Weighted Inequalities of Weak Type for the Fractional Integral Operator

Y. Rakotondratsimba

Abstract. Sufficient conditions on weights *u(.)* and *v(.)* are given in order that the usual fractional integral operator I_o ($0 < \alpha < n$) is bounded from the weighted Lebesgue space $L^p(v(x)dx)$ into weak- $L^p(u(x)dx)$, with $1 \leq p < \infty$. As a consequence a characterization for this boundedness is obtained for a large class of weight functions which particularly contains radial monotone weights.

Keywords: *Fractional integral operators, weighted weak- type inequalities*

AMS subject classification: 42 B 25

1. Introduction

The fractional integral operator I_{α} of order α acts on locally integrable functions $f(\cdot)$ of \mathbb{R}^n as

on
egral operator
$$
I_{\alpha}
$$
 of order α acts on locally integer:

$$
(I_{\alpha}f)(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha - n} f(y) dy \qquad (0 < \alpha < n).
$$

The purpose of this paper is to determine weight functions $u(\cdot)$ and $v(\cdot)$ for which I_{α} is The purpose of this paper is to determine weight functions $u(\cdot)$ and $v(\cdot)$ for which I_{α} is bounded from $L_v^p = L^p(\mathbb{R}^n, v(x)dx)$ into weak- $L^p(\mathbb{R}^n, u(x)dx) = L_u^p$ with $1 \leq p < \infty$. This means that for some constant $C > 0$ ^{*JRn*}
 *AP*_{*I*} *a c cf cm L_c^{<i>n*} = *LP*(\mathbb{R}^n , $v(x)dx$) into weak-*LP*(\mathbb{R}^n , $u(x)dx$) = *L*_{*k*}² with $1 \le p < \infty$.
 *AP*_{$\left\{x | (I_\alpha f)(x) > \lambda\right\}}$ $u(x)dx \le C \int_{\mathbb{R}^n} f^p(x)v(x) dx$ for all $\lambda > 0$ and}

$$
\lambda^p \int_{\{x \mid (\{I_\alpha f\}(x) > \lambda\})} u(x) dx \le C \int_{\mathbb{R}^n} f^p(x) v(x) dx \quad \text{for all } \lambda > 0 \text{ and } f(\cdot) \ge 0. \tag{1.1}
$$

For convenience this boundedness will be denoted by $I_{\alpha}: L_v^p \to L_v^{p\infty}$.

Such an inequality takes an important part in Analysis. For instance, it is wellknown [5] that (1.1) is a main point to get Sobolev inequalities with weights. Moreover, applications on the estimates of eigenvalue of some Schrödinger operators can be derived from (1.1) (see $[2]$). t (1.1) is a main point to get Sobol

in the estimates of eigenvalue of son

e [2]).

] proved that if $1 < p < \infty$, then
 $\int_{Q} (I_{\alpha}u\mathbf{1}_Q)^{p'}(x)v^{1-p'}(x) dx \leq A \int_{Q}$

ttsimba: Institut Polytechnique St.-1 *u* is the Marian Control of the Marian Control of the Schrödinger operators can be defined $I_{\alpha}: L_v^p \to L_u^{p\infty}$ if and only if for s
 $u(x) dx$ for all cubes Q

Sawyer [3] proved that if $1 < p < \infty$, then $I_{\alpha}: L_v^p \to L_u^{p\infty}$ if and only if for some *A>0*

on the estimates of eigenvalue of some Schrödinger operators can be derived
ee [2]).
3] proved that if
$$
1 < p < \infty
$$
, then $I_{\alpha} : L_v^p \to L_u^{p\infty}$ if and only if for some

$$
\int_Q (I_{\alpha}u\mathbf{1}_Q)^{p'}(x)v^{1-p'}(x)dx \leq A \int_Q u(x)dx \quad \text{for all cubes } Q \qquad (1.2)
$$

ISSN 0232-2064 / \$ 2.50 © Heldermann Verlag Berlin

Y. Rakotondratsimba: Institut Polytechnique St-Louis, E.P.M.I., 13 bd de l'Hautil, 95092 Cergy-Pontoise cedex, France

where $p' = \frac{p}{p-1}$ and $1_Q(\cdot)$ is the characteristic function of the cube *Q*. Although (1.2) is a characterizing condition, it is not easy in general to check it for given weight functions. Indeed, a main difficulty comes from the fact that (1.2) is expressed in term of I_{α} , and the integrations over arbitrary cubes are also hard to compute. So it is a challenge problem to derive conditions which ensure inequality (1.1) but more easily verifiable than (1.2). ondratsimba

and $1_Q(\cdot)$ is the

condition, it is r

ilifficulty comes

over arbitrary

ve conditions w
 $q < \infty$, Gabidza
 $\lambda \left(\int_{\{x \mid (I_\alpha f)(x) > \cdot \}} f(\cdot) \geq 0 \text{ if and} \right)$ interior of the cube *Q*. Although (1.2) is

ral to check it for given weight functions.

rat (1.2) is expressed in term of I_{α} , and

hard to compute. So it is a challenge

quality (1.1) but more easily verifiable

il

For $1 < p < q < \infty$, Gabidzashvili and Kokilashvili [1] proved that

$$
\lambda \Bigl(\int_{\{x \mid (I_{\alpha}f)(x) > \lambda\}} u(x) \, dx\Bigr)^{\frac{1}{q}} \le C \Bigl(\int_{\mathbb{R}^n} f^p(x) \, v(x) \, dx\Bigr)^{\frac{1}{p}} \tag{1.3}
$$

for all $\lambda > 0$ and $f(\cdot) \ge 0$ if and only if

$$
0 \text{ and } f(\cdot) \ge 0 \text{ if and only if}
$$

$$
\left(\int_{Q} u(y) dy\right)^{\frac{1}{q}} \left(\int_{y \in \mathbb{R}^{n}} \left(|y - x_{Q}| + |Q|^{\frac{1}{n}}\right)^{(\alpha - n)p'} v^{1 - p'}(y) dy\right)^{\frac{1}{p'}} \le A
$$
 (1.4)

for all cubes *Q*. Here x_Q denotes the centre of *Q*, and $|Q| = \int_Q dx$. The proof of this result does not work for the case $p = q$, so the problem of finding a similar characterization for $I_{\alpha}: L^p_{\alpha} \to L^p_{\alpha}$ remains open. According to Sawyer and Wheeden [5], inequality (1.1) holds if for some $A > 0$ and $1 < t < \frac{n}{\alpha}$

$$
|Q|^{\frac{\alpha}{n}} \Big(|Q|^{-1} \int_Q u^t(y) \, dy\Big)^{\frac{1}{tp}} \Big(|Q|^{-1} \int_Q v^{t(1-p')}(y) \, dy\Big)^{\frac{1}{tp'}} \leq A \quad \text{for all cubes } Q. \quad (1.5)
$$

In fact, in [5] it is seen that (1.5) implies the strong inequality $I_{\alpha}: L_v^p \to L_u^p$ associated to (1.1) , so a weaker sufficient condition than (1.5) , for the weak-type inequality (1.1) , is not known. Bumping $u(\cdot)$ or $v^{1-p'}(\cdot)$ as in (1.5) is not always satisfactory. Indeed, taking $v^{1-p'}(x) = |x|^{-n} \ln^{-p'}(|x|^{-1})$ for $|x| < \frac{1}{2}$, then $\int_{|x| < R} v^{t(1-p')}(x) dx = \infty$, for all $t>1$ and $R < \frac{1}{2}$ though $\int_{|x|. But for such a weight function $v(\cdot)$$ (see Corollary 2.8) the boundedness $I_{\alpha}: L_v^p \to L_v^{p\infty}$ holds true. All of these reasons lead us to consider and study again problem (1.1).

In this work, we first state in Theorem 2.1 a necessary and sufficient condition for $I_{\alpha}: L_v^p \to L_u^{pos}$. Next, in Theorem 2.6 we will prove that with an additional pointwise inequality, the necessary condition (1.4) [with $p = q$] becomes also sufficient to ensure In this work, we first state in Theorem 2.1 a necessary and sufficient condition for $I_{\alpha}: L_v^p \to L_w^{\infty}$. Next, in Theorem 2.6 we will prove that with an additional pointwise inequality, the necessary condition (1.4) [wi $I_{\alpha}: L_{\nu}^p \to L_{\nu}^{p\infty}$. Moreover, it will be shown that the test condition could be restricted to balls centered at the origin, rather on arbitrary cubes. For radial and monotone weights we will see (in the same theorem) that $I_{\alpha}: L_v^p \to L_u^{p\infty}$ is equivalent to $x = |x|^{-n} \ln^{-p'}(|x|)$
 $y = |x|^{-n} \ln^{-p'}(|x|)$
 $y \leq \frac{1}{2}$ though $\int_{|x| < R}$
 $r \geq 2.8$) the bounded

sider and study agency condition
 ∞ . Next, in Theore
 ∞ encessary condition
 ∞ . Moreover, it will

red at

$$
\begin{aligned}\n\text{ared at the origin, rather on arbitrary cubes. For radial and monotone}\n\text{ill see (in the same theorem) that } I_{\alpha}: L_v^p \to L_u^{p\infty} \text{ is equivalent to} \\
\left(\int_{|x| < R} u(x) \, dx \right)^{\frac{1}{p}} \left(\int_{y \in \mathbb{R}^n} (|y| + R)^{(\alpha - n)p'} v^{1 - p'}(y) \, dy \right)^{\frac{1}{p'}} \leq A\n\end{aligned} \tag{1.6}
$$

for all $R > 0$. For $p = 1$ a substitute of (1.6) will be given in Theorem 2.7.

Actually, in Section 3, we will be able to study the more general weighted weak-type inequalities $I_{\alpha}: L_v^p \to wL_u^{p\infty}$, i.e.

$$
\lambda^p \int_{\{z \in \mathbb{R}^n \, | \, w(z)(I_\alpha f)(z) > \lambda\}} u(x) \, dx \leq C \int_{\mathbb{R}^n} f^p(x) \, v(x) \, dx
$$

for all $\lambda > 0$ and $f(\cdot) \ge 0$. The last Section 4 is devoted to the proofs of our results.

h

2. Results for classical weighted weak-type inequalities

In this paper it will be assumed that

Weighted Weak-Type Inequalities for classical weighted weak-type inequalities. The second way, we can use the same result, we can use the same result. The second way, we can use the following equations:\n
$$
0 < \alpha < n, \qquad 1 \leq p < \infty \quad \text{and} \quad p' = \frac{p}{p-1} \text{ if } p > 1
$$

and

 $u(\cdot), v(\cdot)$ are weight functions

with $v^{1-p'}(\cdot) \in L^1_{loc}(\mathbb{R}^n, dx)$ if $p > 1$ else $v^{-1}(\cdot) \in L^\infty_{loc}(\mathbb{R}^n, v(x) dx)$.

Now the first main result, about a necessary and sufficient condition for $I_{\alpha}: L_v^p \to$ $L_n^{p\infty}$, can be stated.

Theorem 2.1. For $p > 1$, the boundedness $I_{\alpha}: L_p^p \to L_p^{p\infty}$ holds if and only if

$$
\widetilde{I}_{\alpha}: L_v^p \to L_u^{p\infty}
$$

and there is a constant $A > 0$ such that, for all $R > 0$,

$$
u(\cdot), v(\cdot) \text{ are weight functions}
$$
\n
$$
o_c(\mathbb{R}^n, dx) \text{ if } p > 1 \text{ else } v^{-1}(\cdot) \in L_{loc}^{\infty}(\mathbb{R}^n, v(x) dx).
$$
\nmain result, about a necessary and sufficient condition for $I_{\alpha}: L_v^p \to \alpha$.

\n1. For $p > 1$, the boundedness $I_{\alpha}: L_v^p \to L_v^{p\infty}$ holds if and only if

\n
$$
\tilde{I}_{\alpha}: L_v^p \to L_v^{p\infty}
$$
\nstant $A > 0$ such that, for all $R > 0$,

\n
$$
R^{\alpha - n} \Big(\int_{|x| < R} u(x) dx \Big)^{\frac{1}{p}} \Big(\int_{|y| < R} v^{1-p'}(y) dy \Big)^{\frac{1}{p'}} \leq A \qquad (2.1)
$$
\n
$$
\int_{\alpha} u(x) dx \Big)^{\frac{1}{p}} \Big(\int_{|y|^{(\alpha - n)p'} v^{1-p'}(y) dy} \Big)^{\frac{1}{p'}} \leq A. \qquad (2.2)
$$

and

$$
\int_{\text{max}}^{\infty} \int_{\text{max}}^{\infty} f(x) dx = \int_{\text{max}}^{\infty} \int_{\text{max}}^{\infty} f(x) dx
$$
\n
$$
= \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx
$$
\n
$$
= \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx = \int_{\alpha}^{\infty} f(x) dx
$$
\n
$$
= \int_{\alpha}^{\infty} \int_{|x| < R} u(x) dx = \int_{\alpha}^{\infty} \int_{|y| < R} u(x) dx = \int_{\alpha}^{\infty} \int_{|y| < R} u(x) dx = \int_{\alpha}^{\infty} \int_{|y| < R} |y|^{(\alpha - n)p'} v^{1 - p'}(y) dy = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx
$$
\n
$$
= \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \int_{|x| < |y|} |x - y|^{(\alpha - n)p'} v^{1 - p'}(y) dy
$$
\n
$$
= \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} f(x) dx = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \int_{\alpha
$$

Here the restricted operator I_{α} *is defined by*

$$
(\widetilde{I}_{\alpha}f)(x)=\int_{\{2^{-1}\left|x\right|<\left|y\right|<2\left|x\right|\}}\left|x-y\right|^{\alpha-n}f(y)\,dy.
$$

For $p = 1$ *a similar equivalence is also true when* (2.1) and (2.2) are replaced, for all *R>0,by*

$$
u(x) dx)^{\nu} \left(\int_{R < |y|} |y|^{\alpha - n/p} v^{1 - p}(y) dy \right)^{\nu} \leq A. \tag{2.2}
$$
\noperator \tilde{I}_{α} is defined by\n
$$
(\tilde{I}_{\alpha}f)(x) = \int_{\{2^{-1} |x| < |y| < 2|x|\}} |x - y|^{\alpha - n} f(y) dy.
$$
\n
$$
u \text{ is given by } \text{arg} \left[\int_{|x| < R} |u(x)| dx \right] \text{ is given by } \text{arg} \left[\int_{R} \tilde{I}_{|y|} \left(\int_{R} u(x) dx \right) \text{ is given by } \left[\int_{R} \tilde{I}_{|y|} \right]_{\{1 \leq R\}} |x - y|^{1/2} \right] \leq A \tag{2.3}
$$
\n
$$
\int_{|x| < R} u(x) dx \text{ is given by } \left[|y|^{\alpha - n} \frac{1}{v(y)} 1_{R < |x|} (y) \right] \leq A, \tag{2.4}
$$

and

$$
\left(\int_{|x|
$$

respectively.

In (2.3) and (2.4) the essential supremum is taken with respect to the measure $v(x)dx$. These conditions can be seen as limiting cases of (2.1) and (2.2) , respectively. Note also that both conditions (2.1) and(2.2) can be summarized by ssential supremum is taken with respect to the measure

n be seen as limiting cases of (2.1) and (2.2), respectively.

ns (2.1) and(2.2) can be summarized by
 $(|y| + R)^{(\alpha - n)p'}v^{1-p'}(y) dy\bigg)^{\frac{1}{p'}} \leq A$ for all $R > 0$. (2.5)

c)dx. These conditions can be seen as limiting cases of (2.1) and (2.2), respectively.
the also that both conditions (2.1) and (2.2) can be summarized by

$$
\left(\int_{|x| 0. \quad (2.5)
$$

Theorem 2.1 means that the weighted weak-inequality problem for I_{α} can be essentially reduced to the corresponding weighted weak-inequality for the restricted operator I_{α} . Although a characterization for $\tilde{I}_{\alpha}: L_v^p \to L_u^{p\infty}$ remains unsolved, surprisingly it is not too hard to derive sufficient conditions ensuring this boundedness.

Proposition 2.2. The boundedness $\widetilde{I}_{\alpha}: L_v^p \to L_v^{p\infty}$ holds if for some constant *A>0*

ondratsimba

\n1 2.2. The boundedness
$$
\widetilde{I}_{\alpha}: L_v^p \to L_u^{p\infty}
$$
 holds if for some constant $|x|^{\alpha} \left(\sup_{4^{-1}|x| < |y| < 4|x|} u(y)\right)^{\frac{1}{p}} \leq A(v(x))^{\frac{1}{p}}$ for a.e. x .

\n(2.6)

\n1. $\int \left(\sup_{4^{-1}|x| < |y| < 4|x|} \frac{1}{v(y)}\right)^{\frac{1}{p}} \leq A$ for a.e. x .

\n(2.6)'

\nor (2.6)'

\n1. $\int \left(\sup_{4^{-1}|x| < |y| < 4|x|} \frac{1}{v(y)}\right)^{\frac{1}{p}} \leq A$ for a.e. x .

\n(2.6)'

\n1. $\int \left(\sup_{4^{-1}|x| < |y| < 4|x|} \frac{1}{v(y)}\right)^{\frac{1}{p}} \leq A$ for a.e. x .

This condition can be replaced by

$$
|x|^{\alpha}(u(x))^{\frac{1}{p}}\Big(\sup_{4^{-1}|x|<|y|<4|x|}\frac{1}{v(y)}\Big)^{\frac{1}{p}}\leq A\qquad\text{for a.e. }x.\tag{2.6}'.
$$

Since (2.6) [or (2.6)'] *is* a pointwise inequality it is in general an easy verifiable condition for given weight functions. And it is an interesting question to determine some situations when (2.6) becomes also a necessary condition for $I_{\alpha}: L_v^p \to L_u^{p\infty}$. For this purpose, growth conditions on weights are needed. So we define $\sigma(\cdot) \in \mathcal{H}$ if $|x|^{\alpha} \left(\sup_{4^{-1}|x| < |y| < 4|x|} u(y)\right)^{\frac{1}{p}} \leq A(v(x))^{\frac{1}{p}}$ for a.e. x.

can be replaced by
 $|x|^{\alpha}(u(x))^{\frac{1}{p}} \left(\sup_{4^{-1}|x| < |y| < 4|x|} \frac{1}{v(y)}\right)^{\frac{1}{p}} \leq A$ for a.e. x.

[or (2.6)'] is a pointwise inequality it is in gene $|x|^{\alpha} \left(\sup_{4^{-1}|x| < |y| < 4}$
 tion can be replaced by
 $|x|^{\alpha}(u(x))^{\frac{1}{p}} \left(\sup_{4^{-1}|x| < |y| < 4} \right)$

2.6) [or (2.6)'] is a poor given weight functions

(or given weight functions come, growth conditions come, growth conditio

$$
\sup_{4^{-1}R < |y| < 4R} \sigma(y) \leq CR^{-n} \int_{2^{-N}R < |y| < 2^N R} \sigma(y) \, dy \qquad \text{for all } R > 0
$$

where $C > 0$ and the non-negative integer N are fixed constants. Many of usual weights satisfy the property H . It is the case of any *radial and monotone weights,* for which N can be taken equal to 3. But the condition $w(\cdot) \in \mathcal{H}$ can be held although $w(\cdot)$ is not necessarily a monotone weight. Indeed, we have the following

Lemma 2.3. *Suppose* $w_1(\cdot)$ and $w_2(\cdot)$ are radial monotone on $(0, R_0)$ and (R_0, ∞) *for some* $R_0 > 0$, *respectively.* Let

$$
w(x) = w_1(|x|) \times \mathbf{1}_{(0,R_0)}(|x|) + w_2(|x|) \times \mathbf{1}_{(R_0,\infty)}(|x|).
$$

Then $w(\cdot) \in \mathcal{H}$, with constants $N = 3$ and $C > 0$ depending on $w_1(\cdot)$ and $w_2(\cdot)$.

An answer to the above question can he given by using the growth condition *N.*

Proposition 2.4. *Suppose* $u(\cdot) \in \mathcal{H}$. *Then:*

(i) For $p > 1$ the pointwise inequality (2.6) (or $(2.6)'$) is satisfied whenever the *Muckenhoupt condition (2.1) holds and* $v^{1-p'}(.) \in \mathcal{H}$.

Another special weight property is the so-called *reverse doubling condition* $w(\cdot) \in$ RD_{ρ} ($\rho > 0$) which means, by definition,

$$
w(x) = w_1(|x|) \times 1_{(0,R_0)}(|x|) + w_2(|x|) \times 1_{(R_0,\infty)}(|x|).
$$

\n
$$
w(\cdot) \in \mathcal{H}, \text{ with constants } N = 3 \text{ and } C > 0 \text{ depending on } w_1(\cdot) \text{ and } w_2(\cdot)
$$

\nAn answer to the above question can be given by using the growth condi
\n**Proposition 2.4.** Suppose $u(\cdot) \in \mathcal{H}$. Then:
\n(i) For $p > 1$ the pointwise inequality (2.6) (or (2.6)') is satisfied when
\nthen *output condition* (2.1) holds and $v^{1-p'}(\cdot) \in \mathcal{H}$.
\n(ii) Similarly, for $p = 1$, inequality (2.6) (or (2.6)') is implied by (2.3).
\nAnother special weight property is the so-called reverse doubling condition,
\n
$$
(\rho > 0) \text{ which means, by definition,\n
$$
\int_{|y| < R} w(y) dy \leq ct^{n\rho} \int_{|y| < R} w(y) dy \quad \text{for all } t \in (0, 1) \text{ and } R > 0
$$

\nfor a fixed constant $c > 0$. The interest in the introduction of this co-
\ncited by
\n**Proposition 2.5.** Assume that $u(\cdot) \in RD$, for some $\rho > 0$. Then
$$

and for a fixed constant $c > 0$. The interest in the introduction of this condition is reflected by

Proposition 2.5. Assume that $u(\cdot) \in RD_p$ for some $\rho > 0$. Then:

(i) For $p > 1$, the dual Hardy condition (2.2) is implied by the Muckenhoupt con*dition (2.1).*

(ii) *Similarly for* $p = 1$, *condition* (2.4) is *implied by* (2.3).

The facts contained in Theorem *2.1* and Propositions *2.4* and *2.5* can be summarized as

Theorem 2.6. *Let* $p > 1$.

A) The boundedness $I_{\alpha}: L_v^p \to L_u^{p\infty}$ implies the Gabidzashvili-Kokilashvili con*dition (2.5). Conversely, this last condition implies* I_{α} : $L_p^p \rightarrow L_u^{p\infty}$ whenever the *pozntwzse inequality (2.6) or (2.6)' is satisfied.*

B) Inequality (2.6) *in part A) can be dropped whenever* $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{H}$.

C) The Gabidzashvili-Kokilashvili condition (2.5) in part A) *can be replaced by the Muckenhoupt condition* (2.1) whenever $u(\cdot) \in RD_a$ for some $\rho > 0$.

Consequently, as announced in the Introduction, for a large class of weight functions (like radial and monotone weights), the Gabidzashvili and Kokilashvili result [1] (valid for $p < q$ can be extended to the case $p = q$. Consequently, as announced in the Introduction, for a large class of weight functions
 $P(A)$ in and monotone weights), the Gabidzashvili and Kokilashvili result [1] (valid
 $P(A)$ an be extended to the case $p = q$.
 A) Th

Theorem 2.7.

Muckenhoupt condition (2.1) *whenever* $u(\cdot) \in RD_{\rho}$ *for some* $\rho > 0$.

Consequently, as announced in the Introduction, for a large class of weight functions

(like radial and monotone weights), the Gabidzashvili and *(2.6)' is satisfied.*

B) *Inequality* (2.6) *in part* A) *can be dropped whenever* $u(\cdot) \in \mathcal{H}$.

C) Condition (2.4) *in parts* A) *and* B) *can be dropped whenever* $u(\cdot) \in RD$ *for some* $\rho > 0$.

Now examples and applications, showing the gain in our results compared with past results, are given.

Corollary 2.8. Let $0 < R_0 < 1 < p < \infty$. Define the weight functions

\n The following inequality holds:\n
$$
\text{log}(\mathbf{x}) = \frac{1}{2} \int_{-\infty}^{\infty} \int
$$

 $v(x) = |x|^{-\gamma} \ln^r (|x|^{-\gamma}) \mathbf{1}_{|x| < R_0}(x) +$
Then, for $\beta > 0$ and $\alpha p < \theta$, $I_\alpha : L_v^p \to L_u^{p\infty}$ whenever

$$
\alpha p + \gamma < n p \leq \alpha p + \beta \qquad \text{and} \qquad \alpha p + \gamma \leq \theta.
$$

Also, set

$$
\alpha p + \gamma < n p \leq \alpha p + \beta \qquad \text{and} \qquad \alpha p + \gamma \leq \theta.
$$
\n
$$
u^*(x) = |x|^{-n} \ln^{-p} |x|^{-1} \mathbf{1}_{|x| < R_0}(x) + |x|^{(1-p)(\theta - n)} \mathbf{1}_{|x| > R_0}(x)
$$
\n
$$
v^*(x) = |x|^{(1-p)(\beta - n)} \mathbf{1}_{|x| < R_0}(x) + |x|^{(1-p)(\gamma - n)} \mathbf{1}_{|x| > R_0}(x).
$$
\n
$$
\Rightarrow 0, \ I_\alpha : L_v^p \to L_u^{p\infty} \text{ whenever}
$$
\n
$$
\alpha p' + \gamma < n p' \leq \alpha p' + \beta \qquad \text{and} \qquad \alpha p' + \gamma \leq \theta.
$$
\nwe mentioned in the Introduction for this example the how

 $Then, for $\beta > 0$, $I_{\alpha}: L_v^p \rightarrow L_u^{p\infty}$ whenever$

$$
v^*(x) = |x|^{(1-p)(\beta-n)} \mathbf{1}_{|x| < R_0}(x) + |x|^{(1-p)(\gamma-n)} \mathbf{1}_{|x| > R_0}(x)
$$
\nthen, for $\beta > 0$, $I_\alpha: L_v^p \to L_u^{p\infty}$ whenever

\n
$$
\alpha p' + \gamma < n p' \leq \alpha p' + \beta \qquad \text{and} \qquad \alpha p' + \gamma \leq \theta.
$$
\nAs we have mentioned in the Introduction for this example the

As we have mentioned in the Introduction, for this example the boundedness I_{α} : $L_r^p \to L_r^{p\infty}$ is not obtainable from the Sawyer-Wheeden criterion (1.6).

Corollary 2.9. *Let* $1 < p < \frac{n}{\alpha}$ and $u(\cdot) \in \mathcal{H} \cap RD_{\rho}$ ($\rho > 0$). *Then, for some* **Corollary 2.9.** *Let* $1 < p < \frac{n}{\alpha}$ and $u(\cdot) \in \mathcal{H} \cap RD_{\rho}$ ($\rho > 0$). *Then, for some* $constant C > 0$,

andratsimba

\n9. Let
$$
1 < p < \frac{n}{\alpha}
$$
 and $u(\cdot) \in \mathcal{H} \cap RD_{\rho}$ ($\rho > 0$). Then, for some

\n
$$
\lambda^{p} \int_{\{z \mid (I_{\alpha}f)(z) > \lambda\}} u(x) \, dx \leq C \int_{\mathbb{R}^{n}} f^{p}(x) \left(M_{\alpha p} u \right)(x) \, dx
$$
 (2.7)

\n
$$
f(\cdot) \geq 0.
$$
 Here $M_{\alpha p}$ is the usual fractional maximal operator

for all $\lambda > 0$ *and* $f(\cdot) \geq 0$. Here $M_{\alpha p}$ is the usual fractional maximal operator

$$
(M_{\alpha p}f)(x)=\sup\Big\{|Q|^{\frac{\alpha p}{n}-1}\int_Q|f(y)|\,dy\,\Big|\,Q\text{ a cube with }Q\ni x\Big\}.
$$

The constant *C* in (2.7) depends only on the constants in properties H and RD_{ρ} but not directly on $u(\cdot)$.

A result like Corollary 2.9 can be used to derive weighted Sobolev inequalities as

Corollary 2.10. Let $1 < p < n$ *and* $u(\cdot) \in \mathcal{H} \cap RD$ *(* $\rho > 0$ *). Then*

\n- (·)
$$
\geq 0
$$
. Here $M_{\alpha p}$ is the usual fractional maximal operator
\n- () $(x) = \sup \left\{ |Q|^{\frac{\alpha p}{n} - 1} \int_Q |f(y)| \, dy \, |Q \, a \, cube \, with \, Q \ni x \right\}.$
\n- 7 in (2.7) depends only on the constants in properties $\mathcal H$ and RD_ρ $u(\cdot).$
\n- orollary 2.9 can be used to derive weighted Sobolev inequalities as
\n- **0.** Let $1 < p < n$ and $u(\cdot) \in \mathcal H \cap RD_\rho$ $(\rho > 0)$. Then
\n- $$
\int_{\mathbb R^n} |g(x)|^p u(x) \, dx \leq C \int_{\mathbb R^n} |(\nabla g)(x)|^p (M_p u)(x) \, dx \tag{2.8}
$$
\n- $\mathbb R^n$) and for some fixed constant $C > 0$.
\n

for all $g(\cdot) \in C_0^{\infty}(\mathbb{R}^n)$ *and for some fixed constant* $C > 0$.

 $\int_{\mathbb{R}^n} |g(x)|^p u(x) dx \leq C \int_{\mathbb{R}^n} |(\nabla g)(x)|^p (M_p u)(x) dx$ (2.8)
for all $g(\cdot) \in C_0^{\infty}(\mathbb{R}^n)$ and for some fixed constant $C > 0$.
Indeed, it is known from [2] (see also [5]) that $I_1 : L_v^p \to L_u^{p\infty}$ implies the Sob

For general weight functions $u(\cdot)$, Pérez [3] proved an inequality like (2.8), with $(M_{\alpha p}u)(\cdot)$ replaced by $(M_{\alpha p}M_0^{[p]}u)(\cdot)$. Here $M^{[p]}$ is the [p]-times iteration of the Hardy-
Littlewood maximal operator M_0 . Since trivially $u(\cdot) \leq M_0^{[p]}u)(\cdot)$, Corollary 2.10 can Inequality $\int_{\mathbb{R}^n} |g(x)|^p u(x) dx \leq C \int_{\mathbb{R}^n} |(\nabla g)(x)|^p v(x) dx$.
For general weight functions $u(\cdot)$, Pérez [3] proved an inequality like (2.8), with $(M_{\alpha p}u)(\cdot)$ replaced by $(M_{\alpha p}M_0^{[p]}u)(\cdot)$. Here $M^{[p]}$ is the be seen as an improvement of this author's result for weights $u(\cdot) \in \mathcal{H} \cap RD_{\rho}$. and by $(M_{\alpha p} M_0^{[p]} u)(\cdot)$. Here $M^{[p]}$ is the [p]-times iteration of the Hardy-
mal operator M_0 . Since trivially $u(\cdot) \le M_0^{[p]} u)(\cdot)$, Corollary 2.10 can
provement of this author's result for weights $u(\cdot) \in \mathcal{H} \cap$ the functions $u(\cdot)$, Pérez [3] proved it
by $(M_{\alpha p} M_0^{[p]} u)(\cdot)$. Here $M^{[p]}$ is the [p]
operator M_0 . Since trivially $u(\cdot) \le$
vement of this author's result for weight
cal **Results**
 $u(\cdot)$, $v(\cdot)$ are assumed as in

3. More General Results

In this section α , p , $u(\cdot)$, $v(\cdot)$ are assumed as in Section 2, and moreover $w(\cdot)$ is a weight function. Our purpose is now to study the more general weighted weak-type inequalities Littlewood maximal
be seen as an improve
3. More Gener.
In this section α , p , u
function. Our purpos
 $I_{\alpha}: L_v^p \to wL_w^p$, i.e.

$$
\lambda^{p} \int_{\{x \in \mathbb{R}^{n} \, | \, w(x)(I_{\alpha}f)(x) > \lambda\}} u(x) \, dx \leq C \int_{\mathbb{R}^{n}} f^{p}(x) \, v(x) \, dx \tag{3.1}
$$

for all $\lambda > 0$ and $f(\cdot) \geq 0$. As usual, *C* is a fixed non-negative constant.

The boundedness $I_{\alpha}: L_v^p \to L_u^{p\infty}$ is a particular case of (1.1), but for $w(\cdot) \neq 1$ the two inequalities are quite different since the weight $w(\cdot)$ cannot be combined with $u(\cdot)$ or $v(\cdot)$. It seems that no result about $I_{\alpha}: L_v^p \to wL_v^{p\infty}$ [with $w(\cdot) \neq 1$] were explicitely written and available elsewhere. Actually, results given in Section 2 are consequences of those we will present in this section.

First a necessary and sufficient condition for $I_{\alpha}: L_v^p \to wL_v^{p\infty}$ is stated.

Theorem 3.1. *For p* > 1, *the boundedness* $I_{\alpha}: L_v^p \to wL_u^{p\infty}$ *holds* if and only if

$$
\widetilde{I}_{\alpha}: L_v^p \to wL_u^{p\infty}
$$

and there is aconstant A > 0 such that

$$
\text{Im } 3.1. \text{ For } p > 1, \text{ the boundedness } I_{\alpha}: L_v^p \to wL_u^{p\infty} \text{ holds if and only if}
$$
\n
$$
\widetilde{I}_{\alpha}: L_v^p \to wL_u^{p\infty}
$$
\n
$$
\text{a constant } A > 0 \text{ such that}
$$
\n
$$
\lambda \Big(\int_{\{w(x)|x|^{q-n} > \lambda\} \cap \{2R < |x|\}} u(x) \, dx \Big)^{\frac{1}{p}} \Big(\int_{|y| < R} v^{1-p'}(y) \, dy \Big)^{\frac{1}{p'}} \leq A \qquad (3.2)
$$
\n
$$
w(x) > \lambda \cap (|x| < R) \qquad u(x) \, dx \Big)^{\frac{1}{p}} \Big(\int_{2R < |y|} |y|^{(\alpha - n)p'} v^{1-p'}(y) \, dy \Big)^{\frac{1}{p'}} \leq A \qquad (3.2)^*
$$

and

$$
I_{\alpha}: L_{v}^{p} \to wL_{u}^{p\infty}
$$
\nhere is a constant $A > 0$ such that

\n
$$
\lambda \Biggl(\int_{\{w(x)|x|^{q-n} > \lambda\} \cap \{2R < |x|\}} u(x) dx \Biggr)^{\frac{1}{p}} \Biggl(\int_{|y| < R} v^{1-p'}(y) dy \Biggr)^{\frac{1}{p'}} \leq A \qquad (3.2)
$$
\n
$$
\lambda \Biggl(\int_{\{w(x) > \lambda\} \cap \{|x| < R\}} u(x) dx \Biggr)^{\frac{1}{p}} \Biggl(\int_{2R < |y|} |y|^{(\alpha-n)p'} v^{1-p'}(y) dy \Biggr)^{\frac{1}{p'}} \leq A \qquad (3.2)^{*}
$$
\nand $R > 0$. The restricted operator \tilde{I}_{α} is defined as in Theorem 2.1.

\nor $p = 1$ a similar equivalence is also true when (3.2) and (3.2)^{*} are replaced by

\n
$$
\lambda \Biggl(\int_{\{w(x)|x|^{q-n} > \lambda\} \cap \{2R < |x|\}} u(x) dx \Biggr) \Biggl(\text{ess sup} \Biggl[\frac{1}{v(y)} 1_{|x| < R}(y) \Biggr] \Biggr) \leq A \qquad (3.3)
$$

for all $\lambda > 0$ and $R > 0$. The restricted operator \widetilde{I}_{α} is defined as in Theorem 2.1.

For p = 1 a similar equivalence is also true when (3.2) and (3.2)' are replaced by

$$
\begin{array}{l}\n\text{(}w(x)>\lambda\text{)}\cap\{|x|
$$

and

$$
\lambda \Big(\int_{\{w(x)|x|^{\alpha-n} > \lambda\} \cap \{2R < |x|\}} u(x) \, dx \Big) \Big(\text{ess sup} \Big[\frac{1}{v(y)} \mathbf{1}_{| \cdot | < R}(y) \Big] \Big) \le A \tag{3.3}
$$
\n
$$
\lambda \Big(\int_{\{w(x) > \lambda\} \cap \{|x| < R\}} u(x) \, dx \Big) \Big(\text{ess sup} \Big[|y|^{\alpha-n} \frac{1}{v(y)} \mathbf{1}_{2R < | \cdot |}(y) \Big] \Big) \le A \tag{3.3}
$$
\n
$$
\lambda > 0 \text{ and } R > 0. \tag{3.3}
$$

for all $\lambda > 0$ *and* $R > 0$.

Theorem 3.1 means that to solve the weighted weak-type inequality (3.1), the real problem is to decide when does the boundedness of the restricted operator \tilde{I}_{α} hold.

Note that in Theorem *3. 1,* the direct Hardy condition (3.2) and its dual version (3.2)' are used. But in Theorem *2.1* the Hardy condition is not appeared and is replaced by the Muckenhoupt condition (2.1). An explanation of this fact will be seen below in Proposition *3.4.* Theorem 3.1, the dire

1 Theorem 2.1 the H_i

1 tot condition (2.1). A

characterization of w

able here to derive a
 $v(\cdot)$ is constant on an

sup $w(z) \le c$
 ${R < |z| < 16R}$

eal inconvenience sin

Although a characterization of weights for which $\tilde{I}_{\alpha}: L_v^p \to wL_u^{p\infty}$ is still an open problem, we are able here to derive an easy sufficient condition ensuring this boundedness whenever $w(\cdot)$ is constant on annuli [or merely $w(\cdot) \in \mathcal{A}$], in the sense that racterization of weights for which $\widetilde{I}_{\alpha}: L_y^p \to wL_y^{p,\infty}$

e here to derive an easy sufficient condition ensuring

is constant on annuli [or merely $w(\cdot) \in \mathcal{A}$], in the set
 $\sup_{|z| < 16R} w(z) \leq c \inf_{\{R < |y| < 16R\}} w(y$

$$
\sup_{\{R<|z|<16R\}}w(z)\leq c\inf_{\{R<|y|<16R\}}w(y)\qquad\text{for all }R>0.
$$

This is not a real inconvenience since many of usual weights $w(\cdot)$ for which (3.1) is considered are constants on annuli. Indeed, for $w(x) = |x|^{\gamma} \ln^{\delta}(e + |x|)$ ($\gamma, \delta \ge 0$), then $w(\cdot) \in \mathcal{A}$. It is also the case of any radial increasing [resp. decreasing] weight $w(\cdot)$ for which $w(4R) \leq c w(R)$ [resp. $w(R) \leq c w(4R)$].

Proposition 3.2. *For w(.)* \in *A*, *the boundedness* $\tilde{I}_{\alpha}: L_v^p \to wL_u^{p\infty}$ *holds if for e constant* $A > 0$ *some constant A >* ⁰

kotondratsimba
\nion 3.2. For
$$
w(\cdot) \in A
$$
, the boundedness $\tilde{I}_{\alpha}: L_v^p \to wL_v^{p\infty}$ holds if for
\n
$$
w(x)|x|^{\alpha} \Big(\sup_{4^{-1}|x| < |y| < 4|x|} u(y)\Big)^{\frac{1}{p}} \leq A\Big(v(x)\Big)^{\frac{1}{p}} \quad \text{for a.e. } x. \tag{3.4}
$$
\n $\text{and} \quad \text{or} \quad \text$

This condition can be replaced by

ikotondratsimba

\ntion 3.2. For
$$
w(\cdot) \in A
$$
, the boundedness $\widetilde{I}_{\alpha} : L_v^p \to wL_v^p \in \mathbb{R}$ holds if for $t A > 0$

\n
$$
w(x)|x|^{\alpha} \left(\sup_{4^{-1}|x| < |y| < 4|x|} u(y) \right)^{\frac{1}{p}} \leq A \left(v(x) \right)^{\frac{1}{p}}
$$
 for a.e. x. (3.4)

\nin can be replaced by

\n
$$
w(x)|x|^{\alpha} \left(u(x) \right)^{\frac{1}{p}} \left(\sup_{4^{-1}|x| < |y| < 4|x|} \frac{1}{v(y)} \right)^{\frac{1}{p}} \leq A
$$
 for a.e. x. (3.4)'

\nhypothesis $w(\cdot) \in A$, the above conclusions are also true whenever I_{α} :

\n
$$
L^p(w^p(x)|x|^{-\alpha p} dx).
$$

Without the hypothesis $w(\cdot) \in A$ *, the above conclusions are also true whenever* I_{α} : $L^p(\mathbb{R}^n, dx) \to L^p(w^p(x)|x|^{-\alpha p}dx).$

In general, (3.4) [or (3.4')] *is* an easy verifiable condition since it is just a pointwise inequality. And the main question remains to determine situations for which (3.4) *is* inequality. And the main question remains to determine situations for which (3.4) is
also a necessary condition for $I_{\alpha}: L_{\nu}^p \to wL_{\nu}^{p\infty}$. In solving this problem, it is useful to note that a necessary condition for such boundedness is the Muckenhoupt condition *Raffluid* λ *A*, *the above conclusion*
 Raffluid λ *RA(fluid in the above conclusion*
 A LP(wP(x)|x|-apdx).
 Ral, (3.4) [or (3.4')] is an easy verifiable condi

And the main question remains to determint

sa *Without the hypothesis* $w(\cdot) \in A$ *, the above conclusions are also true whenever* I_{α} *:
* $L^p(\mathbb{R}^n, dx) \to L^p(\psi^p(x)|x|^{-\alpha p}dx)$ *.

In general, (3.4) [or (3.4')] is an easy verifiable condition since it is just a pointwise
 e* hypothesis $w(\cdot) \in$
 $\rightarrow L^p(w^p(x)|x|^{-\alpha p}dx$

ral, (3.4) [or (3.4')] is

And the main quest

ssary condition for I_c

necessary condition
 $R^{\alpha-n}\lambda\left(\int_{\{w(x)>\lambda\}\cap\{|t\}}\right)$
 >0 and where $A > 0$
 $R^{\alpha-n}\lambda\left(\int_{\{w(x)>\lambda\}\cap$

\n (3.5)
$$
R^{\alpha-n} \lambda \left(\int_{\{w(x) > \lambda\} \cap \{|x| < R\}} u(x) \, dx \right) \left(\text{ess} \sup \left[\frac{1}{v(y)} \right] \right) \leq A
$$
 (3.6) $\text{and all } R > 0$. The mentioned implies:\n

for all λ , $R > 0$ and where $A > 0$ is a fixed constant. The replacement of (3.5) for $p = 1$

$$
R^{\alpha-n}\lambda\Big(\int_{\{w(x)>\lambda\}\cap\{|x|
$$

for all $\lambda > 0$ and all $R > 0$. The mentioned implication can be easily proved similarly as the necessary part in Theorem *2.1.* Obviously, the above question can he reduced to get (3.4) from (3.5) or (3.6). all $\lambda, R > 0$ and where $A > 0$ is a fixed constant. The re
 $R^{\alpha - n} \lambda \left(\int_{\{w(x) > \lambda\} \cap \{|x| < R\}} u(x) dx \right) \left(\text{ess sup} \left[\frac{1}{v(y)} \right] \text{ all } \lambda > 0 \text{ and all } R > 0.$ The mentioned implication ca

he necessary part in Theorem 2.1. Obviousl ${u(x) dx}$ (ess sup $\left\{ \frac{1}{v(y)} \right\}$
and all $R > 0$. The mentioned implication carry part in Theorem 2.1. Obviously, the above 1 (3.5) or (3.6).
ion 3.3. For $p > 1$ and $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H}$, it is field if for some cons $u(x) dx$ $u(x) dx$ $v(x) = F$
 $v(x) = F$
 $v(x) = F$

The mentioned implication can be easily proved similarly

sorem 2.1. Obviously, the above question can be reduced to
 $p > 1$ and $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{H}$, the pointwise inequalit

Proposition 3.3. For $p > 1$ and $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{H}$, the pointwise inequality (3.4) *or* (3.4) ^{*'*} *is satisfied if for some constant* $A > 0$

The necessary part in Theorem 2.1. Obviously, the above question can be reduced to
\n(3.4) from (3.5) or (3.6).
\nProposition 3.3. For
$$
p > 1
$$
 and $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{H}$, the pointwise inequality (3.4)
\n(3.4)' is satisfied if for some constant $A > 0$
\n $R^{\alpha-n}\lambda \Bigl(\int_{\{w(x)>\lambda\}\cap\{R<|x|<2^{2N}R\}} u(x)x\Bigr)^{\frac{1}{p}} \Bigl(\int_{R<|y|<2^{2N}R} v^{1-p'}(y) dy\Bigr)^{\frac{1}{p'}} \leq A$ (3.5)'
\nall $\lambda, R > 0$ and whenever $w(\cdot) \in \mathcal{A}$, in the sense that
\n $\sup_{\{R<|z|<2^{2N}R\}} w(z) \leq c \inf_{\{R<|y|<2^{2N}R\}} w(y)$. (3.7)
\nwe N is the integer from assumption H. In particular, the Muchenhoupt condition)
\nimplies (3.4) or (3.4)'.

for all $\lambda, R > 0$ *and whenever w* $(\cdot) \in \mathcal{A}$ *, in the sense that*

$$
\sup_{R < |z| < 2^{2N} \, R\}} w(z) \le c \inf_{\{R < |y| < 2^{2N} \, R\}} w(y). \tag{3.7}
$$

Here N is the integer from assumption R. In particular, the Muckenhoupt condition (3.5) implies (3.4) or (3.4)'.

Similarly, for $p = 1$ *and* $u(\cdot) \in H$ *, inequality* (3.4) (*or* $(3.4')$) is satisfied if for *some constant* $A > 0$

r all
$$
\lambda, R > 0
$$
 and whenever $w(\cdot) \in A$, in the sense that
\n
$$
w(z) \le c \inf_{\{R < |z| < 2^{2N} R\}} w(y).
$$
\n(3.7)
\nare N is the integer from assumption H. In particular, the Muckenhoupt condition
\n5) implies (3.4) or (3.4').
\nSimilarly, for $p = 1$ and $u(\cdot) \in H$, inequality (3.4) (or (3.4')) is satisfied if for
\nme constant $A > 0$
\n
$$
R^{\alpha - n} \lambda \Big(\int_{\{w(z) > \lambda\} \cap \{R < |z| < 2^{2N} R\}} u(x) dx \Big) \Big(\text{ess sup} \Big[\frac{1}{v(y)} 1_{R < |z| < 2^{2N} R}(y) \Big] \Big) \le A
$$
\n(3.6)'
\nr all $\lambda, R > 0$ and $w(\cdot) \in A$. In particular, condition (3.6) implies (3.4) (or (3.4)').
\nSince for $n > 1$ the Hunks result is $\mu(2.2) \lambda(2.2N) \lambda(1.2) \lambda(1.2$

for all $\lambda, R > 0$ *and w(.)* $\in \mathcal{A}$. In particular, condition (3.6) implies (3.4) (or (3.4)').

Since for $p > 1$ the Hardy conditions (3.2) , (3.2^*) and the Muckenhoupt condition (3.5) are both necessary conditions for the boundedness $I_{\alpha}: L_v^p \to wL_v^{p\infty}$, then it is a natural question to precise some relations between these three conditions whenever the weights have a special property like the reverse doubling condition RD_{ρ} .

Proposition 3.4.

A) For $w(x) = |x|^{-\gamma}$, with $\gamma \geq 0$, the Muckenhoupt condition (3.5) implies the *Hardy condition (3.2), and similarly condition (3.6) implies (3.3).*

B) For general weights $w(\cdot)$ and $p > 1$, then (3.5) implies (3.2) wheneve $v^{1-p'}(\cdot) \in$ RD_{ρ} for some $\rho > 0$.

This result yields an explanation why, for the boundedness $I_{\alpha}: L_v^p \to L_v^{p\infty}$, the Hardy condition does not appear in Theorem 2.1.

Facts described in Propositions 3.2 - 3.4 and in Theorem 3.1 can be now summarized.

Theorem 3.5. Let $p > 1$.

A) The boundedness $I_{\alpha}: L_v^p \to wL_u^{p\infty}$ implies the Hardy conditions (3.2), (3.2)^{*} *and the Muckenhoupt condition (3.5). Conversely, the conditions (3.2), (3.2)' and (3.5) imply* $I_{\alpha}: L_v^p \to wL_v^{p\infty}$ whenever $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H}$ and $w(\cdot) \in \mathcal{A}$ as in (3.7).

B) If $u(\cdot)$ and $v(\cdot)$ are radial and monotone functions and $w(\cdot) \in A$, then $I_0: L_v^p \to$ $wL^{p\infty}$ if and only if (3.2) and (3.2)^{*} are satisfied.

C) In parts A) *and* B), *the Hardy condition (3.2) can be replaced by the Muckenhoupt condition* (3.5) whenever $w(x) = |x|^{-\gamma}, \gamma \ge 0$ or $v^{1-p'}(\cdot) \in RD_\rho$ for some $\rho > 0$.

Consequently, as announced in the Introduction, we obtained a characterization of weight functions and $w(\cdot) \in \mathcal{A}$.

Theorem 3.6.

Consequently, as announced in the introduction, we obtained a differentially the
the boundedness $I_{\alpha}: L_v^n \to wL_v^{p\infty}$ whenever $u(\cdot)$ and $v^{1-p'}(\cdot)$ are radial monotone
weight functions and $w(\cdot) \in A$.
Theorem 3.6.
A) The **A)** *The boundedness* $I_{\alpha}: L_v^1 \to wL_w^{1\infty}$ *implies conditions* (3.3), (3.3)^{*} and (3.6). *Conversely, these conditions* (3.3), (3.3)^{*} and (3.6) *imply* $I_{\alpha}: L_v^1 \to wL_w^{1\infty}$ *whenever* $u(\cdot) \in \mathcal{H}$ and $w(\cdot) \in \mathcal{A}$ as in (3.7).

B) If $u(\cdot)$ and $v(\cdot)$ are radial and monotone functions and $w(\cdot) \in A$, then $I_{\alpha}: L_v^1 \to L_v^2$ $wL_n^{1\infty}$ if and only if (3.3) and (3.3)^{*} are satisfied.

C) In parts A) *and* B), *condition* (3.3) *can be replaced by* (3.6) *whenever* $w(x)$ = $|x|^{-\gamma}, \gamma \geq 0.$

4. Proofs of the results

First we prove results in Section 3 and next outline proofs of those stated in Section 2.

Proof of Theorem 3.1. *Necessary Part*: We first assume that $I_{\alpha}: L_v^p \rightarrow$ which is equivalent to alts in Section 3 and next outline pro

eorem 3.1. Necessary Part: We firs

t to
 $||w(\cdot)(I_{\alpha}f)(\cdot)||_{L_{\alpha}^{\text{pos}}} \leq C||f(\cdot)||_{L_{\alpha}^{\text{pos}}}$ (3.6) whenever $w(x) =$
hose stated in Section 2.
e that $I_{\alpha}: L_v^p \to wL_u^{p\infty}$,
 $(\cdot) \ge 0$ (4.1) $x|^{-\gamma}, \gamma \ge 0.$
4. Proofs of γ
First we prove respectively proof of The which is equivaler
where

$$
\|w(\cdot)(I_{\alpha}f)(\cdot)\|_{L_{\alpha}^{\mathbf{p}\infty}} \leq C \|f(\cdot)\|_{L_{\alpha}^{\mathbf{p}}} \qquad \text{for all } f(\cdot) \geq 0 \tag{4.1}
$$

$$
(\cdot) \|_{L^p_x} \le C \| f(\cdot) \|_{L^p_x} \quad \text{for a}
$$

$$
\| f(\cdot) \|_{L^p_x}^p = \int_{\mathbb{R}^n} |f(x)|^p v(x) dx
$$

and

tsimba
\n
$$
||g(\cdot)||_{L_x^p\infty}^p = \sup_{\lambda>0} \left\{\lambda^p \int_{\{x: |g(x)|>\lambda\}} u(x) dx \right\}.
$$

The main point for condition (3.2) is the existence of a constant $C > 0$ such that

124 Y. Rakotondratsimba
\nand
\n
$$
||g(\cdot)||_{L_x^{\infty}}^p = \sup_{\lambda>0} \left\{ \lambda^p \int_{\{x: |g(x)|>\lambda\}} u(x) dx \right\}.
$$
\nThe boundedness $\widetilde{I}_{\alpha}: L_v^p \to w L_v^p{}^{\infty}$ appears clearly, since trivially $(\widetilde{I}_{\alpha}f)(\cdot) \leq (I_{\alpha}f)(\cdot).$
\nThe main point for condition (3.2) is the existence of a constant $C > 0$ such that
\n
$$
\left(\int_{|y|\n(4.2)
$$

for all $f(\cdot) \ge 0$ and $R > 0$. Indeed, in the case $p > 1$, taking $f(\cdot) = 0$ in this inequality and if $0 < \int_{\{|y| \le R} v^{1-p'}(y) dy < \infty}$, then

$$
(y) dy \Big) \| |\cdot|^{\alpha - n} w(\cdot) \mathbf{1}_{2R < |\cdot|}(\cdot) \|_{L_{\alpha}^{\mathbb{P}^{\infty}}} \leq C \Big(\int_{\mathbb{R}^n} f^p(y) v(y) \Big)
$$

and $R > 0$. Indeed, in the case $p > 1$, taking $f(\cdot) = v^1$
 y and if $0 < \int_{|y| < R} v^{1-p'}(y) dy < \infty$, then

$$
\| |\cdot|^{\alpha - n} w(\cdot) \mathbf{1}_{2R < |\cdot|}(\cdot) \|_{L_{\alpha}^{\mathbb{P}^{\infty}}} \Big(\int_{|y| < R} v^{1-p'}(y) dy \Big)^{\frac{1}{p'}} \leq C
$$

which is nothing else than (3.2). This is obviously satisfied if $0 = \int_{|y| < R} v^{1-p'}(y) dy$. And the fact that $\int_{|y| < R} v^{1-p'}(y) dy < \infty$ is ensured by (4.2) or the hypothesis on $v(\cdot)$. Condition (3.3) (i.e. for $p = 1$) appears by taking $p \to 1$ in this last inequality, and since the constant $C > 0$ in (4.1) or (4.2) does not depend on p. Inequality (4.2) is a direct consequence of the boundedness *I,, : - wL,'* and at $\int_{|y| < R} v^{1-p'}(y) dy < \infty$

(i.e. for $p = 1$) appears l

int $C > 0$ in (4.1) or (4.2)

4.2) is a direct consequence
 $|x| > 2R$
 $x|^{\alpha-n} \int_{|y| < R} f(y) dy \le |x|^{\alpha}$

Inequality (4.2) is a direct consequence of the boundedness $I_{\alpha}: L_v^p \to wL_w^{p\infty}$ and the fact that for $|x| > 2R$

which is nothing else than (3.2). This is obviously satisfied if
$$
0 = \int_{|y| < R} v^{1-p'}(y)
$$
. And the fact that $\int_{|y| < R} v^{1-p'}(y) dy < \infty$ is ensured by (4.2) or the hypothesis on a Condition (3.3) (i.e. for $p = 1$) appears by taking $p \to 1$ in this last inequality, since the constant $C > 0$ in (4.1) or (4.2) does not depend on p . Inequality (4.2) is a direct consequence of the boundedness I_{α} : $L_v^p \to w L_v^p \infty$ the fact that for $|x| > 2R$ $|x|^{\alpha-n} \int_{|y| < R} f(y) dy \le |x|^{\alpha-n} \int_{\frac{1}{2}|x| < |x-y| < \frac{3}{2}|x|} f(y) dy$ $\leq 2^{n-\alpha} \int_{|x-y| < 2|x|} |x-y|^{\alpha-n} f(y) dy$ $\leq 2^{n-\alpha} (I_{\alpha}f)(x).$ Similarly as above, the main point for condition (3.2)^{*} is $\left(\int_{2R < |y|} |y|^{\alpha-n} g(y) dy \right) ||w(\cdot) 1_{|\cdot| < R}(\cdot) ||_{L_v^p} \leq C \left(\int_{\mathbb{R}^n} g^p(y) v(y) dy \right)^{\frac{1}{p}}.$ (4. For all $g(\cdot) \geq 0$ and $R > 0$. Setting $g(\cdot) = | \cdot |^{(\alpha-n)(p'-1)} v^{1-p'}(\cdot) 1_{\{2R < |x| < |R_1\}}(\cdot),$ then $\int_{2R < |y|} |y|^{\alpha-n} g(y) dy = \int_{2R < |y| < R_1} |y|^{\alpha-n} p' v^{1-p'}(y) dy$

Similarly as above, the main point for condition $(3.2)^*$ is

$$
-\int_{|x-y| < 2|x|} |x-y| < 2y \, dy
$$
\n
$$
\leq 2^{n-\alpha} (I_{\alpha}f)(x).
$$
\n1ly as above, the main point for condition (3.2)^{*} is

\n
$$
\left(\int_{2R < |y|} |y|^{\alpha-n} g(y) \, dy \right) \|w(\cdot) \mathbf{1}_{|x| < R}(\cdot) \|_{L_{\alpha}^p} \leq C \left(\int_{\mathbb{R}^n} g^p(y) \, v(y) \, dy \right)^{\frac{1}{p}}.
$$
\n(4.2)^{*}

$$
\int_{2R<|y|} |y|^{\alpha-n} g(y) \, dy = \int_{2R<|y|
$$

and

$$
R < |y|
$$
\n
$$
J_{2R < |y| < R_{1}}
$$
\n
$$
\int_{\mathbb{R}^{n}} g^{p}(y)v(y)dy = \int_{2R < |y| < R_{1}} |y|^{(\alpha - n)p'} v^{1 - p'}(y) dy
$$
\n
$$
< R^{(\alpha - n)p'} \int_{|y| < R_{1}} v^{1 - p'}(y) dy
$$
\n
$$
< \infty.
$$

So taking such a function $g(\cdot)$ in inequality (4.2)^{*} and assuming that

Weighted Weak-Type 1

$$
g(\cdot)
$$
 in inequality (4.2)^{*} and assumption

$$
\int_{2R<|y| 0
$$

it appears that

$$
\int_{2R<|y| 0
$$

$$
||w(\cdot)\mathbf{1}_{|\cdot|
$$

after letting $R_1 \rightarrow \infty$. This is condition $(3.2)^*$.

Inequality (4.2)^{*} is also a direct consequence of $I_{\alpha}: L_v^p \to wL_v^{p\infty}$ and the fact that for $|x| < R$.

the first term is given by:

\n
$$
\|w(\cdot)\mathbf{1}_{|\cdot|\nLet

\n
$$
\text{Hence}
$$
\n
$$

 $\mathit{Sufficient\ Part:}$ To get $I_{\boldsymbol{\alpha}}:\ L^{\boldsymbol{p}}_v\to wL^{\boldsymbol{p}\infty}_u$ first observe that for some constant $c>0$

$$
|y|^{\alpha - n} f(y) dy \le 2^{\alpha - \alpha} \int_{|x - y| < 2|y|} |x - y|^{\alpha - \alpha} f(y) dy \le 2^{\alpha - \alpha} (1 + \alpha) \int_{|x - y| < 2} |x - y|^{2\alpha} dx
$$
\n
$$
|F(x)| \le C \int_{\alpha}^{p} f(x) dx \le C \int_{\alpha}^{p} f(x) dx \quad \text{for all } f(\cdot) \ge 0
$$

where

$$
\infty. \text{ This is condition (3.2)}^{\bullet}.
$$
\n
$$
\bullet \text{ is also a direct consequence of } I_{\alpha}: L_{v}^{p} \to wL_{u}^{p}
$$
\n
$$
\bullet \text{ if } I_{v} \text{ is also a direct consequence of } I_{\alpha}: L_{v}^{p} \to wL_{u}^{p}
$$
\n
$$
\bullet \text{ if } I_{\alpha}: L_{v}^{p} \to wL_{u}^{p} \text{ for all } f(y) \text{ and } Y(y) \
$$

So it is sufficient to bound each S_i $(i \in \{1,2,3\})$ by $CA||f(\cdot)||_{L^p_\alpha}$ where C and A are non-negative constants which do not depend on $f(\cdot)$.

Since $\widetilde{I}_{\alpha}: L_v^p \to wL_u^{p\infty}$, then

$$
\mathcal{S}_2 = ||w(\cdot)(\widetilde{I}_{\alpha}f)(\cdot)||_{L_{\alpha}^{\mathcal{P}^{\infty}}} \le cA||f(\cdot)||_{L_{\alpha}^{\mathcal{P}}}.
$$

Here $A > 0$ is taken as the constant in the Hardy conditions (3.2) and (3.2)^{*}.

Arguing as in (4.2) and (4.2)^{*}, estimates of S_1 and S_2 are reduced to get the Hardy inequalities type

$$
\left\|w(\cdot)|\cdot|^{\alpha-n}\left(\int_{|y|<\frac{1}{2}|\cdot|}f(y)\,dy\right)\right\|_{L^{p,\infty}}\leq cA\big\|f(\cdot)\big\|_{L^{p}_{\nu}}\quad\text{for all }f(\cdot)\geq 0\tag{4.3}
$$

$$
\left\|w(\cdot)\Big(\int_{2|\cdot|<|y|}|y|^{\alpha-n}f(y)\,dy\Big)\right\|_{L^{p,\infty}_x} \le cA\big\|f(\cdot)\big\|_{L^p_v} \quad \text{for all } f(\cdot)\ge 0. \tag{4.3}.
$$

Since the arguments are similar, the proof is limited to that of (4.3).

One of the point keys is the inequality

126 Y. Rakotondratsimba
\nOne of the point keys is the inequality\n
$$
\|\sum_{k} f(\cdot) \mathbf{1}_{\mathcal{E}_{k}}(\cdot) \|_{L_{\alpha}^p}^p \leq \sum_{k} \|f(\cdot) \mathbf{1}_{\mathcal{E}_{k}}(\cdot) \|_{L_{\alpha}^p}^p
$$
\nwhere the \mathcal{E}_k 's are disjoint sets. This cutting summation is valid for $1 \leq p < \infty$ and

can be directly seen by using the definition of $\|\cdot\|_{L^p_u}$ and the fact that

$$
\mathcal{E}_k
$$
's are disjoint sets. This cutting summation is valid for $1 \leq p$
ectly seen by using the definition of $\| \cdot \|_{L^p_\epsilon^\infty}$ and the fact that

$$
\left\{ x \in \mathbb{R}^n \middle| \sum_k f(x) \mathbf{1}_{\mathcal{E}_k}(x) > \lambda \right\} = \left\{ x \in \bigcup_j \mathcal{E}_j \middle| \sum_k f(x) \mathbf{1}_{\mathcal{E}_k}(x) > \lambda \right\}
$$

$$
= \bigcup_j \left\{ x \in \mathcal{E}_j \middle| f(x) > \lambda \right\}.
$$

To prove (4.3) it can be assumed that $f(\cdot) \ge 0$ is a bounded function with compact support, since the further estimates do not depend on the bound of $f(.)$, and the monotone convergence theorem will yield the conclusion for general non-negative functions. Since $0 < \int_{\mathbb{R}^n} f(y) dy < \infty$, then $2^N < \int_{\mathbb{R}^n} f(y) dy \le 2^{N+1}$ for some integer *N*. By the fact that $r \in [0,\infty) \to \int_{|y| < \frac{1}{2}r} f(y) dy$ defines an increasing and continuous function, there $\left\{\sum_k f(x) \mathbf{1}_{\mathcal{E}_k}(x) > \lambda\right\} = \left\{x \in \bigcup_j \mathcal{E}_j \Big| \sum_k f(x) \mathbf{1}_{\mathcal{E}_k}(x) > \lambda\right\}$
 $= \bigcup_j \left\{x \in \mathcal{E}_j \Big| f(x) > \lambda\right\}.$

an be assumed that $f(\cdot) \ge 0$ is a bounded function with compact sup-

her estimates do not depend

$$
2^{m} = \int_{|y| < \frac{1}{2}r_m} f(y) \, dy = 2 \int_{\frac{1}{2}r_{m-1} < |y| < \frac{1}{2}r_m} f(y) \, dy \tag{4.5}
$$

and $2^N = \int_{|y| < \frac{1}{2}r_N} f(y) dy$. Let

that
$$
V \in [0, \infty) \rightarrow J_{|y| < \frac{1}{2}r} J(y) dy
$$
 defines an increasing and continuous function, there is an increasing sequence $(r_m)_{m=-\infty}^N$ of non-negative reals such that\n
$$
2^m = \int_{|y| < \frac{1}{2}r_m} f(y) dy = 2 \int_{\frac{1}{2}r_{m-1} < |y| < \frac{1}{2}r_m} f(y) dy \qquad (4.5)
$$
\nand
$$
2^N = \int_{|y| < \frac{1}{2}r_N} f(y) dy
$$
. Let\n
$$
E_m = \left\{ x \in \mathbb{R}^n \mid 2^m < \int_{|y| < \frac{1}{2}z} f(y) dy \le 2^{m+1} \right\} \qquad (m \leq N-1). \qquad (4.6)
$$
\n
$$
= \left\{ x \in \mathbb{R}^n \mid r_m < |x| \leq r_{m+1} \right\}
$$
\nSetting $r_{N+1} = \infty$, then\n
$$
\text{the } E_k \text{ are pairwise disjoint sets and } \mathbb{R}^n = \bigcup_{m=-\infty}^N E_m. \qquad (4.7)
$$
\nNow we are ready to give the chain of estimates which yields inequality (4.3). Indeed,\n
$$
\prod_{m=1}^N E_m = \
$$

Setting $r_{N+1} = \infty$, then

the
$$
E_k
$$
 are pairwise disjoint sets and $\mathbb{R}^n = \bigcup_{m=-\infty}^N E_m$. (4.7)

Now we are ready to give the chain of estimates which yields inequality (4.3). Indeed,

ing
$$
r_{N+1} = \infty
$$
, then
\nthe E_k are pairwise disjoint sets and $\mathbb{R}^n = \bigcup_{m=-\infty}^N E_m$.
\n
\n*(v we are ready to give the chain of estimates which yields inequality (4.3). Inde $\left\|w(\cdot)\right| \cdot \left|^{\alpha-n} \left(\int_{|y| < \frac{1}{2}| \cdot |} f(y) \, dy\right) \right\|_{L^{\infty}_{\infty}}^p$
\n
$$
= \left\| \sum_{m=-\infty}^N w(\cdot) \right| \cdot \left|^{\alpha-n} \left(\int_{|y| < \frac{1}{2}| \cdot |} f(y) \, dy\right) \mathbf{1}_{E_m}(\cdot) \right\|_{L^{\infty}_{\infty}}^p \quad (by (4.7))
$$

\n
$$
\leq c_1 \sum_{m=-\infty}^N 2^{mp} \left\|w(\cdot)\right| \cdot \left|^{\alpha-n} \mathbf{1}_{E_m}(\cdot) \right\|_{L^{\infty}_{\infty}}^p \quad (by (4.4) \text{ and } (4.6))
$$

\n
$$
\leq c_2 \sum_{m=-\infty}^N \left(\int_{\{\frac{1}{2}r_{m-1} < |\cdot| \leq \frac{1}{2}r_m\}} f(y) \, dy\right)^p \left\|w(\cdot)| \cdot \left|^{\alpha-n} \mathbf{1}_{\{r_m < |\cdot| \leq r_{m+1}\}}(\cdot) \right\|_{L^{\infty}_{\infty}}^p
$$

\n
$$
(by (4.5))
$$*

Weighted Weak-Type Inequalities for
$$
I_{\alpha}
$$
\n
$$
\leq c_2 \sum_{m=-\infty}^{N} \left(\int_{\left\{\frac{1}{2}r_{m-1} < |\cdot| \leq \frac{1}{2}r_m\right\}} f^p(y)v(y) \, dy \right)
$$
\n
$$
\times \left(\int_{\left\{\left| |\cdot| \leq \frac{1}{2}r_m\right\}} v^{1-p'}(y) \, dy \right)^{\frac{p}{p'}} \left\| w(\cdot) |\cdot|^{o-n} 1_{\left\{2\left(\frac{1}{2}r_m\right) < |\cdot|\right\}} (\cdot) \right\|_{L^p_{\alpha}}^p
$$
\n(here the Hölder inequality is applied if $p > 1$

\nand for $p = 1$:

\n
$$
\left(\int_{\left\{\left| |\cdot| \leq \frac{1}{2}r_m\right\}} v^{1-p'}(y) \, dy \right)^{\frac{p}{p'}} \text{ is replaced by essay} \left(\frac{1}{v(1)} 1_{\left\{ |\cdot| \leq \frac{1}{2}r_m\right\}} (\cdot) \right) \right)
$$
\n
$$
\leq c_2 A^p \sum_{m=-\infty}^{N} \int_{\left\{\frac{1}{2}r_{m-1} < |\cdot| \leq \frac{1}{2}r_m\right\}} f^p(y)v(y) \, dy
$$
\n(since the sets $\left\{\frac{1}{2}r_{m-1} < |\cdot| \leq \frac{1}{2}r_m\right\} (y) \right\} f^p(y)v(y) \, dy$

\n
$$
\left(\text{since the sets } \left\{\frac{1}{2}r_{m-1} < |\cdot| \leq \frac{1}{2}r_m\right\} \text{ are disjoint}\right)
$$
\n
$$
\leq c_2 A^p \int_{\mathbb{R}^n} f^p(y)v(y) \, dy
$$
\n
$$
= c_2 A^p \left\| f(\cdot) \right\|_{L^p_{\alpha}}^p.
$$
\nProof of Proposition 3.2. It is suitable to introduce the notations

\n
$$
E_k = \left\{ y \in \mathbb{R}^n | 2^k < |y| \leq 2^{k+1} \right\}
$$
\n

Proof of Proposition 3.2. It is suitable to introduce the notations

$$
\leq c_2 A^p \int_{\mathbb{R}^n} f^p(y) v(y) dy
$$
\n
$$
= c_2 A^p ||f(\cdot)||_{L_v^p}^p.
$$
\nProof of Proposition 3.2. It is suitable to introduce the notations\n
$$
E_k = \{y \in \mathbb{R}^n | 2^k < |y| \leq 2^{k+1} \} \quad \text{and} \quad F_k = \{y \in \mathbb{R}^n | 2^{k-1} < |y| \leq 2^{k+2} \}
$$
\n
$$
W_k = \sup_{z \in E_k} w(z) \quad \text{and} \quad U_k = \sup_{y \in E_k} u(y).
$$
\nusing the property $w(\cdot) \in \mathcal{A}$, then\n
$$
W_k \leq \sup_{4^{-1} |z| < |z| \leq 4|x|} w(z) \leq c_1 w(x) \quad \text{for each } x \in F_k.
$$
\n(4.8)\ne\n
$$
c_1 > 0 \text{ is a constant which only depends on the fact that } w(\cdot) \in \mathcal{A}. \text{ Similarly,}
$$
\n
$$
U_k \leq \sup_{4^{-1} |z| < |z| \leq 4|x|} u(z) \quad \text{for each } x \in F_k.
$$
\n(4.9)\ne\n
$$
e_1 \text{ for the crucial points to get } \widetilde{I}_{\alpha}: L_v^p \to w L_v^{p\infty} \text{ is the inequality}
$$

So using the property $w(\cdot) \in \mathcal{A}$, then

I

IL

N

$$
\mathcal{W}_k \leq \sup_{4^{-1}\vert x\vert < \vert x\vert < 4\vert x\vert} w(z) \leq c_1 w(x) \quad \text{for each } x \in F_k. \tag{4.8}
$$

Here $c_1 > 0$ is a constant which only depends on the fact that $w(\cdot) \in \mathcal{A}$. Similarly,

$$
\mathcal{W}_k = \sup_{z \in E_k} w(z) \quad \text{and} \quad \mathcal{U}_k = \sup_{y \in E_k} u(y).
$$
\n
$$
\mathcal{W}(\cdot) \in \mathcal{A}, \text{ then}
$$
\n
$$
\sup_{4^{-1}|z| < |z| < 4|z|} w(z) \leq c_1 w(x) \quad \text{for each } x \in F_k. \tag{4.8}
$$
\n
$$
\text{tant which only depends on the fact that } w(\cdot) \in \mathcal{A}. \text{ Similarly,}
$$
\n
$$
\mathcal{U}_k \leq \sup_{4^{-1}|z| < |z| < 4|z|} u(z) \quad \text{for each } x \in F_k. \tag{4.9}
$$

Here
$$
c_1 > 0
$$
 is a constant which only depends on the fact that $w(\cdot) \in A$. Similarly,
\n
$$
\mathcal{U}_k \le \sup_{4^{-1} |x| < |z| < 4|x|} u(z) \quad \text{for each } x \in F_k.
$$
\n(4.9)
\nOne of the crucial points to get $\tilde{I}_{\alpha}: L_v^p \to w L_v^{p\infty}$ is the inequality
\n
$$
\int_{E_k} (I_{\alpha} f 1_{F_k})^p(x) |x|^{-\alpha p} dx \le C \int_{F_k} f^p(x) dx \quad \text{for all } f(\cdot) \ge 0
$$
\n(4.10)
\nwhere $C > 0$ is a fixed constant. Inequality (4.10) is true for $p = 1$ since
\n
$$
\int_{E_k} (I_{\alpha} f 1_{F_k})(x) |x|^{-\alpha} dx \sim 2^{-k\alpha} \int_{E_k} (I_{\alpha} f 1_{F_k}(x) dx)
$$

where $C > 0$ is a fixed constant. Inequality (4.10) is true for $p = 1$ since

$$
\int_{E_k} (I_{\alpha} f 1_{F_k})(x) |x|^{-\alpha} dx \sim 2^{-k\alpha} \int_{E_k} (I_{\alpha} f 1_{F_k}(x) dx
$$

128 Y. Rakotondratsimba $\begin{array}{c} \hline \end{array}$

$$
=2^{-k\alpha}\int_{F_k}f(x)(I_{\alpha}1_{E_k})(x)\,dx\leq c2^{-k\alpha}\int_{F_k}f(x)2^{k\alpha}dx=c\int_{F_k}f(x)\,dx.
$$

For $p > 1$, (4.10) is also true since $I_{\alpha}: L^p(dx) \to L^p(|x|^{-\alpha p} dx)$. This last boundedness can be seen by applying one of the well-known boundedness criteria for I_{α} on weighted L^p -spaces (see [5]). Y. Rakotondratsimba
 $= 2^{-k\alpha} \int_{F_k} f(x)(I_{\alpha}1_{E_k})(x) dx \le c2^{-k\alpha} \int_{F_k} f(x)2^{k\alpha} dx = c \int_{F_k} f(x) dx$.
 $p > 1$, (4.10) is also true since $I_{\alpha} : L^p(dx) \to L^p(|x|^{-\alpha p} dx)$. This last boundedness

be seen by applying one of the well-known

as follows: Now assuming hypothesis (3.4), the boundedness $\widetilde{I}_{\alpha}: L_v^p \to wL_u^{p\infty}$ can be obtained

$$
= 2^{-k\alpha} \int_{F_k} f(x) (I_0 I_{E_k})(x) dx \leq c2^{-k\alpha} \int_{F_k} f(x) 2^{k\alpha} dx = c \int_{F_k} f(x) dx.
$$

\n>1, (4.10) is also true since $I_{\alpha}: L^p(dx) \to L^p(|x|^{-\alpha p} dx)$. This last bounded
\ne seen by applying one of the well-known boundedness criteria for I_{α} on weight
\naxes (see [5]).
\now assuming hypothesis (3.4), the boundedness $\tilde{I}_{\alpha}: L_v^p \to wL_w^{p\infty}$ can be obtain
\n*w*(·)($\tilde{I}_{\alpha}f$)(·)|| $\tilde{I}_{L_v^{p\infty}}$
\n
$$
= \Big\|\sum_k w(\cdot) \Big(\int_{\frac{1}{2}|\cdot|<|y|<2|\cdot|} |\cdot-y|^{\alpha-n} f(y) dy\Big) \mathbf{1}_{E_k}(\cdot) \Big\|_{L_v^{p\infty}}^p
$$
\n
$$
\leq \sum_k W_k^p \mathcal{U}_k \Big\| (I_{\alpha}f \mathbf{1}_{F_k})(\cdot) \mathbf{1}_{E_k}(\cdot) \Big\|_{L_v^{p\infty}}^p
$$
\n(*by the definition of* ||*g*(*y*) $\tilde{I}_{E_k} \sim$)
\n
$$
\leq c_2 \sum_k 2^{\alpha k} W_k^p \mathcal{U}_k \int_{E_k} (I_{\alpha}f \mathbf{1}_{F_k})^p(x) |x|^{-\alpha p} dx
$$
\n(*recall that* $L_i^p \subset L_i^{\infty}$)
\n
$$
\leq c_2 \sum_k 2^{\alpha k} W_k^p \mathcal{U}_k \int_{F_k} f^p(x) dx \quad (see (4.10))
$$

\n
$$
\leq 2^{\alpha} c_3 \sum_k \int_{F_k} f^p(x) \Big[W_k |x|^{\alpha} U_k^{\frac{1}{k}} \Big]^p dx
$$

\n
$$
\leq c_4 \sum_k \int_{F_k} f^p(x) \Big[w(x) |x|^{\alpha} \Big(\sup_{4^{-1}|x|<|<|<|<|x|} u(z)\Big)^{\frac{1}{p}} \Big]^p dx \quad (
$$

If instead of (3.4) condition (3.4)' is assumed, then the modifications in obtaining the conclusion are as follows:

ц.

$$
\|w(\cdot)(I_{\alpha}f)(\cdot)\|_{L_{\epsilon}^p\infty}^p
$$

\n
$$
\leq c_5 \sum_{k} \int_{F_k} f^p(x) \Big[W_k 2^{\alpha k} U_k^{\frac{1}{p}} \Big]^p dx \quad \text{(see the above estimates)}
$$

\n
$$
\leq c_6 \sum_{k} \int_{F_k} f^p(x) \sup_{y \in E_k} \Big\{ W_k |y|^{\alpha} \Big(u(y)\Big)^{\frac{1}{p}} \Big\}^p dx
$$

Weighted Weak-Type Inequalities for
$$
I_{\alpha}
$$
 129
\n
$$
\leq c_7 \sum_{k} \int_{F_k} f^p(x) v(x) \sup_{y \in E_k} \left\{ w(y) |y|^{\alpha} (u(y))^{\frac{1}{p}} \left(\frac{1}{v(x)} \right)^{\frac{1}{p}} \right\}^p dx
$$
\n
$$
\leq c_7 \sum_{k} \int_{F_k} f^p(x) v(x) \sup_{y \in E_k} \left\{ w(y) |y|^{\alpha} (u(y))^{\frac{1}{p}} \left(\sup_{4^{-1} |y| < |x| < |x|} \frac{1}{v(x)} \right)^{\frac{1}{p}} \right\}^p dx
$$
\n
$$
\leq c_8 A^p \sum_{k} \int_{F_k} f^p(x) v(x) dx \quad (by condition (3.4)')
$$
\n
$$
\leq 3c_8 A^p \int_{\mathbb{R}^n} f^p(y) v(y) dy
$$
\n
$$
= 3c_8 A^p ||f(\cdot)||_{L^p_v}^p.
$$
\nIf no assumption like $w(\cdot) \in \mathcal{A}$ is assumed, the boundedness $\tilde{I}_{\alpha} : L^p_v \to w L^p_w$ can be similarly obtained if instead of (4.10) then\n
$$
\int_{\alpha} (I_{\alpha} f_1) v(x) \cdot w(x) \cdot \int_{\alpha}^p f(x) \cdot w(x) \cdot \int_{\alpha}^p f(x) \cdot \int_{\alpha}^p f
$$

similarly obtained if instead of (4.10) then

larry obtained it instead of (4.10) then
\n
$$
\int_{E_k} (I_{\alpha} f 1_{F_k})^p(x) w^p(x) |x|^{-\alpha p} dx \leq C \int_{\sqrt{F_k}} f^p(x) dx \quad \text{for all } f(\cdot) \geq 0
$$
\nch is true whenever $I_{\alpha}: L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^n, w^p(x) |x|^{-\alpha p} dx)$.

\nProof of Proposition 3.3. To prove the implication (3.5)' \implies (3.4), take $p > 1$.

\nlog $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H}$ and $w(\cdot) \in \mathcal{A}$, then

which is true whenever $I_{\alpha}: L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^n, w^p(x) |x|^{-\alpha p} dx).$

 $\text{Using } u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H} \text{ and } w(\cdot) \in \mathcal{A}, \text{ then}$ $\mathbb{R}^n, dx \to C$
 1 To prove the
 \Rightarrow **1** To prove the
 \Rightarrow **1** \Rightarrow **1**

$$
\leq 3c_8A^p \int_{\mathbb{R}^n} f^p(y) v(y) dy
$$
\n
$$
= 3c_8A^p ||f(\cdot)||_{L_c^p}^p.
$$
\nfor no assumption like $w(\cdot) \in A$ is assumed, the boundedness $\tilde{I}_{\alpha}: L_v^p \to wL_v^p \infty$ can
\nsimilarly obtained if instead of (4.10) then
\n
$$
\int_{E_k} (I_{\alpha} f 1_{F_k})^p(x) w^p(x) |x|^{-\alpha p} dx \leq C \int_{\sqrt{F_k}} f^p(x) dx \quad \text{for all } f(\cdot) \geq 0
$$
\nwhich is true whenever $I_{\alpha}: L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^n, w^p(x) |x|^{-\alpha p} dx).$

\nProof of Proposition 3.3. To prove the implication (3.5)' \implies (3.4), take $p >$
\nUsing $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H}$ and $w(\cdot) \in A$, then

\n
$$
w(x)|x|^{\alpha} \left(\sup_{4^{-1}|x| < |y| < 4|x|} u(y)\right)^{\frac{1}{p}} \left(v^{1-p'}(x)\right)^{\frac{1}{p'}} \approx c_1 w(x)|x|^{\alpha-n} \left(\int_{2^{-N}|x| < |y| < 2^N|x|} u(y) dy\right)^{\frac{1}{p}} \left(\int_{2^{-N}|x| < |y| < 2^N|x|} v^{1-p'}(y) dy\right)^{\frac{1}{p'}}
$$
\n
$$
= c_2 w(x)|x|^{\alpha-n} \|1_{2^{-N}|x| < |x| < 2^N|x|} (\cdot) \|_{L_{\alpha}^p} \left(\int_{2^{-N}|x| < |y| < 2^N|x|} v^{1-p'}(y) dy\right)^{\frac{1}{p'}}
$$
\n
$$
\leq c_3 |x|^{\alpha-n} \|w(\cdot)1_{2^{-N}|x| < |x| < 2^N|x|} (\cdot) \|_{L_{\alpha}^p} \left(\int_{2^{-N}|x| < |y| < 2^N|x|} v^{1-p'}(y) dy\right)^{\frac{1}{p'}}
$$
\n
$$
\leq c_3 |x|^{\alpha-n} \|w(\cdot)1_{2
$$

The implication $(3.6)' \implies (3.4)$ can be proved by using a similar argument, except that no growth condition on *v(.)* is needed. Therefore Proposition 3.3 is proved since $\leq c_3 A$ (by using condition (3.5)').

The implication (3.6)' \implies (3.4) can be prove

that no growth condition on $v(\cdot)$ is needed. The

trivially (3.5) \implies (3.5)' and (3.6) \implies (3.6)'.
 Proof of Proposition 3.4. trivially (3.5) \implies (3.5)' and (3.6) \implies (3.6)'.
Proof of Proposition 3.4. Part A: Since $w(x) = |x|^{-\gamma}$ for $\gamma \ge 0$, then {x :

 $w(x)|x|^{\alpha-n} > \lambda$ } = { $x : |x| < \lambda^{\frac{1}{\alpha-n-1}}$ } and

$$
||w(\cdot)| \cdot |^{\alpha - n} \mathbf{1}_{2R<|\cdot|}(\cdot)||_{L_{\nu}^{\mathbf{p}\infty}} = \sup_{r>>2R} \left\{ \tau^{(\alpha - n - \gamma)} \Biggl(\int_{2R<|x|

$$
= \sup_{r>>2R} \left\{ \tau^{(\alpha - n - \gamma)} ||\mathbf{1}_{2R<|\cdot|

$$
\leq \sup_{r>>2R} \left\{ \tau^{(\alpha - n)} ||\cdot|^{-\gamma} \mathbf{1}_{2R<|\cdot|

$$
\leq \sup_{r>>2R} \left\{ \tau^{(\alpha - n)} ||w(\cdot)\mathbf{1}_{|\cdot|
$$
$$
$$
$$

The implications (3.5) \implies (3.2) and (3.6) \implies (3.3) appear immediately from this last inequality.

condition (3.2) is satisfied since

30 Y. Rakotondratsimba
\nThe implications (3.5)
$$
\implies
$$
 (3.2) and (3.6) \implies (3.3) appear immediately from this
\nast inequality.
\nPart B: Assuming the Muchenhoupt condition (3.5) and $v^{1-p'}(\cdot) \in RD_{\rho}$, the Hardy
\ncondition (3.2) is satisfied since
\n
$$
||w(\cdot)| \cdot |^{\alpha-n} \mathbf{1}_{2R<|\cdot|}(\cdot) ||^p_{L^{\infty}_u} \left(\int_{|y|\n
$$
\leq c_2 \sum_{k=1}^{\infty} 2^{-kn(p-1)\rho} (2^{k+1} R)^{(\alpha-n)p} ||w(\cdot)\mathbf{1}_{|\cdot|<2^{k+1}R}(\cdot) ||^p_{L^{\infty}_u} \left(\int_{|y|<2^{k+1}R} v^{1-p'}(y) dy \right)^{\frac{p}{p'}}
$$
\n(since $v^{1-p'}(\cdot) \in RD_{\rho}$)
\n
$$
\leq c_2 A^p \sum_{k=1}^{\infty} 2^{-kn(p-1)\rho}
$$

\n
$$
= c_3 A^p
$$
 (by condition (3.5) and p>1).
$$

Proof of Theorem 3.5. Part A: The necessary part is essentially described in Theorem 3.1. The sufficient part is a consequence of Proposition 3.3 and Theorem 3.1.

Part B: In view of Proposition 3.3 and Theorem 3.1, to prove $I_{\alpha}: L_v^p \to L_v^{p\infty}$ it remains to check condition (3.5)' by using (3.2) or (3.2)^{*}. Note that $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H}$ by Lemma 2.3. Suppose for instance that $v^{1-p'}(.)$ /. Then

$$
\sum_{z_2 A^p} \sum_{k=1}^{\infty} 2^{-kn(p-1)\rho}
$$
\n
$$
\sum_{z_3 A^p} (by condition (3.5) and p>1).
$$
\n
$$
\sum_{z_3 A^p} (by condition (3.5) and p>1).
$$
\n
$$
\sum_{z_3 A^p} (by condition (3.5) and p>1).
$$
\n
$$
\sum_{z_3 A^p} (by condition (3.5) and p>1).
$$
\n
$$
\sum_{z_3 A^p} (by condition (3.5) and p>1).
$$
\n
$$
\sum_{z_3 A^p} (by condition (3.5) and p>1).
$$
\n
$$
\sum_{z_3 A^p} (by condition (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\sum_{z_3 A^p} (by equation (3.5) by using (3.2) or (3.2)^*.
$$
\n
$$
\
$$

Consequently,

$$
\leq c_2 R^{-(\alpha-n)p'} \Big(\int_{2(2^N R) < |y|} |y|^{(\alpha-n)p'} v^{1-p'}(y) \, dy \Big)
$$
\nently,

\n
$$
Q(R) = R^{\alpha-n} \|w(\cdot)1_{R < |x| \leq 2^{2N} R} (\cdot) \|_{L^p_\alpha} \Big(\int_{R < |y| < 2^{2N} R} v^{1-p'}(y) \, dy \Big)^{\frac{1}{p'}} \Big|
$$
\n
$$
\leq c_3 \|w(\cdot)1_{|\cdot| < 2^{2N} R} (\cdot) \|_{L^p_\alpha} \Big(\int_{2(2^{2N} R) < |y|} |y|^{(\alpha-n)p'} v^{1-p'}(y) \, dy \Big)^{\frac{1}{p'}} \Big|
$$
\n
$$
\leq c_3 A \quad \text{(by condition (3.2)^*)}.
$$

For $v^{1-p'}(\cdot) \searrow$ then $\int_{R < |y| < 2^N R} v^{1-p'}(y) dy \le c_4 \int_{|y| < \frac{1}{2}R} v^{1-p'}(y) dy$ and hence, by using condition (3.2),
 $Q(R) \le c_5 ||w(\cdot)| \cdot |^{\alpha - n} 1_{2(\frac{1}{2}R) < |\cdot|}(\cdot) ||_{L_u^{p_\infty}} \left(\int_{|y| < \frac{1}{2}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \le c_5 A.$ condition (3.2),

$$
\leq c_3 A \quad (by condition (3.2)^*).
$$
\n
$$
p'(\cdot) \searrow \text{ then } \int_{R < |y| < 2^N R} v^{1-p'}(y) dy \leq c_4 \int_{|y| < \frac{1}{2}R} v^{1-p'}(y) dy \text{ and hence, } 0
$$
\n
$$
Q(R) \leq c_5 \|w(\cdot)| \cdot \left| \alpha^{-n} \mathbf{1}_{2(\frac{1}{2}R) < |\cdot|}(\cdot) \right\|_{L_v^p} \left(\int_{|y| < \frac{1}{2}R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_5 A.
$$

Part C: This statement is an immediate consequence of Proposition *3.4.*

Proof of Theorem 3.6. The arguments are the same as those used in the proof of Theorem *3.5.*

Proof of Theorem 2.1. The necessity part is immediately given by parts A in Theorems *3.5* and *3.6.* The sufficient part can be seen by applying Theorem *3.1* and Part A in Proposition *3.4.* Part C: This statement is a
 Proof of Theorem 3

of Theorem 3.5.
 Proof of Theorem 2

Theorems 3.5 and 3.6. Theorems 3.5 and 3.6. Theorems 3.4.
 Proof of Proposition
 $w(\cdot) = 1$.
 Proof of Lemma 2.3.

First conside in immediate consequence of Proposition 3.4.

6. The arguments are the same as those used in the proof

1.1. The necessity part is immediately given by parts A in

2.2. This result is just a statement of Proposition 3.2 w of Theorem 3.6. The arguments are the same as those used in 3.5.

of Theorem 2.1. The necessity part is immediately given by

3.5 and 3.6. The sufficient part can be seen by applying Theor

Proposition 3.4.

of Propositio 1. The necessity part is immediately g
sufficient part can be seen by applyin
2.2. This result is just a statement of **F**
It remains to estimate $w(y)$ for each y v
re *R* is small, i.e. $R < \frac{1}{8}R_0$. Since 8*R*
 $\frac{c}{R^n$

Proof of Proposition 2.2. This result is just a statement of Proposition *3.2* with $w(\cdot)=1.$

Proof of Lemma 2.3. It remains to estimate $w(y)$ for each y with $\frac{1}{4}R < |y| < 4R$. First consider the case where *R* is small, i.e. $R < \frac{1}{8}R_0$. Since $8R < R_0$, for $w_1(\cdot)$ 1. The necessity part is immed
sufficient part can be seen by
2.2. This result is just a state:
It remains to estimate $w(y)$ for
re R is small, i.e. $R < \frac{1}{8}R_0$.
 $R_n = \frac{c}{R^n} \int_{4R < |z| < \frac{1}{4}R} w_1(z) dz = \frac{c}{R^n} \int_{\frac{1}{8}$ **composition 3.4.**
 of Proposition 2.2. This rest
 of Lemma 2.3. It remains to the case where *R* is small,
 $w(y) = w_1(y) \leq \frac{c}{R^n} \int_{4R < |z| < 8R}$
 $w(y) = w_1(y) \leq \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < \frac{1}{4}R}$

if *R* is big, i.e. alt is just a statement

estimate $w(y)$ for each

i.e. $R < \frac{1}{8}R_0$. Since
 $w_1(z) dz = \frac{c}{R^n} \int_{4R < |z|}$
 $w_1(z) dz = \frac{c}{R^n} \int_{\frac{1}{8}R < |z|}$

ith $w_2(\cdot) \nearrow$ or $w_2(\cdot)$ Theorems 3.5 and 3.6. The sufficient part can be seen by applying Theorem 3.1 and

Proof of Proposition 3.4.
 $w(\cdot) = 1$.

Proof of Deposition 2.2. This result is just a statement of Proposition 3.2 with
 $w(\cdot) = 1$.

Proof

$$
w(y) = w_1(y) \leq \frac{c}{R^n} \int_{4R < |z| < 8R} w_1(z) \, dz = \frac{c}{R^n} \int_{4R < |z| < 8R} w(z) \, dz.
$$

And for $w_1(\cdot) \searrow$ then

$$
w(y) = w_1(y) \leq \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < \frac{1}{4}R} w_1(z) \, dz = \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < \frac{1}{4}R} w(z) \, dz.
$$

$$
w(y) = w_1(y) \le \frac{c}{R^n} \int_{4R < |z| < 8R} w_1(z) \, dz = \frac{c}{R^n} \int_{4R < |z| < 8R} w(z) \, dz.
$$
\nAnd for $w_1(\cdot) \setminus \text{ then}$ \n
$$
w(y) = w_1(y) \le \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < \frac{1}{4}R} w_1(z) \, dz = \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < \frac{1}{4}R} w(z) \, dz.
$$
\nSimilarly, if R is big, i.e. $8R_0 < R$, and with $w_2(\cdot) \nearrow$ or $w_2(\cdot) \searrow$ then\n
$$
w(y) = w_2(y) \le \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < 8R} w_2(z) \, dz = \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < 8R} w(z) \, dz.
$$
\nFinally, for $R \approx R_0$, i.e. $\frac{1}{8}R_0 \le R \le 8R_0$, then $w(y) \le C$ for a fixed const

Finally, for $R \approx R_0$, i.e. $\frac{1}{8}R_0 \leq R \leq 8R_0$, then $w(y) \leq C$ for a fixed constant $C > 0$
which depends only on $w(\cdot)$. Assume for instance that $\frac{1}{8}R_0 \leq R \leq R_0$. If $w_1(\cdot) \nearrow$, then
 $w(y) \leq \frac{c}{w_1(\frac{1}{32}R$

$$
w(y) = w_1(y) \le \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < \frac{1}{4}R} w_1(z) dz = \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < \frac{1}{4}R} w(z) dz.
$$
\nIf R is big, i.e. 8R₀ < R, and with $w_2(\cdot) \nearrow$ or $w_2(\cdot) \searrow$ then\n
$$
w(y) = w_2(y) \le \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < 8R} w_2(z) dz = \frac{c}{R^n} \int_{\frac{1}{8}R < |z| < 8R} w(z) dz.
$$
\nr $R \approx R_0$, i.e. $\frac{1}{8}R_0 \le R \le 8R_0$, then $w(y) \le C$ for a fixed const\nend so only on $w(\cdot)$. Assume for instance that $\frac{1}{8}R_0 \le R \le R_0$. If $w_1(w(y)) \le \frac{c}{w_1(\frac{1}{32}R_0)R^n} \int_{\frac{1}{4}R < |z| < R} w_1(z) dz \le \frac{C}{R^n} \int_{\frac{1}{4}R < |z| < 4R} w(z) dz.$ \n
$$
w_1(\cdot) \searrow
$$
, then\n
$$
w(y) \le \frac{c}{w_1(R_0)R^n} \int_{\frac{1}{4}R < |z| < R} w_1(z) dz \le \frac{C}{R^n} \int_{\frac{1}{4}R < |z| < 4R} w(z) dz.
$$
\n
$$
dy
$$
, for $R_0 \le R < 8R_0$ and $w_2(\cdot) \nearrow$ or $w_2(\cdot) \searrow$, then\n
$$
w(y) \le \frac{C}{R^n} \int_{R < |z| < 4R} w_2(z) dz \le \frac{C}{R^n} \int_{\frac{1}{8}R < |z| < 8R} w(z) dz.
$$
\n
$$
f
$$
 of Proposition 2.4. Proposition 2.4 is just a consequence of 1\n
$$
y(\cdot) = 1.
$$

And when $w_1(\cdot) \setminus$, then

$$
w(y) \leq \frac{c}{w_1(\frac{1}{32}R_0)R^n} \int_{\frac{1}{4}R < |z| < R} w_1(z) \, dz \leq \frac{C}{R^n} \int_{\frac{1}{4}R < |z| < 4R} w(z) \, dz
$$
\n
$$
w_1(\cdot) \searrow, \text{ then}
$$
\n
$$
w(y) \leq \frac{c}{w_1(R_0)R^n} \int_{\frac{1}{4}R < |z| < R} w_1(z) \, dz \leq \frac{C}{R^n} \int_{\frac{1}{4}R < |z| < 4R} w(z) \, dz.
$$
\n
$$
y, \text{ for } R_0 \leq R < 8R_0 \text{ and } w_2(\cdot) \nearrow \text{ or } w_2(\cdot) \searrow, \text{ then}
$$
\n
$$
w(y) \leq \frac{C}{R^n} \int_{R < |z| < 4R} w_2(z) \, dz \leq \frac{C}{R^n} \int_{\frac{1}{8}R < |z| < 8R} w(z) \, dz.
$$
\nof Proposition 2.4. Proposition 2.4 is just a consequence of

Analogously, for $R_0 \leq R < 8R_0$ and $w_2(\cdot) \nearrow$ or $w_2(\cdot) \searrow$, then

$$
w(y) \leq \frac{C}{R^n}\int_{R<|z|<4R}w_2(z)\,dz \leq \frac{C}{R^n}\int_{\frac{1}{8}R<|z|<8R}w(z)\,dz.
$$

Proof of Proposition 2.4. Proposition *2.4 is* just a consequence of Proposition 3.3 with $w(\cdot) = 1$.

Proof of Proposition 2.5. The fact that the Muckenhoupt condition (2.1) implies the dual Hardy condition (2.2) for $u(\cdot) \in RD_{\rho}$ can be seen as in the proof of part B of 132 Y. Rakotondratsimba
Lemma 2.6 (see also $(5: p: 832 - 833)$). The implication $(2.3) \implies (2.4)$ is also true for the same hypothesis on $u(\cdot)$. Indeed,

132 Y. Rakotondratsimba
\nLemma 2.6 (see also [5: p: 832 - 833]). The implication (2.3)
$$
\implies
$$
 (2.4) is also true
\nthe same hypothesis on $u(.)$. Indeed,
\n
$$
\left(\int_{|x|\n
$$
\leq c_1 \sum_{k=1}^{\infty} (2^{k+1}R)^{\alpha-n} \left(\int_{|x|\n
$$
\leq c_2 \sum_{k=1}^{\infty} 2^{-kn\rho} (2^{k+1}R)^{\alpha-n} \left(\int_{|x|<2^{k+1}R} u(x)dx\right) \times \operatorname{ess} \sup\left[\frac{1}{v(y)} 1_{|\cdot|\leq 2^{k+1}R}(y)\right]
$$
\n
$$
\leq c_2 A \sum_{k=1}^{\infty} 2^{-kn\rho}
$$
\n
$$
= c_3 A.
$$
$$
$$

Proof of Theorem 2.6. Part A: The sufficient part is just a consequence of Theorem *2.1* and Proposition *2.4.* Part B is true by Proposition *2.4.* For Part C it is sufficient to apply Proposition *2.5.*

Proof of Theorem 2.7. It is sufficient to follow the same argument as in Theorem *2.6.*

Proof of Corollary 2.8. Recall that

$$
\text{ry 2.8. Recall that}
$$
\n
$$
u(x) = \begin{cases} u_1(x) = |x|^{(\beta - n)} & \text{for } |x| < R_0 \\ u_2(x) = |x|^{\gamma - n} & \text{for } |x| > R_0 \end{cases}
$$

and

to apply Proposition 2.5.
\nProof of Theorem 2.7. It is sufficient to follow the same argument
\n2.6.
\nProof of Corollary 2.8. Recall that
\n
$$
u(x) = \begin{cases} u_1(x) = |x|^{(\beta-n)} & \text{for } |x| < R_0 \\ u_2(x) = |x|^{\gamma-n} & \text{for } |x| > R_0 \end{cases}
$$
\nand
\n
$$
\sigma(x) = \begin{cases} v_1^{-\frac{1}{p-1}}(x) = |x|^{-n} \ln^{-p'}(|x|^{-1}) & \text{for } |x| < R_0 \\ v_2^{-\frac{1}{p-1}}(x) = |x|^{-\frac{\beta-n}{p-1}} & \text{for } |x| > R_0. \end{cases}
$$
\nThe dual Hardy condition (2.2) is satisfied when

$$
u(x) = \begin{cases} u_1(x) = |x|^{(\beta - n)} & \text{for } |x| < R_0 \\ u_2(x) = |x|^{\gamma - n} & \text{for } |x| > R_0 \end{cases}
$$
\n
$$
\sigma(x) = \begin{cases} v_1^{-\frac{1}{p-1}}(x) = |x|^{-n} \ln^{-p'}(|x|^{-1}) & \text{for } |x| < R_0 \\ v_2^{-\frac{1}{p-1}}(x) = |x|^{-\frac{\beta - n}{p-1}} & \text{for } |x| > R_0. \end{cases}
$$
\nand Hardy condition (2.2) is satisfied when

\n
$$
\left(\mathcal{U}_1(R_0) + \int_{R_0 < |x| < R} u_2(x) dx \right)^{\frac{1}{p}} \left(\int_{R < |x|} |x|^{(\alpha - n)p'} \sigma_2(x) dx \right)^{\frac{1}{p'}} \le C \qquad (4.11)
$$
\n
$$
< R
$$
\nand

\n
$$
\left(\int_{|x| < R} u_1(x) dx \right)^{\frac{1}{p}} \left(\sum_{2} (R_0) + \int_{R < |x| < R_0} |x|^{(\alpha - n)p'} \sigma_1(x) dx \right)^{\frac{1}{p'}} \le C \qquad (4.12)
$$
\n
$$
< R_0. \text{ The Mukenhoupt condition (2.1) is equivalent to}
$$
\n
$$
R^{\alpha - n} \left(\int_{|x| < R} u_1(x) dx \right)^{\frac{1}{p}} \left(\int_{|x| < R} \sigma_1(x) dx \right)^{\frac{1}{p'}} \le C \qquad (4.13)
$$

for $R_0 < R$, and

$$
\langle R, \text{ and}
$$
\n
$$
\left(\int_{|x| < R} u_1(x) dx\right)^{\frac{1}{p}} \left(\overline{\Sigma}_2(R_0) + \int_{R < |x| < R_0} |x|^{(\alpha - n)p'} \sigma_1(x) dx\right)^{\frac{1}{p'}} \leq C \qquad (4.12)
$$
\n
$$
\langle R_0. \text{ The Muckenhoupt condition (2.1) is equivalent to}
$$
\n
$$
R^{\alpha - n} \left(\int_{|x| < R} u_1(x) dx\right)^{\frac{1}{p}} \left(\int_{|x| < R_0} \sigma_1(x) dx\right)^{\frac{1}{p'}} \leq C \qquad (4.13)
$$

for $R < R_0$. The Muckenhoupt condition (2.1) is equivalent to

$$
R^{\alpha-n}\left(\int_{|x|
$$

for $R < R_0$ and

Weighted Weak-Type Inequalities for
$$
I_{\alpha}
$$
 133
\nfor $R < R_0$ and\n
$$
R^{\alpha - n} \left(\mathcal{U}_1(R_0) + \int_{R_0 < |x| < R} u_2(x) dx \right)^{\frac{1}{p}} \left(\Sigma_1(R_0) + \int_{R_0 < |x| < R} \sigma_2(x) dx \right)^{\frac{1}{p'}} \le C \quad (4.14)
$$
\nfor $R_0 < R$. In these conditions\n
$$
\Sigma_1(R) = \int_{|x| < R} \sigma_1(x) dx, \quad \mathcal{U}_1(R) = \int_{|x| < R} u_1(x) dx, \quad \overline{\Sigma}_2(R) = \int_{R < |x|} |x|^{(\alpha - n)p'} \sigma_2(x) dx.
$$

for $R_0 < R$. In these conditions

$$
\Sigma_1(R) = \int_{|x| < R} \sigma_1(x) \, dx, \quad U_1(R) = \int_{|x| < R} u_1(x) \, dx, \quad \overline{\Sigma}_2(R) = \int_{R < |x|} |x|^{(\alpha - n)p'} \sigma_2(x) \, dx.
$$
\nFor $R \le R_0$ and by standard calculations then

\n
$$
U_1(R) \approx R^{\beta}, \quad \Sigma_1(R) \approx \ln^{-\frac{p'}{p}}(R^{-1}), \quad \int_{R < |x| < R_0} |x|^{(\alpha - n)p'} \sigma_1(x) \, dx \le cR^{(\alpha - n)p'}.
$$
\nOn the other hand, for $R_0 < R$

\n
$$
\int_{R_0 < |x| < R} u_2(x) \, dx \le c \times \begin{cases} R^{\gamma} & \text{for } \gamma > 0 \\ \ln R + \ln(R_0^{-1}) & \text{for } \gamma = 0 \\ R_0^{\gamma} & \text{for } \gamma < 0 \end{cases}
$$
\nand since $\alpha p < \theta$, then $\overline{\Sigma}_2(R) \approx R^{p'[\alpha - \frac{\theta}{p}]}$ and

For $R \leq R_0$ and by standard calculations then

$$
J_{|x|\n
$$
R \leq R_0 \text{ and by standard calculations then}
$$
\n
$$
U_1(R) \approx R^{\beta}, \quad \Sigma_1(R) \approx \ln^{-\frac{p'}{p}}(R^{-1}), \quad \int_{R<|x|
$$
$$

On the other hand, for $R_0 < R$

$$
+ \int_{R_0 < |x| < R} u_2(x) dx \int \left(\Sigma_1(R_0) + \int_{R_0 < |x| < R} \sigma_2(x) dx \right)
$$
\nthese conditions\n
$$
\sigma_1(x) dx, \quad U_1(R) = \int_{|x| < R} u_1(x) dx, \quad \overline{\Sigma}_2(R) = \int_{R < |x|} |x|
$$
\nby standard calculations then\n
$$
\sum_1(R) \approx \ln^{-\frac{p'}{p}}(R^{-1}), \quad \int_{R < |x| < R_0} |x|^{(\alpha - n)p'} \sigma_1(x) dx
$$
\nand, for $R_0 < R$ \n
$$
\int_{R_0 < |x| < R} u_2(x) dx \le c \times \begin{cases} R^\gamma & \text{for } \gamma > 0 \\ \ln R + \ln(R_0^{-1}) & \text{for } \gamma = 0 \\ R_0^\gamma & \text{for } \gamma < 0 \end{cases}
$$
\n
$$
\theta, \text{ then } \overline{\Sigma}_2(R) \approx R^{p'[\alpha - \frac{\theta}{p}]} \text{ and}
$$

$$
R \t J|z| < R \t JR < |I|
$$

and by standard calculations then

$$
\beta, \quad \Sigma_1(R) \approx \ln^{-\frac{p'}{p}}(R^{-1}), \quad \int_{R < |z| < R_0} |x|^{(\alpha - n)p'} \sigma_1(x) dx
$$

and, for $R_0 < R$

$$
\int_{R_0 < |z| < R} u_2(x) dx \leq c \times \begin{cases} R^{\gamma} & \text{for } \gamma > 0 \\ \ln R + \ln(R_0^{-1}) & \text{for } \gamma = 0 \\ R_0^{\gamma} & \text{for } \gamma < 0 \end{cases}
$$

$$
\vdots \theta, \text{ then } \overline{\Sigma}_2(R) \approx R^{p'[\alpha - \frac{\theta}{p}]} \text{ and}
$$

$$
\int_{R_0 < |z| < R} \sigma_2(x) dx \leq c \times \begin{cases} R^{p'[n - \frac{\theta}{p}]} & \text{for } \theta < np \\ \ln R + \ln(R_0^{-1}) & \text{for } \theta = np \\ R_0^{p'[n - \frac{\theta}{p}]} & \text{for } np < \theta. \end{cases}
$$

$$
\text{, we have } u(\cdot), v^{-\frac{1}{p-1}}(\cdot) \in \mathcal{H}. \text{ In view of the above calculus}
$$

$$
\text{. This last inequality also leads to (4.12) is ensured}
$$

By Lemma 2.3, we have $u(\cdot),v^{-\frac{1}{p-1}}(\cdot) \in \mathcal{H}$. In view of the above calculations, condition (4.11) is true since $\alpha p < \theta$ and $\alpha p + \gamma \le \theta$. Also, (4.12) is ensured by $\beta > 0$ and $np \leq \alpha p + \beta$. This last inequality also leads to (4.13). Finally, condition (4.14) is appeared by using $\alpha p < \theta$ and $\alpha p + \gamma \leq \theta$. Consequently, by Part B in Theorem 2.6, then $I_{\alpha}: L_v^p \to L_u^{p\infty}$.

The boundedness $I_{\alpha}: L_v^p \to L_v^{p\infty}$ can be obtained by using similar arguments. The details are omitted.

Proof of Corollary 2.9. The conclusion will be obtained from an application of Part C in Theorem 2.6 with $v(\cdot) = (M_p u)(\cdot)$. Precisely the main key is to check the Muckenhoupt condition (2.1) and the pointwise condition (2.6)' because $u(\cdot) \in RD_{\rho}$. *v(x) <c(ROP_n°^J u(y)dy)* for all In *<R.* (4.13)

The Muckenhoupt condition (2.1) appears immediatly once for a fixed constant $c>0$

$$
v^{-\frac{1}{p-1}}(x) \le c \Big(R^{\alpha p - n} \int_{|y| < R} u(y) \, dy \Big)^{1-p'} \qquad \text{for all } |x| < R. \tag{4.13}
$$

Indeed, by this inequality

___ *R(f u(y)dy)(/ v* _ P-I *(x) dx)' I y I<I JI1I<R ' I—p'l* <c ¹ *R°" (L< u(y) d) [R n (R"` fjyj<R u(y) dy)* ^j*R*

 $\emph{Inequality (4.13) is true since for $|x| < R$}$

Y. Rakotondratsimba
\nty (4.13) is true since for
$$
|x| < R
$$

\n
$$
R^{\alpha p-n} \int_{|y| < R} u(y) dy \le R^{\alpha p-n} \int_{|x-y| < 2R} u(y) dy \le c(M_{\alpha p}u)(x) = c v(x).
$$
\ne (2.6)' it is sufficient to find a fixed constant $C > 0$ for which
\n
$$
|x|^{\alpha p} u(x) \frac{1}{v(y)} \le C \quad \text{for } 4^{-1}|x| < |y| < 4|x|.
$$
\nivalent to write
\n
$$
u(x) \le c_2 |y|^{-\alpha p} v(y) = c_2 |y|^{-\alpha p} (M_{\alpha p}u)(y).
$$
\nty (4.14) is an easy consequence of the fact that $u(\cdot) \in RD_{\rho}$. Indeed, f
\nis c₃, c₄ > 0

To prove (2.6)' it is sufficient to find a fixed constant $C > 0$ for which

$$
u(y) dy \le R^{-r} \int_{|x-y| < 2R} u(y) dy \le c(M_{\alpha p}u)(
$$
\nsubject to find a fixed constant $C > 0$ for w :

\n
$$
|x|^{\alpha p} u(x) \frac{1}{v(y)} \le C \qquad \text{for } 4^{-1} |x| < |y| < 4|x|.
$$

It is equivalent to write

$$
u(x) \le c_2|y|^{-\alpha p}v(y) = c_2|y|^{-\alpha p}(M_{\alpha p}u)(y). \tag{4.14}
$$

Inequality (4.14) is an easy consequence of the fact that $u(\cdot) \in RD$. Indeed, for some constants $c_3, c_4 > 0$

sufficient to find a fixed constant
$$
C > 0
$$
 for wh:
\n
$$
|x|^{\alpha p}u(x)\frac{1}{v(y)} \leq C \quad \text{for } 4^{-1}|x| < |y| < 4|x|.
$$
\n
\nrite
\n
$$
u(x) \leq c_2|y|^{-\alpha p}v(y) = c_2|y|^{-\alpha p}(M_{\alpha p}u)(y).
$$
\n
\nIn easy consequence of the fact that $u(\cdot) \in RL$
\n
$$
u(x) \leq c_3|y|^{-n} \int_{c_4^{-1}|y| < |z| < c_4|y|} u(z) dz
$$
\n
$$
\leq c_3|y|^{-\alpha p}|y|^{\alpha p-n} \int_{|x-y| < (1+c_4)|y|} u(z) dz
$$
\n
$$
\leq c_5|y|^{-\alpha p}(M_{\alpha p}u)(y).
$$

References

- [1] Gabidzhashvili, M. and V. Kokilashvi!i: *Two weight weak type inequalities for fractional type integrals.* Ceskoslovenska Akademie Ved 45 (1989), 1 - 11.
- *[2] Long,* R. and F. Nb: *Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators.* Lect. Notes Math. 1494 (1990), 131 - 141.
- [3] Perez, C.: *Sharp LP -weighted Sobolev inequalities.* Ann. Inst. Fourier 45 (1995), 809 824.
- [4] Sawyer, E.: *A two weight weak type inequality for fractional integrals.* Trans. Amer. Math. Soc. 281 (1984), 339 - 345.
- *[5] Sawyer, E. and R. Wheeden: Weighted inequalities for fractional integrals on euclidean* and homogeneous spaces. Amer. J. Math. 114 (1992), 813 - 874.

Received 17.12.1996; in revised form 06.11.1997