Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 17 (1998), No.1, 135 - 157

A Characterization of the Dependence of the Riemannian Metric on the Curvature Tensor by Young Symmetrizers

B. Fiedler

Abstract. In differential geometry several differential equation systems are known which allow the determination of the Riemannian metric from the curvature tensor in normal coordinates. We consider two of such differential equation systems. The first system used by Günther [8] yields a power series of the metric the coefficients of which depend on the covariant derivatives of the curvature tensor symmetrized in a certain manner. The second system, the so-called Herglotz relations [9], leads to a power series of the metric depending on symmetrized partial derivatives of the curvature tensor.

We determine a left ideal of the group ring $\mathbb{C}[\mathcal{S}_{r+4}]$ of the symmetric group $\mathcal{S}_{r+4}$ which is associated with the partial derivatives $\partial^{(r)} R$ of the curvature tensor R of order r and construct a decomposition of this left ideal into three minimal left ideals using Young symmetrizers and the Littlewood-Richardson rule. Exactly one of these minimal left ideals characterizes the so-called essential part of $\partial^{(r)} R$ on which the metric really depends via the Herglotz relations. We give examples of metrics with and without a non-essential part of $\partial^{(r)} R$. Applying our results to the covariant derivatives of the curvature tensor we can show that the algebra of tensor polynomials $\mathcal{R}$ generated by $\nabla_{(i_1} \ldots \nabla_{i_r)} R_{ijkl}$ and the algebra $\mathcal{R}^s$ generated by $\nabla_{(i_1} \ldots \nabla_{i_r} R_{|k|i_{r+1}i_{r+2})l}$ fulfil $\mathcal{R} = \mathcal{R}^s$.

Keywords: *Calculation of a metric, curvature tensor, partial derivatives of the curvature tensor, covariant derivatives of the curvature tensor, algebras of tensor polynomials, Herglotz relations, power series method, minimal left ideals, Young symmetrizers, Littlewood-Richardson rule, use of computer algebra systems.*

AMS subject classification: Primary 05 E 10, 53 C 21, secondary 20 C 05, 53 A 55, 53 - 04

1. Introduction

Several investigations in differential geometry and general relativity theory make use of certain differential equation systems which allow to determine a pseudo-Riemannian metric from its Riemannian curvature tensor in normal coordinates. P. Günther has established the following construction of a differential equation system of such a type in [8: Appendix I].

Let (M, g) be an n-dimensional analytic pseudo-Riemannian manifold with metric g and Levi-Civita connection ∇, and let $\{U, x\}$ be a normal coordinate system of (M, g)

B. Fiedler: Universität Leipzig, Mathematisches Institut, Augustuspl. 10/11, D - 04109 Leipzig, Germany

ISSN 0232-2064 / $ 2.50

around the centre $P_0 \in U \subseteq M$, i.e. $x(P_0) = 0$. If we choose an orthonormal basis $\{v_1, \ldots, v_n\} \subset M_{P_0}$ of the tangent space M_{P_0} of the manifold M in the point P_0 and carry out a parallel transport of this basis along every geodesic starting in P_0, we obtain n smooth vector fields $\{X_1, \ldots, X_n\}$ on a suitable open neighbourhood $U' \subseteq U$ of P_0 which form an n-frame in every point of U'. We denote by $T_{A_1 \ldots A_r} := T(X_{A_1}, \ldots, X_{A_r})$ the coordinates of a covariant tensor field T of order r with respect to $\{X_1, \ldots, X_n\}$ and by $T_{i_1 \ldots i_r} := T(\partial_{i_1}, \ldots, \partial_{i_r})$ the coordinates of the same tensor field with respect to the basis vector fields $\partial_i := \partial/\partial x^i$ of the normal coordinate system $\{U, x\}$. Then there hold true the relations

$$g_{ij} = \sigma_i^A \sigma_j^B \, g_{AB} \quad , \quad g_{AB} = \sigma_A^i \sigma_B^j \, g_{ij} \tag{1.1}$$

with the transformation matrices[1] $\sigma := (\sigma_i^A)$ and $\sigma^{-1} := (\sigma_A^i)$ defined by

$$\partial_i = \sigma_i^A \, X_A \quad , \quad X_A = \sigma_A^i \, \partial_i \ .$$

The coordinates g_{AB} in (1.1) fulfil

$$g_{AB} = \text{const} = \begin{cases} \pm 1 & \text{if} \quad A = B \\ 0 & \text{if} \quad A \neq B \end{cases} \tag{1.2}$$

where the number of $+1$ and -1 in (1.2) is given by the signature of the metric g.

P. Günther has shown in [8: Appendix I] that the matrix σ satisfies on an open neighbourhood of P_0 the relation[2]

$$XX(\sigma) + X(\sigma) + \sigma \cdot Q = 0 \ . \tag{1.3}$$

Here X denotes the vector field $X := x^i \partial_i$ formed from the normal coordinates x^i. Further, Q is an analytic $(n \times n)$-matrix-valued function with power series $Q = \sum_{l=2}^{\infty} Q_{(l)}$ the summands $Q_{(l)}$ of which are obtained by the equation $Q_{(l)} = \sigma^{-1}(P_0) \cdot R_{(l)} \cdot \sigma(P_0)$ from analytic $(n \times n)$-matrices $R_{(l)}$ which depend on the covariant derivatives of the Riemannian curvature tensor[3] R_{ijkl} according to

$$R_{(2)} := \left(R_{a i_1 i_2 b}(P_0) \, x^{i_1} x^{i_2} \right)_{a,b = 1, \ldots, n} \tag{1.4}$$

$$R_{(l)} := \left(\tfrac{1}{(l-2)!} (\nabla_{i_1} \ldots \nabla_{i_{l-2}} R_{a i_{l-1} i_l b})(P_0) \, x^{i_1} \ldots x^{i_l} \right)_{a,b = 1, \ldots, n} \quad , \ l \geq 3 \ . \tag{1.5}$$

Often, investigations in differential geometry use the algebra

$$\mathcal{R} := \langle \, g_{ij} \, , \, g^{ij} \, ; \, R_{ijkl} \, ; \, \nabla_{i_1} \ldots \nabla_{i_r} R_{ijkl} \, , \, r \geq 1 \, \rangle \tag{1.6}$$

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[1] The matrix (σ_A^i) can be regarded as the matrix of the parallel transport along the family of geodesics, described above, with respect to the basis vector fields ∂_i. A vector field Z which is parallel along this family of geodesics fulfils $Z = z^A X_A = (z^A \sigma_A^i) \partial_i$ with $z^A = \text{const}$.

[2] Important results on relations of type (1.3) have been published by P. Günther in [7].

[3] We use the convention $R_{ijk}{}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{is}^l \Gamma_{jk}^s - \Gamma_{js}^l \Gamma_{ik}^s$ with the connection coefficients $\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

of all such tensor expressions which are complex linear combinations of expressions formed from the tensor coordinates in $\langle\ldots\rangle$ by arbitrary multiplications and index contractions. Taking into account the so-called Ricci identities for the Riemannian curvature tensor

$$\begin{aligned}\nabla_{[a}\nabla_{b]}\nabla_{i_1}\ldots\nabla_{i_r}R_{j_1\ldots j_4} &= -\frac{1}{2}\sum_{t=1}^{r} R_{abi_t}{}^{m_t}\nabla_{i_1}\ldots\nabla_{m_t}\ldots\nabla_{i_r}R_{j_1\ldots j_4}\\ &\quad -\frac{1}{2}\sum_{t=1}^{4} R_{abj_t}{}^{m_t}\nabla_{i_1}\ldots\nabla_{i_r}R_{j_1\ldots m_t\ldots j_4}\end{aligned}$$

we see that the algebra $\mathcal{R}$ is generated already from g_{ij} , g^{ij} , R_{ijkl} and the symmetrized covariant derivatives of the curvature tensor,

$$\mathcal{R} = \langle\, g_{ij}\,,\, g^{ij}\,,\, R_{ijkl}\,;\, \nabla_{(i_1}\ldots\nabla_{i_r)}R_{ijkl}\,,\, r\geq 1\,\rangle \quad , \tag{1.7}$$

because the Ricci identities yield

$$\begin{aligned}\nabla_{i_1}\ldots\nabla_{i_r}R_{j_1\ldots j_4} &= \nabla_{(i_1}\ldots\nabla_{i_r)}R_{j_1\ldots j_4}\\ &\quad + \text{terms with covariant derivatives of } R \text{ of order } r' \leq r-2 \quad .\end{aligned}$$

(We denote by $(\ldots)$ or $[\ldots]$ the symmetrization or anti-symmetrization, respectively.)

Considering (1.5) we find out that the analytic matrix function Q in (1.3) depends only on the stronger symmetrized covariant derivatives

$$\nabla_{(i_1}\ldots\nabla_{i_r}R_{|a|i_{r+1}i_{r+2})b}$$

of the curvature tensor which lie in the algebra

$$\mathcal{R}^s \;:=\; \langle\, g_{ij}\,,\, g^{ij}\,;\, \nabla_{(i_1}\ldots\nabla_{i_r}R_{|a|i_{r+1}i_{r+2})b}\,,\, r\geq 0\,\rangle \tag{1.8}$$

formed from the generating tensor coordinates by the same operations like $\mathcal{R}$. (The notation $|a|$ means that the index a is excluded from the symmetrization.)

Obviously, $\mathcal{R}^s$ is a subalgebra of $\mathcal{R}$. Now the question arises whether the algebra $\mathcal{R}^s$ is equal to the algebra $\mathcal{R}$. We show the equality of these two algebras by considering a more general situation.

Besides (1.3), another differential equation system allowing the calculation of the Riemannian metric from the curvature tensor in normal coordinates is given by the so-called Herglotz relations [9] which we describe in Section 2. The Herglotz relations are non-linear differential equations and yield power series of the metric which are determined by the symmetrized partial derivatives of the curvature tensor

$$\partial_{(i_1}\ldots\partial_{i_r}R_{|a|i_{r+1}i_{r+2})b}(P_0) \quad .$$

The partial derivatives of the curvature tensor $\partial_{i_1}\ldots\partial_{i_r}R_{ijkl}$ satisfy the same symmetry properties like $\nabla_{(i_1}\ldots\nabla_{i_r)}R_{ijkl}$ with the exception of the second Bianchi identity

$$\nabla_h R_{ijkl} + \nabla_i R_{jhkl} + \nabla_j R_{hikl} = 0$$

such that the situation given by the Herglotz relations is algebraically more general than the situation in the case of (1.3).

Using the representation theory of the symmetric group $\mathcal{S}_r$, we can clear up the connection between $\partial_{i_1}\ldots\partial_{i_r}R_{ijkl}$ and $\partial_{(i_1}\ldots\partial_{i_r}R_{|a|i_{r+1}i_{r+2})b}$. The partial derivatives $\partial_{i_1}\ldots\partial_{i_r}R_{ijkl}$ induce group ring elements which lie in the direct sum

$$J_{(r)}\oplus\hat{J}_{(r)}\oplus\check{J}_{(r)}$$

of three minimal left ideals of $\mathbb{C}[\mathcal{S}_{r+4}]$ and the transition to the symmetrized partial derivatives $\partial_{(i_1}\ldots\partial_{i_r}R_{|a|i_{r+1}i_{r+2})b}$ corresponds to a linear mapping

$$J_{(r)}\oplus\hat{J}_{(r)}\oplus\check{J}_{(r)}\to J_{(r)}\cdot\epsilon\quad,\quad f\mapsto f\cdot\epsilon\quad,\quad \epsilon\in\mathbb{C}[\mathcal{S}_{r+4}]$$

which maps $\hat{J}_{(r)}\oplus\check{J}_{(r)}$ to 0. In the case of $\nabla_{(i_1}\ldots\nabla_{i_r)}R_{ijkl}$ and $\nabla_{(i_1}\ldots\nabla_{i_r}R_{|a|i_{r+1}i_{r+2})b}$ only the ideals $J_{(r)}$ and $J_{(r)}\cdot\epsilon$ are associated with these covariant derivatives. The inverse mapping $J_{(r)}\cdot\epsilon\to J_{(r)}$ gives us a relation between $\nabla_{(i_1}\ldots\nabla_{i_r)}R_{ijkl}$ and $\nabla_{(i_1}\ldots\nabla_{i_r}R_{|a|i_{r+1}i_{r+2})b}$ which yields $\mathcal{R}=\mathcal{R}^s$.

2. The Herglotz relations

In this section we give a short summary of the paper [9] in which G. Herglotz states his method of determination of a Riemannian metric from the coordinates of the Riemannian curvature tensor in normal coordinates.

Proposition 2.1. *Let (M,g) be an n-dimensional pseudo-Riemannian manifold with metric g and Levi-Civita connection ∇, and let $\{U,x\}$ be a system of normal coordinates on a normal neighbourhood $U\subseteq M$ with centre $P_0\in U$, i.e. $x(P_0)=0$. If we form the differential operator $X:=x^i\partial_i$ and the $(n\times n)$-matrices*

$$G:=(g_{ij})\quad,\quad K:=(R_{iklj}x^kx^l)\quad,\quad i\text{ row index}\ ,\ j\text{ column index}$$

from the coordinates g_{ij}, R_{iklj} of the metric g and the Riemannian curvature tensor R with respect to $\{U,x\}$, then on U there holds true the so-called Herglotz relation[1]

$$XX(G)+X(G)-\tfrac{1}{2}X(G)\cdot G^{-1}\cdot X(G)\ =\ -2K\ . \tag{2.1}$$

Now we assume the g_{ij} to be analytic functions on U and make use of the facts that $\partial_i g_{jk}(P_0)=0$ in normal coordinates $\{U,x\}$ and that the metric coordinates $g_{ij}(P_0)$ in P_0 may be transformed into

$$G(P_0)\ =\ F\ :=\ \mathrm{diag}(1,\ldots,1,-1,\ldots,-1) \tag{2.2}$$

by an allowed linear coordinate transformation. The numbers of 1 and -1 in the diagonal matrix F are determined by the signature of the metric g. Thus we can write G as a matrix-valued power series

$$G=F(E-\Gamma)\quad,\quad\Gamma\ =\ \sum_{k=2}^{\infty}\Gamma_k \tag{2.3}$$

[1] The dot "·" denotes the matrix product in (2.1).

where E denotes the unit matrix and the Γ_k are matrix-valued homogeneous polynomials of order k. Equations (2.1) and (2.3) lead to

$$XX(\Gamma) + X(\Gamma) + \tfrac{1}{2} X(\Gamma) \cdot (E - \Gamma)^{-1} \cdot X(\Gamma) \;=\; 2F \cdot K \; . \tag{2.4}$$

If we use the formulas

$$\begin{aligned} X(\Gamma_k) &= k\,\Gamma_k \\ XX(\Gamma_k) &= k^2\,\Gamma_k \;, \end{aligned}$$

the Frobenius series

$$G^{-1} = (E - \Gamma)^{-1} F \;=\; \Big(E + \sum_{l=1}^{\infty} \Gamma^l \Big) F \;,$$

the formula

$$\begin{aligned} X(\Gamma) \cdot (E - \Gamma)^{-1} \cdot X(\Gamma) &= \sum_{k,l=2}^{\infty} kl\, \Gamma_k \cdot (E - \Gamma)^{-1} \cdot \Gamma_l \\ &= \sum_{m=4}^{\infty} \sum_{2 \le k \le [\frac{m}{2}]} \sum_{\substack{l_1 + \dots + l_k = m \\ l_i \ge 2}} l_1 l_k\, \Gamma_{l_1} \cdot \ldots \cdot \Gamma_{l_k} \end{aligned}$$

and the power series development of K

$$K \;=\; \sum_{k=2}^{\infty} K_k \tag{2.5}$$

with matrix-valued homogeneous polynomials K_k of order k, then we obtain the recursive relations

$$\begin{aligned} m = 2,3: &\qquad m(m+1)\,\Gamma_m = 2F \cdot K_m \\ m \ge 4: &\qquad m(m+1)\,\Gamma_m = 2F \cdot K_m - \frac{1}{2} \sum_{2 \le k \le [\frac{m}{2}]} \sum_{\substack{l_1 + \dots + l_k = m \\ l_i \ge 2}} l_1 l_k\, \Gamma_{l_1} \cdot \ldots \cdot \Gamma_{l_k} \end{aligned} \tag{2.6}$$

In [9] G. Herglotz has proved the following facts about a metric g which is determined by (2.6).

Theorem 2.1. *Let $\{U, x\}$ be a chart of an n-dimensional differentiable manifold M with $x(P_0) = 0$ for $P_0 \in U$. Further let K_{ijkl} be the coordinates of a covariant tensor field of order 4 which are analytic functions with respect to $\{U, x\}$ and which possess the symmetry properties of the Riemannian curvature tensor, i.e. K_{ijkl} satisfies*

$$K_{ijkl} = -K_{jikl} = -K_{ijlk} = K_{klij} \tag{2.7}$$

and the first Bianchi identity

$$K_{ijkl} + K_{iklj} + K_{iljk} \;=\; 0 \; . \tag{2.8}$$

If we consider the Herglotz relation (2.1) with a right-hand side $K := (K_{ijkl} x^j x^k)$ and search for a solution G by means of an ansatz (2.3), then there hold true:

1. *The equations* (2.6) *yield a uniquely determined formal power series solution* (2.3) *of* (2.1).
2. *The convergence of this formal power series solution* (2.3) *follows from the convergence of the power series K on a suitable open neighbourhood $U' \subseteq U$ of P_0 by means of a comparison method.*
3. *The Riemannian metric g_{ij} given by the calculated solution of* (2.1) *fulfils*

$$(g_{ij} - g_{ij}(P_0))x^j = 0 ,$$

that means the coordinates x^i are normal coordinates with respect to the constructed metric g_{ij} if we restrict us to a star-shaped open neighbourhood $U'' \subseteq U'$ of P_0. The centre of these normal coordinates is P_0.

If we calculate the Riemannian curvature tensor R_{ijkl} of the metric g_{ij} which we have determined according to Theorem 2.1, then the Herglotz relations (2.1) hold true with R_{ijkl} too such that

$$R_{ijkl}x^j x^k = K_{ijkl}x^j x^k \tag{2.9}$$

follows. But we will have $R_{ijkl} \neq K_{ijkl}$ in general. In the next sections we work out a characterization of the difference between R_{ijkl} and K_{ijkl}.

3. The decomposition of the partial derivatives of the Riemannian curvature tensor

Although a motive of our investigations arises from techniques of differential geometry which use normal coordinates, the considerations of this paper do not require normal coordinates. If a special coordinate system is not explicitely defined, we assume always that our coordinates belong to an arbitrary chart $\{U, x\}$ of a differentiable manifold M.

In the following, we use statements about the connection between covariant tensors of order r and the group ring $\mathbb{C}[\mathcal{S}_r]$ of the symmetric group $\mathcal{S}_r$ which we have given in [5].

Let T be a covariant complex-valued tensor on a vector space V on $\mathbb{C}$ and $b := \{v_1, \ldots, v_r\} \subset V$ an arbitrary subset of r vectors from V. Then T and b induce a complex-valued function T_b on the symmetric group $\mathcal{S}_r$

$$T_b : \mathcal{S}_r \to \mathbb{C} \quad , \quad T_b : p \mapsto T_b(p) := T(v_{p(1)}, \ldots, v_{p(r)})$$

which we will identify with the group ring element $\sum_{p \in \mathcal{S}_r} T_b(p)p$ denoted by T_b too. If T is a differentiable tensor field on a differentiable manifold M, then we obtain a group ring element T_b for every subset $b = \{v_1, \ldots, v_r\} \subset M_P$ of the tangent space M_P of any point $P \in M$.

The action of a group ring element $a = \sum_{p \in \mathcal{S}_r} a(p)p \in \mathbb{C}[\mathcal{S}_r]$ on a tensor or a tensor field T is defined by

$$a : T \mapsto aT \quad , \quad (aT)_{i_1 \ldots i_r} := \sum_{p \in \mathcal{S}_r} a(p)\, T_{i_{p(1)} \ldots i_{p(r)}} .$$

Further, we use the mapping

$$* : \mathbb{C}[\mathcal{S}_r] \to \mathbb{C}[\mathcal{S}_r] \quad , \quad a = \sum_{p\in\mathcal{S}_r} a(p)p \;\mapsto\; a^* := \sum_{p\in\mathcal{S}_r} a(p)p^{-1} \ .$$

Then there holds true the relation[1] [5]

$$(aT)_b \;=\; T_b \cdot a^* \ . \tag{3.1}$$

The power series[2]

$$R_{ijkl} \;=\; \sum_{r=0}^{\infty} \tfrac{1}{r!}\, \partial_{i_1}\ldots\partial_{i_r} R_{ijkl}(P_0)\, x^{i_1}\ldots x^{i_r} \tag{3.2}$$

of the Riemannian curvature tensor R around $P_0 \in U$ is determined by the partial derivatives

$$(\partial^{(r)}R)_{i_1 i_2 i_3 i_4 i_5 \ldots i_{r+4}} \;:=\; \partial_{i_5}\ldots\partial_{i_{r+4}} R_{i_1 i_2 i_3 i_4} \quad , \quad \partial^{(0)}R := R \tag{3.3}$$

of R in $P \in U$. Since we will not make any coordinate transformation, we can consider the $(\partial^{(r)}R)_{i_1 i_2 i_3 i_4, i_5 \ldots i_{r+4}}$ as the coordinates of a 'covariant tensor field' of order $r+4$ on U with respect to the basis $\{\partial_1, \ldots, \partial_n\}$ of the given chart $\{U, x\}$. Now we will investigate the left ideals of the group ring $\mathbb{C}[\mathcal{S}_{r+4}]$ in which the group ring elements $(\partial^{(r)}R)_b$ lie which correspond to the $\partial^{(r)}R$.

Let $r \geq 1$. We consider the stability subgroups

$$\grave{\mathcal{S}}_4 := (\mathcal{S}_{r+4})_{5,\ldots,r+4} \quad , \quad \acute{\mathcal{S}}_r := (\mathcal{S}_{r+4})_{1,\ldots,4} \tag{3.4}$$

of $\mathcal{S}_{r+4}$ which fix the numbers $5, \ldots, r+4$ or $1, \ldots, 4$, respectively. We denote by $\grave{y}, \acute{y}_r$ the group ring elements $\grave{y} \in \mathbb{C}[\grave{\mathcal{S}}_4]$, $\acute{y}_r \in \mathbb{C}[\acute{\mathcal{S}}_r]$ which are obtained from the Young symmetrizers of the standard tableaux[3]

$$\begin{matrix} 1\,3 \\ 2\,4 \end{matrix} \quad , \quad 1\,2 \ldots (r-1)\,r \tag{3.5}$$

of $\mathcal{S}_4$, $\mathcal{S}_r$ by means of the natural embeddings $\mathcal{S}_4 \to \mathcal{S}_{r+4}$ and $\mathcal{S}_r \to \mathcal{S}_{r+4}$

$$\begin{pmatrix} 1 \ldots 4 \\ i_1 \ldots i_4 \end{pmatrix} \mapsto \begin{pmatrix} 1 \ldots 4\; 5 \ldots r+4 \\ i_1 \ldots i_4\; 5 \ldots r+4 \end{pmatrix} ,$$
$$\begin{pmatrix} 1 \ldots r \\ i_1 \ldots i_r \end{pmatrix} \mapsto \begin{pmatrix} 1 \ldots 4 & 5 & \ldots r+4 \\ 1 \ldots 4 & i_1+4 & \ldots i_r+4 \end{pmatrix} .$$

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[1] We use the convention $(p \circ q) : i \mapsto (p \circ q)(i) := p(q(i))$ for the multiplication of permutations.

[2] In (3.2) we add up on the indices $i_1, \ldots, i_r$ according to Einstein's summation convention.

[3] About Young symmetrizers and Young tableaux see, for instance, [2, 5, 6, 10, 11, 13, 14, 15, 16, 17]. We use the definition $y_t := \sum_{p\in\mathcal{H}_t} \sum_{q\in\mathcal{V}_t} \chi(q)\, p \circ q$ of a Young symmetrizer of a Young tableau t. Here $\mathcal{H}_t$, $\mathcal{V}_t$ are the groups of the horizontal and vertical permutations of t and $\chi(q)$ denotes the signature of the permutation q.

Obviously, we have

$$\grave{y} = (id + (1\,3)) \cdot (id + (2\,4)) \cdot (id - (1\,2)) \cdot (id - (3\,4)) \qquad (3.6)$$
$$\acute{y}_r = \sum_{\acute{p} \in \acute{\mathcal{S}}_r} \acute{p} \qquad (3.7)$$

where we have used the cyclic form of the permutations in (3.6). If $r = 0$, we consider only $\grave{\mathcal{S}}_4 = \mathcal{S}_4$.

Proposition 3.1. *Let $\{U, x\}$ be a chart and $r \geq 1$. Then the group ring element $(\partial^{(r)} R)_b \in \mathbb{C}[\mathcal{S}_{r+4}]$ is contained in the left ideal*

$$I_{(r)} := \mathbb{C}[\mathcal{S}_{r+4}] \cdot \grave{y} \cdot \acute{y}_r \qquad (3.8)$$

of $\mathbb{C}[\mathcal{S}_{r+4}]$ for every set of vectors $b = \{v_1, \ldots, v_{r+4}\} \subset M_P$, $P \in U$. If $r = 0$, then every $(\partial^{(0)} R)_b = R_b \in \mathbb{C}[\mathcal{S}_4]$ lies in

$$I_{(0)} := \mathbb{C}[\mathcal{S}_4] \cdot \grave{y} \ . \qquad (3.9)$$

Proof. Let $r \geq 1$. Obviously, the symmetry of $(\partial^{(r)} R)_{i_1 i_2 i_3 i_4, i_5 \ldots i_{r+4}}$ in $i_5, \ldots, i_{r+4}$ and (3.7) yield

$$\acute{y}_r^*(\partial^{(r)} R) = \acute{y}_r(\partial^{(r)} R) = r!\, \partial^{(r)} R \ . \qquad (3.10)$$

From equation (3.6) we obtain $\grave{y}^*(\partial^{(r)} R)$ as a sum of 16 summands for $r \geq 0$. Then we find

$$\grave{y}^*(\partial^{(r)} R) = 12\, \partial^{(r)} R \qquad (3.11)$$

by expressing all summands of $\grave{y}^*(\partial^{(r)} R)$ by the two terms

$$(\partial^{(r)} R)_{i_1 i_2 i_3 i_4, i_5 \ldots i_{r+4}} \ , \quad (\partial^{(r)} R)_{i_1 i_3 i_2 i_4, i_5 \ldots i_{r+4}}$$

using the identities (2.7) and (2.8). Thus there follows from (3.1), (3.10) and (3.11) for $r \geq 1$

$$12 r!\, (\partial^{(r)} R)_b = (\acute{y}_r^* \grave{y}^* (\partial^{(r)} R))_b = (\partial^{(r)} R)_b \cdot \grave{y} \cdot \acute{y}_r$$

and for $r = 0$

$$12\, R_b = (\grave{y}^* R)_b = R_b \cdot \grave{y} \quad \blacksquare$$

An other proof of (3.11) follows from [6: Theorem 2.1 and remark at page 1162] (see Section 6).

Let be $r \geq 1$. We consider the representations

$$\grave{\alpha} : \grave{\mathcal{S}}_4 \rightarrow GL(\mathbb{C}[\grave{\mathcal{S}}_4] \cdot \grave{y}) \ , \quad \grave{\alpha}_{\grave{p}}(\grave{f}) := \grave{p} \cdot \grave{f} \qquad (3.12)$$
$$\acute{\alpha} : \acute{\mathcal{S}}_r \rightarrow GL(\mathbb{C}[\acute{\mathcal{S}}_r] \cdot \acute{y}_r) \ , \quad \acute{\alpha}_{\acute{p}}(\acute{f}) := \acute{p} \cdot \acute{f} \qquad (3.13)$$
$$\gamma : \grave{\mathcal{S}}_4 \cdot \acute{\mathcal{S}}_r \rightarrow GL\Big((\mathbb{C}[\grave{\mathcal{S}}_4] \cdot \grave{y}) \otimes (\mathbb{C}[\acute{\mathcal{S}}_r] \cdot \acute{y}_r)\Big) \ , \quad \gamma_{\grave{p} \cdot \acute{p}}(\grave{f} \cdot \acute{f}) := \grave{p} \cdot \acute{p} \cdot \grave{f} \cdot \acute{f} \qquad (3.14)$$
$$\beta : \mathcal{S}_{r+4} \rightarrow GL(\mathbb{C}[\mathcal{S}_{r+4}] \cdot \grave{y} \cdot \acute{y}_r) \ , \quad \beta_p(f) := p \cdot f \ . \qquad (3.15)$$

Obviously, the subgroup $H := \dot{\mathcal{S}}_4 \cdot \acute{\mathcal{S}}_r \subset \mathcal{S}_{r+4}$ is the direct product of the subgroups $\dot{\mathcal{S}}_4$, $\acute{\mathcal{S}}_r \subset \mathcal{S}_{r+4}$. The tensor product in (3.14) is realized by the group ring multiplication $(\dot{f}, \acute{f}) \mapsto \dot{f} \cdot \acute{f}$. This tensor product fulfils

$$\mathbb{C}[\dot{\mathcal{S}}_4 \cdot \acute{\mathcal{S}}_r] \cdot \dot{y} \cdot \acute{y}_r = (\mathbb{C}[\dot{\mathcal{S}}_4] \otimes \mathbb{C}[\acute{\mathcal{S}}_r]) \cdot \dot{y} \cdot \acute{y}_r = (\mathbb{C}[\dot{\mathcal{S}}_4] \cdot \dot{y}) \otimes (\mathbb{C}[\acute{\mathcal{S}}_r] \cdot \acute{y}_r) \quad .$$

The representation γ is the outer tensor product of the representations $\dot{\alpha}$, $\acute{\alpha}$ (i.e. $\gamma = \dot{\alpha} \# \acute{\alpha}$ in the notation of [11]) since there holds true

$$\gamma_{\dot{p} \cdot \acute{p}}(\dot{f} \cdot \acute{f}) \;=\; (\dot{p} \cdot \dot{f}) \cdot (\acute{p} \cdot \acute{f}) \;=\; \dot{\alpha}_{\dot{p}}(\dot{f}) \cdot \acute{\alpha}_{\acute{p}}(\acute{f}) \quad .$$

Further, the representations $\dot{\alpha}$, $\acute{\alpha}$ are irreducible because their representation spaces are left ideals generated by Young symmetrizers. Now the following lemma says that the representation β is induced by the representation γ (i.e. $\beta = \gamma \uparrow \mathcal{S}_{r+4}$).

Lemma 3.1. *Let G be a finite group, $H \subseteq G$ a subgroup of G and $a \in \mathbb{C}[H]$ an element of the group ring of H. If we consider the representations*

$$\begin{array}{ll} \beta : G \to GL(V) & , \quad \beta_g(v) := g \cdot v \\ \alpha : H \to GL(W) & , \quad \alpha_h(w) := h \cdot w \quad , \end{array}$$

with the representation spaces $V := \mathbb{C}[G] \cdot a$, $W := \mathbb{C}[H] \cdot a$, then the representation β is induced by the representation α, i.e. $\beta = \alpha \uparrow G$.

Proof. Obviously, there holds true $\beta_h(W) \subseteq W$ for all $h \in H$. We choose a system of representatives $\mathcal{R}$ of the left cosets $p \cdot H$ of G relative to H. Let $W_a := \mathcal{L}\{a\}$ be the 1-dimensional vector space on $\mathbb{C}$ spanned by a. Then we can write

$$V = \sum_{g \in G} g \cdot W_a = \sum_{p \in \mathcal{R}} \sum_{h \in H} p \cdot h \cdot W_a = \sum_{p \in \mathcal{R}} p \cdot W = \bigoplus_{p \in \mathcal{R}} \beta_p(W) \quad .$$

The last calculation step is correct because $p \cdot W \subseteq p \cdot \mathbb{C}[H] = \mathcal{L}\{p \cdot H\}$ for all $p \in \mathcal{R}$ and since $\mathbb{C}[G] = \bigoplus_{p \in \mathcal{R}} \mathcal{L}\{p \cdot H\}$ ∎

Obviously, (3.14) and (3.15) satisfy the assumptions of Lemma 3.1 since

$$\dot{y} \cdot \acute{y}_r \in \Big((\mathbb{C}[\dot{\mathcal{S}}_4] \cdot \dot{y}) \otimes (\mathbb{C}[\acute{\mathcal{S}}_r] \cdot \acute{y}_r) \Big) = \mathbb{C}[\dot{\mathcal{S}}_4 \cdot \acute{\mathcal{S}}_r] \cdot \dot{y} \cdot \acute{y}_r \quad .$$

Thus we obtain $\beta = \gamma \uparrow \mathcal{S}_{r+4} = (\dot{\alpha} \# \acute{\alpha}) \uparrow \mathcal{S}_{r+4}$. Now we will determine a decomposition of the left ideal $I_{(r)}$ into a direct sum of minimal left ideals (or, equivalently, a decomposition of β into irreducible representations).

Because the representations $\dot{\alpha}$, $\acute{\alpha}$ are irreducible we can determine the Young frames of the irreducible subrepresentations in the decomposition of β from the Young frames (3.5) of $\dot{\alpha}$, $\acute{\alpha}$ by means of the Littlewood-Richardson rule (see [13: pp. 94], [11: Vol. I, p. 84], [14: pp. 68] and [6]). From (3.5) the Littlewood-Richardson rule yields exactly the three frames

$$\overbrace{\begin{array}{llllll} \square & \square & a & a & \ldots & a \\ \square & \square & & & & \end{array}}^{r+2} \quad , \quad \overbrace{\begin{array}{llllll} \square & \square & a & a & \ldots & a \\ \square & \square & & & & \\ a & & & & & \end{array}}^{r+1} \quad , \quad \overbrace{\begin{array}{llllll} \square & \square & a & a & \ldots & a \\ \square & \square & & & & \\ a & a & & & & \end{array}}^{r} \quad .$$

Thus we have

Proposition 3.2. *Let $r \geq 2$. Then the representation β according to (3.15) can be decomposed in exactly three mutually inequivalent irreducible subrepresentations which are characterized by the partitions*

$$(r+2\ \ 2)\,,\ (r+1\ \ 2\ \ 1)\,,\ (r\ \ 2\ \ 2)\ \vdash\ r+4\ . \tag{3.16}$$

In the case $r = 1$ we have only two irreducible subrepresentations given by the partitions

$$(3\ \ 2)\,,\ (2\ \ 2\ \ 1)\ \vdash\ 5\ . \tag{3.17}$$

Corollary 3.1. *From Proposition 3.2 there follows:*

- *For $r \geq 2$ the left ideal $I_{(r)}$ can be decomposed into three mutually inequivalent minimal left ideals the equivalence classes of which are characterized by (3.16).*
- *For $r = 1$ the left ideal $I_{(1)}$ can be decomposed into two mutually inequivalent minimal left ideals the equivalence classes of which are characterized by (3.17).*
- *The left ideal $I_{(0)}$ is minimal since it is generated by a Young symmetrizer.*

The minimal left subideal of $I_{(r)}$ corresponding to the partition $(r+2\ \ 2)$ can be explicitly determined.

Proposition 3.3. *Let $r \geq 0$. Then the Young symmetrizer $y_{t_r} \in \mathbb{C}[\mathcal{S}_{r+4}]$ of the standard tableau*

$$t_0 := \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}\quad,\quad t_r := \begin{array}{cccccc} 1 & 3 & 5 & 6 & \ldots & (r+4) \\ 2 & 4 & & & & \end{array}\quad,\quad r \geq 1 \tag{3.18}$$

generates that minimal left subideal $J_{(r)}$ of $I_{(r)}$ which corresponds to the partition $(r+2\ \ 2)$ of $r+4$.

Proof. A proof is necessary only for $r \geq 1$. We show that there is a $c = \text{const} \neq 0$ such that

$$y_{t_r} \cdot \dot{y} \cdot \acute{y}_r \ = \ c\, y_{t_r}\ . \tag{3.19}$$

Then there follows from (3.19) that the minimal left ideal $K_{(r)} := \mathbb{C}[\mathcal{S}_{r+4}] \cdot y_{t_r}$ is a subideal of $I_{(r)}$. But because the decomposition of $I_{(r)}$ into a direct sum of minimal left ideals contains exactly one minimal left ideal $J_{(r)}$ corresponding to the partition $(r+2\ \ 2)$, the ideal $K_{(r)}$ has to coincide with that ideal $J_{(r)}$.

Let us prove (3.19). We denote by $P_{\{i_1,\ldots,i_k\}}$ the subgroup of $\mathcal{S}_{r+4}$ consisting of all those permutations from $\mathcal{S}_{r+4}$ which fix all numbers in $\{1,\ldots,r+4\} \setminus \{i_1,\ldots,i_k\}$. Now let $\mathcal{H}_{t_r}$ be the group of the horizontal permutations of the tableaux t_r and let $\mathcal{R}$ be a system of representatives of the left cosets of $P_{\{1,3,5,6,\ldots,r+4\}}$ relative to $P_{\{1,3\}}$. Then we can write

$$\sum_{p\in\mathcal{H}_{t_r}} p \ = \ \sum_{s\in\mathcal{R}} s\cdot(id+(1\,3))\cdot(id+(2\,4))$$

and

$$y_{t_r} = \sum_{s\in\mathcal{R}} s\cdot(id+(1\,3))\cdot(id+(2\,4))\cdot(id-(1\,2))\cdot(id-(3\,4)) \;,$$
$$y_{t_r} = \sum_{s\in\mathcal{R}} s\cdot\dot{y} \;. \tag{3.20}$$

Since $\dot{y}\cdot\dot{y} = \mu\,\dot{y}$ with a constant $\mu \neq 0$, we obtain from (3.20)

$$y_{t_r}\cdot\dot{y}\cdot\acute{y}_r = \sum_{s\in\mathcal{R}} s\cdot\dot{y}\cdot\dot{y}\cdot\acute{y}_r = \mu\sum_{s\in\mathcal{R}} s\cdot\dot{y}\cdot\acute{y}_r = \mu\,y_{t_r}\cdot\acute{y}_r \;. \tag{3.21}$$

Now let $\tilde{\mathcal{R}}$ be a system of representatives of the left cosets of $P_{\{1,3,5,6,\ldots,r+4\}}$ relative to $P_{\{5,6,\ldots,r+4\}}$. Then there holds

$$\sum_{p\in\mathcal{H}_{t_r}} p = \sum_{\tilde{s}\in\tilde{\mathcal{R}}} \tilde{s}\cdot\acute{y}_r\cdot(id+(2\,4)) = (id+(2\,4))\cdot\sum_{\tilde{s}\in\tilde{\mathcal{R}}} \tilde{s}\cdot\acute{y}_r \;.$$

Denoting the group of vertical permutations of t_r by $\mathcal{V}_{t_r}$ and taking into account that $\acute{y}_r\cdot q = q\cdot\acute{y}_r$ for all $q\in\mathcal{V}_{t_r}$, we can write

$$y_{t_r} = \sum_{p\in\mathcal{H}_{t_r}}\sum_{q\in\mathcal{V}_{t_r}} \chi(q)\,p\cdot q = (id+(2\,4))\cdot\sum_{\tilde{s}\in\tilde{\mathcal{R}}}\sum_{q\in\mathcal{V}_{t_r}} \chi(q)\,\tilde{s}\cdot q\cdot\acute{y}_r \;.$$

Then this relation and (3.21) yield

$$\begin{aligned} y_{t_r}\cdot\dot{y}\cdot\acute{y}_r &= \mu\,(id+(2\,4))\cdot\sum_{\tilde{s}\in\tilde{\mathcal{R}}}\sum_{q\in\mathcal{V}_{t_r}} \chi(q)\,\tilde{s}\cdot q\cdot\acute{y}_r\cdot\acute{y}_r \\ &= \mu\,r!\,(id+(2\,4))\cdot\sum_{\tilde{s}\in\tilde{\mathcal{R}}}\sum_{q\in\mathcal{V}_{t_r}} \chi(q)\,\tilde{s}\cdot q\cdot\acute{y}_r \\ &= \mu\,r!\,y_{t_r} \quad\blacksquare \end{aligned}$$

4. The essential part of the partial derivatives of the Riemannian curvature tensor

Since the right-hand side of the Herglotz relation is the matrix with elements $R_{ijkl}x^j x^k$, the Riemannian metric g does not depend on the partial derivatives $\partial_{i_1}\ldots\partial_{i_r}R_{ijkl}(P_0)$ of the Riemannian curvature tensor but on the symmetrized partial derivatives

$$(\partial^{(r)}\breve{R})_{i_1 i_2 i_3 i_4 i_5\ldots i_{r+4}} := \partial_{(i_5}\ldots\partial_{i_{r+4}}R_{|i_1|i_2 i_3)i_4} \tag{4.1}$$

$$(\partial^{(0)}\breve{R})_{i_1\ldots i_4} := \breve{R}_{i_1\ldots i_4} := R_{i_1(i_2 i_3)i_4} \tag{4.2}$$

of the curvature tensor at the centre P_0 of the normal neighbourhood U.

Let now $\{U,x\}$ be an arbitrary chart which do not have to be a normal coordinate system. In this section we investigate the left ideal of $\mathbb{C}[\mathcal{S}_{r+4}]$ which contains the group ring elements $(\partial^{(r)}\breve{R})_b$ induced by $\partial^{(r)}\breve{R}$ and a vector set $b = \{v_1,\ldots v_{r+4}\} \subset M_P$, $P\in U$.

Lemma 4.1. *Let be $r \geq 0$. We denote by C the subgroup of $\mathcal{S}_{r+4}$ which fixes the numbers 1 and 4 and by ϵ the sum of all elements of C,*

$$C := P_{\{2,3,5,\ldots,r+4\}} \quad , \quad \epsilon := \sum_{c\in C} c \quad . \tag{4.3}$$

Then the group ring element $(\partial^{(r)}\breve{R})_b$ induced by $\partial^{(r)}\breve{R}$ and a set $b = \{v_1, \ldots, v_{r+4}\} \subset M_P$ of vectors of the tangent space M_P lies in the left ideal $\breve{I}_{(r)} := I_{(r)} \cdot \epsilon$ of $\mathbb{C}[\mathcal{S}_{r+4}]$ for every vector set b.

Proof. Because there holds true $\partial^{(r)}\breve{R} = \epsilon(\partial^{(r)}R)/(r+2)!$ and $\epsilon^* = \epsilon$ we obtain the assertion from

$$(\partial^{(r)}\breve{R})_b = \tfrac{1}{(r+2)!}\left(\epsilon(\partial^{(r)}R)\right)_b = \tfrac{1}{(r+2)!}(\partial^{(r)}R)_b \cdot \epsilon \quad \blacksquare$$

We consider the decomposition of $I_{(r)}$ into minimal left ideals

$$I_{(r)} = J_{(r)} \oplus \hat{J}_{(r)} \oplus \check{J}_{(r)} \tag{4.4}$$

according to Corollary 3.1. Let the correspondence between the minimal left ideals and their characterizing partitions be

$$\begin{aligned} J_{(r)} &\Leftrightarrow (r+2\ \ 2)\ , \\ \hat{J}_{(r)} &\Leftrightarrow (r+1\ \ 2\ \ 1)\ , \\ \check{J}_{(r)} &\Leftrightarrow (r\ \ 2\ \ 2)\ . \end{aligned}$$

If $r = 1$, then $\check{J}_{(r)}$ does not occur in (4.4).

From (4.4) there follows a decomposition of $\breve{I}_{(r)}$

$$\breve{I}_{(r)} = (J_{(r)} \cdot \epsilon) \oplus (\hat{J}_{(r)} \cdot \epsilon) \oplus (\check{J}_{(r)} \cdot \epsilon) \tag{4.5}$$

which is certainly a direct sum since the minimal left ideals are mutually inequivalent. Now the question arises whether one of the ideals $(J_{(r)} \cdot \epsilon)$, $(\hat{J}_{(r)} \cdot \epsilon)$, $(\check{J}_{(r)} \cdot \epsilon)$ vanishes.

Theorem 4.1. *For $r \geq 0$ there holds true*

$$\breve{I}_{(r)} = J_{(r)} \cdot \epsilon = \mathbb{C}[\mathcal{S}_{r+4}] \cdot y_{t_r} \cdot \epsilon$$

that means all other minimal left ideals in (4.4) are mapped to 0 by $f \mapsto f \cdot \epsilon$.

Proof. *Step 1 :* First we show that $y_{t_r} \cdot \epsilon \neq 0$. We use the notations t_r, $\mathcal{H}_{t_r}$, $\mathcal{V}_{t_r}$ of Section 3. Denoting $C' := P_{\{1,3\}}$ if $r = 0$, $C' := P_{\{1,3,5,\ldots,r+4\}}$ if $r \geq 1$ and taking into account $C' = (1\,2) \cdot C \cdot (1\,2)$ we can write for the sum of the horizontal permutations of the tableaux t_r (3.18)

$$\sum_{p\in\mathcal{H}_{t_r}} p = \sum_{s\in C'} s + \sum_{s\in C'} s \cdot (2\,4) = (1\,2) \cdot \epsilon \cdot (1\,2) \cdot (id + (2\,4)) \quad .$$

Because (1 2) is a vertical permutation of t_r, there follows on the other hand

$$y_{t_r} \cdot (1\,2) = \sum_{q \in \mathcal{V}_{t_r}} \sum_{p \in \mathcal{H}_{t_r}} \chi(q) p \cdot q \cdot (1\,2) = \chi((1\,2)) \sum_{q \in \mathcal{V}_{t_r}} \sum_{p \in \mathcal{H}_{t_r}} \chi(q) p \cdot q = -y_{t_r} \ .$$

Thus we obtain

$$\begin{aligned} y_{t_r} \cdot y_{t_r} &= \sum_{q \in \mathcal{V}_{t_r}} \sum_{p \in \mathcal{H}_{t_r}} \chi(q) y_{t_r} \cdot p \cdot q \\ &= y_{t_r} \cdot (1\,2) \cdot \epsilon \cdot (1\,2) \cdot (id + (2\,4)) \cdot \sum_{q \in \mathcal{V}_{t_r}} \chi(q) q \\ &= -y_{t_r} \cdot \epsilon \cdot (1\,2) \cdot (id + (2\,4)) \cdot \sum_{q \in \mathcal{V}_{t_r}} \chi(q) q \ . \end{aligned} \tag{4.6}$$

But this yields $y_{t_r} \cdot \epsilon \neq 0$ since $y_{t_r} \cdot y_{t_r} \neq 0$. Consequently, the ideal $J_{(r)} \cdot \epsilon$ has to occur in the decomposition (4.5).

If $r = 0$, Theorem 4.1 follows from $I_{(0)} = J_{(0)}$. Thus we can assume $r \geq 1$ in the following.

Step 2: Using the hook length formula (see [11: Vol I, p. 81], [1: pp. 101] and [6]) we can calculate the dimensions of the left ideals $J_{(r)}, \hat{J}_{(r)}, \check{J}_{(r)}$ from the Young frames of these ideals or, equivalently, from the partitions (3.16). The results are

$$r \geq 0 \quad \Rightarrow \quad d_r := \dim J_{(r)} = \frac{(r+4)(r+1)}{2} \ , \tag{4.7}$$

$$r \geq 1 \quad \Rightarrow \quad \hat{d}_r := \dim \hat{J}_{(r)} = \frac{(r+4)(r+2)r}{3} \ , \tag{4.8}$$

$$r \geq 2 \quad \Rightarrow \quad \check{d}_r := \dim \check{J}_{(r)} = \frac{(r+4)(r+3)r(r-1)}{12} \ . \tag{4.9}$$

Furthermore, the left ideal $L_{(r)} := \mathbb{C}[\mathcal{S}_{r+4}] \cdot \epsilon$ has the dimension

$$l_r := \dim L_{(r)} = (r+4)(r+3) \ . \tag{4.10}$$

Consider a system of representatives $\mathcal{R}$ of the left cosets of $\mathcal{S}_{r+4}$ relative to C. Then $\mathcal{B} := \{p \cdot \epsilon \mid p \in \mathcal{R}\}$ is a system of generating vectors of $L_{(r)}$. But on the other hand $\mathcal{B}$ is a system of linearly independent vectors since the vectors $p \cdot \epsilon$ lie in pairwise distinct cosets. Thus B has a basis of $|\mathcal{R}| = (r+4)(r+3)$ vectors.

The left ideal $\check{I}_{(r)}$ is a subideal of $L_{(r)}$ such that $\dim \check{I}_{(r)} \leq \dim L_{(r)}$. Further, the linear mapping $f \mapsto f \cdot \epsilon$ maps a minimal left ideal either onto 0 or onto an equivalent minimal left ideal. In Table 1 we have listed the first values of the dimensions d_r , $\hat{d}_r$, $\check{d}_r$, l_r. Since these dimensions are monotonically increasing functions of r and $\check{I}_{(r)}$ has a subideal of dimension d_r for all $r \geq 1$, we read from Table 1 that for $r \geq 4$ subideals of dimensions $\hat{d}_r$, $\check{d}_r$ can not occur in $\check{I}_{(r)}$. Moreover, for $r = 3$ a subideal of $\check{I}_{(r)}$ of dimension $\hat{d}_3 = 35$ is impossible.

Step 3: We handle the remaining cases of the left ideals $\hat{J}_{(1)}, \hat{J}_{(2)}, \check{J}_{(2)}, \check{J}_{(3)}$ by a

Table 1. The dimensions d_r, $\hat{d}_r$, $\check{d}_r$, l_r for low r.

r	d_r	$\hat{d}_r$	$\check{d}_r$	l_r
1	5	5	/	20
2	9	16	5	30
3	14	35	21	42
4	20	64	56	56
5	27	105	120	72

computer calculation applying our Mathematica package PERMS [4]. To determine generating idempotents of these left ideals we consider the Young standard tableaux

$$\begin{matrix}1 & 3 \\ 2 & 4 \\ 5 & \end{matrix} \;,\quad \begin{matrix}1 & 3 & 6 \\ 2 & 4 & \\ 5 & & \end{matrix} \;,\quad \begin{matrix}1 & 3 \\ 2 & 4 \\ 5 & 6\end{matrix} \;,\quad \begin{matrix}1 & 3 & 7 \\ 2 & 4 & \\ 5 & 6 & \end{matrix} \quad . \tag{4.11}$$

Let y run through the set of the four Young symmetrizers of the tableaux (4.11). Then we find by means of PERMS

$$y \cdot \grave{y} \cdot \acute{y}_r \neq 0 \quad \text{and} \quad y \cdot \grave{y} \cdot \acute{y}_r \cdot y \neq 0$$

for all those four Young symmetrizers y. There follows from the second of these relations that $y \cdot \grave{y} \cdot \acute{y}_r$ is an essentially idempotent element generating a minimal left subideal of $I_{(r)}$ of the equivalence class of y[1]. But since $I_{(r)}$ has at most one subideal from the equivalence class of y, these essentially idempotent elements are generating elements of the left ideals $\hat{J}_{(1)}, \hat{J}_{(2)}, \check{J}_{(2)}, \check{J}_{(3)}$. Now another calculation with PERMS yields

$$y \cdot \grave{y} \cdot \acute{y}_r \cdot \epsilon \;=\; 0$$

for all y. Thus the ideals $(\hat{J}_{(1)} \cdot \epsilon), (\hat{J}_{(2)} \cdot \epsilon), (\check{J}_{(2)} \cdot \epsilon), (\check{J}_{(3)} \cdot \epsilon)$ vanish ∎

Definition 4.1. Let y_{t_r} be the Young symmetrizer of the standard tableau (3.18). We call $y^*_{t_r}(\partial^{(r)}R)$ the *essential part* of $\partial^{(r)}R$ and $\partial^{(r)}R - y^*_{t_r}(\partial^{(r)}R)$ the *non-essential part* of $\partial^{(r)}R$.

Obviously, the mapping $f \mapsto f \cdot \epsilon$ is an isomorphism of the minimal left ideals $J_{(r)}$ and $(J_{(r)} \cdot \epsilon)$, describing the equivalence of these ideals. From this fact there follows

$$\partial^{(r)}\breve{R} \;=\; \text{const}\; \epsilon(y^*_{t_r}(\partial^{(r)}R)) \quad , \quad \text{const} \neq 0 \quad . \tag{4.12}$$

We finish this section with a formula for the inverse of this mapping.

Proposition 4.1. *Let $r \geq 0$ and denote y_{t_r} the Young symmetrizer of the Young tableau* (3.18) *and ϵ the group ring element according to* (4.3). *Let further be*[2]

$$\eta \;:=\; (1\,2) \cdot (id + (2\,4)) \cdot (id - (1\,2)) \cdot (id - (3\,4)) \quad . \tag{4.13}$$

[1] This situation is a special case of Proposition 3.1 in [5].

[2] In the cyclic form of a permutation we write the image of a number left from the inverse image.

Then there holds true

$$y_{t_r}\cdot\epsilon\cdot\eta \;=\; -\mu_r\, y_{t_r} \qquad \text{with} \qquad \mu_r := 2\,(r+3)(r+2)\,r! \tag{4.14}$$

such that the mapping $J_{(r)}\cdot\epsilon \to J_{(r)}$, $h \mapsto -(1/\mu_r)\,h\cdot\eta$ *is the inverse of the mapping* $J_{(r)} \to J_{(r)}\cdot\epsilon$, $f \mapsto f\cdot\epsilon$. *From* (4.13) , (4.14) *there follows*

$$\begin{aligned}
&\tfrac{1}{(r+2)!}\,(y^*_{t_r}(\partial^{(r)}R))_{i_1 i_2 i_3 i_4 i_5\ldots i_{r+4}} \;=\\
&+(\partial^{(r)}\check{R})_{i_1 i_2 i_3 i_4 i_5\ldots i_{r+4}} - (\partial^{(r)}\check{R})_{i_2 i_1 i_3 i_4 i_5\ldots i_{r+4}} - (\partial^{(r)}\check{R})_{i_1 i_2 i_4 i_3 i_5\ldots i_{r+4}}\\
&+(\partial^{(r)}\check{R})_{i_4 i_2 i_3 i_1 i_5\ldots i_{r+4}} + (\partial^{(r)}\check{R})_{i_2 i_1 i_4 i_3 i_5\ldots i_{r+4}} - (\partial^{(r)}\check{R})_{i_4 i_1 i_3 i_2 i_5\ldots i_{r+4}}\\
&-(\partial^{(r)}\check{R})_{i_3 i_2 i_4 i_1 i_5\ldots i_{r+4}} + (\partial^{(r)}\check{R})_{i_3 i_1 i_4 i_2 i_5\ldots i_{r+4}} \;.
\end{aligned} \tag{4.15}$$

Proof. Equation (4.14) follows from (4.6), definition (4.13), equation (4.7) and

$$y_{t_r}\cdot y_{t_r} \;=\; \mu_r\, y_{t_r} \qquad \text{with} \qquad \mu_r := \frac{(r+4)!}{d_r} \quad ; \quad d_r := \dim J_{(r)} \;. \tag{4.16}$$

The formula for μ_r in (4.16) is given, e.g., in [1: p. 103].

We denote by $e, \hat{e}, \check{e}$ the generating idempotents of $J_{(r)}, \hat{J}_{(r)}, \check{J}_{(r)}$ corresponding to the decomposition (4.4) of $I_{(r)}$. These idempotents fulfil

$$e = \tfrac{1}{\mu_r}\, y_{t_r} \quad , \quad \hat{e}\cdot\epsilon = 0 \quad , \quad \check{e}\cdot\epsilon = 0 \;.$$

Furthermore, we can write for every vector set $b = \{v_1,\ldots,v_{r+4}\} \subset M_P$ of the tangent space M_P

$$(\partial^{(r)}R)_b \;=\; (\partial^{(r)}R)_b\cdot e + (\partial^{(r)}R)_b\cdot\hat{e} + (\partial^{(r)}R)^{(r)}_b\cdot\check{e} \;. \tag{4.17}$$

Then using equation (4.14), (4.17) and $\epsilon^*(\partial^{(r)}R) = (r+2)!\,\partial^{(r)}\check{R}$ we obtain

$$\begin{aligned}
(y^*_{t_r}(\partial^{(r)}R))_b &= (\partial^{(r)}R)_b\cdot y_{t_r} \;=\; -(\partial^{(r)}R)_b\cdot e\cdot\epsilon\cdot\eta \;=\; -(\partial^{(r)}R)_b\cdot\epsilon\cdot\eta\\
&= -(\epsilon^*(\partial^{(r)}R))_b\cdot\eta \;=\; -(r+2)!\,(\partial^{(r)}\check{R})_b\cdot\eta\\
&= -(r+2)!\,(\eta^*(\partial^{(r)}\check{R}))_b
\end{aligned}$$

and consequently

$$y^*_{t_r}(\partial^{(r)}R) \;=\; -(r+2)!\;\eta^*(\partial^{(r)}\check{R}) \;.$$

This together with

$$\eta^* \;=\; -id + (12) + (34) - (14) - (12)(34) + (124) + (143) - (1243)$$

yields (4.15) ∎

5. The occurrence of the non-essential part of the partial derivatives of the Riemannian curvature tensor

In this section we discuss the question whether examples of metrics can be found for which the $(\partial^{(r)}R)_b$ of the partial derivatives of the curvature tensor possesses non-vanishing parts lying at least in one of the left ideals $\hat{J}_{(r)}$ or $\check{J}_{(r)}$. First we give a case for which the $(\partial^{(r)}R)_b$ are contained exclusively in $J_{(r)}$.

Proposition 5.1. *We assume that the Riemannian metric g is decomposable into a sum of 2-dimensional metrics $g^{(i)}$, $i = 1, \ldots, m$, that means around every point P_0 of the underlying manifold M a chart $\{U, x\}$ can be found such that the metric takes the form*

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = \sum_{i=1}^{m} g^{(i)}_{\alpha_i \beta_i}(x^{\gamma_i})\, dx^{\alpha_i} dx^{\beta_i} \qquad (5.1)$$
$$\alpha, \beta \in \{1, \ldots, 2m\} \qquad \alpha_i, \beta_i, \gamma_i \in \{2i-1, 2i\} \; .$$

Then there holds true with respect to $\{U, x\}$

$$(\partial^{(r)}R)_b \in J_{(r)} = \mathbb{C}[\mathcal{S}_r] \cdot y_{t_r} \qquad (5.2)$$

for $r \geq 1$ and every $b = \{v_1, \ldots, v_{r+4}\} \subset M_P$, $P \in U$. In particular, a 2-dimensional Riemannian manifold fulfils (5.2).

Proof. If we calculate the Christoffel symbols and the coordinates of the curvature tensor and its partial derivatives for a decomposable metric (5.1), we obtain that at most those coordinates

$$\Gamma^{\kappa_i}_{\mu_i \nu_i}(x^{\gamma_i}) \;,\; R_{\kappa_i \lambda_i \mu_i \nu_i}(x^{\gamma_i}) \;,\; \partial_{\alpha_i} R_{\kappa_i \lambda_i \mu_i \nu_i}(x^{\gamma_i}) \;,\; \partial_{\alpha_i} \partial_{\beta_i} R_{\kappa_i \lambda_i \mu_i \nu_i}(x^{\gamma_i}) \;, \ldots$$

do not vanish, the indices of which lie in one of the sets $\{2i-1, 2i\}$, i.e.

$$\alpha_i\,,\, \beta_i\,,\, \gamma_i\,,\, \kappa_i\,,\, \lambda_i\,,\, \mu_i\,,\, \nu_i \in \{2i-1, 2i\} \quad , \quad i = 1, \ldots, m \; .$$

As in the proof of Proposition 4.1 we denote by $e, \hat{e}, \check{e}$ the generating idempotents of $J_{(r)}, \hat{J}_{(r)}, \check{J}_{(r)}$ corresponding to the decomposition (4.4) of $I_{(r)}$. The left ideal $\hat{J}_{(r)}$ belongs to the equivalence class of minimal left ideals of the partition $\lambda = (r+1\ 2\ 1)$. The left ideal

$$I_\lambda := \bigoplus_{t \in \mathcal{ST}_\lambda} \mathbb{C}[\mathcal{S}_r] \cdot y_t \qquad (5.3)$$

contains all minimal left ideals of the class of λ (see, e.g., [1: p.58 and p.102]). In (5.3) $\mathcal{ST}_\lambda$ denotes the set of all standard tableaux of the partition λ and y_t is the Young symmetrizer of the standard tableau t. Since $\hat{e} \in I_\lambda$, we can write

$$\hat{e} = \sum_{t \in \mathcal{ST}_\lambda} x_t \cdot y_t \qquad (5.4)$$

with certain group ring elements $x_t \in \mathbb{C}[\mathcal{S}_r]$.

Now, equation (5.4) yields

$$\hat{e}^*(\partial^{(r)} R) = \sum_{t \in \mathcal{ST}_\lambda} y_t^*(x_t^*(\partial^{(r)} R)) .$$

$x_t^*(\partial^{(r)} R)$ is a linear combination of certain coordinates of $\partial^{(r)} R$ with permuted indices. The application of y_t^* to $x_t^*(\partial^{(r)} R)$ brings an anti-symmetrization of three indices about every summand of $x_t^*(\partial^{(r)} R)$ because every standard tableaux $t \in \mathcal{ST}_\lambda$ has three rows. But a non-vanishing coordinate of $\partial^{(r)} R$ can not have more than two values among its indices, so $y_t^*(x_t^*(\partial^{(r)} R)) = 0$ for all $t \in \mathcal{ST}_\lambda$. Consequently, there follows $\hat{e}^*(\partial^{(r)} R) = 0$ and $(\partial^{(r)} R)_b \cdot \hat{e} = 0$ for all vector sets $b = \{v_1, \ldots, v_{r+4}\} \subset M_P$.

By the same arguments we can show that $(\partial^{(r)} R)_b \cdot \check{e} = 0$ for all $b = \{v_1, \ldots, v_{r+4}\} \subset M_P$. Taking into account (4.17), we obtain $(\partial^{(r)} R)_b = (\partial^{(r)} R)_b \cdot e \in J_{(r)}$ ∎

An example of a metric such that $(\partial^{(r)} R)_b$ have a part in the ideal $\hat{J}_{(r)} \oplus \check{J}_{(r)}$ can be found in the class of Riemannian manifolds for which the $R_{ijkl} x^j x^k$ are polynomials in normal coordinates x^i.

Proposition 5.2. *Let $\{U, x\}$ be a chart of a 3-dimensional analytic manifold with $x(P_0) = 0$ for a point $P_0 \in U$. Consider the Herglotz relations (2.1) with a right-hand side*

$$K = \left(K_{ijkl} x^j x^k\right) \quad \text{with} \quad K_{ijkl} := \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl} \quad , \quad \delta_{ij} := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{5.5}$$

If we determine the formal power series solution G of (2.1) to a positive definite metric g from (2.6) and choose[1] an open neighbourhood $U' \subseteq U$ of $P_0 \in U$ such that the series of G converges on U' and the chart $\{U', x\}$ is a normal coordinate system of the metric g, then the Riemannian curvature tensor R of the calculated metric g fulfils

$$\forall r \geq 1 \; , \; \forall b = \{v_1, \ldots, v_{r+4}\} \subset M_{P_0} \; : \qquad (\partial^{(r)} R)_b \in \hat{J}_{(r)} \oplus \check{J}_{(r)} \; . \tag{5.6}$$

Furthermore, there holds $(\partial^{(r)} R)_b \neq 0$ at least for $r = 2, 4, 6$ and for suitable chosen vector sets $b = \{v_1, \ldots, v_{r+4}\} \subset M_{P_0}$.

Proof. Obviously, the matrix K from (5.5) satisfies

$$K \cdot K = r^2 K \quad \text{with} \quad r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \; . \tag{5.7}$$

Taking into account (5.7) and $F = E = (\delta_{ij})$, $K_2 = K$, $K_m = 0$ for $m \geq 3$ we obtain from (2.6)

$$\left.\begin{array}{lcl} \Gamma_{2m+1} & = & 0 \\ \Gamma_{2m} & = & c_m r^{2m-2} K \quad , \quad c_m = \text{const} \end{array}\right\} \quad , \qquad m = 1, 2, \ldots \tag{5.8}$$

[1] This is possible on the basis of Theorem 2.1.

This yields

$$G = E + f(r)\,K \tag{5.9}$$

with a convergent power series $f(r)$ for wich a more precise calculation[1] gives

$$f(r) = -\tfrac{1}{3} + \tfrac{1}{90}r^2 + \tfrac{1}{945}r^4 + \tfrac{43}{340200}r^6 + \ldots \tag{5.10}$$

The metric g_{ij} defined by (5.9) is centrally symmetric and turns into

$$ds^2 = dr^2 + h(r)\{d\theta^2 + \sin^2\theta\, d\phi^2\} \quad , \quad h(r) := r^2 + r^4 f(r) \tag{5.11}$$

if we introduce spherical coordinates

$$x^1 = r\cos\phi\sin\theta \quad , \quad x^2 = r\sin\phi\sin\theta \quad , \quad x^3 = r\cos\theta \quad .$$

The non-vanishing Christoffel symbols of a metric (5.11) are

$$\begin{array}{lll} \Gamma^r_{\theta\theta} = -\frac{1}{2}h'(r) & ; \ \Gamma^\theta_{r\theta} = \frac{h'(r)}{2h(r)} & ; \ \Gamma^\phi_{r\phi} = \frac{h'(r)}{2h(r)} \\ \Gamma^r_{\phi\phi} = -\frac{1}{2}h'(r)\sin^2\theta & , \ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta & ; \ \Gamma^\phi_{\theta\phi} = \cot\theta \quad . \end{array}$$

The only non-vanishing coordinates of the curvature tensor of (5.11) read[2]

$$R_{r\theta r\theta} = \frac{h''(r)}{2} - \frac{h'(r)^2}{4h(r)} \ ; \ R_{r\phi r\phi} = R_{r\theta r\theta}\sin^2\theta \ ; \ R_{\theta\phi\theta\phi} = \left(\frac{h'(r)^2}{4} - h(r)\right)\sin^2\theta \ .$$

Now we calculate from (5.10) and (5.11)

$$\frac{h'(r)^2}{4} - h(r) = -r^4 + \tfrac{1}{2}r^6 - \tfrac{1}{27}r^8 - \tfrac{11}{3240}r^{10} + \ldots \tag{5.12}$$

Since the coordinate transformation

$$R_{\theta\phi\theta\phi} = \partial_\theta x^i \partial_\phi x^j \partial_\theta x^k \partial_\phi x^l \, R_{ijkl} \qquad ; \qquad i,j,k,l \in \{1,2,3\}$$

produces a multiplication of the coordinates R_{ijkl} relating to $\{U', x\}$ by a factor r^4, we see from (5.12) that the power series of the coordinates R_{ijkl} contain homogeneous polynomials of orders 2 , 4 and 6 in the coordinates x^1, x^2, x^3. From this there follows $\partial^{(m)}R|_{P_0} \neq 0$ for $m = 2, 4, 6$.

But because $R_{ijkl}x^j x^k$ is a quadratic polynomial in the coordinates x^i we have $\partial^{(m)}\breve{R}|_{P_0} = 0$ for $m \geq 1$. Then (4.15) yields $y^*_{t_m}(\partial^{(m)}R)|_{P_0} = 0$ and consequently $(\partial^{(m)}R)_b \in \dot{J}_{(m)} \ominus \hat{J}_{(m)}$ for all $b = \{v_1, \ldots, v_{m+4}\} \subset M_{P_0}$ and $m \geq 1$. Furthermore, there exist non-vanishing group ring elements $(\partial^{(m)}R)_b$ for at least $m = 2, 4, 6$ since $\partial^{(m)}R|_{P_0} \neq 0$ for these m-values ∎

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[1] We have done the calculations of (5.9) and (5.12) by means of **Mathematica** [18].

[2] The $\Gamma^\kappa_{\mu\nu}$ and the $R_{\lambda\kappa\mu\nu}$ have been calculated by means of the **Mathematica** package **MathTensor** [3].

Remark 5.1. The metric (5.11), (5.10) possesses a non-constant scalar curvature τ. Using Mathematica and MathTensor one obtains

$$\tau := g^{ik}g^{jl}R_{ijkl} = \frac{-4\,h(r) - h'(r)^2 + 4h(r)\,h''(r)}{2\,h(r)^2} ,$$

and the replacement of h by its power series development, determined from (5.11) and (5.10), leads to

$$\tau = -6 - \tfrac{5}{3}r^2 - \tfrac{58}{135}r^4 - \tfrac{1213}{11340}r^6 + O(r^7) .$$

Consequently, the metric (5.11) is not contained in several classes of Riemannian spaces which require a constant scalar curvature τ. Obviously, (5.11) is not an Einstein space or a space of constant curvature. Furthermore, (5.11) is not a D'Atri space (see [12: p. 250]); thus the properties of local symmetry and local isotropy are also excluded (see [12: p. 251]). Finally, metric (5.11) can not be locally homogeneous, too.

Remark 5.2. For all dimensions $\dim M > 3$ there exist also examples (M,g) of Riemannian manifolds such that the $(\partial^{(r)}R)_b$ have a part in the ideal $\hat{J}_{(r)} \oplus \check{J}_{(r)}$. For instance, such an example is given by a product manifold $(M,g) = (M',g') \times (M'',g'')$ which is formed from a 3-dimensional Riemannian manifold (M',g') according to Proposition 5.2 and a flat Riemannian manifold (M'',g''). Let us assume that $\{M',x'\}$ is a normal coordinate system according to Proposition 5.2 with centre $P' \in M'$. Then we can determine a product chart $x = x' \times x''$ of $M' \times M''$ around any point $(P',P'') \in M' \times M''$ which is a normal coordinate system with respect to g. At most the coordinates

$$R_{i'j'k'l'}(x^{a'}) \quad , \quad a',i',j',k',l' = 1,2,3 ,$$

of the curvature tensor do not vanish with respect to x. We see from the proof of Proposition 5.2 that the $R_{i'j'k'l'}$ contain homogeneous polynomials of orders 2, 4 and 6 in x^1, x^2, x^3 such that there holds $\partial^{(m)}R|_{(P',P'')} \neq 0$ for $m = 2,4,6$. On the other hand, the expressions $R_{i'jkl'}x^jx^k = R_{i'j'k'l'}x^{j'}x^{k'}$ are quadratic polynomials in the coordinates x^1, x^2, x^3, and the expressions $R_{ijkl}x^jx^k$ vanish if $i > 3$ or $l > 3$. Thus we obtain $\partial^{(m)}\check{R}|_{(P',P'')} = 0$ for $m \geq 1$. But then the same arguments which we used in the proof of Proposition 5.2 tell us that $(\partial^{(m)}R)_b \in \hat{J}_{(m)} \oplus \check{J}_{(m)}$ for all $b = \{v_1,\ldots,v_{m+4}\} \subset (M' \times M'')_{(P',P'')}$ and $m \geq 1$, and that non-vanishing $(\partial^{(m)}R)_b$ exist for at least $m = 2,4,6$.

6. The equality of the tensor algebras $\mathcal{R}$ and $\mathcal{R}^s$

Now we return to the question whether the tensor algebra $\mathcal{R}$ (1.6) is equal to the tensor algebra $\mathcal{R}^s$ (1.8). To answer this question, we use the following proposition which follows easily from results of [6].

Proposition 6.1. *Let $\nabla^{(r)}_{()}R$ denote the symmetrized covariant derivative of order r of the Riemannian curvature tensor with coordinates $\nabla_{(i_5}\ldots\nabla_{i_{r+4})}R_{i_1\ldots i_4}$. Further, we put $\nabla^{(0)}_{()}R := R$. Then there holds true for $r \geq 0$*

$$y^*_{t_r}\nabla^{(r)}_{()}R = \mu_r\,\nabla^{(r)}_{()}R \quad , \quad \mu_r = 2\,(r+3)\,(r+2)\,r! \qquad (6.1)$$

if y_{t_r} is the Young symmetrizer of the standard tableau t_r (3.18).

Proof. We will carry out here those steps of the proof which are not given explicitely in [6].

In the case $r = 0$ the assertion follows from Proposition 3.1, (3.9). Thus we can assume $r \geq 1$ in the following.

Definition 6.1. We denote by $\mathcal{T}_{r,\mathcal{B}}V$ the vector space of complex-valued covariant tensors T of order $r+4$ on a vector space V over $\mathbb{C}$ which have the following properties:

1. Every $T \in \mathcal{T}_{r,\mathcal{B}}V$ possesses the symmetry properties of the Riemannian curvature tensor relating to the indices $i_1, \ldots, i_4$, i.e.

$$T_{i_1 i_2 i_3 i_4 i_5 \ldots i_{r+4}} = -T_{i_2 i_1 i_3 i_4 i_5 \ldots i_{r+4}} = -T_{i_1 i_2 i_4 i_3 i_5 \ldots i_{r+4}} = T_{i_3 i_4 i_1 i_2 i_5 \ldots i_{r+4}} .$$

2. Every $T \in \mathcal{T}_{r,\mathcal{B}}V$ satisfies the first Bianchi identity relating to the indices i_2, i_3, i_4 and the second Bianchi identity relating to the indices i_3, i_4, i_5, i.e.

$$T_{i_1 i_2 i_3 i_4 i_5 \ldots i_{r+4}} + T_{i_1 i_3 i_4 i_2 i_5 \ldots i_{r+4}} + T_{i_1 i_4 i_2 i_3 i_5 \ldots i_{r+4}} = 0$$

and

$$T_{i_1 i_2 i_3 i_4 i_5 \ldots i_{r+4}} + T_{i_1 i_2 i_4 i_5 i_3 \ldots i_{r+4}} + T_{i_1 i_2 i_5 i_3 i_4 \ldots i_{r+4}} = 0 .$$

3. Every $T \in \mathcal{T}_{r,\mathcal{B}}V$ is symmetric in $i_5, \ldots, i_{r+4}$.

Furthermore, we assume that there is given an order relation $<$ in the set of the $r+4$ index names of a $T \in \mathcal{T}_{r,\mathcal{B}}V$. Let $a < b < c < d < e$ be the 5 smallest index names. Then there is proved in [6: p. 1154]:

Proposition 6.2. *Every coordinate $T_{i_1 \ldots i_{r+4}}$ of a tensor $T \in \mathcal{T}_{r,\mathcal{B}}V$ with an arbitrary arrangement of its index names can be expressed as a linear combination of the following types of coordinates:*

$T_{abcde\ldots}$	
$T_{abcid\ldots}$	*with* $d < i$
$T_{acbde\ldots}$	
$T_{acbid\ldots}$	*with* $d < i$
$T_{aibjc\ldots}$	*with* $c < i < j$.

The dots represent the ordered sequence of the remaining index names. The number of these special coordinates is

$$1 + r + 1 + r + \frac{r(r+1)}{2} = \frac{(r+1)(r+4)}{2} . \tag{6.2}$$

Another result of [6: p. 1162] reads:

Proposition 6.3. *If T is an arbitrary covariant tensor of order $r+4$ on V, then $y^*_{t_r} T$ lies in $\mathcal{T}_{r,\mathcal{B}}V$.*

Let $Q \subset \mathcal{S}_{r+4}$ be the set of all permutations which transform the ordered sequence of the $r+4$ index names of a covariant tensor T of order $r+4$ into the index arrangements given in Proposition 6.2. Then there follows from Proposition 6.2 that every $T \in \mathcal{T}_{r,\mathcal{B}}V$ satisfies

$$\forall p \in \mathcal{S}_{r+4} : \quad pT \ = \ \sum_{q \in Q} a_{pq}\, qT \quad , \quad a_{pq} \in \mathbb{C} \tag{6.3}$$

with coefficients a_{pq} which are independent on T. Taking into account the relation

$$\forall b = \{v_1, \ldots, v_{r+4}\} \subset V \ , \ \forall p, s \in \mathcal{S}_{r+4} : \quad (sT)_b(p) \ = \ T_b(p \circ s) \ ,$$

which is a consequence of

$$(sT)_b \ = \ T_b \cdot s^* \ = \ \sum_{p' \in \mathcal{S}_{r+4}} T_b(p')\, p' \circ s^{-1} \ = \ \sum_{p \in \mathcal{S}_{r+4}} T_b(p \circ s)\, p \ ,$$

we obtain from (6.3)

$$T_b \ = \ \sum_{s \in \mathcal{S}_{r+4}} (sT)_b(id)\, s \ = \ \sum_{s \in \mathcal{S}_{r+4}} \sum_{q \in Q} a_{sq}\, (qT)_b(id)\, s \ = \ \sum_{q \in Q} T_b(q)\, u_q \tag{6.4}$$

where $u_q := \sum_{s \in \mathcal{S}_{r+4}} a_{sq}\, s$.

Now, let $W_{\mathcal{B}}(V) := \mathcal{L}\{T_b \mid T \in \mathcal{T}_{r,\mathcal{B}}V \ , \ b = \{v_1, \ldots, v_{r+4}\} \subset V\}$ be the vector subspace of $\mathbb{C}[\mathcal{S}_{r+4}]$ generated by all T_b of the tensors $T \in \mathcal{T}_{r,\mathcal{B}}V$. Then equation (6.4) yields $W_{\mathcal{B}}(V) \subseteq \mathcal{L}\{u_q \mid q \in Q\}$ and $\dim W_{\mathcal{B}}(V) \leq |Q| = (r+4)(r+1)/2$.

Proposition 6.3 means that $(y^*_{t_r} T)_b \in W_{\mathcal{B}}(V)$ for all subsets $b = \{v_1, \ldots, v_{r+4}\} \subset V$. In the following we assume $\dim V \geq r+4$. Then there exists a vector set $b_0 = \{v_1, \ldots, v_{r+4}\} \subset V$ such that $\mathbb{C}[\mathcal{S}_{r+4}]$ is generated by the T_{b_0} of all covariant tensors T of order $r+4$ (see [5: Lemma 2.1]) and consequently the left ideal $J_{(r)} = \mathbb{C}[\mathcal{S}_{r+4}] \cdot y_{t_r}$ is spanned by the $T_{b_0} \cdot y_{t_r} = (y^*_{t_r} T)_{b_0}$ of all covariant tensors T of order $r+4$. Thus we obtain $J_{(r)} \subseteq W_{\mathcal{B}}(V)$. But since $\dim J_{(r)} = (r+4)(r+1)/2$ because of (4.7), there follows $J_{(r)} = W_{\mathcal{B}}(V)$.

In the case $m := \dim V < r+4$ we introduce an $(r+4)$-dimensional vector space $\tilde{V}$ which we map linearly onto V by means of a linear mapping $\phi : \tilde{V} \to V$ defined on given bases $\{u_1, \ldots, u_m\}$ of V and $\{\tilde{u}_1, \ldots, \tilde{u}_{r+4}\}$ of $\tilde{V}$ by the rule

$$\phi(\tilde{u}_i) \ := \ \begin{cases} u_i & \text{if } i = 1, \ldots, m \\ 0 & \text{if } i = m+1, \ldots, r+4 \end{cases} .$$

Then the pull back $(\phi^* T)(\tilde{v}_1, \ldots, \tilde{v}_{r+4}) := T(\phi(\tilde{v}_1), \ldots, \phi(\tilde{v}_{r+4}))$, $\tilde{v}_i \in \tilde{V}$, of every tensor $T \in \mathcal{T}_{r,\mathcal{B}}V$ lies in $\mathcal{T}_{r,\mathcal{B}}\tilde{V}$. Every vector set $b = \{v_1, \ldots, v_{r+4}\} \subset V$ corresponds to a uniquely determined vector set $\tilde{b} = \{\tilde{v}_1, \ldots, \tilde{v}_{r+4}\} \subset \mathcal{L}\{\tilde{u}_1, \ldots, \tilde{u}_m\}$ via $v_i = \phi(\tilde{v}_i)$. Thus there holds true $T_b = (\phi^* T)_{\tilde{b}} \in W_{\mathcal{B}}(\tilde{V}) = J_{(r)}$ for every $T \in \mathcal{T}_{r,\mathcal{B}}V$, $b \subset V$.

Let now $V = M_P$ be a tangent space of our differentiable manifold M in a point $P \in M$. Then there is $\nabla^{(r)}_{()} R \in \mathcal{T}_{r,\mathcal{B}} M_P$. This leads to $(\nabla^{(r)}_{()} R)_b \in J_{(r)} = \mathbb{C}[\mathcal{S}_{r+4}] \cdot y_{t_r}$, that means $(\nabla^{(r)}_{()} R)_b = x \cdot y_{t_r}$ with some $x \in \mathbb{C}[\mathcal{S}_{r+4}]$. Now taking into account (4.16) and (4.14) we obtain

$$(y^*_{t_r} \nabla^{(r)}_{()} R)_b \ = \ x \cdot y_{t_r} \cdot y_{t_r} \ = \ \mu_r (\nabla^{(r)}_{()} R)_b$$

for every vector set $b = \{v_1, \ldots, v_{r+4}\} \subset M_p$ by which Proposition 6.1 is proved ∎

Now the version of Proposition 3.1 for $\nabla^{(r)}_{()}R$ reads

Corollary 6.1. *Let $r \geq 0$. Then the group ring element $(\nabla^{(r)}_{()}R)_b \in \mathbb{C}[\mathcal{S}_{r+4}]$ is contained in the left ideal*

$$J_{(r)} = \mathbb{C}[\mathcal{S}_{r+4}] \cdot y_{t_r}$$

of $\mathbb{C}[\mathcal{S}_{r+4}]$ for every set of vectors $b = \{v_1, \ldots, v_{r+4}\} \subset M_P$, $P \in M$.

Since $J_{(r)}$ is minimal, the problem of decomposition of $J_{(r)}$ does not arise.

Theorem 6.1. *We denote by $\nabla^{(r)}\check{R}$ the 'stronger' symmetrized covariant derivative of the Riemannian curvature tensor of order r the coordinates of which have the form*

$$(\nabla^{(r)}\check{R})_{i_1 i_2 i_3 i_4 i_5 \ldots i_{r+4}} := \nabla_{(i_5} \ldots \nabla_{i_{r+4}} R_{|i_1|i_2 i_3) i_4}$$

$$(\nabla^{(0)}\check{R})_{i_1 \ldots i_4} := \check{R}_{i_1 \ldots i_4} := R_{i_1 (i_2 i_3) i_4}$$

Then there holds true for $r \geq 0$

$$\begin{aligned} 2\,\tfrac{r+3}{r+1}\,(\nabla^{(r)}_{()}R)_{i_1 i_2 i_3 i_4 i_5 \ldots i_{r+4}} &= \\ +(\nabla^{(r)}\check{R})_{i_1 i_2 i_3 i_4 i_5 \ldots i_{r+4}} - (\nabla^{(r)}\check{R})_{i_2 i_1 i_3 i_4 i_5 \ldots i_{r+4}} - (\nabla^{(r)}\check{R})_{i_1 i_2 i_4 i_3 i_5 \ldots i_{r+4}} & \qquad (6.5) \\ +(\nabla^{(r)}\check{R})_{i_4 i_2 i_3 i_1 i_5 \ldots i_{r+4}} + (\nabla^{(r)}\check{R})_{i_2 i_1 i_4 i_3 i_5 \ldots i_{r+4}} - (\nabla^{(r)}\check{R})_{i_4 i_1 i_3 i_2 i_5 \ldots i_{r+4}} & \\ -(\nabla^{(r)}\check{R})_{i_3 i_2 i_4 i_1 i_5 \ldots i_{r+4}} + (\nabla^{(r)}\check{R})_{i_3 i_1 i_4 i_2 i_5 \ldots i_{r+4}} & \ . \end{aligned}$$

As a consequence of (6.5), we obtain $\mathcal{R} = \mathcal{R}^s$.

Proof. For every subset $b = \{v_1, \ldots, v_{r+4}\} \subset M_P$ of the tangent space in an arbitrary point $P \in M$ there holds true

$$(y^*_{t_r}(\nabla^{(r)}_{()}R))_b = \mu_r(\nabla^{(r)}_{()}R)_b \quad \text{and} \quad (\nabla^{(r)}\check{R})_b = \frac{1}{(r+2)!}(\epsilon^*(\nabla^{(r)}_{()}R))_b \ .$$

Then using (4.14) we can write

$$\begin{aligned} \mu_r(\nabla^{(r)}_{()}R)_b &= (y^*_{t_r}(\nabla^{(r)}_{()}R))_b = (\nabla^{(r)}_{()}R)_b \cdot y_{t_r} = -\frac{1}{\mu_r}(\nabla^{(r)}_{()}R)_b \cdot y_{t_r} \cdot \epsilon \cdot \eta \\ &= -\frac{1}{\mu_r}(\eta^* \cdot \epsilon^* \cdot y^*_{t_r}(\nabla^{(r)}_{()}R))_b = -(\eta^* \cdot \epsilon^*(\nabla^{(r)}_{()}R))_b \\ &= -(r+2)!\,(\eta^*(\nabla^{(r)}\check{R}))_b \ . \end{aligned}$$

Now equation (6.5) can be proved by the same arguments which we applied to show (4.15) ∎

Acknowledgements. The investigations of this paper originated from a suggestion of Professor P. Günther. I am greatly indebted to him for the constant support and scientific guidance which he has given me during my whole professional development.

I wish to express my sincerest thanks to Prof. G. Eisenreich for checks of my work and valuable hints.

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Received 19.6.1997.