Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 17 (1998), No. 1, 115–134

Weighted Inequalities of Weak Type for the Fractional Integral Operator

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Abstract. Sufficient conditions on weights $u(\cdot)$ and $v(\cdot)$ are given in order that the usual fractional integral operator I_α $(0 < \alpha < n)$ is bounded from the weighted Lebesgue space $L^p(v(x)dx)$ into weak-$L^p(u(x)dx)$, with $1 \le p < \infty$. As a consequence a characterization for this boundedness is obtained for a large class of weight functions which particularly contains radial monotone weights.

Keywords: *Fractional integral operators, weighted weak-type inequalities*

AMS subject classification: 42 B 25

1. Introduction

The fractional integral operator I_α of order α acts on locally integrable functions $f(\cdot)$ of $\mathbb{R}^n$ as

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,dy \qquad (0 < \alpha < n).$$

The purpose of this paper is to determine weight functions $u(\cdot)$ and $v(\cdot)$ for which I_α is bounded from $L^p_v = L^p(\mathbb{R}^n, v(x)dx)$ into weak-$L^p(\mathbb{R}^n, u(x)dx) = L^{p\infty}_u$ with $1 \le p < \infty$. This means that for some constant $C > 0$

$$\lambda^p \int_{\{x \mid (I_\alpha f)(x) > \lambda\}} u(x)\,dx \le C \int_{\mathbb{R}^n} f^p(x) v(x)\,dx \qquad \text{for all } \lambda > 0 \text{ and } f(\cdot) \ge 0. \qquad (1.1)$$

For convenience this boundedness will be denoted by $I_\alpha : L^p_v \to L^{p\infty}_u$.

Such an inequality takes an important part in Analysis. For instance, it is well-known [5] that (1.1) is a main point to get Sobolev inequalities with weights. Moreover, applications on the estimates of eigenvalue of some Schrödinger operators can be derived from (1.1) (see [2]).

Sawyer [3] proved that if $1 < p < \infty$, then $I_\alpha : L^p_v \to L^{p\infty}_u$ if and only if for some $A > 0$

$$\int_Q (I_\alpha u \mathbf{1}_Q)^{p'}(x)\, v^{1-p'}(x)\,dx \le A \int_Q u(x)\,dx \qquad \text{for all cubes } Q \qquad (1.2)$$

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ISSN 0232-2064 / \$ 2.50

where $p' = \frac{p}{p-1}$ and $1_Q(\cdot)$ is the characteristic function of the cube Q. Although (1.2) is a characterizing condition, it is not easy in general to check it for given weight functions. Indeed, a main difficulty comes from the fact that (1.2) is expressed in term of I_α, and the integrations over arbitrary cubes are also hard to compute. So it is a challenge problem to derive conditions which ensure inequality (1.1) but more easily verifiable than (1.2).

For $1 < p < q < \infty$, Gabidzashvili and Kokilashvili [1] proved that

$$\lambda\Big(\int_{\{x \mid (I_\alpha f)(x)>\lambda\}} u(x)\,dx\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)\,dx\Big)^{\frac{1}{p}} \tag{1.3}$$

for all $\lambda > 0$ and $f(\cdot) \ge 0$ if and only if

$$\Big(\int_Q u(y)\,dy\Big)^{\frac{1}{q}}\Big(\int_{y\in\mathbb{R}^n}(|y-x_Q|+|Q|^{\frac{1}{n}})^{(\alpha-n)p'}v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le A \tag{1.4}$$

for all cubes Q. Here x_Q denotes the centre of Q, and $|Q| = \int_Q dx$. The proof of this result does not work for the case $p = q$, so the problem of finding a similar characterization for $I_\alpha : L^p_v \to L^{p\infty}_u$ remains open. According to Sawyer and Wheeden [5], inequality (1.1) holds if for some $A > 0$ and $1 < t < \frac{n}{\alpha}$

$$|Q|^{\frac{\alpha}{n}}\Big(|Q|^{-1}\int_Q u^t(y)\,dy\Big)^{\frac{1}{tp}}\Big(|Q|^{-1}\int_Q v^{t(1-p')}(y)\,dy\Big)^{\frac{1}{tp'}} \le A \quad \text{for all cubes } Q. \tag{1.5}$$

In fact, in [5] it is seen that (1.5) implies the strong inequality $I_\alpha : L^p_v \to L^p_u$ associated to (1.1), so a weaker sufficient condition than (1.5), for the weak-type inequality (1.1), is not known. Bumping $u(\cdot)$ or $v^{1-p'}(\cdot)$ as in (1.5) is not always satisfactory. Indeed, taking $v^{1-p'}(x) = |x|^{-n}\ln^{-p'}(|x|^{-1})$ for $|x| < \frac{1}{2}$, then $\int_{|x|<R} v^{t(1-p')}(x)\,dx = \infty$, for all $t > 1$ and $R < \frac{1}{2}$ though $\int_{|x|<R} v^{1-p'}(x)\,dx < \infty$. But for such a weight function $v(\cdot)$ (see Corollary 2.8) the boundedness $I_\alpha : L^p_v \to L^{p\infty}_u$ holds true. All of these reasons lead us to consider and study again problem (1.1).

In this work, we first state in Theorem 2.1 a necessary and sufficient condition for $I_\alpha : L^p_v \to L^{p\infty}_u$. Next, in Theorem 2.6 we will prove that with an additional pointwise inequality, the necessary condition (1.4) [with $p = q$] becomes also sufficient to ensure $I_\alpha : L^p_v \to L^{p\infty}_u$. Moreover, it will be shown that the test condition could be restricted to balls centered at the origin, rather on arbitrary cubes. For radial and monotone weights we will see (in the same theorem) that $I_\alpha : L^p_v \to L^{p\infty}_u$ is equivalent to

$$\Big(\int_{|x|<R} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{y\in\mathbb{R}^n}(|y|+R)^{(\alpha-n)p'}v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le A \tag{1.6}$$

for all $R > 0$. For $p = 1$ a substitute of (1.6) will be given in Theorem 2.7.

Actually, in Section 3, we will be able to study the more general weighted weak-type inequalities $I_\alpha : L^p_v \to wL^{p\infty}_u$, i.e.

$$\lambda^p\int_{\{x\in\mathbb{R}^n \mid w(x)(I_\alpha f)(x)>\lambda\}} u(x)\,dx \le C\int_{\mathbb{R}^n} f^p(x)v(x)\,dx$$

for all $\lambda > 0$ and $f(\cdot) \ge 0$. The last Section 4 is devoted to the proofs of our results.

2. Results for classical weighted weak-type inequalities

In this paper it will be assumed that

$$0 < \alpha < n, \qquad 1 \le p < \infty \quad \text{and} \quad p' = \frac{p}{p-1} \text{ if } p > 1$$

and

$$u(\cdot), v(\cdot) \quad \text{are weight functions}$$

with $v^{1-p'}(\cdot) \in L^1_{loc}(\mathbb{R}^n, dx)$ if $p > 1$ else $v^{-1}(\cdot) \in L^\infty_{loc}(\mathbb{R}^n, v(x)\,dx)$.

Now the first main result, about a necessary and sufficient condition for $I_\alpha : L^p_v \to L^{p\infty}_u$, can be stated.

Theorem 2.1. *For $p > 1$, the boundedness $I_\alpha : L^p_v \to L^{p\infty}_u$ holds if and only if*

$$\widetilde{I}_\alpha : L^p_v \to L^{p\infty}_u$$

and there is a constant $A > 0$ such that, for all $R > 0$,

$$R^{\alpha-n}\Big(\int_{|x|<R} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{|y|<R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le A \tag{2.1}$$

and

$$\Big(\int_{|x|<R} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{R<|y|} |y|^{(\alpha-n)p'} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le A. \tag{2.2}$$

Here the restricted operator $\widetilde{I}_\alpha$ is defined by

$$(\widetilde{I}_\alpha f)(x) = \int_{\{2^{-1}|x|<|y|<2|x|\}} |x-y|^{\alpha-n} f(y)\,dy.$$

For $p = 1$ a similar equivalence is also true when (2.1) and (2.2) are replaced, for all $R > 0$, by

$$R^{\alpha-n}\Big(\int_{|x|<R} u(x)\,dx\Big)\cdot \operatorname{ess\,sup}\Big[\frac{1}{v(y)}\mathbf{1}_{|\cdot|<R}(y)\Big] \le A \tag{2.3}$$

and

$$\Big(\int_{|x|<R} u(x)\,dx\Big)\cdot \operatorname{ess\,sup}\Big[|y|^{\alpha-n}\frac{1}{v(y)}\mathbf{1}_{R<|\cdot|}(y)\Big] \le A, \tag{2.4}$$

respectively.

In (2.3) and (2.4) the essential supremum is taken with respect to the measure $v(x)dx$. These conditions can be seen as limiting cases of (2.1) and (2.2), respectively. Note also that both conditions (2.1) and(2.2) can be summarized by

$$\Big(\int_{|x|<R} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{y\in\mathbb{R}^n} (|y|+R)^{(\alpha-n)p'} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0. \tag{2.5}$$

Theorem 2.1 means that the weighted weak-inequality problem for I_α can be essentially reduced to the corresponding weighted weak-inequality for the restricted operator $\widetilde{I}_\alpha$. Although a characterization for $\widetilde{I}_\alpha : L^p_v \to L^{p\infty}_u$ remains unsolved, surprisingly it is not too hard to derive sufficient conditions ensuring this boundedness.

Proposition 2.2. *The boundedness* $\widetilde{I}_\alpha : L^p_v \to L^{p\infty}_u$ *holds if for some constant* $A > 0$

$$|x|^\alpha \Big(\sup_{4^{-1}|x|<|y|<4|x|} u(y) \Big)^{\frac{1}{p}} \le A(v(x))^{\frac{1}{p}} \qquad \textit{for a.e. } x. \tag{2.6}$$

This condition can be replaced by

$$|x|^\alpha (u(x))^{\frac{1}{p}} \Big(\sup_{4^{-1}|x|<|y|<4|x|} \frac{1}{v(y)} \Big)^{\frac{1}{p}} \le A \qquad \textit{for a.e. } x. \tag{2.6$'$}$$

Since (2.6) [or (2.6)′] is a pointwise inequality it is in general an easy verifiable condition for given weight functions. And it is an interesting question to determine some situations when (2.6) becomes also a necessary condition for $I_\alpha : L^p_v \to L^{p\infty}_u$. For this purpose, growth conditions on weights are needed. So we define $\sigma(\cdot) \in \mathcal{H}$ if

$$\sup_{4^{-1}R<|y|<4R} \sigma(y) \le CR^{-n} \int_{2^{-N}R<|y|<2^N R} \sigma(y)\, dy \qquad \text{for all } R > 0$$

where $C > 0$ and the non-negative integer N are fixed constants. Many of usual weights satisfy the property $\mathcal{H}$. It is the case of any *radial and monotone weights*, for which N can be taken equal to 3. But the condition $w(\cdot) \in \mathcal{H}$ can be held although $w(\cdot)$ is not necessarily a monotone weight. Indeed, we have the following

Lemma 2.3. *Suppose* $w_1(\cdot)$ *and* $w_2(\cdot)$ *are radial monotone on* $(0, R_0)$ *and* (R_0, ∞) *for some* $R_0 > 0$*, respectively. Let*

$$w(x) = w_1(|x|) \times \mathbf{1}_{(0,R_0)}(|x|) + w_2(|x|) \times \mathbf{1}_{(R_0,\infty)}(|x|).$$

Then $w(\cdot) \in \mathcal{H}$*, with constants* $N = 3$ *and* $C > 0$ *depending on* $w_1(\cdot)$ *and* $w_2(\cdot)$*.*

An answer to the above question can be given by using the growth condition $\mathcal{H}$.

Proposition 2.4. *Suppose* $u(\cdot) \in \mathcal{H}$*. Then:*

(i) *For* $p > 1$ *the pointwise inequality* (2.6) *(or* (2.6)′*) is satisfied whenever the Muckenhoupt condition* (2.1) *holds and* $v^{1-p'}(\cdot) \in \mathcal{H}$*.*

(ii) *Similarly, for* $p = 1$*, inequality* (2.6) *(or* (2.6)′*) is implied by* (2.3)*.*

Another special weight property is the so-called *reverse doubling condition* $w(\cdot) \in RD_\rho$ $(\rho > 0)$ which means, by definition,

$$\int_{|y|<tR} w(y)\, dy \le c\, t^{n\rho} \int_{|y|<R} w(y)\, dy \qquad \text{for all } t \in (0, 1) \text{ and } R > 0$$

and for a fixed constant $c > 0$. The interest in the introduction of this condition is reflected by

Proposition 2.5. *Assume that* $u(\cdot) \in RD_\rho$ *for some* $\rho > 0$*. Then:*

(i) *For* $p > 1$*, the dual Hardy condition* (2.2) *is implied by the Muckenhoupt condition* (2.1)*.*

(ii) *Similarly for* $p = 1$*, condition* (2.4) *is implied by* (2.3)*.*

The facts contained in Theorem 2.1 and Propositions 2.4 and 2.5 can be summarized as

Theorem 2.6. *Let $p > 1$.*

A) *The boundedness $I_\alpha : L^p_v \to L^{p\infty}_u$ implies the Gabidzashvili-Kokilashvili condition* (2.5). *Conversely, this last condition implies $I_\alpha : L^p_v \to L^{p\infty}_u$ whenever the pointwise inequality* (2.6) *or* (2.6)′ *is satisfied.*

B) *Inequality* (2.6) *in part* A) *can be dropped whenever $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{H}$.*

C) *The Gabidzashvili-Kokilashvili condition* (2.5) *in part* A) *can be replaced by the Muckenhoupt condition* (2.1) *whenever $u(\cdot) \in RD_\rho$ for some $\rho > 0$.*

Consequently, as announced in the Introduction, for a large class of weight functions (like radial and monotone weights), the Gabidzashvili and Kokilashvili result [1] (valid for $p < q$) can be extended to the case $p = q$.

Theorem 2.7.

A) *The boundedness $I_\alpha : L^1_v \to L^{1\infty}_u$ implies conditions* (2.3) *and* (2.4). *Conversely, these two conditions imply $I_\alpha : L^1_v \to L^{1\infty}_u$ whenever the pointwise inequality* (2.6) *or* (2.6)′ *is satisfied.*

B) *Inequality* (2.6) *in part* A) *can be dropped whenever $u(\cdot) \in \mathcal{H}$.*

C) *Condition* (2.4) *in parts* A) *and* B) *can be dropped whenever $u(\cdot) \in RD_\rho$ for some $\rho > 0$.*

Now examples and applications, showing the gain in our results compared with past results, are given.

Corollary 2.8. *Let $0 < R_0 < 1 < p < \infty$. Define the weight functions*

$$u(x) = |x|^{\beta-n}\mathbf{1}_{|x|<R_0}(x) + |x|^{\gamma-n}\mathbf{1}_{|x|>R_0}(x)$$

$$v(x) = |x|^{n(p-1)}\ln^p(|x|^{-1})\mathbf{1}_{|x|<R_0}(x) + |x|^{\theta-n}\mathbf{1}_{|x|>R_0}(x).$$

Then, for $\beta > 0$ and $\alpha p < \theta$, $I_\alpha : L^p_v \to L^{p\infty}_u$ whenever

$$\alpha p + \gamma < np \le \alpha p + \beta \qquad \text{and} \qquad \alpha p + \gamma \le \theta.$$

Also, set

$$u^*(x) = |x|^{-n}\ln^{-p}|x|^{-1}\mathbf{1}_{|x|<R_0}(x) + |x|^{(1-p)(\theta-n)}\mathbf{1}_{|x|>R_0}(x)$$

$$v^*(x) = |x|^{(1-p)(\beta-n)}\mathbf{1}_{|x|<R_0}(x) + |x|^{(1-p)(\gamma-n)}\mathbf{1}_{|x|>R_0}(x).$$

Then, for $\beta > 0$, $I_\alpha : L^p_{v^*} \to L^{p\infty}_{u^*}$ whenever

$$\alpha p' + \gamma < np' \le \alpha p' + \beta \qquad \text{and} \qquad \alpha p' + \gamma \le \theta.$$

As we have mentioned in the Introduction, for this example the boundedness $I_\alpha : L^p_v \to L^{p\infty}_u$ is not obtainable from the Sawyer-Wheeden criterion (1.6).

Corollary 2.9. *Let $1 < p < \frac{n}{\alpha}$ and $u(\cdot) \in \mathcal{H} \cap RD_\rho$ $(\rho > 0)$. Then, for some constant $C > 0$,*

$$\lambda^p \int_{\{x \mid (I_\alpha f)(x) > \lambda\}} u(x)\,dx \le C \int_{\mathbb{R}^n} f^p(x)\,(M_{\alpha p} u)(x)\,dx \tag{2.7}$$

for all $\lambda > 0$ and $f(\cdot) \ge 0$. Here $M_{\alpha p}$ is the usual fractional maximal operator

$$(M_{\alpha p} f)(x) = \sup\Big\{ |Q|^{\frac{\alpha p}{n} - 1} \int_Q |f(y)|\,dy \Big| \, Q \text{ a cube with } Q \ni x \Big\}.$$

The constant C in (2.7) depends only on the constants in properties $\mathcal{H}$ and RD_ρ but not directly on $u(\cdot)$.

A result like Corollary 2.9 can be used to derive weighted Sobolev inequalities as

Corollary 2.10. *Let $1 < p < n$ and $u(\cdot) \in \mathcal{H} \cap RD_\rho$ $(\rho > 0)$. Then*

$$\int_{\mathbb{R}^n} |g(x)|^p u(x)\,dx \le C \int_{\mathbb{R}^n} |(\nabla g)(x)|^p (M_p u)(x)\,dx \tag{2.8}$$

for all $g(\cdot) \in C_0^\infty(\mathbb{R}^n)$ and for some fixed constant $C > 0$.

Indeed, it is known from [2] (see also [5]) that $I_1 : L^p_v \to L^{p\infty}_u$ implies the Sobolev inequality $\int_{\mathbb{R}^n} |g(x)|^p u(x)\,dx \le C \int_{\mathbb{R}^n} |(\nabla g)(x)|^p v(x)\,dx$.

For general weight functions $u(\cdot)$, Pérez [3] proved an inequality like (2.8), with $(M_{\alpha p} u)(\cdot)$ replaced by $(M_{\alpha p} M_0^{[p]} u)(\cdot)$. Here $M^{[p]}$ is the $[p]$-times iteration of the Hardy-Littlewood maximal operator M_0. Since trivially $u(\cdot) \le M_0^{[p]} u)(\cdot)$, Corollary 2.10 can be seen as an improvement of this author's result for weights $u(\cdot) \in \mathcal{H} \cap RD_\rho$.

3. More General Results

In this section α, p, $u(\cdot)$, $v(\cdot)$ are assumed as in Section 2, and moreover $w(\cdot)$ is a weight function. Our purpose is now to study the more general weighted weak-type inequalities $I_\alpha : L^p_v \to wL^{p\infty}_u$, i.e.

$$\lambda^p \int_{\{x \in \mathbb{R}^n \mid w(x)(I_\alpha f)(x) > \lambda\}} u(x)\,dx \le C \int_{\mathbb{R}^n} f^p(x)\,v(x)\,dx \tag{3.1}$$

for all $\lambda > 0$ and $f(\cdot) \ge 0$. As usual, C is a fixed non-negative constant.

The boundedness $I_\alpha : L^p_v \to L^{p\infty}_u$ is a particular case of (1.1), but for $w(\cdot) \ne 1$ the two inequalities are quite different since the weight $w(\cdot)$ cannot be combined with $u(\cdot)$ or $v(\cdot)$. It seems that no result about $I_\alpha : L^p_v \to wL^{p\infty}_u$ [with $w(\cdot) \ne 1$] were explicitely written and available elsewhere. Actually, results given in Section 2 are consequences of those we will present in this section.

First a necessary and sufficient condition for $I_\alpha : L^p_v \to wL^{p\infty}_u$ is stated.

Theorem 3.1. *For $p > 1$, the boundedness $I_\alpha : L^p_v \to wL^{p\infty}_u$ holds if and only if*

$$\widetilde{I}_\alpha : L^p_v \to wL^{p\infty}_u$$

and there is aconstant $A > 0$ such that

$$\lambda\Big(\int_{\{w(x)|x|^{\alpha-n}>\lambda\}\cap\{2R<|x|\}} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{|y|<R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le A \tag{3.2}$$

and

$$\lambda\Big(\int_{\{w(x)>\lambda\}\cap\{|x|<R\}} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{2R<|y|} |y|^{(\alpha-n)p'}v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le A \tag{3.2)*$$

for all $\lambda > 0$ and $R > 0$. The restricted operator $\widetilde{I}_\alpha$ is defined as in Theorem 2.1.

For $p = 1$ a similar equivalence is also true when (3.2) and (3.2) are replaced by*

$$\lambda\Big(\int_{\{w(x)|x|^{\alpha-n}>\lambda\}\cap\{2R<|x|\}} u(x)\,dx\Big)\Big(\operatorname{ess\,sup}\Big[\frac{1}{v(y)}\mathbf{1}_{|\cdot|<R}(y)\Big]\Big) \le A \tag{3.3}$$

and

$$\lambda\Big(\int_{\{w(x)>\lambda\}\cap\{|x|<R\}} u(x)dx\Big)\Big(\operatorname{ess\,sup}\Big[|y|^{\alpha-n}\frac{1}{v(y)}\mathbf{1}_{2R<|\cdot|}(y)\Big]\Big) \le A \tag{3.3)*$$

for all $\lambda > 0$ and $R > 0$.

Theorem 3.1 means that to solve the weighted weak-type inequality (3.1), the real problem is to decide when does the boundedness of the restricted operator $\widetilde{I}_\alpha$ hold.

Note that in Theorem 3.1, the direct Hardy condition (3.2) and its dual version (3.2)* are used. But in Theorem 2.1 the Hardy condition is not appeared and is replaced by the Muckenhoupt condition (2.1). An explanation of this fact will be seen below in Proposition 3.4.

Although a characterization of weights for which $\widetilde{I}_\alpha : L^p_v \to wL^{p\infty}_u$ is still an open problem, we are able here to derive an easy sufficient condition ensuring this boundedness whenever $w(\cdot)$ is constant on annuli [or merely $w(\cdot) \in \mathcal{A}$], in the sense that

$$\sup_{\{R<|z|<16R\}} w(z) \le c \inf_{\{R<|y|<16R\}} w(y) \qquad \text{for all } R > 0.$$

This is not a real inconvenience since many of usual weights $w(\cdot)$ for which (3.1) is considered are constants on annuli. Indeed, for $w(x) = |x|^\gamma \ln^\delta(e + |x|)$ $(\gamma, \delta \ge 0)$, then $w(\cdot) \in \mathcal{A}$. It is also the case of any radial increasing [resp. decreasing] weight $w(\cdot)$ for which $w(4R) \le c\,w(R)$ [resp. $w(R) \le c\,w(4R)$].

Proposition 3.2. *For $w(\cdot) \in \mathcal{A}$, the boundedness $\widetilde{I}_\alpha : L^p_v \to wL^{p\infty}_u$ holds if for some constant $A > 0$*

$$w(x)|x|^\alpha\Big(\sup_{4^{-1}|x|<|y|<4|x|} u(y)\Big)^{\frac{1}{p}} \leq A\Big(v(x)\Big)^{\frac{1}{p}} \qquad \text{for a.e. } x. \tag{3.4}$$

This condition can be replaced by

$$w(x)|x|^\alpha\Big(u(x)\Big)^{\frac{1}{p}}\Big(\sup_{4^{-1}|x|<|y|<4|x|} \frac{1}{v(y)}\Big)^{\frac{1}{p}} \leq A \qquad \text{for a.e. } x. \tag{3.4'}$$

Without the hypothesis $w(\cdot) \in \mathcal{A}$, the above conclusions are also true whenever I_α : $L^p(\mathbb{R}^n, dx) \to L^p(w^p(x)|x|^{-\alpha p}dx)$.

In general, (3.4) [or (3.4′)] is an easy verifiable condition since it is just a pointwise inequality. And the main question remains to determine situations for which (3.4) is also a necessary condition for $I_\alpha : L^p_v \to wL^{p\infty}_u$. In solving this problem, it is useful to note that a necessary condition for such boundedness is the Muckenhoupt condition

$$R^{\alpha-n}\lambda\Big(\int_{\{w(x)>\lambda\}\cap\{|x|<R\}} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{|y|<R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \leq A \tag{3.5}$$

for all $\lambda, R > 0$ and where $A > 0$ is a fixed constant. The replacement of (3.5) for $p = 1$ is

$$R^{\alpha-n}\lambda\Big(\int_{\{w(x)>\lambda\}\cap\{|x|<R\}} u(x)\,dx\Big)\Big(\operatorname{ess\,sup}\Big[\frac{1}{v(y)}\mathbf{1}_{|\cdot|<R}(y)\Big]\Big) \leq A \tag{3.6}$$

for all $\lambda > 0$ and all $R > 0$. The mentioned implication can be easily proved similarly as the necessary part in Theorem 2.1. Obviously, the above question can be reduced to get (3.4) from (3.5) or (3.6).

Proposition 3.3. *For $p > 1$ and $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{H}$, the pointwise inequality (3.4) or (3.4)′ is satisfied if for some constant $A > 0$*

$$R^{\alpha-n}\lambda\Big(\int_{\{w(x)>\lambda\}\cap\{R<|x|<2^{2N}R\}} u(x)\,x\Big)^{\frac{1}{p}}\Big(\int_{R<|y|<2^{2N}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \leq A \tag{3.5'}$$

for all $\lambda, R > 0$ and whenever $w(\cdot) \in \mathcal{A}$, in the sense that

$$\sup_{\{R<|z|<2^{2N}R\}} w(z) \leq c \inf_{\{R<|y|<2^{2N}R\}} w(y). \tag{3.7}$$

Here N is the integer from assumption $\mathcal{H}$. In particular, the Muckenhoupt condition (3.5) implies (3.4) or (3.4)′.

Similarly, for $p = 1$ and $u(\cdot) \in \mathcal{H}$, inequality (3.4) (or (3.4′)) is satisfied if for some constant $A > 0$

$$R^{\alpha-n}\lambda\Big(\int_{\{w(x)>\lambda\}\cap\{R<|x|<2^{2N}R\}} u(x)dx\Big)\Big(\operatorname{ess\,sup}\Big[\frac{1}{v(y)}\mathbf{1}_{R<|\cdot|<2^{2N}R}(y)\Big]\Big) \leq A \tag{3.6'}$$

for all $\lambda, R > 0$ and $w(\cdot) \in \mathcal{A}$. In particular, condition (3.6) implies (3.4) (or (3.4)′).

Since for $p > 1$ the Hardy conditions (3.2), (3.2*) and the Muckenhoupt condition (3.5) are both necessary conditions for the boundedness $I_\alpha : L^p_v \to wL^{p\infty}_u$, then it is a natural question to precise some relations between these three conditions whenever the weights have a special property like the reverse doubling condition RD_p.

Proposition 3.4.

A) *For* $w(x) = |x|^{-\gamma}$, *with* $\gamma \geq 0$, *the Muckenhoupt condition* (3.5) *implies the Hardy condition* (3.2), *and similarly condition* (3.6) *implies* (3.3).

B) *For general weights* $w(\cdot)$ *and* $p > 1$, *then* (3.5) *implies* (3.2) *wheneve* $v^{1-p'}(\cdot) \in RD_\rho$ *for some* $\rho > 0$.

This result yields an explanation why, for the boundedness $I_\alpha : L^p_v \to L^{p\infty}_u$, the Hardy condition does not appear in Theorem 2.1.

Facts described in Propositions 3.2 - 3.4 and in Theorem 3.1 can be now summarized.

Theorem 3.5. *Let* $p > 1$.

A) *The boundedness* $I_\alpha : L^p_v \to wL^{p\infty}_u$ *implies the Hardy conditions* (3.2), (3.2)* *and the Muckenhoupt condition* (3.5). *Conversely, the conditions* (3.2), (3.2)* *and* (3.5) *imply* $I_\alpha : L^p_v \to wL^{p\infty}_u$ *whenever* $u(\cdot)$, $v^{1-p'}(\cdot) \in \mathcal{H}$ *and* $w(\cdot) \in \mathcal{A}$ *as in* (3.7).

B) *If* $u(\cdot)$ *and* $v(\cdot)$ *are radial and monotone functions and* $w(\cdot) \in \mathcal{A}$, *then* $I_\alpha : L^p_v \to wL^{p\infty}_u$ *if and only if* (3.2) *and* (3.2)* *are satisfied.*

C) *In parts* A) *and* B), *the Hardy condition* (3.2) *can be replaced by the Muckenhoupt condition* (3.5) *whenever* $w(x) = |x|^{-\gamma}$, $\gamma \geq 0$ *or* $v^{1-p'}(\cdot) \in RD_\rho$ *for some* $\rho > 0$.

Consequently, as announced in the Introduction, we obtained a characterization of the boundedness $I_\alpha : L^p_v \to wL^{p\infty}_u$ whenever $u(\cdot)$ and $v^{1-p'}(\cdot)$ are radial monotone weight functions and $w(\cdot) \in \mathcal{A}$.

Theorem 3.6.

A) *The boundedness* $I_\alpha : L^1_v \to wL^{1\infty}_u$ *implies conditions* (3.3), (3.3)* *and* (3.6). *Conversely, these conditions* (3.3), (3.3)* *and* (3.6) *imply* $I_\alpha : L^1_v \to wL^{1\infty}_u$ *whenever* $u(\cdot) \in \mathcal{H}$ *and* $w(\cdot) \in \mathcal{A}$ *as in* (3.7).

B) *If* $u(\cdot)$ *and* $v(\cdot)$ *are radial and monotone functions and* $w(\cdot) \in \mathcal{A}$, *then* $I_\alpha : L^1_v \to wL^{1\infty}_u$ *if and only if* (3.3) *and* (3.3)* *are satisfied.*

C) *In parts* A) *and* B), *condition* (3.3) *can be replaced by* (3.6) *whenever* $w(x) = |x|^{-\gamma}, \gamma \geq 0$.

4. Proofs of the results

First we prove results in Section 3 and next outline proofs of those stated in Section 2.

Proof of Theorem 3.1. *Necessary Part*: We first assume that $I_\alpha : L^p_v \to wL^{p\infty}_u$, which is equivalent to

$$\|w(\cdot)(I_\alpha f)(\cdot)\|_{L^{p\infty}_u} \leq C\|f(\cdot)\|_{L^p_v} \qquad \text{for all } f(\cdot) \geq 0 \tag{4.1}$$

where

$$\|f(\cdot)\|^p_{L^p_v} = \int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx$$

and

$$\|g(\cdot)\|^p_{L^{p\infty}_u} = \sup_{\lambda>0}\Big\{\lambda^p \int_{\{x:\,|g(x)|>\lambda\}} u(x)\,dx\Big\}.$$

The boundedness $\widetilde{I}_\alpha : L^p_v \to wL^{p\infty}_u$ appears clearly, since trivially $(\widetilde{I}_\alpha f)(\cdot) \le (I_\alpha f)(\cdot)$. The main point for condition (3.2) is the existence of a constant $C > 0$ such that

$$\Big(\int_{|y|<R} f(y)\,dy\Big)\big\|\,|\cdot|^{\alpha-n}w(\cdot)\mathbf{1}_{2R<|\cdot|}(\cdot)\,\big\|_{L^{p\infty}_u} \le C\Big(\int_{\mathbb{R}^n} f^p(y)\,v(y)\,dy\Big)^{\frac{1}{p}} \tag{4.2}$$

for all $f(\cdot) \ge 0$ and $R > 0$. Indeed, in the case $p > 1$, taking $f(\cdot) = v^{1-p'}(\cdot)\mathbf{1}_{\{|\cdot|<R\}}(\cdot)$ in this inequality and if $0 < \int_{|y|<R} v^{1-p'}(y)\,dy < \infty$, then

$$\big\|\,|\cdot|^{\alpha-n}w(\cdot)\mathbf{1}_{2R<|\cdot|}(\cdot)\,\big\|_{L^{p\infty}_u}\Big(\int_{|y|<R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le C$$

which is nothing else than (3.2). This is obviously satisfied if $0 = \int_{|y|<R} v^{1-p'}(y)\,dy$. And the fact that $\int_{|y|<R} v^{1-p'}(y)\,dy < \infty$ is ensured by (4.2) or the hypothesis on $v(\cdot)$. Condition (3.3) (i.e. for $p = 1$) appears by taking $p \to 1$ in this last inequality, and since the constant $C > 0$ in (4.1) or (4.2) does not depend on p.

Inequality (4.2) is a direct consequence of the boundedness $I_\alpha : L^p_v \to wL^{p\infty}_u$ and the fact that for $|x| > 2R$

$$\begin{aligned}|x|^{\alpha-n}\int_{|y|<R} f(y)\,dy &\le |x|^{\alpha-n}\int_{\frac{1}{2}|x|<|x-y|<\frac{3}{2}|x|} f(y)\,dy\\ &\le 2^{n-\alpha}\int_{|x-y|<2|x|} |x-y|^{\alpha-n} f(y)\,dy\\ &\le 2^{n-\alpha}(I_\alpha f)(x).\end{aligned}$$

Similarly as above, the main point for condition $(3.2)^*$ is

$$\Big(\int_{2R<|y|} |y|^{\alpha-n} g(y)\,dy\Big)\big\|\,w(\cdot)\mathbf{1}_{|\cdot|<R}(\cdot)\,\big\|_{L^{p\infty}_u} \le C\Big(\int_{\mathbb{R}^n} g^p(y)\,v(y)\,dy\Big)^{\frac{1}{p}} \tag{4.2*}$$

for all $g(\cdot) \ge 0$ and $R > 0$. Setting $g(\cdot) = |\cdot|^{(\alpha-n)(p'-1)}v^{1-p'}(\cdot)\mathbf{1}_{\{2R<|\cdot|<R_1\}}(\cdot)$, then

$$\int_{2R<|y|} |y|^{\alpha-n} g(y)\,dy = \int_{2R<|y|<R_1} |y|^{(\alpha-n)p'} v^{1-p'}(y)\,dy$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} g^p(y)v(y)dy &= \int_{2R<|y|<R_1} |y|^{(\alpha-n)p'} v^{1-p'}(y)\,dy\\ &< R^{(\alpha-n)p'}\int_{|y|<R_1} v^{1-p'}(y)\,dy\\ &< \infty.\end{aligned}$$

So taking such a function $g(\cdot)$ in inequality (4.2)* and assuming that

$$\int_{2R<|y|<R_1} |y|^{(\alpha-n)p'} v^{1-p'}(y)\,dy > 0$$

it appears that

$$\| w(\cdot)\mathbf{1}_{|\cdot|<R}(\cdot)\|_{L_u^{p\infty}} \Big(\int_{2R<|y|} |y|^{(\alpha-n)p'} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \le C$$

after letting $R_1 \to \infty$. This is condition (3.2)*.

Inequality (4.2)* is also a direct consequence of $I_\alpha : L_v^p \to wL_u^{p\infty}$ and the fact that for $|x| < R$

$$\int_{2R<|y|} |y|^{\alpha-n} f(y)\,dy \le 2^{n-\alpha} \int_{|x-y|<2|y|} |x-y|^{\alpha-n} f(y)\,dy \le 2^{n-\alpha}(I_\alpha f)(x).$$

Sufficient Part: To get $I_\alpha : L_v^p \to wL_u^{p\infty}$ first observe that for some constant $c > 0$

$$\|w(\cdot)(I_\alpha f)(\cdot)\|^p_{L_u^{p\infty}} \le c\Big(\mathcal{S}_1^p + \mathcal{S}_2^p + \mathcal{S}_3^p\Big) \qquad \text{for all } f(\cdot) \ge 0$$

where

$$\mathcal{S}_1 = \Big\|w(\cdot)\Big(\int_{|y|<\frac{1}{2}|\cdot|} |\cdot - y|^{\alpha-n} f(y)\,dy\Big)\Big\|_{L_u^{p\infty}}$$

$$\mathcal{S}_2 = \Big\|w(\cdot)\Big(\int_{\frac{1}{2}|\cdot|<|y|<2|\cdot|} |\cdot - y|^{\alpha-n} f(y)\,dy\Big)\Big\|_{L_u^{p\infty}}$$

$$\mathcal{S}_3 = \Big\|w(\cdot)\Big(\int_{2|\cdot|<|y|} |\cdot - y|^{\alpha-n} f(y)\,dy\Big)\Big\|_{L_u^{p\infty}}.$$

So it is sufficient to bound each $\mathcal{S}_i$ $(i \in \{1,2,3\})$ by $CA\|f(\cdot)\|_{L_v^p}$ where C and A are non-negative constants which do not depend on $f(\cdot)$.

Since $\widetilde{I}_\alpha : L_v^p \to wL_u^{p\infty}$, then

$$\mathcal{S}_2 = \|w(\cdot)(\widetilde{I}_\alpha f)(\cdot)\|_{L_u^{p\infty}} \le cA\|f(\cdot)\|_{L_v^p}.$$

Here $A > 0$ is taken as the constant in the Hardy conditions (3.2) and (3.2)*.

Arguing as in (4.2) and (4.2)*, estimates of $\mathcal{S}_1$ and $\mathcal{S}_2$ are reduced to get the Hardy inequalities type

$$\Big\|w(\cdot)|\cdot|^{\alpha-n}\Big(\int_{|y|<\frac{1}{2}|\cdot|} f(y)\,dy\Big)\Big\|_{L_u^{p\infty}} \le cA\|f(\cdot)\|_{L_v^p} \quad \text{for all } f(\cdot) \ge 0 \qquad (4.3)$$

$$\Big\|w(\cdot)\Big(\int_{2|\cdot|<|y|} |y|^{\alpha-n} f(y)\,dy\Big)\Big\|_{L_u^{p\infty}} \le cA\|f(\cdot)\|_{L_v^p} \quad \text{for all } f(\cdot) \ge 0. \qquad (4.3)^*$$

Since the arguments are similar, the proof is limited to that of (4.3).

One of the point keys is the inequality

$$\Big\|\sum_k f(\cdot)\mathbf{1}_{\mathcal{E}_k}(\cdot)\Big\|^p_{L^{p\infty}_u} \le \sum_k \big\|f(\cdot)\mathbf{1}_{\mathcal{E}_k}(\cdot)\big\|^p_{L^{p\infty}_u} \tag{4.4}$$

where the $\mathcal{E}_k$'s are disjoint sets. This cutting summation is valid for $1 \le p < \infty$ and can be directly seen by using the definition of $\|\cdot\|_{L^{p\infty}_u}$ and the fact that

$$\Big\{x \in \mathbb{R}^n \Big| \sum_k f(x)\mathbf{1}_{\mathcal{E}_k}(x) > \lambda\Big\} = \Big\{x \in \bigcup_j \mathcal{E}_j \Big| \sum_k f(x)\mathbf{1}_{\mathcal{E}_k}(x) > \lambda\Big\} = \bigcup_j \big\{x \in \mathcal{E}_j \big| f(x) > \lambda\big\}.$$

To prove (4.3) it can be assumed that $f(\cdot) \ge 0$ is a bounded function with compact support, since the further estimates do not depend on the bound of $f(\cdot)$, and the monotone convergence theorem will yield the conclusion for general non-negative functions. Since $0 < \int_{\mathbb{R}^n} f(y)\,dy < \infty$, then $2^N < \int_{\mathbb{R}^n} f(y)\,dy \le 2^{N+1}$ for some integer N. By the fact that $r \in [0,\infty) \to \int_{|y|<\frac{1}{2}r} f(y)\,dy$ defines an increasing and continuous function, there is an increasing sequence $(r_m)_{m=-\infty}^N$ of non-negative reals such that

$$2^m = \int_{|y|<\frac{1}{2}r_m} f(y)\,dy = 2\int_{\frac{1}{2}r_{m-1}<|y|<\frac{1}{2}r_m} f(y)\,dy \tag{4.5}$$

and $2^N = \int_{|y|<\frac{1}{2}r_N} f(y)\,dy$. Let

$$E_m = \Big\{x \in \mathbb{R}^n \Big| 2^m < \int_{|y|<\frac{1}{2}|x|} f(y)\,dy \le 2^{m+1}\Big\} = \Big\{x \in \mathbb{R}^n \Big| r_m < |x| \le r_{m+1}\Big\} \qquad (m \le N-1). \tag{4.6}$$

Setting $r_{N+1} = \infty$, then

$$\text{the } E_k \text{ are pairwise disjoint sets and } \mathbb{R}^n = \cup_{m=-\infty}^N E_m. \tag{4.7}$$

Now we are ready to give the chain of estimates which yields inequality (4.3). Indeed,

$$\begin{aligned}
&\Big\|w(\cdot)|\cdot|^{\alpha-n}\Big(\int_{|y|<\frac{1}{2}|\cdot|} f(y)\,dy\Big)\Big\|^p_{L^{p\infty}_u} \\
&= \Big\|\sum_{m=-\infty}^{N} w(\cdot)|\cdot|^{\alpha-n}\Big(\int_{|y|<\frac{1}{2}|\cdot|} f(y)\,dy\Big)\mathbf{1}_{E_m}(\cdot)\Big\|^p_{L^{p\infty}_u} \quad (\text{by } (4.7)) \\
&\le c_1 \sum_{m=-\infty}^{N} 2^{mp}\Big\|w(\cdot)|\cdot|^{\alpha-n}\mathbf{1}_{E_m}(\cdot)\Big\|^p_{L^{p\infty}_u} \quad (\text{by } (4.4) \text{ and } (4.6)) \\
&\le c_2 \sum_{m=-\infty}^{N} \Big(\int_{\{\frac{1}{2}r_{m-1}<|\cdot|\le\frac{1}{2}r_m\}} f(y)\,dy\Big)^p \Big\|w(\cdot)|\cdot|^{\alpha-n}\mathbf{1}_{\{r_m<|\cdot|\le r_{m+1}\}}(\cdot)\Big\|^p_{L^{p\infty}_u} \\
&(\text{by } (4.5))
\end{aligned}$$

$$\le c_2 \sum_{m=-\infty}^{N} \Big(\int_{\{\frac{1}{2}r_{m-1}<|\cdot|\le\frac{1}{2}r_m\}} f^p(y)\,v(y)\,dy \Big)$$

$$\times \Big(\int_{\{|\cdot|\le\frac{1}{2}r_m\}} v^{1-p'}(y)dy \Big)^{\frac{p}{p'}} \Big\| w(\cdot)|\cdot|^{\alpha-n}\mathbf{1}_{\{2(\frac{1}{2}r_m)<|\cdot|\}}(\cdot)\Big\|^p_{L^{p\infty}_u}$$

(*here the Hölder inequality is applied if* $p>1$

and for $p=1$: $\left(\int_{\{|\cdot|\le\frac{1}{2}r_m\}} v^{1-p'}(y)\,dy\right)^{\frac{p}{p'}}$ *is replaced by* $\operatorname{ess\,sup}\left(\frac{1}{v(\cdot)}\mathbf{1}_{|\cdot|<\frac{1}{2}r_m}(\cdot)\right)$)

$$\le c_2 A^p \sum_{m=-\infty}^{N} \int_{\{\frac{1}{2}r_{m-1}<|\cdot|\le\frac{1}{2}r_m\}} f^p(y)v(y)\,dy$$

(*by using condition* (3.2) *if* $p>1$ *and* (3.3) *for* $p=1$)

$$= c_2 A^p \int_{\mathbb{R}^n} \Big[\sum_{m=-\infty}^{N} \mathbf{1}_{\{\frac{1}{2}r_{m-1}<|\cdot|\le\frac{1}{2}r_m\}}(y) \Big] f^p(y)\,v(y)\,dy$$

(*since the sets* $\{\frac{1}{2}r_{m-1}<|\cdot|\le\frac{1}{2}r_m\}$ *are disjoint*)

$$\le c_2 A^p \int_{\mathbb{R}^n} f^p(y)\,v(y)\,dy$$

$$= c_2 A^p \|f(\cdot)\|^p_{L^p_v}.$$

Proof of Proposition 3.2. It is suitable to introduce the notations

$$E_k = \{y\in\mathbb{R}^n \,|\, 2^k<|y|\le 2^{k+1}\} \qquad \text{and} \qquad F_k = \{y\in\mathbb{R}^n \,|\, 2^{k-1}<|y|\le 2^{k+2}\}$$

$$\mathcal{W}_k = \sup_{z\in E_k} w(z) \qquad \text{and} \qquad \mathcal{U}_k = \sup_{y\in E_k} u(y).$$

So using the property $w(\cdot)\in\mathcal{A}$, then

$$\mathcal{W}_k \le \sup_{4^{-1}|x|<|z|<4|x|} w(z) \le c_1 w(x) \qquad \text{for each } x\in F_k. \tag{4.8}$$

Here $c_1>0$ is a constant which only depends on the fact that $w(\cdot)\in\mathcal{A}$. Similarly,

$$\mathcal{U}_k \le \sup_{4^{-1}|x|<|z|<4|x|} u(z) \qquad \text{for each } x\in F_k. \tag{4.9}$$

One of the crucial points to get $\widetilde{I}_\alpha : L^p_v \to wL^{p\infty}_u$ is the inequality

$$\int_{E_k} (I_\alpha f\mathbf{1}_{F_k})^p(x)\,|x|^{-\alpha p}dx \le C\int_{F_k} f^p(x)\,dx \qquad \text{for all } f(\cdot)\ge 0 \tag{4.10}$$

where $C>0$ is a fixed constant. Inequality (4.10) is true for $p=1$ since

$$\int_{E_k} (I_\alpha f\mathbf{1}_{F_k})(x)\,|x|^{-\alpha}dx \sim 2^{-k\alpha}\int_{E_k} (I_\alpha f\mathbf{1}_{F_k}(x)\,dx$$

$$= 2^{-k\alpha}\int_{F_k} f(x)(I_\alpha 1_{E_k})(x)\,dx \le c2^{-k\alpha}\int_{F_k} f(x)2^{k\alpha}dx = c\int_{F_k} f(x)\,dx.$$

For $p > 1$, (4.10) is also true since $I_\alpha : L^p(dx) \to L^p(|x|^{-\alpha p}\,dx)$. This last boundedness can be seen by applying one of the well-known boundedness criteria for I_α on weighted L^p-spaces (see [5]).

Now assuming hypothesis (3.4), the boundedness $\widetilde{I}_\alpha : L^p_v \to wL^{p\infty}_u$ can be obtained as follows:

$$\|w(\cdot)(\widetilde{I}_\alpha f)(\cdot)\|^p_{L^{p\infty}_u}$$

$$= \Big\|\sum_k w(\cdot)\Big(\int_{\frac{1}{2}|\cdot|<|y|<2|\cdot|} |\cdot - y|^{\alpha-n} f(y)\,dy\Big) 1_{E_k}(\cdot)\Big\|^p_{L^{p\infty}_u}$$

$$\le \sum_k \mathcal{W}^p_k \mathcal{U}_k \|(I_\alpha f 1_{F_k})(\cdot)1_{E_k}(\cdot)\|^p_{L^{p\infty}_1}$$

$$\big(\textit{by the definition of } \|g(\cdot)\|^p_{L^{p\infty}_u}\big)$$

$$\le c_2 \sum_k 2^{\alpha k p}\mathcal{W}^p_k \mathcal{U}_k \int_{E_k} (I_\alpha f 1_{F_k})^p(x)\,|x|^{-\alpha p}dx$$

$$\big(\textit{recall that } L^p_1 \subset L^{p\infty}_1\big)$$

$$\le c_3 \sum_k 2^{\alpha k p}\mathcal{W}^p_k \mathcal{U}_k \int_{F_k} f^p(x)\,dx \quad \textit{(see (4.10))}$$

$$\le 2^\alpha c_3 \sum_k \int_{F_k} f^p(x)\Big[\mathcal{W}_k |x|^\alpha \mathcal{U}_k^{\frac{1}{p}}\Big]^p dx$$

$$\le c_4 \sum_k \int_{F_k} f^p(x)\Big[w(x)|x|^\alpha\Big(\sup_{4^{-1}|x|<|z|<4|x|} u(z)\Big)^{\frac{1}{p}}\Big]^p dx \quad \textit{(by (4.8) and (4.9))}$$

$$\le c_4 A^p \sum_k \int_{2^{k-1}<|x|\le 2^{k+2}} f^p(x)\,v(x)\,dx \quad \textit{(by condition (3.4))}$$

$$\le 3c_4 A^p \sum_k \int_{2^k<|x|\le 2^{k+1}} f^p(x)\,v(x)\,dx$$

$$= 3c_4 A^p \int_{\mathbb{R}^n} f^p(y)\,v(y)\,dy$$

$$= 3c_4 A^p \|f(\cdot)\|^p_{L^p_v}.$$

If instead of (3.4) condition (3.4)$'$ is assumed, then the modifications in obtaining the conclusion are as follows:

$$\|w(\cdot)(\widetilde{I}_\alpha f)(\cdot)\|^p_{L^{p\infty}_u}$$

$$\le c_5 \sum_k \int_{F_k} f^p(x)\Big[\mathcal{W}_k 2^{\alpha k}\mathcal{U}_k^{\frac{1}{p}}\Big]^p dx \quad \textit{(see the above estimates)}$$

$$\le c_6 \sum_k \int_{F_k} f^p(x) \sup_{y\in E_k}\Big\{\mathcal{W}_k |y|^\alpha \big(u(y)\big)^{\frac{1}{p}}\Big\}^p dx$$

$$\le c_7 \sum_k \int_{F_k} f^p(x)\, v(x) \sup_{y\in E_k} \Big\{ w(y)|y|^\alpha \big(u(y)\big)^{\frac1p} \Big(\frac{1}{v(x)}\Big)^{\frac1p} \Big\}^p dx$$

$$\le c_7 \sum_k \int_{F_k} f^p(x) v(x) \sup_{y\in E_k} \Big\{ w(y)|y|^\alpha \big(u(y)\big)^{\frac1p} \Big(\sup_{4^{-1}|y|<|z|<4|y|} \frac{1}{v(z)}\Big)^{\frac1p} \Big\}^p dx$$

$$\le c_8 A^p \sum_k \int_{F_k} f^p(x)\, v(x)\, dx \quad \text{(by condition (3.4)')}$$

$$\le 3c_8 A^p \int_{\mathbb{R}^n} f^p(y)\, v(y)\, dy$$

$$= 3c_8 A^p \|f(\cdot)\|^p_{L^p_v}.$$

If no assumption like $w(\cdot) \in \mathcal{A}$ is assumed, the boundedness $\widetilde{I}_\alpha : L^p_v \to wL^{p\infty}_u$ can be similarly obtained if instead of (4.10) then

$$\int_{E_k} (I_\alpha f \mathbf{1}_{F_k})^p(x)\, w^p(x)\, |x|^{-\alpha p} dx \le C \int_{F_k} f^p(x)\, dx \qquad \text{for all } f(\cdot) \ge 0$$

which is true whenever $I_\alpha : L^p(\mathbb{R}^n, dx) \to L^p(\mathbb{R}^n, w^p(x)\, |x|^{-\alpha p} dx)$.

Proof of Proposition 3.3. To prove the implication (3.5)$'$ $\Longrightarrow$ (3.4), take $p > 1$. Using $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H}$ and $w(\cdot) \in \mathcal{A}$, then

$$w(x)|x|^\alpha \Big(\sup_{4^{-1}|x|<|y|<4|x|} u(y)\Big)^{\frac1p} \Big(v^{1-p'}(x)\Big)^{\frac{1}{p'}}$$

$$\le c_1 w(x)|x|^{\alpha-n} \Big(\int_{2^{-N}|x|<|y|<2^N|x|} u(y)\, dy\Big)^{\frac1p} \Big(\int_{2^{-N}|x|<|y|<2^N|x|} v^{1-p'}(y)\, dy\Big)^{\frac{1}{p'}}$$

$$= c_2 w(x)|x|^{\alpha-n} \big\|\mathbf{1}_{2^{-N}|x|<|\cdot|<2^N|x|}(\cdot)\big\|_{L^{p\infty}_u} \Big(\int_{2^{-N}|x|<|y|<2^N|x|} v^{1-p'}(y)\, dy\Big)^{\frac{1}{p'}}$$

$$\le c_3 |x|^{\alpha-n} \big\|w(\cdot)\mathbf{1}_{2^{-N}|x|<|\cdot|<2^N|x|}(\cdot)\big\|_{L^{p\infty}_u} \Big(\int_{2^{-N}|x|<|y|<2^N|x|} v^{1-p'}(y)\, dy\Big)^{\frac{1}{p'}}$$

$$\le c_3 A \quad \text{(by using condition (3.5)')}.$$

The implication (3.6)$'$ $\Longrightarrow$ (3.4) can be proved by using a similar argument, except that no growth condition on $v(\cdot)$ is needed. Therefore Proposition 3.3 is proved since trivially (3.5) $\Longrightarrow$ (3.5)$'$ and (3.6) $\Longrightarrow$ (3.6)$'$.

Proof of Proposition 3.4. Part A: Since $w(x) = |x|^{-\gamma}$ for $\gamma \ge 0$, then $\{x : w(x)|x|^{\alpha-n} > \lambda\} = \{x : |x| < \lambda^{\frac{1}{\alpha-n-\gamma}}\}$ and

$$\begin{aligned}
\big\|w(\cdot)|\cdot|^{\alpha-n}\mathbf{1}_{2R<|\cdot|}(\cdot)\big\|_{L^{p\infty}_u} &= \sup_{r>2R} \Big\{ r^{(\alpha-n-\gamma)} \Big(\int_{2R<|x|<r} u(x)\, dx\Big)^{\frac1p} \Big\} \\
&= \sup_{r>2R} \Big\{ r^{(\alpha-n-\gamma)} \big\|\mathbf{1}_{2R<|\cdot|<r}(\cdot)\big\|_{L^{p\infty}_u} \Big\} \\
&\le \sup_{r>2R} \Big\{ r^{(\alpha-n)} \big\||\cdot|^{-\gamma}\mathbf{1}_{2R<|\cdot|<r}(\cdot)\big\|_{L^{p\infty}_u} \Big\} \\
&\le \sup_{r>2R} \Big\{ r^{(\alpha-n)} \big\|w(\cdot)\mathbf{1}_{|\cdot|<r}(\cdot)\big\|_{L^{p\infty}_u} \Big\}.
\end{aligned}$$

The implications (3.5) $\Longrightarrow$ (3.2) and (3.6) $\Longrightarrow$ (3.3) appear immediately from this last inequality.

Part B: Assuming the Muckenhoupt condition (3.5) and $v^{1-p'}(\cdot) \in RD_\rho$, the Hardy condition (3.2) is satisfied since

$$\begin{aligned}
&\big\| w(\cdot)|\cdot|^{\alpha-n}\mathbf{1}_{2R<|\cdot|}(\cdot)\big\|^p_{L^{p\infty}_u}\Big(\int_{|y|<R} v^{1-p'}(y)\,dy\Big)^{\frac{p}{p'}} \\
&\leq c_1\sum_{k=1}^{\infty}(2^kR)^{(\alpha-n)p}\big\|w(\cdot)\mathbf{1}_{2^kR<|\cdot|<2^{k+1}R}(\cdot)\big\|^p_{L^{p\infty}_u}\Big(\int_{|y|<R} v^{1-p'}(y)\,dy\Big)^{\frac{p}{p'}} \\
&\qquad \textit{(by using the cutting summation (4.4))} \\
&\leq c_2\sum_{k=1}^{\infty}2^{-kn(p-1)\rho}(2^{k+1}R)^{(\alpha-n)p}\big\|w(\cdot)\mathbf{1}_{|\cdot|<2^{k+1}R}(\cdot)\big\|^p_{L^{p\infty}_u}\Big(\int_{|y|<2^{k+1}R} v^{1-p'}(y)\,dy\Big)^{\frac{p}{p'}} \\
&\qquad \Big(\textit{since } v^{1-p'}(\cdot)\in RD_\rho\Big) \\
&\leq c_2A^p\sum_{k=1}^{\infty}2^{-kn(p-1)\rho} \\
&= c_3A^p \quad \textit{(by condition (3.5) and } p>1\textit{)}.
\end{aligned}$$

Proof of Theorem 3.5. Part A: The necessary part is essentially described in Theorem 3.1. The sufficient part is a consequence of Proposition 3.3 and Theorem 3.1.

Part B: In view of Proposition 3.3 and Theorem 3.1, to prove $I_\alpha : L^p_v \to L^{p\infty}_u$ it remains to check condition (3.5)′ by using (3.2) or (3.2)*. Note that $u(\cdot), v^{1-p'}(\cdot) \in \mathcal{H}$ by Lemma 2.3. Suppose for instance that $v^{1-p'}(\cdot) \nearrow$. Then

$$\begin{aligned}
\int_{R<|y|<2^NR} v^{1-p'}(y)\,dy &\leq c_1\int_{2(2^NR)<|y|<4(2^NR)} v^{1-p'}(y)\,dy \\
&\leq c_2R^{-(\alpha-n)p'}\Big(\int_{2(2^NR)<|y|} |y|^{(\alpha-n)p'}v^{1-p'}(y)\,dy\Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
Q(R) &= R^{\alpha-n}\big\|w(\cdot)\mathbf{1}_{R<|\cdot|<2^{2N}R}(\cdot)\big\|_{L^{p\infty}_u}\Big(\int_{R<|y|<2^{2N}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \\
&\leq c_3\big\|w(\cdot)\mathbf{1}_{|\cdot|<2^{2N}R}(\cdot)\big\|_{L^{p\infty}_u}\Big(\int_{2(2^{2N}R)<|y|} |y|^{(\alpha-n)p'}v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \\
&\leq c_3A \quad \textit{(by condition (3.2)*)}.
\end{aligned}$$

For $v^{1-p'}(\cdot) \searrow$ then $\int_{R<|y|<2^NR} v^{1-p'}(y)dy \leq c_4\int_{|y|<\frac{1}{2}R} v^{1-p'}(y)\,dy$ and hence, by using condition (3.2),

$$Q(R) \leq c_5\big\|w(\cdot)|\cdot|^{\alpha-n}\mathbf{1}_{2(\frac{1}{2}R)<|\cdot|}(\cdot)\big\|_{L^{p\infty}_u}\Big(\int_{|y|<\frac{1}{2}R} v^{1-p'}(y)\,dy\Big)^{\frac{1}{p'}} \leq c_5A.$$

Part C: This statement is an immediate consequence of Proposition 3.4.

Proof of Theorem 3.6. The arguments are the same as those used in the proof of Theorem 3.5.

Proof of Theorem 2.1. The necessity part is immediately given by parts A in Theorems 3.5 and 3.6. The sufficient part can be seen by applying Theorem 3.1 and Part A in Proposition 3.4.

Proof of Proposition 2.2. This result is just a statement of Proposition 3.2 with $w(\cdot) = 1$.

Proof of Lemma 2.3. It remains to estimate $w(y)$ for each y with $\frac{1}{4}R < |y| < 4R$. First consider the case where R is small, i.e. $R < \frac{1}{8}R_0$. Since $8R < R_0$, for $w_1(\cdot) \nearrow$ then

$$w(y) = w_1(y) \leq \frac{c}{R^n}\int_{4R<|z|<8R} w_1(z)\,dz = \frac{c}{R^n}\int_{4R<|z|<8R} w(z)\,dz.$$

And for $w_1(\cdot) \searrow$ then

$$w(y) = w_1(y) \leq \frac{c}{R^n}\int_{\frac{1}{8}R<|z|<\frac{1}{4}R} w_1(z)\,dz = \frac{c}{R^n}\int_{\frac{1}{8}R<|z|<\frac{1}{4}R} w(z)\,dz.$$

Similarly, if R is big, i.e. $8R_0 < R$, and with $w_2(\cdot) \nearrow$ or $w_2(\cdot) \searrow$ then

$$w(y) = w_2(y) \leq \frac{c}{R^n}\int_{\frac{1}{8}R<|z|<8R} w_2(z)\,dz = \frac{c}{R^n}\int_{\frac{1}{8}R<|z|<8R} w(z)\,dz.$$

Finally, for $R \approx R_0$, i.e. $\frac{1}{8}R_0 \leq R \leq 8R_0$, then $w(y) \leq C$ for a fixed constant $C > 0$ which depends only on $w(\cdot)$. Assume for instance that $\frac{1}{8}R_0 \leq R \leq R_0$. If $w_1(\cdot) \nearrow$, then

$$w(y) \leq \frac{c}{w_1(\frac{1}{32}R_0)R^n}\int_{\frac{1}{4}R<|z|<R} w_1(z)\,dz \leq \frac{C}{R^n}\int_{\frac{1}{4}R<|z|<4R} w(z)\,dz.$$

And when $w_1(\cdot) \searrow$, then

$$w(y) \leq \frac{c}{w_1(R_0)R^n}\int_{\frac{1}{4}R<|z|<R} w_1(z)\,dz \leq \frac{C}{R^n}\int_{\frac{1}{4}R<|z|<4R} w(z)\,dz.$$

Analogously, for $R_0 \leq R < 8R_0$ and $w_2(\cdot) \nearrow$ or $w_2(\cdot) \searrow$, then

$$w(y) \leq \frac{C}{R^n}\int_{R<|z|<4R} w_2(z)\,dz \leq \frac{C}{R^n}\int_{\frac{1}{8}R<|z|<8R} w(z)\,dz.$$

Proof of Proposition 2.4. Proposition 2.4 is just a consequence of Proposition 3.3 with $w(\cdot) = 1$.

Proof of Proposition 2.5. The fact that the Muckenhoupt condition (2.1) implies the dual Hardy condition (2.2) for $u(\cdot) \in RD_\rho$ can be seen as in the proof of part B of

Lemma 2.6 (see also [5: p. 832 - 833]). The implication (2.3) $\Longrightarrow$ (2.4) is also true for the same hypothesis on $u(\cdot)$. Indeed,

$$\begin{aligned}
&\left(\int_{|x|<R} u(x)dx\right) \times \operatorname{ess\,sup}\left[|y|^{\alpha-n}\frac{1}{v(y)}\mathbf{1}_{2R<|\cdot|}(y)\right]\\
&\leq c_1\sum_{k=1}^{\infty}(2^{k+1}R)^{\alpha-n}\left(\int_{|x|<R} u(x)dx\right)\times \operatorname{ess\,sup}\left[\frac{1}{v(y)}\mathbf{1}_{2^kR<|\cdot|\leq 2^{k+1}R}(y)\right]\\
&\leq c_2\sum_{k=1}^{\infty}2^{-kn\rho}(2^{k+1}R)^{\alpha-n}\left(\int_{|x|<2^{k+1}R} u(x)dx\right)\times \operatorname{ess\,sup}\left[\frac{1}{v(y)}\mathbf{1}_{|\cdot|\leq 2^{k+1}R}(y)\right]\\
&\leq c_2A\sum_{k=1}^{\infty}2^{-kn\rho}\\
&= c_3A.
\end{aligned}$$

Proof of Theorem 2.6. Part A: The sufficient part is just a consequence of Theorem 2.1 and Proposition 2.4. Part B is true by Proposition 2.4. For Part C it is sufficient to apply Proposition 2.5.

Proof of Theorem 2.7. It is sufficient to follow the same argument as in Theorem 2.6.

Proof of Corollary 2.8. Recall that

$$u(x) = \begin{cases} u_1(x) = |x|^{(\beta-n)} & \text{for } |x| < R_0 \\ u_2(x) = |x|^{\gamma-n} & \text{for } |x| > R_0 \end{cases}$$

and

$$\sigma(x) = \begin{cases} v_1^{-\frac{1}{p-1}}(x) = |x|^{-n}\ln^{-p'}(|x|^{-1}) & \text{for } |x| < R_0 \\ v_2^{-\frac{1}{p-1}}(x) = |x|^{-\frac{\theta-n}{p-1}} & \text{for } |x| > R_0. \end{cases}$$

The dual Hardy condition (2.2) is satisfied when

$$\left(\mathcal{U}_1(R_0) + \int_{R_0<|x|<R} u_2(x)\,dx\right)^{\frac{1}{p}}\left(\int_{R<|x|}|x|^{(\alpha-n)p'}\sigma_2(x)dx\right)^{\frac{1}{p'}} \leq C \tag{4.11}$$

for $R_0 < R$, and

$$\left(\int_{|x|<R} u_1(x)dx\right)^{\frac{1}{p}}\left(\Sigma_2(R_0) + \int_{R<|x|<R_0}|x|^{(\alpha-n)p'}\sigma_1(x)\,dx\right)^{\frac{1}{p'}} \leq C \tag{4.12}$$

for $R < R_0$. The Muckenhoupt condition (2.1) is equivalent to

$$R^{\alpha-n}\left(\int_{|x|<R} u_1(x)dx\right)^{\frac{1}{p}}\left(\int_{|x|<R}\sigma_1(x)dx\right)^{\frac{1}{p'}} \leq C \tag{4.13}$$

for $R < R_0$ and

$$R^{\alpha-n}\left(\mathcal{U}_1(R_0)+\int_{R_0<|x|<R} u_2(x)dx\right)^{\frac{1}{p}}\left(\Sigma_1(R_0)+\int_{R_0<|x|<R}\sigma_2(x)\,dx\right)^{\frac{1}{p'}} \leq C \quad (4.14)$$

for $R_0 < R$. In these conditions

$$\Sigma_1(R)=\int_{|x|<R}\sigma_1(x)\,dx,\quad \mathcal{U}_1(R)=\int_{|x|<R}u_1(x)\,dx,\quad \overline{\Sigma}_2(R)=\int_{R<|x|}|x|^{(\alpha-n)p'}\sigma_2(x)\,dx.$$

For $R \leq R_0$ and by standard calculations then

$$\mathcal{U}_1(R)\approx R^{\beta},\quad \Sigma_1(R)\approx \ln^{-\frac{p'}{p}}(R^{-1}),\quad \int_{R<|x|<R_0}|x|^{(\alpha-n)p'}\sigma_1(x)\,dx \leq cR^{(\alpha-n)p'}.$$

On the other hand, for $R_0 < R$

$$\int_{R_0<|x|<R} u_2(x)dx \leq c\times\begin{cases} R^{\gamma} & \text{for } \gamma>0\\ \ln R+\ln(R_0^{-1}) & \text{for } \gamma=0\\ R_0^{\gamma} & \text{for } \gamma<0\end{cases}$$

and since $\alpha p < \theta$, then $\overline{\Sigma}_2(R) \approx R^{p'[\alpha-\frac{\theta}{p}]}$ and

$$\int_{R_0<|x|<R}\sigma_2(x)\,dx \leq c\times\begin{cases} R^{p'[n-\frac{\theta}{p}]} & \text{for } \theta<np\\ \ln R+\ln(R_0^{-1}) & \text{for } \theta=np\\ R_0^{p'[n-\frac{\theta}{p}]} & \text{for } np<\theta.\end{cases}$$

By Lemma 2.3, we have $u(\cdot), v^{-\frac{1}{p-1}}(\cdot) \in \mathcal{H}$. In view of the above calculations, condition (4.11) is true since $\alpha p < \theta$ and $\alpha p + \gamma \leq \theta$. Also, (4.12) is ensured by $\beta > 0$ and $np \leq \alpha p + \beta$. This last inequality also leads to (4.13). Finally, condition (4.14) is appeared by using $\alpha p < \theta$ and $\alpha p + \gamma \leq \theta$. Consequently, by Part B in Theorem 2.6, then $I_\alpha : L^p_v \to L^{p\infty}_u$.

The boundedness $I_\alpha : L^p_{v^*} \to L^{p\infty}_{u^*}$ can be obtained by using similar arguments. The details are omitted.

Proof of Corollary 2.9. The conclusion will be obtained from an application of Part C in Theorem 2.6 with $v(\cdot) = (M_p u)(\cdot)$. Precisely the main key is to check the Muckenhoupt condition (2.1) and the pointwise condition (2.6)$'$ because $u(\cdot) \in RD_\rho$.

The Muckenhoupt condition (2.1) appears immediatly once for a fixed constant $c > 0$

$$v^{-\frac{1}{p-1}}(x) \leq c\left(R^{\alpha p-n}\int_{|y|<R}u(y)\,dy\right)^{1-p'} \qquad \text{for all } |x|<R. \qquad (4.13)$$

Indeed, by this inequality

$$\begin{aligned} &R^{\alpha-n}\left(\int_{|y|<R}u(y)\,dy\right)^{\frac{1}{p}}\left(\int_{|x|<R}v^{-\frac{1}{p-1}}(x)\,dx\right)^{\frac{1}{p'}}\\ &\leq c_1R^{\alpha-n}\left(\int_{|y|<R}u(y)\,dy\right)^{\frac{1}{p}}\left[R^n\left(R^{\alpha p-n}\int_{|y|<R}u(y)\,dy\right)^{1-p'}\right]^{\frac{1}{p'}}\\ &\leq c_1.\end{aligned}$$

Inequality (4.13) is true since for $|x| < R$

$$R^{\alpha p-n}\int_{|y|<R} u(y)\,dy \leq R^{\alpha p-n}\int_{|x-y|<2R} u(y)\,dy \leq c(M_{\alpha p}u)(x) = c\,v(x).$$

To prove $(2.6)'$ it is sufficient to find a fixed constant $C > 0$ for which

$$|x|^{\alpha p}u(x)\frac{1}{v(y)} \leq C \qquad \text{for } 4^{-1}|x| < |y| < 4|x|.$$

It is equivalent to write

$$u(x) \leq c_2|y|^{-\alpha p}v(y) = c_2|y|^{-\alpha p}(M_{\alpha p}u)(y). \tag{4.14}$$

Inequality (4.14) is an easy consequence of the fact that $u(\cdot) \in RD_\rho$. Indeed, for some constants $c_3, c_4 > 0$

$$\begin{aligned} u(x) &\leq c_3|y|^{-n}\int_{c_4^{-1}|y|<|z|<c_4|y|} u(z)\,dz \\ &\leq c_3|y|^{-\alpha p}|y|^{\alpha p-n}\int_{|z-y|<(1+c_4)|y|} u(z)\,dz \\ &\leq c_5|y|^{-\alpha p}(M_{\alpha p}u)(y). \end{aligned}$$

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Received 17.12.1996; in revised form 06.11.1997