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On the Solution of an Integral-Functional Equation with a Parameter

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Abstract. For a homogeneous integral-functional equation containing a parameter, we show existence and uniqueness of a compactly supported solution with given value for its integral. The solution is infinitely often differentiable, symmetric with respect to the point $\frac{1}{2}$, monotonous at both sides of $\frac{1}{2}$, and satisfies further functional equations. The Fourier series of the periodic continuation is determined. We also investigate spectral properties of the integral equation and find surprising connections between the Laplace transform of the eigenfunction and the eigenfunctions of the adjoint equation, and also directly between different eigenfunctions both in the compact and in the non-compact case. Moreover, asymptotic considerations are made.

Keywords: *Integral-functional equations, Fourier series, spectral properties, biorthogonal sequences, Appell polynomials, asymptotic approximations*

AMS subject classification: 39 B 22, 45 D 05, 42 A 16, 41 A 60

This paper deals with the homogeneous integral-functional equation

$$\phi(t) = L\phi(t) \ , \quad L\phi(t) = b \int\limits_{at-a+1}^{at} \phi(\tau)d\tau \qquad (b = \tfrac{a}{a-1}) \ , \tag{1}$$

where a is a fixed parameter with $a > 1$ and $t \in \mathbb{R}$. We mainly look for compactly supported solutions. By means of the Laplace transform, we show that such solutions exist uniquely up to a constant factor c. Obviously, they are infinitely often differentiable and, for $c \neq 0$, the support is contained in the interval [0,1]. Hence, they can be used as test functions in the theory of distributions. Such functions are never holomorphic, and they satisfy $\phi^{(n)}(0) = \phi^{(n)}(1) = 0$ for all $n \in \mathbb{N}_0$. The constant factor c can be fixed by the condition

$$\int\limits_0^1 \phi(t)\,dt = 1 \ , \tag{2}$$

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which implies that the solution is a real one. Solutions with non-compact support are only considered in order to obtain new conclusions concerning compactly supported ones.

In Section 6, we shall see that (1) has only the trivial solution $\phi(t) = 0$ for $0 < a < 1$, which explains our assumption $a > 1$. The case $a = 2$ was investigated by W. Volk in [7], where ϕ is a function with a polynomial behaviour in all dyadic points, i.e. more precisely,

$$\phi^{(m)}\left(\frac{\nu}{2^n}\right) = 0 \quad \text{for} \quad m \geq n \quad \text{and} \quad \nu = 0(1)2^n \ .$$

The case $a = 3$ was investigated by G. J. Wirsching in [8], where ϕ is the density of a certain transition probability. As we shall see in Section 3, ϕ is also non-negative in the general case $a > 1$, so that in view of (2) it always can be used as a probability density.

Of course, (1) can also be written as a Fredholm integral equation on $[0,1] \times [0,1]$ with kernel $k(t,\tau)$ defined by $k(t,\tau) = b$ for $0 \leq at - a + 1 \leq \tau \leq at \leq 1$ and $k(t,\tau) = 0$ elsewhere on $[0,1] \times [0,1]$, but it is more natural, to consider it as a Volterra integral-functional equation. Equations of type (1) do not seem to appear in relevant books on Volterra functional equations (cf., e.g., [5]).

In the last sections, we also consider the eigenvalue problems belonging to equation (1) and also to the adjoint equation, and find surprising connections between different eigenfunctions.

1. Existence and uniqueness

We begin with the basic existence theorem.

Theorem 1.1. *For each $a > 1$, equation* (1) *has a unique compactly supported solution ϕ, which fulfills condition* (2).

Proof. First, we assume that (1) has an integrable solution with a support contained in [0,1]. By (finite) Laplace transform of (1), we obtain

$$\Phi(p) = \frac{1 - e^{-p/b}}{p/b} \, \Phi\left(\frac{p}{a}\right) \tag{1.1}$$

where $\Phi(p) = \mathcal{L}\{\phi(t)\}$. In view of $a > 1$ and (2), we find for $n \to \infty$

$$\Phi(a^{-n}p) = \int_0^1 e^{-a^{-n}pt}\phi(t)\,dt \to \int_0^1 \phi(t)\,dt = 1 \ .$$

Hence, the function

$$\Phi(p) = \prod_{k=0}^{\infty} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)} \tag{1.2}$$

is the unique solution of (1.1) with $\Phi(0) = 1$, so far as the product converges. This implies that (1) has at most one compactly supported solution satisfying (2). On the other side, for $\Re z \geq 0$, we have the estimates

$$\left|\frac{1-e^{-z}}{z}\right| = \left|\int_0^1 e^{-zt}dt\right| \leq \int_0^1 e^{-\Re zt}dt \leq 1$$

and

$$\left|1 - \frac{1-e^{-z}}{z}\right| = \left|\int_0^1 (1-e^{-zt})\,dt\right| \leq \int_0^1 |1-e^{-zt}|\,dt \leq \int_0^1 |z|t\,dt = \frac{1}{2}|z|\,.$$

For $z = p/(ba^k)$ with $0 \leq \Re p$ and $|p| \leq r$ with arbitrary fixed $r > 0$, this implies that the product (1.2) is absolutely and uniformly convergent, and consequently a holomorphic function for $\Re p > 0$ with $|\Phi(p)| \leq 1$ (cf. Knopp [6: p. 451 - 452]). The last estimate can be sharpened by means of (1.1) to

$$\Phi(p) = \frac{ab^2}{p^2}(1-e^{-p/b})(1-e^{-p/(ab)})\,\Phi\left(\frac{p}{a^2}\right) = O\left(\frac{1}{p^2}\right)$$

for $\Re p \geq 0$, so that Φ possesses an original function ϕ satisfying (1), (2) (cf. L. Berg [1: p. 30]) ∎

Remark. In the case of $a = 3$, G. Wirsching has shown in [8] that the operator L is contractive on the convex space of functions from $L^1[0,1]$ satisfying (2). The proof can be transferred to the case $2 < a$, but not to the case $1 < a \leq 2$. For the case $a = 2$, W. Volk has shown in [7] that the operator L is contractive on a certain subspace from $C^1[0,1]$ equipped with the norm $\|f\| := \alpha\|f\|_\infty + \beta\|Df\|_\infty$, where $0 < \alpha < \beta$.

2. Laplace transform and convolution

As a finite Laplace transform, Φ is in fact an entire function. The analytic continuation to the left half plane can also be seen from the formula

$$\Phi(p) = e^{-p}\Phi(-p)\,, \tag{2.1}$$

which follows from (1.2) in view of

$$\prod_{k=0}^{\infty} \frac{1-e^{-p/(ba^k)}}{p/(ba^k)} = \prod_{k=0}^{\infty} e^{-p/(ba^k)} \prod_{k=0}^{\infty} \frac{e^{p/(ba^k)}-1}{p/(ba^k)}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{a^k} = b\,. \tag{2.2}$$

The function Φ has the further representation

$$\Phi(p) = e^{-\frac{p}{2}} \prod_{k=0}^{\infty} \frac{\sinh(p/(2ba^k))}{p/(2ba^k)} , \tag{2.3}$$

which follows from (1.2) in view of

$$\frac{1-e^{-z}}{z} = e^{-\frac{z}{2}} \frac{\sinh(z/2)}{z/2}$$

with $z = p/(ba^k)$ and (2.2). For $a \to 1$, representation (2.3) implies that $\Phi(p) \to e^{-p/2}$, which is the Laplace transform of the distribution $\delta(t-1/2)$.

From (2.3) with $p = ix$, we obtain

$$\Phi(ix) = e^{-\frac{ix}{2}} P(x) , \tag{2.4}$$

where

$$P(x) = \prod_{k=1}^{\infty} \frac{\sin\left(\frac{a-1}{2a^k} x\right)}{\frac{a-1}{2a^k} x} . \tag{2.5}$$

Note that

$$\ln \frac{\sin x}{x} = \sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)!} \frac{x^{2n}}{2n} \qquad (|x| < \pi)$$

where B_n are the Bernoulli numbers

$$B_0 = 1 , \quad B_1 = -\frac{1}{2} , \quad B_2 = \frac{1}{6} , \quad B_3 = 0 , \quad B_4 = -\frac{1}{30} , \quad \dots \tag{2.6}$$

(cf. [3: p. 513], with $-B_n$ instead of $(-1)^n B_{2n}$, $n \geq 1$). Here, we have chosen the notation of B_n from [6: p. 185] resp. [1: p. 103]. From (2.5), we get

$$\ln P(x) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)!\,2n} \left(\frac{a-1}{2a^k} x\right)^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!\,2n} \frac{(a-1)^{2n}}{a^{2n}-1} x^{2n}$$

for $|x| < 2b\pi$, $b = a/(a-1)$. According to (2.4) with $x = -ip$ and $B_1 = -1/2$ as well as $B_{2n+1} = 0$, $n \geq 1$, this implies the representation

$$\ln \Phi(p) = \sum_{n=1}^{\infty} \frac{B_n}{n!\,n} \frac{(a-1)^n}{a^n - 1} p^n , \quad |p| < 2b\pi . \tag{2.7}$$

For the next results in this section, it is useful to denote the solution of (1), (2) by $\phi(t,a)$, and its Laplace transform by $\Phi(p,a)$.

Proposition 2.1. *The solution $\phi(t,a)$ has the convolution property*

$$\phi(t,a) = \alpha\,\beta \int_0^t \phi\left(\alpha(t-\tau), a^2\right) \phi(\beta\tau, a^2)\, d\tau \tag{2.8}$$

where $\alpha = 1 + 1/a$, $\beta = 1 + a$.

Proof. We separate the factors in (1.2) in those with only even k and those with only odd k, so that we obtain

$$\Phi(p,a) = \prod_{\nu=0}^{\infty} \frac{1 - e^{-p/(ba^{2\nu})}}{p/(ba^{2\nu})} \prod_{\mu=0}^{\infty} \frac{1 - e^{-p/(ba^{2\mu+1})}}{p/(ba^{2\mu+1})}.$$

In view of $b = a/(a-1)$ and $\alpha a^2/(a^2-1) = b$, $\beta a^2/(a^2-1) = ab$, the foregoing equation can be written as

$$\Phi(p,a) = \Phi\left(\frac{p}{\alpha}, a^2\right) \Phi\left(\frac{p}{\beta}, a^2\right). \tag{2.9}$$

This equation immediately implies (2.8) by means of the inverse Laplace transform, using the convolution theorem and the well known property $\mathcal{L}\{\gamma\phi(\gamma t)\} = \Phi(p/\gamma)$ for arbitrary $\gamma > 0$ ∎

Let us mention that the function $\phi(\alpha t)$ has the support $[0, 1/\alpha]$ and $\phi(\beta t)$ the support $[0, 1/\beta]$, so that in view of $1/\alpha + 1/\beta = 1$, the convolution (2.8) has in fact the support [0,1] (cf. the first three propositions in Section 4).

Writing the convolution as usual by means of a star, we can generalize (2.8) by

$$\phi(t,a) = \alpha_1\alpha_2 \cdots \alpha_n\, \phi(\alpha_1 t, a^n) * \phi(\alpha_2 t, a^n) * \ldots * \phi(\alpha_n t, a^n)$$

where $\alpha_\nu = a^\nu(1 - a^{-n})/(a-1)$ for $\nu = 1, \ldots, n$ and $n \in \mathbb{N}$.

Though we are actually only interested in the case $a > 1$, the power series (2.7) is also convergent in the case of $0 \le a \le 1$ for $|p| < 2\pi/(1-a)$. From (2.7), we get

$$\ln \Phi(p, 1/a) = \sum_{n=1}^{\infty} \frac{B_n}{n!\,n} \frac{(1-a)^n}{1-a^n} p^n = -\ln \Phi(-p, a)$$

for $|p| < 2\pi b$ in case of $a > 1$. Thus, for these p, we have $\Phi(p,a)\Phi(-p,1/a) = 1$ and in view of (2.1) the equation

$$\Phi(p,a)\,\Phi(p,1/a) = e^{-p}. \tag{2.10}$$

In particular, we find

$$\Phi(p,0) = \frac{p}{e^p - 1}, \qquad \Phi(p,1) = e^{-p/2}, \qquad \Phi(p,\infty) = \frac{1 - e^{-p}}{p} \tag{2.11}$$

(cf. (1.1)) and $\Phi(0,p) = 1$. A product representation for $\Phi(p,a)$ in case of $0 < a < 1$ can be derived, writing (1.1) as

$$\Phi(p,a) = \frac{ap/b}{1-e^{-ap/b}}\,\Phi(ap,a)$$

and iterating this equation. The result also follows from (2.10).

Later on, we need the coefficients of the expansion

$$\Phi(p,a) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!}\,p^n \,. \tag{2.12}$$

These coefficients can be determined by means of the recursion formula

$$\begin{aligned} &\rho_0(a) = 1 \\ &\rho_n(a) = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\rho_{n-k}(a)\,B_k\,\frac{(a-1)^k}{a^k-1} \quad quad(n \geq 1) \end{aligned} \tag{2.13}$$

which follows from (2.7) and (2.12) in view of $\Phi' = \Phi(\ln\Phi)'$ by comparison of coefficients. From (2.13), we find that the coefficients $\rho_n(a)$ are rational functions with respect to a, in particular

$$\rho_1(a) = -\frac{1}{2}\,, \qquad \rho_2(a) = \frac{2a+1}{6(a+1)}\,, \qquad \rho_3(a) = \frac{-a}{4(a+1)}\,,$$

$$\rho_4(a) = \frac{6a^4+3a^3+5a^2+2a-1}{30(a+1)^2(a^2+1)}\,, \qquad \rho_5(a) = \frac{-2a^4-2a^2+a}{12(a+1)^2(a^2+1)}\,, \;\ldots\,.$$

According to (2.11), we have the special values

$$\rho_n(0) = B_n, \qquad \rho_n(1) = \frac{(-1)^n}{2^n}, \qquad \rho_n(\infty) = \frac{(-1)^n}{n+1}, \tag{2.14}$$

where B_n are the Bernoulli numbers (cf. (2.6)). Let us mention that (2.13), for $a \to 0$, turns over into a nonlinear recursion formula for the Bernoulli numbers.

For $a > 1$, the coefficients of the expansion (2.12) possess the representation

$$\rho_n(a) = (-1)^n \int_0^1 t^n \phi(t,a)\,dt \tag{2.15}$$

i.e. $(-1)^n\rho_n(a)$ are the n-th moments of the solution ϕ of (1), (2) (cf. [1: p. 87]). In view of (2.14) and the original functions of (2.11) $\phi(t,1) = \delta(t-1/2)$ as well as $\phi(t,\infty) = 1$ for $0 < t < 1$ and $\phi(t,\infty) = 0$ elsewhere, the formula (2.15) is also true both for $a = 1$ and for $a = \infty$.

Considering $\phi(t,a)$ as probability density, the corresponding random variable has the expected value $1/2$ and the variance $(a-1)/(12(a+1))$.

Without proof, we mention that for $a > 1$ the n-th moments $(-1)^n\rho_n(a)$ satisfy the estimate

$$\frac{1}{2^n} \leq (-1)^n\rho_n(a) \leq \frac{1}{n+1}\,,$$

so that in case of $a \geq 1$ the power series (2.12) converges in fact for all $p \in \mathbb{C}$.

3. Approximations

The solution ϕ of (1), (2), $a > 1$, can also be obtained by means of iteration.

Theorem 3.1. *For every non-negative L-integrable function f_0 on the interval $[0,1]$ with $f_0(t) = 0$ for $t \notin [0,1]$ and the property*

$$\int_0^1 f_0(t)\,dt = 1 , \tag{3.1}$$

the sequence $f_n = Lf_{n-1}$ $(n \geq 1)$ converges uniformly on $[0,1]$ to the unique solution ϕ of (1), (2), which is also non-negative.

Proof. Obviously, the iterates f_n $(n \geq 1)$ are continuous, non-negative, and $f_n(t) = 0$ for $t \notin (0,1)$. It can easily be seen that (3.1) is invariant concerning L, i.e. we also have

$$\int_0^1 f_n(t)\,dt = 1 \qquad (n \geq 1) . \tag{3.2}$$

In order to prove convergence of the sequence f_n, we wish to apply the selection theorem due to Arzelà and Ascoli. For this reason, we have to show that the functions f_n are uniformly bounded on $[0,1]$ and equicontinuous. Since $f_n(t)$ is non-negative, and satisfies $f_n(t) = 0$ for $t \notin (0,1)$ as well as (3.2), we obtain for each $t \in [0,1]$

$$0 \leq f_n(t) = b \int_{at-a+1}^{at} f_{n-1}(\tau)\,d\tau \leq b \int_0^1 f_{n-1}(\tau)\,d\tau = b ,$$

i.e. the boundedness of the sequence. The inequality

$$|f_n(t+h) - f_n(t)| = b \left| \int_{at}^{a(t+h)} f_{n-1}(\tau)\,d\tau - \int_{at-a+1}^{a(t+h)-a+1} f_{n-1}(\tau)\,d\tau \right| \leq 2ab^2|h|$$

shows the equicontinuity. Hence, we can select a subsequence $f_{k(n)}$ which is uniformly convergent to a non-negative function f.

Next, we prove that this subsequence $f_{k(n)}$ converges to ϕ. For this reason we consider the (finite) Lapace transform $F_n(p) = \int_0^1 e^{-pt} f_n(t)\,dt$. Hence, from $f_n = Lf_{n-1}$ and $f_n(t) = 0$ for $t \leq 0$, we get $F_n(p) = \frac{b}{p}(1 - e^{-p/b})\,F_{n-1}\left(\frac{p}{a}\right)$ $(n \geq 1)$ and it follows that

$$F_n(p) = F_0(a^{-n}p) \prod_{k=0}^{n} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)} .$$

Since $a > 1$, we obtain for $n \to \infty$

$$F_0(a^{-n}p) = \int_0^1 e^{-a^{-n}pt} f_0(t)\,dt \to \int_0^1 f_0(t)\,dt = 1 .$$

In view of (1.2), we see that the sequence F_n converges to the Laplace transform of the solution ϕ, and it follows that $f = \phi$.

Now, we can show that the sequence f_n is uniformly convergent to ϕ. From above, we see that every subsequence of f_n, as sequence in $C[0,1]$, has a subsequence which converges in the maximum norm to $\phi \in C[0,1]$. Hence, by a convergence principle in Banach spaces (cf. Zeidler [9: p. 480]) the sequence f_n converges in this norm to ϕ, i.e. the sequence f_n is uniformly convergent to ϕ ∎

Remark. The condition $f_0 \geq 0$ is not necessary. Namely, if f_0 is an arbitrary L-integrable function on $[0,1]$ with $f_0(t) = 0$ for $t \notin (0,1)$, then we can introduce a sequence g_n by $g_0 = |f_0|$ and $g_n = Lg_{n-1}$, so that $|f_n(t)| \leq g_n(t)$, and the case is actually reduced to the foregoing one with some further modifications.

4. Symmetry and Monotony

We know that the solution ϕ of (1), $a > 1$, is differentiable so that it also satisfies the functional-differential equation

$$\phi'(t) = ab\left[\phi(at) - \phi(at - a + 1)\right] \qquad \left(b = \tfrac{a}{a-1}\right). \tag{4.1}$$

Proposition 4.1. *The solution ϕ of* (1), (2) *has the property $\phi(t) = \phi(1-t)$, i.e. ϕ is symmetric (with respect to $1/2$).*

Proof. It is easy to check that if a function f has the property $f(t) = f(1-t)$, then $g = Lf$ has this property too. Thus, choosing f_0 symmetric, this property transfers to all iterates $f_n = Lf_{n-1}$ and also to the limit function ϕ ∎

Remark. Note that the symmetry of ϕ is also a consequence of (2.1).

Proposition 4.2. *In case of $1 < a < 2$, the solution ϕ of* (1), (2) *is strongly increasing in $(0, 1/2)$.*

Proof. Let $1 < a < 2$. At first, we show that $\phi(t)$ is not decreasing for $0 < t < 1/2$. We choose f_0 from Proposition 3.1 symmetric with respect to $1/2$, continuous and strongly increasing in (0,1/2). Then the iterates f_n $(n \geq 1)$ are also symmetric and differentiable, and we show by induction that they are also strongly increasing in (0,1/2). By differentiation of $f_{n+1} = Lf_n$, we get

$$f'_{n+1}(t) = ab\left[f_n(at) - f_n(at - a + 1)\right].$$

Since f_n is strongly increasing in (0,1/2) and $f_n(t) = 0$ for $t \leq 0$, it follows

$$f'_{n+1}(t) = ab\, f_n(at) > 0$$

for $0 < t < 1 - 1/a$. Furthermore, each c with $f'_{n+1}(c) = 0$ satisfies $f_n(ac) = f_n(ac - a + 1)$. Since $ac > ac - a + 1$, it follows in view of $f_n(1-t) = f_n(t)$ that $1 - ac = ac - a + 1$ and therefore $c = 1/2$. This means that $f'_{n+1}(t) > 0$ for $t \in (0, 1/2)$. As $n \to \infty$, we obtain $\phi'(t) \geq 0$, so that $\phi(t)$ is not decreasing for these t.

Next, we show that $\phi(t)$ is strongly positive for $t \in (0,1)$. Assume that $\phi(t) = 0$ for $0 \le t \le \varepsilon$ where ε is maximal. From (1) it follows that also $\phi(t) = 0$ for $t \le a\varepsilon$ and thus $\varepsilon = 0$ in view of $a > 1$. The monotony of ϕ implies $\phi(t) > 0$ for $t \in (0, 1/2]$, and the symmetry transfers this property to the whole interval (0,1).

Now, we are able to show that ϕ is strongly increasing in (0,1/2). In view of (4.1), we have $\phi'(t) = ab\,\phi(at) > 0$ for $0 < t < 1 - 1/a$, and in view of the symmetry, we have $\phi'(1/2) = 0$. Let c be the smallest point in $[1 - 1/a, 1/2]$ with $\phi'(c) = 0$ which exists in view of the continuity of ϕ'. From (4.1) and the symmetry of ϕ it follows that

$$\phi(ac - a + 1) = \phi(ac) = \phi(1 - ac). \tag{4.2}$$

With the notation $d = \min(ac, 1 - ac)$ where $d \le 1/2$, we have obviously $ac - a + 1 \le d$, so that there arise two cases. In the case $ac - a + 1 < d$ the monotony and (4.2) imply $\phi(t) = \text{const}$ for $ac - a + 1 \le t \le d$ and hence $\phi'(ac - a + 1) = 0$. In view of $0 < ac - a + 1 < c$, this is a contradiction to the choice of c. It remains the case $ac - a + 1 = d$, which in view of $a > 1$ is only possible for $ac - a + 1 = ac$ and hence $c = 1/2$, i.e. $\phi'(t) > 0$ for $t \in (0, 1/2)$ ∎

Proposition 4.3. *In case of $a \ge 2$, the solution ϕ of (1),(2) has the following properties:*

(i) *$\phi(t)$ is strongly increasing in $(0, 1/a)$.*

(ii) *$\phi(t) = b$ for $t \in [1/a, 1 - 1/a]$.*

Proof. Let $a \ge 2$. In case of $t \le 1/a$, we have $at - a + 1 \le 0$. Hence, (4.1) implies that $\phi'(t) = ab\,\phi(at) \ge 0$, and $\phi(t)$ is not decreasing for these t. Analogously to the foregoing proof it follows that $\phi(t)$ is strongly positive for $0 < t < 1$. For $0 < t < 1/a$, we have $0 < at < 1$ and $at - a + 1 < 0$, hence, (4.1) implies $\phi'(t) > 0$ for these t.

On the other side, for $1/a \le t \le 1 - 1/a$, we have $at - a + 1 \le 0$ and $at \ge 1$, hence $\phi = L\phi$, $\phi(\tau) = 0$ for $\tau \notin (0,1)$ and (2) imply for these t that $\phi(t) = b \int_0^1 \phi(\tau)\,d\tau = b$ and the assertion is proved ∎

5. Fourier expansions

For the Laplace transform Φ of the solution ϕ of (1), (2), $a > 1$, we have in case of $p = ix$

$$\Phi(ix) = \int_0^1 e^{-ixt}\phi(t)\,dt = \int_0^1 \cos(xt)\phi(t)\,dt - i\int_0^1 \sin(xt)\phi(t)\,dt\,.$$

Comparision with (2.4) and separation in real and imaginary part yields

$$\int_0^1 \cos(xt)\phi(t)\,dt = \cos\left(\frac{x}{2}\right) P(x) \tag{5.1}$$

and

$$\int_0^1 \sin(xt)\phi(t)\,dt = \sin\left(\frac{x}{2}\right) P(x)$$

where P is given by (2.5). These formulas can be used to expand ϕ in a Fourier series. For this reason, let f be the 1-periodic function with $f(t) = \phi(t)$ for $0 \le t \le 1$. In view of Proposition 4.1, the function f is an even one and has the Fourier series

$$f(t) = 1 + \sum_{n=1}^{\infty} a_n \cos(n\pi t)$$

where in view of (5.1)

$$a_n = 2\int_0^1 f(t)\cos(n\pi t)\,dt = 2\cos\left(\frac{n\pi}{2}\right) P(n\pi)\,, \tag{5.2}$$

i.e. $a_{2n-1} = 0$ and $a_{2n} = 2(-1)^n P(2n\pi)$ for all $n \ge 1$. Hence,

$$f(t) = 1 + 2\sum_{n=1}^{\infty}(-1)^n P(2n\pi)\cos(2n\pi t)\,. \tag{5.3}$$

The foregoing way to calculate the Fourier coefficients is simpler than the usual one via the theorem of residues (cf. Berg [1: p. 33]) because it saves us to check the premises of Jordan's Lemma. In view of

$$\frac{\sin\alpha}{\alpha} = \prod_{n=1}^{\infty} \cos\frac{\alpha}{2^n} \qquad (\alpha \ne 0)$$

(cf. [3: p. 109]) we have the further representation

$$P(x) = \prod_{n=1}^{\infty}\prod_{k=1}^{\infty} \cos\left(\frac{a-1}{2^{n+1}a^k}\,x\right)\,.$$

The last results in this section are only valid in the case of $a = 2$, where the foregoing formula simplifies to

$$P(x) = \prod_{k=1}^{\infty} \frac{\sin\frac{x}{2^{k+1}}}{\frac{x}{2^{k+1}}} = \prod_{\nu=1}^{\infty} \cos^{\nu}\left(\frac{x}{2^{\nu+2}}\right)\,. \tag{5.4}$$

Consequently, $P(4n\pi) = 0$ and therefore also $a_{4n} = 0$ (cf. (5.2)). Hence, the Fourier series (5.3) simplifies to

$$f(t) = 1 + \sum_{n=1}^{\infty} a_{4n-2}\cos((4n-2)\pi t)$$

with $a_{4n-2} = -2P((4n-2)\pi)$, i.e.

$$a_{4n-2} = -2\prod_{k=1}^{\infty} \frac{\sin\frac{(2n-1)\pi}{2^k}}{\frac{(2n-1)\pi}{2^k}} = -2\prod_{\nu=1}^{\infty} \cos^{\nu}\left(\frac{2n-1}{2^{\nu+1}}\pi\right). \tag{5.5}$$

Calculations by means of a computer yield for the following first coefficients:

$$\begin{array}{llll} a_2 = -1.10754255179 & & a_6 = 0.09240459827 \\ a_{10} = 0.01731373586 & & a_{14} = -0.00209896165 \\ a_{20} = 0.00071230572 & & a_{24} = -0.00059673667 \\ a_{28} = -0.00018280996 & & a_{32} = 0.00001311091 \\ a_{36} = 0.00000676768 & & a_{40} = -0.00002464224. \end{array}$$

Since the sign from $\sin(\pi x)$ is equal to $(-1)^k$ for $x \in (k, k+1)$, it follows from (5.5) that the sign ε_{4n-2} of $a_{4n-2} = -P((4n-2)\pi)$ is equal to

$$\varepsilon_{4n-2} = -(-1)^{\sum_{k=1}^{\infty}\left[\frac{2n-1}{2^k}\right]} = (-1)^{\sum_{k=0}^{\infty}\left[\frac{2n-1}{2^k}\right]}. \tag{5.6}$$

Assume that $2n-1$ has the dyadic representation $2n-1 = d_s \cdots d_1 d_0$ with $d_s = 1$ and $d_\nu \in \{0,1\}$. Since the number $[(2n-1)/2^k] = d_s \cdots d_k$ is odd if and only if $d_k = 1$, it follows from (5.6) that $\varepsilon_{4n-2} = (-1)^{\nu(2n-1)}$, where $\nu(n)$ denotes the number of ones in the dyadic representation of n, i.e. $\nu(n)$ is the binary sum-of-digits function (cf. [4: p. 293]).

In order to obtain an estimate for $|P(2n\pi)|$, we choose K subject to $\frac{n\pi}{2^{K+1}} < 1 \le \frac{n\pi}{2^K}$, i.e. $K = [\ln(n\pi)/\ln 2]$. Then we get from (5.4) the inequality

$$|P(2n\pi)| \le \prod_{k=1}^{K} \frac{2^k}{n\pi} = \frac{2^{K(K+1)2}}{n^K\pi^K} = \left(\frac{2}{n\pi}\frac{2^K}{n\pi}\right)^{K/2} \le \left(\frac{2}{n\pi}\right)^{K/2},$$

which matches to the asymptotic considerations in Section 9.

6. Spectral properties

Generalizing (1), we consider the integral equation

$$\lambda\,\phi(t) = \int_{at-a+1}^{at} \phi(\tau)\,d\tau \tag{6.1}$$

with $a > 1$ and $\operatorname{supp}\phi \subset [0,1]$, and the corresponding adjoint equation

$$\lambda\,\psi(\tau) = \int_{\tau/a}^{(\tau+a-1)/a} \psi(t)\,dt \tag{6.2}$$

for $0 \le \tau \le 1$. Both equations are eigenvalue problems for adjoint operators which are linear and compact. The compactness can be shown as in Section 2 by means of the selection theorem of Arzelà and Ascoli. Hence by the Fredholm alternative, both equations must have the same non-zero eigenvalues with the same multiplicity (cf. [9: p. 789]). First, we consider equation (6.2).

Proposition 6.1. *For $|\lambda| > 1/b$, the integral equation* (6.2) *with $\tau \in [0,1]$ has only the trivial solution $\psi(\tau) \equiv 0$.*

Proof. A solution ψ of (6.2) is continuous. Assume that $K = \max|\psi(\tau)|$ on $[0,1]$ and $d \in [0,1]$ with $|\psi(d)| = K$. From (6.2), we get

$$|\lambda|K = |\lambda|\,|\psi(d)| \le \int_{d/a}^{(d+a-1)/a} |\psi(t)|\,dt \le \frac{a-1}{a}K .$$

Since $|\lambda| > 1/b = (a-1)/a$, this implies that $K = 0$, i.e. $\psi(\tau) \equiv 0$ ∎

Proposition 6.2. *The integral equation* (6.2) *has exactly the non-zero eigenvalues $\lambda_n = 1/(ba^n)$, $n \in \mathbb{N}_0$, which are all simple. Corresponding eigenfunctions are the polynomials*

$$\psi_n(\tau) = \sum_{\nu=0}^{n} \binom{n}{\nu} r_{n-\nu}(a)\,\tau^\nu , \tag{6.3}$$

where $r_0(a) = 1$ and the coefficients $r_\nu(a)$ for $\nu \ge 1$ are determined recursively by

$$r_\nu(a) = \frac{-1}{a^\nu - 1}\,\frac{1}{\nu+1}\sum_{k=0}^{\nu-1}\binom{\nu+1}{k} r_k(a)(a-1)^{\nu-k}a^k . \tag{6.4}$$

Additionally, also $\lambda = 0$ is an eigenvalue of (6.2) *and each integrable function ψ with period $1 - 1/a$ and average 0, restricted to the interval $[0,1]$, is solution of* (6.2) *with $\lambda = 0$.*

Proof. Assume that ψ is a solution of (6.2) with $\lambda \ne 0$. Then ψ is differentiable. From (6.2), we find by differentiation

$$\lambda\,\psi'(\tau) = \frac{1}{a}\int_{\tau/a}^{(\tau+a-1)/a} \psi'(t)\,dt ,$$

i.e. ψ' is a solution of (6.2) with eigenvalue parameter $a\lambda$ instead of λ. It follows that $\psi^{(n)}$, $n \in \mathbb{N}$, is a solution of (6.2) with eigenvalue parameter $a^n\lambda$. Since $a > 1$, for each $\lambda \ne 0$ there exists an integer n such that $a^n|\lambda| > 1/b$. Proposition 6.1 implies that $\psi^{(n)} = 0$, i.e. ψ must be a polynomial. If we make the ansatz

$$\psi(\tau) = \sum_{\nu=0}^{n} c_{n\nu}\tau^\nu , \qquad c_{nn} = 1 ,$$

equation (6.2) turns over into the equation

$$\lambda\sum_{\nu=0}^{n} c_{n\nu}\tau^\nu = \sum_{\nu=0}^{n} c_{n\nu}\frac{1}{\nu+1}\,\frac{(\tau+a-1)^{\nu+1} - \tau^{\nu+1}}{a^{\nu+1}} ,$$

from which we obtain by comparison of the coefficients of τ^n that $\lambda = \frac{a-1}{a^{n+1}}$, i.e. at most $\lambda = 1/(ba^n)$ can be an eigenvalue of (6.2). With this value for λ, we obtain once more by comparison of coefficients recursively

$$c_{n\mu} = \sum_{\nu=\mu}^{n} \frac{1}{\nu+1} \binom{\nu+1}{\mu} c_{n\nu}(a-1)^{\nu-\mu} a^{n-\nu}$$

for $\mu = n-1, n-2, \dots, 0$. Now, with the further ansatz $c_{n\mu} = \binom{n}{\mu} r_{n-\mu}(a)$ and the substitution $\nu = n-k$, we get

$$r_{n-\mu}(a) = \frac{1}{n-\mu+1} \sum_{k=0}^{n-\mu} \binom{n-\mu+1}{k} r_k(a)(a-1)^{n-\mu-k} a^k . \tag{6.5}$$

Writing $n - \mu = \nu$ and solving this equations with respect to $r_\nu(a)$, we immediately obtain (6.4). Of course, all steps can be done in the opposite direction, so that $\lambda = 1/(ba^n)$ is in fact an eigenvalue of (6.2) with the eigenfunction (6.3), (6.4). This eigenvalue is simple, since all coefficients of ψ_n are uniquely determined. The statement with respect to $\lambda = 0$ is obvious, so that the proposition is proved ∎

Since (6.1) is the adjoint equation of (6.2), the Fredholm theory implies that equation (6.1) also possesses exactly the simple non-zero eigenvalues $\lambda_n = 1/(a^n b)$, $n \in \mathbb{N}_0$, but $\lambda = 0$ is not an eigenvalue of (6.1). From (1), we find by differentiation

$$\phi^{(n)}(t) = a^n b \int_{at-a+1}^{at} \phi^{(n)}(\tau)\, d\tau \tag{6.6}$$

so that $\phi^{(n)}$ is an eigenfunction to λ_n. Hence, we have the

Proposition 6.3. *Equation* (6.1) *with* $t \in [0,1]$ *has exactly the eigenvalues* $\lambda_n = 1/(a^n b)$, $n \in \mathbb{N}_0$. *Corresponding eigenfunctions are* $\phi^{(n)}$, *where* ϕ *is the solution of* (1),(2).

Equation (6.2) has also in case of $\tau \in \mathbb{R}$ only the eigenvalues $\lambda_n = 1/(a^n b)$ with $n \in \mathbb{N}_0$, since the proof of Proposition 6.1 can be transferred to the case $K = \max|\psi(\tau)|$ on $[-M, M]$ with an arbitrary $M > 1$. This implies that the assertion of Proposition 6.2 is also true for arbitrary real τ, and the polynomials (6.3) as well as periodic functions with period $1 - 1/a$ are also corresponding eigenfunctions of (6.2) in this case.

However, if we admit also solutions of (6.1) with non-compact support, then there appear further eigenvalues, which are unbounded, i.e. then the corresponding operator cannot be compact, and the Fredholm alternative cannot be applied. In order to show this, it is useful to denote the polynomials (6.3) by $\psi_n(\tau, a)$ and to introduce functions ϕ_n, $n \in \mathbb{N}_0$, defined by $\phi_0 = \phi$ from (1), (2) and

$$\phi_{n+1}(t) = \int_0^t \phi_n(\tau)\, d\tau \tag{6.7}$$

for $n \geq 0$.

Proposition 6.4. *Equation* (6.1) *with* $a > 1$ *has besides* $1/(a^n b)$ *also the eigenvalues* $\lambda_{-n-1} = a^{n+1}/b, n \in \mathbb{N}_0$, *with corresponding eigenfunctions* $\psi_n(t, 1/a)$ *and* $\phi_{n+1}(t)$, *if we admit that the solutions have non-compact support.*

Proof. In the proof of Proposition 6.2, we never have used explicitely the assumption $a > 1$. Hence, the polynomials $\psi_n(\tau, a)$ $(n \in \mathbb{N}_0)$ are also solutions of (6.2) with $\lambda = 1/(ba^n)$ in case of $0 < a < 1$, if τ is not restricted to [0,1]. This means with the substitution $a = 1/\alpha$ for $\alpha > 1$ that

$$\alpha^n(1-\alpha)\,\psi_n(t, 1/\alpha) = \int_{\alpha t}^{\alpha t-\alpha+1} \psi_n(\tau, 1/\alpha)\,d\tau \qquad (n \in \mathbb{N}_0), \tag{6.8}$$

i.e. $\psi_n(t, 1/a)$ is an eigenfunction of equation (6.1) with $t \in \mathbb{R}$ and $a > 1$ to the eigenvalue $\lambda_{-n-1} = a^n(a-1) = a^{n+1}/b, n \in \mathbb{N}_0$. It is clear as in (6.6) that ϕ_{n+1} is also an eigenfunction to the eigenvalue λ_{-n-1} ∎

Later on in Section 9 we shall see that equation (6.1) with $t \in \mathbb{R}$ and $a > 1$ has even all non-negative numbers λ as multiple eigenvalues. Analogously as in Proposition 6.4, we can conclude by means of the substitution $a = 1/\alpha$ that problem (1), (2) with $\tau \in [0, 1]$ is unsolvable for $0 < a < 1$.

7. Connections between eigenfunctions

The next proposition shows a surprising connection between the eigenfunctions ψ_n of (6.2) and the Laplace transform Φ of the solution ϕ of (1), (2), i.e. of the eigenfunction ϕ of (6.1) with $\lambda = 1/b$ and $a > 1$. The result, however, is also valid for $0 \le a \le 1$ and all $z \in \mathbb{C}$.

Proposition 7.1. *The function* $g(\tau, z) = e^{(\tau-1)z}/\Phi(z)$ *is the generating function of the eigenfunctions* $\psi_n(\tau)$, *i.e. we have*

$$\frac{e^{(\tau-1)z}}{\Phi(z)} = \sum_{n=0}^{\infty} \frac{\psi_n(\tau)}{n!}\, z^n\ , \tag{7.1}$$

where the power series converges for all $\tau \in \mathbb{R}$, *and* $|z| < 2b\pi$, $b = a/(a-1)$.

Proof. Introducing the generating function of the coefficients $r_\nu(a)$

$$g(z) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} r_\nu(a)\, z^\nu$$

as formal power series, equation (6.5) with $n - \mu = \nu$ yields

$$g(z) = \frac{e^{z(a-1)} - 1}{z(a-1)}\, g(az)\ .$$

Hence, in view of $g(0) = 1$, (1.2) and (2.1), we obtain the equation

$$g(z) = \prod_{n=0}^{\infty} \frac{z/(ba^n)}{e^{z/(ba^n)} - 1} = \frac{1}{\Phi(-z)} = \frac{e^{-z}}{\Phi(z)},$$

which implies also the convergence of (7.1) for $|z| < 2b\pi$ (cf. Knopp [6: p. 451]). By multiplication with the power series of $e^{\tau z}$, we find

$$e^{\tau z} \frac{e^{-z}}{\Phi(z)} = \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \frac{1}{\nu!\,(n-\nu)!} r_{n-\nu}(a) \tau^{\nu} z^n ,$$

so that, in view of (6.3), the proposition is proved ∎

According to (7.1), the eigenfunctions ψ_n are so-called *Appell polynomials* (cf. [2: p. 22]). By comparison of (2.12) and (7.1), we find in view of (2.10) and $\psi_n(0,a) = r_n(a)$ the relation

$$r_n(a) = \rho_n\left(\frac{1}{a}\right), \tag{7.2}$$

so that according to the first coefficients $\rho_n(a)$ (cf. Section 2) we obtain for the eigenfunctions $\psi_n(\tau) = \psi_n(\tau, a)$ of (6.2) the representations

$$\psi_n(\tau) = \tau^n - \frac{n}{2}\tau^{n-1} + \frac{n(n-1)}{12}\frac{a+2}{a+1}\tau^{n-2} - \frac{n(n-1)(n-2)}{24(a+1)}\tau^{n-3} + \dots .$$

In particular, equation (7.1) implies in view of (2.11) that

$$\psi_n(\tau, 0) = \frac{\tau^{n+1} - (\tau-1)^{n+1}}{n+1}, \quad \psi_n(\tau, 1) = \left(\tau - \frac{1}{2}\right)^n, \quad \psi_n(\tau, \infty) = B_n(\tau),$$

where B_n are the Bernoulli polynomials. Moreover, the special values (2.14) and also the recursion formula (2.13) can easily be transferred by means of (7.2).

The expansion (7.1) implies some properties of the eigenfunctions ψ_n, resp. of the coefficients $r_n(a) = \psi_n(0)$, which are generalizations of well known properties of Bernoulli polynomials, resp. Bernoulli numbers.

Proposition 7.2. *The eigenfunctions ψ_n of equation* (6.2) *with $a > 1$ have the properties*

(i) $\psi_n'(\tau) = n\psi_{n-1}(\tau)$

(ii) $\psi_n(1-\tau) = (-1)^n \psi_n(\tau)$

and they form with the derivatives $\phi^{(m)}$ a biorthogonal system in $L^2[0,1]$, i.e.

(iii) $\int_0^1 \phi^{(m)}(\tau)\psi_n(\tau)\,d\tau = (-1)^m m!\,\delta_{m,n} \quad (m, n \in \mathbb{N}_0)$.

Proof. Property (i) is a general one for Appell polynomials. According to (2.1), we have the equation $g(1-t, z) = g(t, -z)$, which in view of (7.1) yields the symmetry property (ii). For $m \neq n$, the functions $\phi^{(m)}(\tau)$ and $\psi_n(\tau)$ are orthogonal, since they are eigenfunctions of adjoint equations to different eigenvalues. Now, let be $m = n$. Obviously, (iii) is clear for $n = 0$ in view of (2) and $\psi_0(\tau) = 1$. Integration by parts

$$\int_0^1 \phi^{(n+1)}(\tau)\psi_{n+1}(\tau)\,d\tau = -\int_0^1 \phi^{(n)}(\tau)\psi'_{n+1}(\tau)\,d\tau$$

and (i) complete the proof by means of induction ∎

Remarks. 1. We can easily check directly that ψ'_n is also an eigenfunction of (6.2) with $\lambda = 1/(a^{n-1}b)$. Since we know that all eigenvalues are simple, it follows that $\psi'_n = k_n\psi_{n-1}$ with a certain constant k_n. Now $\psi_n^{(n)} = n!$ implies $k_n = n$.

2. Since $\psi_n(1-\tau)$ is also an eigenfunction of (6.2) with the simple eigenvalue $\lambda = 1/(a^n b)$, it follows that $\psi_n(1-\tau) = c_n\psi_n(\tau)$ with $c_n^2 = 1$, and therefore $c_n = 1$ or $c_n = -1$.

3. Note that the properties (i) and (ii) are also valid for the polynomials $\psi_n(t, 1/a)$, $a > 1$.

4. Of course, also the periodic eigenfunctions of (6.2) with respect to the eigenvalue 0 are orthogonal to the eigenfunctions of (6.1).

Formula (6.3) with $\tau = 1$ implies the relation

$$r_n(a) = (-1)^n \sum_{\nu=0}^{n} \binom{n}{\nu} r_\nu(a)\,,$$

which can be used to check the already calculated $r_\nu(a)$. The last relation means that the functions $r_\nu(a)$ are linearly dependent where more precisely

$$\sum_{\nu=0}^{n} \binom{n}{\nu}\left(2 - \frac{\nu}{n}\right) r_{2n-1-\nu}(a) = 0 \qquad (n \geq 1).$$

From (2.9), we can derive an addition theorem, namely

$$\sum_{\nu=0}^{n} \binom{n}{\nu} a^\nu \psi_\nu(\sigma, a^2)\,\psi_{n-\nu}(\tau, a^2) = (a+1)^n \psi_n\left(\frac{\sigma}{\alpha} + \frac{\tau}{\beta}, a\right), \tag{7.3}$$

which for $\sigma = \tau = 0$ turns over into a nonlinear formula for $r_n(a)$. Moreover, from (7.1) and (2.10) it follows that

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \psi_\nu(\sigma, a)\psi_{n-\nu}(\tau, 1/a) = (\sigma + \tau - 1)^n,$$

and the general formula for Appell polynomials

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \sigma^\nu \psi_{n-\nu}(\tau, a) = \psi_n(\sigma + \tau, a),$$

which is equivalent to Proposition 7.2/(i) in view of Taylor's formula.

8. Polynomial relations

By means of certain relations between eigenfunctions, we can derive some polynomial relations for the solution ϕ of the initial problem (1), (2).

Proposition 8.1. *The eigenfunctions* ϕ_n $(n \in \mathbb{N})$ *of* (6.1) *with* $a > 1$ *have the representations*

$$\phi_n(t) = \begin{cases} a^{\frac{n(n-3)}{2}}(a-1)^n \phi(a^{-n}t, a) & \text{for } 0 \le t \le a-1 \\ \dfrac{1}{(n-1)!}\psi_{n-1}(t, 1/a) & \text{for } 1 \le t. \end{cases} \tag{8.1}$$

Proof. The first equation in (8.1) can easily be proved by induction, since it is an identity for $n = 0$, and the induction step can be performed by means of (6.6) and

$$\phi\left(\frac{t}{a}\right) = b\int_0^t \phi(\tau)\,d\tau \qquad (0 \le t \le a-1). \tag{8.2}$$

The second equation follows from $\phi_n^{(n)}(t) = 0$ for $t \ge 1$, so that ϕ_n for these t is a polynomial of degree at most $n-1$. This polynomial remains an eigenfunction of (6.1) to the eigenvalue a^n/b also under the restriction $t \ge 1$, i.e. in view of (6.7) $\phi_n(t) = k_n\,\psi_{n-1}(t, 1/a)$ since, as in the proof of Proposition 6.2, polynomials which are eigenfunctions of (6.1) are uniquely determined up to a constant factor. The factor is fixed by

$$k_n\,(n-1)! = k_n\,\psi_{n-1}^{(n-1)}(t, 1/a) = \phi_n^{(n-1)}(t) = \int_0^1 \phi(\tau)\,d\tau = 1$$

for $t \ge 1$ (cf. (2)) ∎

The two representations of ϕ_n in (8.1) imply the

Corollary. *For* $a \ge 2$, *the solution of* (1), (2) *has the representation*

$$\phi(t) = \frac{\psi_{n-1}(a^n t, 1/a)}{(n-1)!\,a^{n(n-3)/2}\,(a-1)^n} \qquad \text{for } \frac{1}{a^n} \le t \le \frac{a-1}{a^n} \tag{8.3}$$

with $n \in \mathbb{N}$.

For $n = 1$, we know this already from Proposition 4.3/(ii) in view of $\psi_0(t) = 1$. In case of $a > 2$, it can be shown by means of (8.2) and

$$\phi\left(\frac{t+a-1}{a}\right) = b\int_t^1 \phi(\tau)\,d\tau \qquad \text{for } 2-a \le t \le 1$$

that ϕ is a polynomial of degree n in all 2^n intervals which can be attained from $(\frac{1}{a}, 1-\frac{1}{a})$ by n applications of the mappings

$$t \longmapsto \frac{t}{a} \qquad \text{and} \qquad t \longmapsto \frac{t+a-1}{a}$$

in an arbitrary order, $n \in \mathbb{N}_0$. The union of these intervals is an open Cantor set of Lebesgue measure 1. For $a = 3$, the last results are well-known from [8].

As generalization of the second representation in (8.1), we have

Proposition 8.2. *The eigenfunctions ϕ_n $(n \in \mathbb{N})$ of* (6.1) *with $a > 1$ possess the property*

$$\phi_n(t) - (-1)^n \phi_n(1-t) = \frac{1}{(n-1)!}\, \psi_{n-1}(t, 1/a) \tag{8.4}$$

for all $t \in \mathbb{R}$.

Proof. First, we prove (8.4) for $0 \le t \le 1$ by induction. For $n = 1$ the equation is true in view of (6.7) and the symmetry property $\phi(t) = \phi(1-t)$ as well as (2) and $\psi_0(t) = 1$. According to (6.8) and Proposition 7.2/(i), we find by integration of (8.4)

$$\phi_{n+1}(t) + (-1)^n \phi_{n+1}(1-t) = \frac{1}{n!}\, \psi_n(t, 1/a) + c_n$$

with a certain constant c_n. For $t = 1$ we get $\phi_{n+1}(1) = \frac{1}{n!}\psi_n(1, 1/a) + c_n$ and, by comparison with (8.1), we obtain $c_n = 0$. Thus equation (8.4) is proved for $0 \le t \le 1$ by means of induction.

For $t \ge 1$, equation (8.4) is already known from Proposition 8.1 and $\phi_n(1-t) = 0$. But (8.4) is also valid for $t \le 0$, in view of Proposition 7.2/(ii) ∎

Remark. Though we have derived (8.4) by means of Proposition 7.2/(i), (ii) and used that (8.1) is already known for $t \ge 1$, representation (8.4) implies conversely in view of definition (6.7) both properties (i), (ii) in Proposition 7.2, and for $t \ge 1$ also the second formula in (8.1).

Corollary. *For $a \ge 3/2$, the solution ϕ of* (1), (2) *has the property*

$$\phi\left(\frac{1}{2a^n} + \tau\right) + (-1)^{n-1}\phi\left(\frac{1}{2a^n} - \tau\right) = \frac{\psi_{n-1}(1/2 + a^n\tau, 1/a)}{(n-1)!\, a^{n(n-3)/2}\,(a-1)^n} \tag{8.5}$$

for $|\tau| \le a^{-n}(a - 3/2)$ and $n \in \mathbb{N}$.

This follows from (8.4) and the first equation in (8.1) with $t = 1/2 \pm a^n\tau$, where $0 \le 1/2 \pm a^n\tau \le a-1$ implies that (8.5) is valid for $|\tau| \le a^{-n}(a-3/2)$ and $|\tau| \le 1/(2a^n)$. But the last condition is unnecessary in view of (8.3).

Generalizing the first equation in (8.1), we have for all t and $n \in \mathbb{N}$ the representation

$$\phi_n(t) = a^{n(n-3)/2}(a-1)^n \sum_{\nu_1,\dots,\nu_n \ge 0} \phi\left(\frac{t}{a^n} - \frac{\nu_1}{a^{n-1}b} - \dots - \frac{\nu_n}{b}\right), \tag{8.6}$$

where the sum has only finitely many non-vanishing terms for fixed t. This equation can be proved by induction as the first equation in (8.1), if we use instead of (8.2) the relation

$$b\int_0^t \phi(\tau)\,d\tau = b\sum_{\nu\ge 0}\int_{t-(\nu+1)(a-1)}^{t-\nu(a-1)} \phi(\tau)\,d\tau = \sum_{\nu \ge 0}\phi\left(\frac{t}{a} - \frac{\nu}{b}\right)$$

which is valid for $t \ge 0$.

Representation (8.6) together with the second equation in (8.1) imply the

Corollary. *For $a > 1$, the solution ϕ of* (1), (2) *satisfies the polynomial relation*

$$\sum_{\nu_1,\dots,\nu_n \geq 0} \phi\left(t - \frac{\nu_1}{a^{n-1}b} - \dots - \frac{\nu_n}{b}\right) = \frac{\psi_{n-1}(a^n t, 1/a)}{(n-1)!\, a^{\frac{n(n-3)}{2}}(a-1)^n} \tag{8.7}$$

for $t \geq 1/a^n$, $n \in \mathbb{N}$.

For $n = 1$, we find in view of $\psi_0(t) = 1$ that the solution ϕ of (1), (2) with arbitrary $a > 1$ satisfies the equation

$$\sum_{\nu=-\infty}^{+\infty} \phi\left(t - \frac{\nu}{b}\right) = b \tag{8.8}$$

for all real t, which for $1/a \leq t \leq 1$ follows directly from (8.7). Equation (8.8) can be used to check the orthogonality property mentioned in Remark 4 of the previous section.

The last results in this section concern some special properties of the polynomials $\psi_n(t, 1/a)$ with $a > 1$.

Proposition 8.3. *The eigenfunctions $\psi_n(t, 1/a)$ $(n \in \mathbb{N}_0)$ of* (6.1) *with $a > 1$ have the representation*

$$\psi_n(t, 1/a) = \int_0^1 (t-\tau)^n \phi(\tau)\, d\tau \tag{8.9}$$

with the solution ϕ of (1), (2). *Moreover, the polynomials $\psi_n(t, 1/a)$ $(n \in \mathbb{N})$ are positive and strongly increasing for $t > 1/2$. For $n \to +\infty$, we have $\psi_n(t, 1/a) \to 0$ uniformly in $0 \leq t \leq 1$ and $|\psi_n(t, 1/a)| \to +\infty$ for each $t \notin [0, 1]$.*

Proof. According to (2.10) and (7.1), we have

$$e^{tz}\Phi(z) = \sum_{n=0}^{\infty} \frac{\psi_n(t, 1/a)}{n!} z^n ,$$

i.e. $e^{tz}\Phi(z)$ is the generating function of the eigenfunctions $\psi_n(t, 1/a)$. On the other hand, we obtain

$$e^{tz}\Phi(z) = \int_0^1 e^{(t-\tau)z} \phi(\tau)\, d\tau = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^1 (t-\tau)^n \phi(\tau)\, d\tau$$

hence, comparison of coefficients yields the representation (8.9).

Since $\phi(t) > 0$ for $0 < t < 1$, equation (8.9) implies that for even n, we have $\psi_n(t, 1/a) > 0$ for all t. In view of $\psi_0(t) = 1$ and $\psi_1(t) = t - 1/2$, Proposition 7.2/(i) and (ii) yield the assertions concerning the monotony.

For $0 \leq t \leq 1$, we get from

$$|\psi_n(t, 1/a)| = \int_0^t (t-\tau)^n \phi(\tau)\, d\tau + \int_t^1 (\tau - t)^n \phi(\tau)\, d\tau$$

in view of $0 \leq \phi(t) \leq b$ (cf. Section 4) the estimate

$$|\psi_n(t, 1/a)| \leq \frac{t^{n+1} + (1-t)^{n+1}}{n+1} b \qquad (0 \leq t \leq 1) \tag{8.10}$$

with $b = a/(a-1)$. Hence, $\psi_n(t, 1/a)$ converges to 0 uniformly in [0,1]. Now let $1 < t < 2$, i.e. $t = 1 + c$ with $0 < c < 1$. From (8.9), we obtain in view of the monotony of $\phi(t)$ for $t \in [0, 1/2]$ that

$$\psi_n\left(1 + c, \frac{1}{a}\right) \geq \int_{c/4}^{c/2} (1 + c - \tau)^n \phi(\tau)\, d\tau \geq \frac{c}{4}\left(1 + \frac{c}{2}\right)^n \phi\left(\frac{c}{4}\right),$$

i.e. $\psi_n(1 + c) \to +\infty$ for $n \to +\infty$. This implies in view of the monotony of $\psi_n(t, 1/a)$ for $t > 1/2$ and Proposition 7.2/(ii) the divergence $|\psi_n(t, 1/a)| \to +\infty$ for $t \notin [0, 1]$ ∎

According to $\psi_n(t, 1) = (t - 1/2)^n$ and $\phi(\tau, 1) = \delta(\tau - 1/2)$, formula (8.9) is true also for $a = 1$, but not the assertion concerning the divergence for $n \to +\infty$. For $t = 0$, formula (8.9) reduced to (2.15).

9. The truncated equation

It is useful to compare the solutions of the differentiated equation (6.1)

$$\lambda\, \phi'(t) = a\big(\phi(at) - \phi(at - a + 1)\big) \tag{9.1}$$

with the solutions of the truncated equation

$$\lambda\, g'(t) = a\, g(at) \qquad (a > 1\ ,\ \lambda > 0). \tag{9.2}$$

We look for solutions with supp $\subset [0, \infty)$, which possess a Laplace transform G. Transformation of (9.2) yields $\lambda p\, G(p) = G(\frac{p}{a})$ with the general solution

$$G(p) = G_0(p)\, Q\left(\frac{\ln p}{\ln a}\right), \qquad G_0(p) = p^\alpha \exp(-\beta \ln^2 p)\,, \tag{9.3}$$

where

$$\alpha = -\frac{1}{2} - \frac{\ln \lambda}{\ln a}, \qquad \beta = \frac{1}{2 \ln a} \tag{9.4}$$

and where Q is an arbitrary 1-periodic function. It can easily be seen that G_0 is a Laplace transform with the original function

$$g_0(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(pt - \beta \ln^2 p) p^\alpha\, dp\,, \tag{9.5}$$

$c > 0$, $g(t) = 0$ for $t \leq 0$, and that further solutions of (9.2) for $t \geq 0$ are

$$g(t) = \int_0^t g_0'(t - \tau) P\left(\frac{\ln \tau}{\ln a}\right) d\tau$$

with an arbitrary (and for simplicity) continuous function $P(t) = P(t + 1)$.

Next, we want to study the asymptotic behaviour of $g_0(t)$ for $t \to +0$.

Proposition 9.1. *For $t \to +0$, the function* (9.5) *has the asymptotic behaviour*

$$g_0(t) \sim \frac{(2\beta)^\varepsilon}{\sqrt{2\pi}}\, t^\gamma (-\ln t)^\delta \exp\left(-\beta \ln^2\left(\frac{t}{-\ln t}\right)\right) \tag{9.6}$$

with

$$\gamma = -2\beta - \delta - \frac{1}{2}\,, \qquad \delta = \frac{1}{2} + \alpha - 2\beta\ln(2\beta)\,, \qquad \varepsilon = \frac{1}{2} + \alpha - \beta\ln(2\beta)\,.$$

Proof. According to L. Berg [1: Theorem 47.3], we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp[pt + f(p)]\,dp \sim \frac{\exp[xt + f(x)]}{\sqrt{2\pi f''(x)}}$$

for $t \to +0$, if $x = x(t) \to \infty$ with $t + f'(x) = o(\sqrt{f''(x)})$, and if some further assumptions are satisfied. Since the check of these assumptions is only a question of routine, we drop it and restrict ourselves to the necessary formal calculations. In view of

$$f(p) = \alpha \ln p - \beta \ln^2 p\,, \quad f'(p) = \frac{1}{p}(\alpha - 2\beta \ln p)\,, \quad f''(p) = \frac{1}{p^2}(2\beta \ln p + 2\beta - \alpha)$$

we choose $x = -(2\beta/t)\ln t$, so that $\ln x \sim -\ln t$ for $t \to +0$ and therefore $f''(x) \sim t^2/(-2\beta \ln t)$. Hence, the condition

$$t + f'(x) = \frac{t}{\ln t}\left[\ln(-2\beta\ln t) - \frac{\alpha}{2\beta}\right] = o\left(\frac{t}{\sqrt{-\ln t}}\right)$$

is satisfied, and in view of

$$\begin{aligned} xt + f(x) &= -2\beta \ln t + \alpha \ln\left(\frac{-2\beta \ln t}{t}\right) - \beta \ln^2\left(\frac{-2\beta\ln t}{t}\right) \\ &= -\beta\ln^2\left(\frac{-t}{\ln t}\right) + (\gamma+1)\ln t + \left(\delta - \frac{1}{2}\right)\ln(-\ln t) + \left(\varepsilon - \frac{1}{2}\right)\ln(2\beta) \end{aligned}$$

and $\sqrt{f''(x)} \sim t/\sqrt{-2\beta \ln t}$, we obtain (9.6) after elementary calculations ∎

It can directly be checked that the right-hand side of (9.6) satisfies equation (9.2) asymptotically for $t \to +0$.

The results about the solutions of (9.2) can be used to obtain new informations concerning the solutions of (9.1). It is clear that all solutions of (9.2) with $g(t) = 0$ for $t \le 0$ are C^∞-functions, they satisfy $g^{(n)}(0) = 0$ for all $n \in \mathbb{N}_0$, and they satisfy also (9.1) for $t \le 1 - 1/a$. Obviously, all these solutions of (9.2) can be continued to solutions of (9.1), defining $\phi(t) = g(t)$ for $t \le a - 1$ and $\phi(t) = \phi(t - a + 1) + \frac{\lambda}{a}\,\phi'(\frac{t}{a})$ for $t > a - 1$. Since all these solutions are also solutions of (6.1) and in case of $\lambda = 0$ also all $(a-1)$-periodic integrable functions with average zero, we have proved the

Proposition 9.2. *Considering integral equation* (6.1), $a > 1$, *in the non-compact case, all* $\lambda \geq 0$ *are multiple eigenvalues of it.*

Finally, we return to equations (1), (2) and the Laplace transform (1.2) of the solution. This corresponds to the special case $\lambda = 1/b$ in (9.1), so that $\alpha = 1/2 - (\ln(a-1))/a$ in (9.4).

Proposition 9.3. *For positive* p, *we have the representation*

$$\Phi(p) = \frac{G(p)}{\prod_{k=1}^{\infty}(1 - e^{-pa^k/b})} \tag{9.7}$$

with G *from* (9.3), *a suitable function* Q *there and* $\alpha = 1/b$ *in* (9.4).

Proof. As already in Section 2, we write equation (1.1) for the (one-sided) Laplace transform Φ of ϕ in the form

$$\Phi(p) = \frac{ap}{b}\frac{\Phi(ap)}{(1 - e^{-pa/b})}. \tag{9.8}$$

Since G from (9.3) with $\lambda = 1/b$ in (9.4) is the general solution of $G(p) = \frac{ap}{b} G(ap)$, the general solution of (9.8) reads (9.7), if we do not specify Q. Hence, also (1.2) has the form (9.7) with a special 1-periodic function Q in (9.3) which can be represented as

$$Q(t) = a^{\frac{1}{2}t^2 - \alpha t} \prod_{k=0}^{\infty} \frac{1 - e^{-a^{t-k}/b}}{a^{t-k}/b} \prod_{l=1}^{\infty}\left(1 - e^{-a^{t+l}/b}\right)$$

and the statement is proved ∎

If we introduce the functions

$$\Gamma_n(p) = \frac{G(p)}{\prod_{k=1}^{n}(1 - e^{-pa^k/b})}$$

with $\Gamma_0(p) = G(p)$, then for $n \in \mathbb{N}$ we have $\Gamma_n(p) = \frac{\Gamma_{n-1}(p)}{1 - e^{-pa^n/b}}$ and the corresponding original functions

$$\gamma_n(t) = \sum_{k=0}^{\infty} \gamma_{n-1}\left(t - \frac{ka^n}{b}\right)$$

converge to $\phi(t)$ for every fixed t after finitely many steps.

For $\Re p \to \infty$ with $|\arg p| \leq \vartheta < \frac{\pi}{2}$ we find from (9.7) that $\Phi(p) \sim G(p)$ with (9.3). Hence, we expect that ϕ has an asymptotic behavior as g_0 in (9.6) up to a factor which is both bounded and bounded away from zero.

References

[1] Berg, L.: *Operatorenrechnung*. Part II: *Funktionentheoretische Methoden*. Berlin: Dt. Verlag Wiss. 1974.

[2] Boas, R. P. and R. C. Buck: *Polynomial Expansions of Analytic Functions*. Berlin - Göttingen - Heidelberg: Springer-Verlag 1958.

[3] Fichtenholz, G. M.: *Differential- und Integralrechnung*, Part II. Berlin: Dt. Verlag Wiss. 1964.

[4] Flajolet, F., Grabner, P., Prodinger, H. and R. F. Tichy: *Mellin transform and asymptotics: digital sums*. Theoret. Comput. Sci. 123 (1994), 291 - 314.

[5] Griepenberg, G., Londen, S.-O. and O. Staffans: *Volterra Integral and Functional Equations*. Cambridge: Univ. Press 1990.

[6] Knopp, K.: *Theorie und Anwendung der unendlichen Reihen*. Berlin - Heidelberg: Springer-Verlag 1947.

[7] Volk, W.: *Properties of subspaces generated by an infinitely often differentiable function and its translates*. Z. Ang. Math. Mech. (ZAMM) 76 (1996), Suppl. 1, 575 - 576.

[8] Wirsching, G. J.: *The Dynamical System on the Natural Numbers Generated by the* $3n+1$ *Function*. Theses. Eichstädt: Kath. Univ. 1995.

[9] Zeidler, E.: *Nonlinear Functional Analysis and its Applications*. Part I: *Fixed Point Theorems*. New York - Berlin - Heidelberg - Tokyo: Springer-Verlag 1986.

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