, 0) =$ $\lambda u_0^{\sigma-1}(x) \ge \lambda \mu^{\sigma-1} > 0$, the maximum principle yields that $u_t(x,t) > 0$ on $\overline{\Omega}_T$. Thus, $u(x,t) \geq \mu$ on $\overline{\Omega}_T$ and the equation in (4.2) gives $u_{xx} > 0$ on $\overline{\Omega}_T$. Now, as $u_{xt}(x,0) =$ $\lambda(\sigma - 1)u_0^{\sigma-2}(x)u_0'(x) > 0$ on (0,1), taking into account (4.5) one can conclude that $u_{xt}(x,t) > 0$ on $(0,1] \times [0,T]$.

Next, because of the smoothness of *u* and due to the fact that $u_0(x) \leq \mathcal{L} = M/2$ there exists a time $s > 0$ such that $u(x,t) \leq M$ on $\overline{\Omega}_s$, i.e. u is also the unique solution of our original problem (1.6) such that 0 on $(0,1] \times [0,T]$.

because of the smoothness of u and due to the fact that $u_0(x) \le$

s a time $s > 0$ such that $u(x,t) \leq M$ on $\overline{\Omega}_s$, i.e. u is also the uniquinal problem (1.6) such that
 $u, u_t, u_{xx} > 0$ on $\overline{\Omega}_s$

$$
u, u_t, u_{xx} > 0
$$
 on $\overline{\Omega}_s$ and $u_x, u_{xt} > 0$ on $(0,1] \times [0,s]$.

Put

$$
T_{max} = \sup \Big\{ s \Big| \text{ there is a solution of Problem (1.6) on } \overline{\Omega}_s \Big\}.
$$

If $T_{max} < \infty$, it is not difficult to show that $u(1, t) \rightarrow \infty$ if $t \rightarrow T_{max}$. Indeed, let us suppose that $u(1, t) \le \text{const}$ for all $t \in [0, T_{\text{max}})$. Then from (4.1), (4.3) and (4.4) one can get

$$
u_x(x,t) + u_t(x,t) + u_{xx}(x,t) \leq C < \infty \tag{4.6}
$$

An Example of Blowup 99

difficult to show that $u(1,t) \rightarrow \infty$ if $t \rightarrow T_{max}$. Indeed, let us

onst for all $t \in [0, T_{max})$. Then from $(4.1), (4.3)$ and (4.4) one
 $u_x(x,t) + u_t(x,t) + u_{xx}(x,t) \leq C < \infty$ (4.6)
 u, T_{max}). Note that each for all $(x, t) \in [0, 1] \times [0, T_{\text{max}})$. Note that each term on the left-hand side of (4.6) is non-negative. Applying now [10: Chapter TV/Theorem *5.3)* for Problem (4.2) and then [10: Chapter IV/Theorem 5.2] for (4.3) we obtain the uniform bound of u, u_x in the norm of $H^{2+\beta,1+\beta/2}(\overline{\Omega}_T)$ with respect to $T, T < T_{\text{max}}$. Denote $v_t(x,t) + u_{xx}(x,t)$:

Note that each term

Chapter IV/Theorer

or (4.3) we obtain the

spect to $T, T < T_{\text{max}}$
 $v(x) \equiv \lim_{t \to T_{\text{max}}} u(x,t)$.

$$
v(x) \equiv \lim_{t \to T_{\max}} u(x,t).
$$

Then $v \in H^{2+\nu}([0,1])$ $(0 < \nu < 1), v_x(0) = 0$ and $a(v_x(1)) = v^{q-1}(1)$, i.e. the solution *u* can be extended beyond *Tmax,* which is a contradiction.

5. Proof of Theorem 3

We shall first deal with the problem

w [10: Chapter IV/Theorem 5.3] for Problem (4.2) and then
\n5.2] for (4.3) we obtain the uniform bound of
$$
u, u_x
$$
 in the
\nwith respect to $T, T < T_{\text{max}}$. Denote
\n $v(x) \equiv \lim_{t \to T_{\text{max}}} u(x, t).$
\n $\langle \nu \times 1 \rangle, v_x(0) = 0$ and $a(v_x(1)) = v^{q-1}(1)$, i.e. the solution
\n T_{max} , which is a contradiction.
\n**m** 3
\ne problem
\n $(a_2(v_x))_x = \lambda v^{q-1} \quad (0 < x < 1)$
\n $v_x(0) = 0$
\n $a_2(v_x(1)) = v^{q-1}(1)$
\n $\in C^3(\mathbb{R}), 0 < \kappa \le a'_2(\xi) \le \kappa^{-1} \quad (\xi \in \mathbb{R})$ for some $0 < \kappa < 1$
\n $c^{\sigma-1}$. Put $A(\xi) = \int_0^{\xi} a'_2(\tau) \tau \, d\tau$ and denote by A^{-1} its inverse
\n $\kappa \xi^2 \le 2A(\xi) \le \kappa^{-1} \xi^2$ and
\n $2\kappa)^{1/2} \sqrt{\xi} \le A^{-1}(\xi) \le \left(\frac{2}{\kappa}\right)^{1/2} \sqrt{\xi}$
\npositive classical solution of Problem (5.1), then $v_x > 0$ on
\nequation in (5.1) by v_x we obtain easily
\n $v_x(x) = A^{-1} \left(\frac{\lambda}{\sigma} (v^{\sigma}(x) - \mu^{\sigma}) \right)$
\n $\ell = v(1), \hat{\ell} = \ell^{\sigma-1}$ and assuming that $\hat{\ell} \ge \eta$, (5.3) for $x = 1$
\n $\lambda = \frac{\sigma A(\hat{\ell})}{\ell^{\sigma} - \mu^{\sigma}}$.
\nWe arrive at

for some $\lambda > 0$, where $a_2 \in C^3(\mathbb{R})$, $0 < \kappa \leq a'_2(\xi) \leq \kappa^{-1}$ ($\xi \in \mathbb{R}$) for some $0 < \kappa < 1$ We shall first deal with the problem
 $(a_2(v_x))_x = \lambda v^{\sigma-1}$ $(0 < x < 1)$
 $v_x(0) = 0$
 $a_2(v_x(1)) = v^{q-1}(1)$

for some $\lambda > 0$, where $a_2 \in C^3(\mathbb{R})$, $0 < \kappa \le a'_2(\xi) \le \kappa^{-1}$ ($\xi \in \mathbb{R}$) for some $0 < \kappa < 1$

and $a_2(\xi) = a(\xi$ (or some $\lambda > 0$, where $a_2 \in C^3(\mathbb{R})$, $0 < \kappa \le a'_2(\xi) \le a_1$
and $a_2(\xi) = a(\xi)$ if $|\xi| \le 2\mathcal{L}^{\sigma-1}$. Put $\mathcal{A}(\xi) = \int_0^{\xi} a'_2(\tau) \tau$
(on $[0, \infty)$). Then we have $\kappa \xi^2 \le 2\mathcal{A}(\xi) \le \kappa^{-1} \xi^2$ and $\begin{aligned} \n\lambda &= \frac{a_2(\zeta)}{2} \\ \n\xi &= \int_0^{\xi} a_2' \\ \n\lambda &= \frac{a_2(\zeta)}{2} \\ \n\lambda &= \frac{a_2(\zeta)}$

$$
^{1}(2\kappa)^{1/2}\sqrt{\xi}\leq\mathcal{A}^{-1}(\xi)\leq\left(\frac{2}{\kappa}\right)^{1/2}\sqrt{\xi}
$$
\n(5.2)

for any $\xi \in \mathbb{R}^+$. If *v* is a positive classical solution of Problem (5.1), then $v_x > 0$ on $(0, 1]$ and multiplying the equation in (5.1) by v_x we obtain easily

$$
\kappa \xi^2 \le 2\mathcal{A}(\xi) \le \kappa^{-1} \xi^2 \text{ and}
$$

\n
$$
\kappa \xi^2 \le 2\mathcal{A}(\xi) \le \kappa^{-1} \xi^2 \text{ and}
$$

\n
$$
\kappa \xi^2 \le \mathcal{A}^{-1}(\xi) \le \left(\frac{2}{\kappa}\right)^{1/2} \sqrt{\xi}
$$
(5.2)
\npositive classical solution of Problem (5.1), then $v_x > 0$ on
\nquation in (5.1) by v_x we obtain easily
\n
$$
v_x(x) = \mathcal{A}^{-1} \left(\frac{\lambda}{\sigma} (v^{\sigma}(x) - \mu^{\sigma})\right)
$$
(5.3)
\n
$$
v_x(x) = v(1), \hat{\ell} = \ell^{\sigma-1} \text{ and assuming that } \hat{\ell} \ge \eta, (5.3) \text{ for } x = 1
$$

\n
$$
\lambda = \frac{\sigma \mathcal{A}(\hat{\ell})}{\ell^{\sigma} - \mu^{\sigma}}.
$$
(5.4)
\n
$$
\text{a} \text{ arrive at}
$$

\n
$$
\frac{dz}{dz} = x \quad \forall \ x \in [0, 1].
$$

where $\mu = v(0)$. Denoting $\ell = v(1)$, $\ell = \ell^{\sigma-1}$ and assuming that $\ell \geq \eta$, (5.3) for $x = 1$ gives (0). Denotin

ation of (5.3)
 $\int_{\mu}^{v_1}$
 $\frac{1}{v_1}$ especially

ready to for

$$
\lambda = \frac{\sigma \mathcal{A}(\hat{\ell})}{\ell^{\sigma} - \mu^{\sigma}}.
$$
\n(5.4)

After integration of (5.3) we arrive at

g the equation in (5.1) by
$$
v_x
$$
 we obtain easily
\n
$$
v_x(x) = A^{-1} \left(\frac{\lambda}{\sigma} (v^{\sigma}(x) - \mu^{\sigma}) \right)
$$
\n
$$
v_{\sigma}(x) = v(1), \hat{\ell} = \ell^{\sigma - 1} \text{ and assuming that } \hat{\ell} \ge \eta, (5.3) \text{ for } x = 1
$$
\n
$$
\lambda = \frac{\sigma A(\hat{\ell})}{\ell^{\sigma} - \mu^{\sigma}}.
$$
\n(5.3) we arrive at\n
$$
\int_{\mu}^{v(x)} \frac{dz}{A^{-1} \left(\frac{\lambda}{\sigma} (z^{\sigma} - \mu^{\sigma}) \right)} = x \quad \forall \ x \in [0, 1].
$$
\n(5.5)

Putting

$$
\lambda = \frac{\sigma \mathcal{A}(\hat{\ell})}{\ell^{\sigma} - \mu^{\sigma}}.
$$
\n(5.4)

\nwe arrive at

\n
$$
\frac{dz}{\mathcal{A}^{-1}(\frac{\lambda}{\sigma}(z^{\sigma} - \mu^{\sigma}))} = x \quad \forall x \in [0, 1].
$$
\n
$$
\mathcal{F}(\ell, \mu) = \int_{\mu}^{\ell} \frac{dz}{\mathcal{A}^{-1}(\frac{\lambda}{\sigma}(z^{\sigma} - \mu^{\sigma}))}
$$
\nields

\n
$$
\mathcal{F}(\ell, \mu) = 1.
$$
\n(5.6)

\nulate the following two statements.

 (5.5) for $x = 1$ especially yie

Now we are ready to formulate the following two statements.

Proposition 1. Assume that $\hat{\ell} \geq \eta$, let constants $0 < \mu < \ell$ be related by equation (5.6) and a function v be given implicitly by (5.5). Then $v \in C^{\infty}([0,1])$ and v is a positive solution of Problem (5.1) with $v(0) = \mu$, $v(1) = \ell$ and λ being given by (5.4).

Proposition 2. Let $\hat{\ell} \geq \eta$ be given. Then there exist a positive λ and $\mu \in (0, \ell)$ *such that (5.6) is satisfied.*

Proof of Proposition 1. Let $0 < \mu < \ell$ be given such that (5.6) is satisfied with $h(y) = \int_{\mu}^{y} \frac{dz}{\mathcal{A}^{-1}(\mathcal{A}(\hat{\ell})\left(\frac{z^{\sigma} - \mu^{\sigma}}{\sigma}\right))}$ for $y \in (\mu, \infty)$. λ being given by (5.4). Put

Let
$$
y(x)
$$
 is a set of $y(x)$ is a set of $y(x)$.

\nLet $y(x)$ is a set of $y(x)$ and $y(x)$ is a set of $y(x)$.

\nLet $y(x)$ is a set of $y(x)$ and $y(x)$ is a set of $y(x)$.

\nLet $y(x)$ is a set of $y(x)$ and $y(x)$ is a set of $y(x)$.

\nand $y(x)$ is a set of $y(x)$.

\nFrom $y(x)$ is a set of $y(x)$.

\nTherefore, $y(x)$

With the assistance of (5.2) one can now obtain for the function *h* the estimate

$$
\mu \mathcal{A} \left(\mathcal{A}(t) \left(\overline{t^{\sigma} - \mu^{\sigma}} \right) \right)
$$

2) one can now obtain for the function *h* the estimate

$$
\left(\frac{\kappa}{2} \right)^{1/2} F(y) \le h(y) \le (2\kappa)^{-1/2} F(y), \tag{5.7}
$$

$$
F(y) = \frac{\mu}{\sqrt{t^{\rho}} \sqrt{\frac{y}{\mu}}} f\left(\frac{y}{\mu} \right) \sqrt{\left(\frac{\ell}{\mu} \right)^{\sigma} - 1}
$$

where

tion 1. Let
$$
0 < \mu < \ell
$$
 be given such that

\nPut

\n
$$
\int_{\mu}^{y} \frac{dz}{\mathcal{A}^{-1} \left(\mathcal{A}(\hat{\ell}) \left(\frac{z^{\sigma} - \mu^{\sigma}}{\ell^{\sigma} - \mu^{\sigma}} \right) \right)} \quad \text{for } y \in \mathbb{R}
$$
\n(5.2) one can now obtain for the function

\n
$$
\left(\frac{\kappa}{2} \right)^{1/2} F(y) \leq h(y) \leq (2\kappa)^{-1/2} F(y)
$$
\n
$$
F(y) = \frac{\mu}{\sqrt{\mathcal{A}(\hat{\ell})}} f\left(\frac{y}{\mu} \right) \sqrt{\left(\frac{\ell}{\mu} \right)^{\sigma} - 1}
$$
\nIf $x = \int_{1}^{x} \frac{dz}{\sqrt{z^{\sigma} - 1}} \quad \text{for } x \in (1, \infty)$

\nwhere $x \in \mathbb{R}$ is the function of $x \in \mathbb{R}$.

and

$$
\int_{\mu}^{y} \frac{dz}{\mathcal{A}^{-1} \left(\mathcal{A}(\hat{\ell}) \left(\frac{z^{\sigma} - \mu^{\sigma}}{\ell^{\sigma} - \mu^{\sigma}} \right) \right)} \quad \text{for } y \in (\mu, \infty).
$$
\n(5.2) one can now obtain for the function *h* the estimate\n
$$
\left(\frac{\kappa}{2} \right)^{1/2} F(y) \le h(y) \le (2\kappa)^{-1/2} F(y), \tag{5.7}
$$
\n
$$
F(y) = \frac{\mu}{\sqrt{\mathcal{A}(\hat{\ell})}} f\left(\frac{y}{\mu} \right) \sqrt{\left(\frac{\ell}{\mu} \right)^{\sigma} - 1}
$$
\n
$$
f(x) = \int_{1}^{x} \frac{dz}{\sqrt{z^{\sigma} - 1}} \quad \text{for } x \in (1, \infty).
$$
\n
$$
\text{verges since } \sqrt{z^{\sigma} - 1} \sim \sqrt{\sigma(z - 1)} \text{ near } z = 1, \text{ hence, } h \text{ is well}
$$
\n
$$
C([\mu, \infty)) \cap C^{\infty}(\mu, \infty), h(\mu) = 0 \text{ and } h(\ell) = 1. \text{ As } h'(y) > 0 \text{ on}
$$

The above integral converges since $\sqrt{z^{\sigma}-1} \sim \sqrt{\sigma(z-1)}$ near $z=1$, hence, *h* is well defined on $[\mu, \infty)$, $h \in C([\mu, \infty)) \cap C^{\infty}(\mu, \infty)$, $h(\mu) = 0$ and $h(\ell) = 1$. As $h'(y) > 0$ on The above integral converges since $\sqrt{z^{\sigma}-1}$ for $x \in (1, \infty)$. (5.8)

The above integral converges since $\sqrt{z^{\sigma}-1} \sim \sqrt{\sigma(z-1)}$ near $z = 1$, hence, h is well

defined on $[\mu, \infty)$, $h \in C([\mu, \infty)) \cap C^{\infty}(\mu, \infty)$, $h(\mu) =$ The above integral converges since $\sqrt{z^{\sigma}-1} \sim \sqrt{\sigma(z-1)}$ near $z = 1$, hence, h is well
defined on $[\mu, \infty)$, $h \in C([\mu, \infty)) \cap C^{\infty}(\mu, \infty)$, $h(\mu) = 0$ and $h(\ell) = 1$. As $h'(y) > 0$ on
 (μ, ∞) , for each $x \in [0, 1]$ there ex $f(x) = \int_1^x \frac{dz}{\sqrt{z^{\sigma} - 1}}$ for $x \in (1, \infty)$. (5.8)

The above integral converges since $\sqrt{z^{\sigma} - 1} \sim \sqrt{\sigma(z - 1)}$ near $z = 1$, hence, *h* is well

defined on $[\mu, \infty)$, $h \in C([\mu, \infty)) \cap C^{\infty}(\mu, \infty)$, $h(\mu) = 0$ and $h(\ell) =$ see that *v* satisfies (5.1) with λ given by (5.4). Thus $a'_1(v_x)v_{xx} = \lambda v^{\sigma-1}(x)$ and the smoothness of *v* follows. *z* $y = \frac{\mu}{\sqrt{A(\hat{\ell})}} f(\frac{y}{\mu}) \sqrt{\left(\frac{\ell}{\mu}\right)^{\sigma} - 1}$
 $= \int_{1}^{x} \frac{dz}{\sqrt{z^{\sigma} - 1}}$ for $x \in (1, \infty)$. (5.8)
 $= \int_{1}^{x} \frac{dz}{\sqrt{z^{\sigma} - 1}}$ for $x \in (1, \infty)$. (5.8)
 $= \int_{1}^{x} \frac{dz}{\sqrt{z^{\sigma} - 1}}$ for $x \in (1, \infty)$. (5.8)
 $= \int$

Proof of Proposition 2. Assume that ℓ , satisfying $\ell \geq \eta$, is given and put $z = \ell/\mu$. As we consider $0 < \mu < \ell$, $z \in (1, \infty)$, and take

As we consider
$$
0 < \mu < \ell
$$
, $z \in (1, \infty)$, and take
\n
$$
W(z) = \frac{\ell z^{\sigma/2 - 1}}{\sqrt{A(\ell)}} f(z) \sqrt{1 - \frac{1}{z^{\sigma}}},
$$
\nwhere f is defined by (5.8). With the assistance of (5.7) one can see that\n
$$
\left(\frac{\kappa}{2}\right)^{1/2} W(z) \leq \mathcal{F}(\ell, \ell/z) \leq (2\kappa)^{-1/2} W(z)
$$
\nfor $z \in (1, \infty)$. (5.10)
\nWe know already that $W(z) \to 0$ as $z \to 1$ and it is enough to show that $W(z) \to \infty$

where f is defined by (5.8) . With the assistance of (5.7) one can see that

$$
\left(\frac{\kappa}{2}\right)^{1/2} W(z) \leq \mathcal{F}(\ell, \ell/z) \leq (2\kappa)^{-1/2} W(z) \quad \text{for } z \in (1, \infty). \quad (5.10)
$$

as $z \to \infty$, because then (5.10) yields the existence of at least one $z_0 \in (1,\infty)$ such that $\mathcal{F}(\ell, \ell/z_0) = 1$, and the assertion of Proposition 2 would follow. The fact that $W(z) \to \infty$ as $z \to \infty$ follows from (5.9) taking into account the fact that $\sigma > 2$.

Proof of Theorem 3. Put $\ell = \mathcal{L}$. As $v_{xx} > 0$ on [0, 1] it follows that $0 \le v_x(x) \le$ $\mathcal{L}^{\sigma-1}$ for all $x \in [0,1]$. Since $a_2(\xi) = a(\xi)$ for $|\xi| \leq 2\mathcal{L}^{\sigma-1}$ (see (5.1) above), the proof of Theorem *3 is* completed.

6. Concluding remarks

Let us finish the paper with some remarks concerning Problem (1.6) for those values of the parameters p and q that we do not concern here.

Remark 1. Let q satisfy (1.3) , i.e. the inequality

An Example of Blowup 101
\nremarks
\nper with some remarks concerning Problem (1.6) for those values of
\nd q that we do not concern here.
\nt q satisfy (1.3), i.e. the inequality
\n
$$
|v(1)|^q \leq \varepsilon \int_0^1 |v_x(x)|^p dx + C_{\varepsilon} \int_0^1 |v(x)|^2 dx + C
$$
\n(6.1)
\n
$$
W_n^1(0,1).
$$
 Multiplying the equation in (1.6) by u and taking into

holds for any $v \in W_p^1(0,1)$. Multiplying the equation in (1.6) by *u* and taking into $|v(1)|^q \leq \varepsilon \int_0^1 |v_x(x)|^p dx + C_{\varepsilon} \int_0^1 |v_x(x)|^p dx$
holds for any $v \in W_p^1(0,1)$. Multiplying the equation
account (6.1) and the fact that $c |\xi|^p - C \leq a(\xi)\xi \leq$
constants c and C, one can easily arrive at $|\xi|^p$ $(\xi \in \mathbb{R})$ for some positive $|v(1)|^q \leq \varepsilon \int_0^1 |v_x(x)|^p dx + C_{\varepsilon} \int_0^1 |v(x)|^2 dx + C$ (6
holds for any $v \in W_p^1(0,1)$. Multiplying the equation in (1.6) by u and taking in
account (6.1) and the fact that $c|\xi|^p - C \leq a(\xi)\xi \leq |\xi|^p$ ($\xi \in \mathbb{R}$) for some

$$
\max_{0\leq t\leq T}\int_0^1 |u(x,t)|^2 dx \leq \left(\int_0^1 |u_0(x)|^2 dx + L\right) e^{Kt} - L,
$$

$$
\max_{0 \le t \le T} \int_0^t |u(x,t)|^2 dx \le \left(\int_0^t |u_0(x)|^2 dx + L\right) e^{kt} - L,
$$
\nwhere $K, L > 0$ do not depend on u. Now, from the energy inequality\n
$$
\int_0^t \int_0^1 u_t^2(x,t) dx dt + J(u(t)) \le J(u_0), \quad J(v) = \int_0^1 \int_0^{v_x} a(\tau) d\tau dx - q^{-1} |v(1)|^q \tag{6.2}
$$
\nwe obtain\n
$$
\int_0^T \int_0^1 u_t^2(x,t) dx dt + \max_{u \in [u_0, u_0]} |u(t)| |_{W^1(0,1)} \le \text{const},
$$

we obtain

$$
\int_0^T \int_0^1 u_t^2(x,t) \, dx \, dt + \max_{0 \le t \le T} ||u(t)||_{W_p^1(0,1)} \le \text{const},
$$

which does not depend on *u*. Therefore, one can conclude that *u* exists globally.

Remark 2. If $p \ge 2$ and $q \le 2$, it is shown in [6] by the same way as outlined in Remark 1 that all solutions of Problems (1.1) and (1.6) exist globally. In the case $q > 2$ all non-decreasing (in x) positive solutions of Problem (1.6) (Problem (1.1)) blow up in a finite time. Indeed, note that $\int_0^T \int_0^1 u_t^2(x, t)$
pend on u. Therefore $\int_0^T f(t) dt$ and q
solutions of P.
(in x) positive
ed, note that
 $\frac{d}{dt} \int_0^1 u(x, t) dt$

$$
\frac{d}{dt} \int_0^1 u(x,t) \, dx = u^{q-1}(1,t) \ge \left(\int_0^1 u(x,t) \, dx \right)^{q-1}
$$

As $q - 1 > 1$, *u* can not exists globally, i.e. $T_{\text{max}} < \infty$. To show that *u* blows up in a finite time we need to show that $u(1, t) \rightarrow \infty$ if $t \rightarrow T_{\text{max}}$. For smooth solutions of Problem (1.6) satisfying (2.1) it is demonstrated in the proof of Theorem 2 above (for weak solutions of (1.1) see, e.g., $[4]$). Remark 1 that all solutions of Problems (1.1) and (1.6) ex
all non-decreasing (in x) positive solutions of Problem (1.6)
a finite time. Indeed, note that
 $\frac{d}{dt} \int_0^1 u(x,t) dx = u^{q-1}(1,t) \ge \left(\int_0^1 u(x+1) dx\right) dt$
As $q-1 > 1$, u c

Remark 3. The regularity of solutions of problems like (1.1) has been intensively

References

- [1] Chipot, M., Fila, M. and P. Quittner: *Stationary solutions, blow up and convergence to stationary solutions for sernilinear parabolic equations with nonlinear boundary conditions.* Acta Math. Univ. Comenianae 40 (1991), 35 - 103.
- [2] DiBenedetto, E.: *Degenerate Parabolic Equations.* New York: Springer-Verlag 1993.
- *[3] Fila, M. and J. Fib: Blow-up on the boundary: a survey. Singularities and Differential Equations.* Banach Center Publications 33 (1996), 67 - 79.
- *[4[* Fib, J.: *On solutions of a parturbed fast diffusion equation.* Aplikace Maternatiky 32 $(1987), 364 - 380.$
- *[5] Fib, J.: Diffusivity versus absorption through the boundary.* J. Duff. Equ. 99 (1992), 281 $-305.$
- $[6]$ Filo, J.: Local existence and L^{∞} -estimate of weak solutions to a nonlinear degenerate *parabolic equation with nonlinear boundary data.* PanAmer. Math. J. 4 (1994)3, 1 - 31.
- *[7] Friedman, A.: Partial Differential Equations of Parabolic Type.* Englewood Cliffs: Prentice-Hall 1964.
- *[8] Friedman, A. and B. McLeod: Blow-up of positive solutions of semilinear heat equations.* Indiana Univ. Math. J. 34 (1985), 425 - 447.
- [9] Levine, H. and L. E. Payne: *Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time.* J. Duff. Equ. 16 (1974), 319 - 334.
- [10] Ladyzenskaja, 0. A., Sobonnikov, V. A. and N. N. Uralceva: *Linear and Quasilinear Equations of Parabolic Type* (Translations of Mathematical Monographs: Vol. 23). Providence (Rhode Island): Amer. Math. Soc. 1968.

Received 28.05.1997