# **On the Solution of an Integral-Functional Equation**  with a Parameter

**L. Berg** and M. Krüppel

Abstract. For a homogeneous integral-functional equation containing a parameter, we show existence and uniqueness of a compactly supported solution with given value for its integral. The solution is infinitely often differentiable, symmetric with respect to the point  $\frac{1}{2}$ , monotonous at both sides of  $\frac{1}{2}$ , and satisfies further functional equations. The Fourier series of the periodic continuation is determined. We also investigate spectral properties of the integral equation and find surprising connections between the Laplace transform of the eigenfunction and the eigenfunctions of the adjoint equation, and also directly between different eigenfunctions both in the compact and in the non-compact case. Moreover, asymptotic considerations are made.

Keywords: *Integral-functional equations, Fourier series, spectral properties, biorthogonal sequences, Appell polynomials, asymptotic approximations* 

AMS subject classification: 39 B 22, 45 D 05, 42 A 16, 41 A 60

This paper deals with the homogeneous integral-functional equation

regral-functional equations, Fourier series, spectral properties, biorthogonal se-  
nces, Appell polynomials, asymptotic approximations

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\nls with the homogeneous integral-functional equation

\n
$$
\phi(t) = L\phi(t), \quad L\phi(t) = b \int_{a t-a+1}^{a t} \phi(\tau) d\tau \qquad (b = \frac{a}{a-1}), \qquad (1)
$$

where a is a fixed parameter with  $a > 1$  and  $t \in \mathbb{R}$ . We mainly look for compactly supported solutions. By means of the Laplace transform, we show that such solutions exist uniquely up to a constant factor *c.* Obviously, they are infinitely often differentiable and, for  $c \neq 0$ , the support is contained in the interval [0,1]. Hence, they can be used as test functions in the theory of distributions. Such functions are never holomorphic, and they satisfy  $\phi^{(n)}(0) = \phi^{(n)}(1) = 0$  for all  $n \in \mathbb{N}_0$ . The constant factor c can be fixed by the condition *0(t) dt = 1, (2)* 

$$
\int_{0}^{1} \phi(t) dt = 1 , \qquad (2)
$$

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which implies that the solution is a real one. Solutions with non-compact support are only considered in order to obtain new conclusions concerning compactly supported ones.

In Section 6, we shall see that (1) has only the trivial solution  $\phi(t) = 0$  for  $0 < a < 1$ , which explains our assumption  $a > 1$ . The case  $a = 2$  was investigated by W. Volk in [7], where  $\phi$  is a function with a polynomial behaviour in all dyadic points, i.e. more precisely,

$$
\phi^{(m)}\left(\frac{\nu}{2^n}\right) = 0
$$
 for  $m \ge n$  and  $\nu = 0(1)2^n$ .

The case  $a = 3$  was investigated by G. J. Wirsching in [8], where  $\phi$  is the density of a certain transition probability. As we shall see in Section 3,  $\phi$  is also non-negative in the general case  $a > 1$ , so that in view of (2) it always can be used as a probability density.

Of course, (1) can also be written as a Fredholm integral equation on  $[0, 1] \times [0, 1]$ with kernel  $k(t,\tau)$  defined by  $k(t,\tau) = b$  for  $0 \le at -a + 1 \le \tau \le at \le 1$  and  $k(t,\tau) = 0$ elsewhere on  $[0, 1] \times [0, 1]$ , but it is more natural, to consider it as a Volterra integralfunctional equation. Equations of type (1) do not seem to appear in relevant books on Volterra functional equations (cf., e.g., [5]).

In the last sections, we also consider the eigenvalue problems belonging to equation (1) and also to the adjoint equation, and find surprising connections between different eigenfunctions.

### **1. Existence and uniqueness**

We begin with the basic existence theorem.

**Theorem 1.1.** *For each a > 1, equation (1) has a unique compactly supported solution q, which fulfills condition (2).* 

**Proof.** First, we assume that (1) has an integrable solution with a support contained in  $[0,1]$ . By (finite) Laplace transform of  $(1)$ , we obtain **Proof.** First, we assume the function of  $[0,1]$ . By (finite) Laplity<br>where  $\Phi(p) = \mathcal{L}\{\phi(t)\}$ . In view  $\Phi(a^{-n}p) =$ <br>Hence, the function

$$
\Phi(p) = \frac{1 - e^{-p/b}}{p/b} \Phi\left(\frac{p}{a}\right)
$$
\nof a single complementary support con-  
\n
$$
\Phi(p) = \frac{1 - e^{-p/b}}{p/b} \Phi\left(\frac{p}{a}\right)
$$
\n(1.1)

$$
\Phi(p) = \frac{1 - e^{-p/b}}{p/b} \Phi\left(\frac{p}{a}\right)
$$
  
where  $\Phi(p) = \mathcal{L}\{\phi(t)\}$ . In view of  $a > 1$  and (2), we find for  $n \to \infty$   

$$
\Phi(a^{-n}p) = \int_0^1 e^{-a^{-n}pt} \phi(t) dt \to \int_0^1 \phi(t) dt = 1
$$
.  
Hence, the function  

$$
\Phi(p) = \prod_{k=0}^\infty \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)}
$$
.

$$
\Phi(p) = \prod_{k=0}^{\infty} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)}.
$$
\n(1.2)

is the unique solution of (1.1) with  $\Phi(0) = 1$ , so far as the product converges. This implies that (1) has at most one compactly supported solution satisfying (2). On the implies that (1) has at most one compactly<br>other side, for  $\Re z \ge 0$ , we have the estimates<br> $\left| \frac{1 - e^{-z}}{z} \right| = \left| \int e^{-zt} dz \right|$ Exploition of an<br>
1960  $\Phi(0) = 1$ , so for pactly support<br>  $\left| \Phi \right|$   $\left| e^{-zt} dt \right| \leq \int_{0}^{1} e^{-zt} dt$ 

On the Solution of an Integral-Func  
of (1.1) with 
$$
\Phi(0) = 1
$$
, so far as the pr  
most one compactly supported solution  
we have the estimates  

$$
\left|\frac{1-e^{-z}}{z}\right| = \left|\int_{0}^{1} e^{-zt} dt\right| \leq \int_{0}^{1} e^{-\Re z t} dt \leq 1
$$

$$
= \left|\int_{0}^{1} (1 - e^{-zt}) dt\right| \leq \int_{0}^{1} |1 - e^{-zt}| dt \leq \int_{0}^{1}
$$

$$
\leq \Re p \text{ and } |p| \leq r \text{ with arbitrary fixed } r >
$$
tely and uniformly convergent, and cons  
the  $|\Phi(n)| \leq 1$  (cf. Knopp [6: p. 451 - 4]

and

 

On the Solution of an Integral-Functional Equation  
\nnique solution of (1.1) with 
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\nle, for  $\Re z \ge 0$ , we have the estimates  
\n
$$
\left|\frac{1-e^{-z}}{z}\right| = \left|\int_0^1 e^{-zt} dt\right| \le \int_0^1 e^{-\Re z t} dt \le 1
$$
\n
$$
\left|1 - \frac{1-e^{-z}}{z}\right| = \left|\int_0^1 (1 - e^{-zt}) dt\right| \le \int_0^1 |1 - e^{-zt}| dt \le \int_0^1 |z| t dt = \frac{1}{2}|z|.
$$
\n
$$
P((\text{for}^k) \text{ with } 0 \le \Re n \text{ and } |x| \le r \text{ with arbitrary fixed } r > 0, \text{ this implies}
$$

For  $z = p/(ba^k)$  with  $0 \leq \Re p$  and  $|p| \leq r$  with arbitrary fixed  $r > 0$ , this implies that the product (1.2) is absolutely and uniformly convergent, and consequently a holomorphic function for  $\Re p > 0$  with  $|\Phi(p)| \le 1$  (cf. Knopp [6: p. 451 - 452]). The last estimate can be sharpened by means of (1.1) to

$$
\Phi(p) = \frac{ab^2}{p^2} (1 - e^{-p/b})(1 - e^{-p/(ab)}) \Phi\left(\frac{p}{a^2}\right) = O\left(\frac{1}{p^2}\right)
$$

for  $\Re p \ge 0$ , so that  $\Phi$  possesses an original function  $\phi$  satisfying (1), (2) (cf. L. Berg [1:  $p. 30)$   $\blacksquare$ 

**Remark.** In the case of  $a = 3$ , G. Wirsching has shown in [8] that the operator  $L$ is contractive on the convex space of functions from  $L^1[0,1]$  satisfying  $(2)$ . The proof can be transferred to the case  $2 < a$ , but not to the case  $1 < a \leq 2$ . For the case  $a = 2$ , W. Volk has shown in [7] that the operator *L* is contractive on a certain subspace from  $C^1[0,1]$  equipped with the norm  $||f|| := \alpha ||f||_{\infty} + \beta ||Df||_{\infty}$ , where  $0 < \alpha < \beta$ . *g*, G. Wirsching has shown in [o] that the operator *D* of functions from  $L^1[0,1]$  satisfying (2). The proof *a*, but not to the case  $1 < a \le 2$ . For the case  $a = 2$ , operator *L* is contractive on a certain subspace fr

## **2. Laplace transform and convolution**

As a finite Laplace transform,  $\Phi$  is in fact an entire function. The analytic continuation to the left half plane can also be seen from the formula

$$
\Phi(p) = e^{-p} \Phi(-p) , \qquad (2.1)
$$

which follows from (1.2) in view of

2. [0, 1] equipped with the norm 
$$
||f|| := \alpha ||f||_{\infty} + p ||Df||_{\infty}
$$
, where  $0 < a < p$ .  
\n2. Laplace transform and convolution  
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\n
$$
\Phi(p) = e^{-p} \Phi(-p), \qquad (2.1)
$$
\nwhich follows from (1.2) in view of  
\n
$$
\prod_{k=0}^{\infty} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)} = \prod_{k=0}^{\infty} e^{-p/(ba^k)} \prod_{k=0}^{\infty} \frac{e^{p/(ba^k)} - 1}{p/(ba^k)}
$$
\nand  
\n
$$
\sum_{k=0}^{\infty} \frac{1}{a^k} = b \qquad (2.2)
$$

$$
\sum_{k=0}^{\infty} \frac{1}{a^k} = b \tag{2.2}
$$

The function  $\Phi$  has the further representation

Def

\nthere representation

\n
$$
\Phi(p) = e^{-\frac{p}{2}} \prod_{k=0}^{\infty} \frac{\sinh(p/(2ba^{k}))}{p/(2ba^{k})},
$$
\nview of

\n
$$
\frac{1 - e^{-z}}{z} = e^{-\frac{z}{2}} \frac{\sinh(z/2)}{z/2}
$$
\nFor  $a \to 1$ , representation (2.3) implies that  $\Phi(p) \to e^{-p/2}$ .

which follows from (1.2) in view of

$$
\frac{1 - e^{-z}}{z} = e^{-\frac{z}{2}} \frac{\sinh(z/2)}{z/2}
$$

with  $z = p/(ba^k)$  and (2.2). For  $a \to 1$ , representation (2.3) implies that  $\Phi(p) \to e^{-p/2}$ , which is the Laplace transform of the distribution  $\delta(t-1/2)$ . f<br> *f*<br>  $\frac{e^{-z}}{z} = e^{-\frac{z}{2}} \frac{\sinh(z/2)}{z/2}$ <br>  $\rightarrow 1$ , representation (2.3) imp<br>
the distribution  $\delta(t - 1/2)$ .<br>
tain<br>  $\Phi(ix) = e^{-\frac{iz}{2}} P(x)$ ,<br>  $\frac{\infty}{\sin(\frac{a-1}{2}\tau)}$ 

From (2.3) with  $p = ix$ , we obtain

$$
\Phi(ix) = e^{-\frac{i\pi}{2}}P(x) \tag{2.4}
$$

where

For 
$$
a \to 1
$$
, representation (2.3) implies that  $\Phi(p) \to e^{-p/2}$ ,  
form of the distribution  $\delta(t - 1/2)$ .  
  

$$
\Phi(ix) = e^{-\frac{i\pi}{2}} P(x),
$$
(2.4)  

$$
P(x) = \prod_{k=1}^{\infty} \frac{\sin(\frac{a-1}{2a^k}x)}{\frac{a-1}{2a^k}x}.
$$
(2.5)  

$$
\frac{\ln x}{x} = \sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)!} \frac{x^{2n}}{2n} \qquad (|x| < \pi)
$$

Note that

$$
\ln \frac{\sin x}{x} = \sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)!} \frac{x^{2n}}{2n} \qquad (|x| < \pi)
$$

where  $B_n$  are the Bernoulli numbers

$$
\ln \frac{\sin x}{x} = \sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)!} \frac{x^{2n}}{2n} \qquad (|x| < \pi)
$$
\nare the Bernoulli numbers

\n
$$
B_0 = 1 \, , \quad B_1 = -\frac{1}{2} \, , \quad B_2 = \frac{1}{6} \, , \quad B_3 = 0 \, , \quad B_4 = -\frac{1}{30} \, , \quad \dots \qquad (2.6)
$$

(cf. [3: p. 513], with  $-B_n$  instead of  $(-1)^n B_{2n}$ ,  $n \ge 1$ ). Here, we have chosen the notation of  $B_n$  from [6: p. 185] resp. [1: p. 103]. From (2.5), we get notation of  $B_n$  from [6: p. 185] resp. [1: p. 103]. From (2.5), we get of  $(-1)^n B_{2n}$ ,  $n \ge 1$ ). Here, we have<br>[1: p. 103]. From (2.5), we get<br> $\left(\frac{a-1}{2-k}x\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!2^n} \frac{(a-1)^{2n}}{(2n)!}$ 

$$
\ln \frac{\sin x}{x} = \sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)!} \frac{x^{2n}}{2n} \qquad (|x| < \pi)
$$
\n
$$
B_n \text{ are the Bernoulli numbers}
$$
\n
$$
B_0 = 1 \qquad B_1 = -\frac{1}{2} \qquad B_2 = \frac{1}{6} \qquad B_3 = 0 \qquad B_4 = -\frac{1}{30} \qquad \dots
$$
\np. 513], with  $-B_n$  instead of  $(-1)^n B_{2n}$ ,  $n \ge 1$ ). Here, we have cho  
\n $\ln 6B_n$  from [6: p. 185] resp. [1: p. 103]. From (2.5), we get

\n
$$
\ln P(x) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)! 2n} \left(\frac{a-1}{2a^k} x\right)^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{(2n)! 2n} \frac{(a-1)^{2n}}{a^{2n} - 1} x^{2n}
$$
\n
$$
< 2b\pi, b = a/(a-1).
$$
 According to (2.4) with  $x = -ip$  and  $B_1 = -1/2$ \n
$$
x + 1 = 0, n \ge 1
$$
, this implies the representation

for  $|x| < 2b\pi$ ,  $b = a/(a-1)$ . According to (2.4) with  $x = -ip$  and  $B_1 = -1/2$  as well as  $B_{2n+1} = 0$ ,  $n \ge 1$ , this implies the representation

$$
n=1
$$
\n
$$
B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad \dots \quad (2.6)
$$
\n
$$
A = -B_n \text{ instead of } (-1)^n B_{2n}, \quad n \ge 1. \text{ Here, we have chosen the}
$$
\n
$$
[6: p. 185] \text{ resp. } [1: p. 103]. \text{ From (2.5), we get}
$$
\n
$$
\sum_{n=1}^{\infty} \frac{(-4)^n B_{2n}}{(2n)! \cdot 2n} \left(\frac{a-1}{2a^k} x\right)^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{(2n)! \cdot 2n} \frac{(a-1)^{2n}}{a^{2n} - 1} x^{2n}
$$
\n
$$
(a-1). \text{ According to (2.4) with } x = -ip \text{ and } B_1 = -1/2 \text{ as well this implies the representation}
$$
\n
$$
\ln \Phi(p) = \sum_{n=1}^{\infty} \frac{B_n}{n! n} \frac{(a-1)^n}{a^n - 1} p^n, \quad |p| < 2b\pi. \quad (2.7)
$$
\n
$$
\text{This is not possible to denote the solution of (1), (2) by}
$$
\n
$$
\text{at least in this section, it is useful to denote the solution of (1), (2) by}
$$

For the next results in this section, it is useful to denote the solution of  $(1)$ ,  $(2)$  by  $\phi(t, a)$ , and its Laplace transform by  $\Phi(p, a)$ .

**Proposition 2.1.** *The solution*  $\phi(t, a)$  has the convolution property

On the Solution of an Integral-Functional Equation 163.  
\n*The solution* 
$$
\phi(t, a)
$$
 *has the convolution property*  
\n
$$
\phi(t, a) = \alpha \beta \int_{0}^{t} \phi(\alpha(t - \tau), a^2) \phi(\beta \tau, a^2) d\tau
$$
\n(2.8)

*where*  $\alpha = 1 + 1/a$ ,  $\beta = 1 + a$ .

Proof. We separate the factors in (1.2) in those with only even *k* and those with only odd *k,* so that we obtain

$$
\phi(t, a) = \alpha \beta \int_{0}^{\infty} \phi(\alpha(t - \tau), a^{-1}) \phi(p\tau, a^{-1}) d\tau
$$
  

$$
\beta = 1 + a.
$$
  
arate the factors in (1.2) in those with only even k  
we obtain  

$$
\Phi(p, a) = \prod_{\nu=0}^{\infty} \frac{1 - e^{-p/(ba^{2\nu})}}{p/(ba^{2\nu})} \prod_{\mu=0}^{\infty} \frac{1 - e^{-p/(ba^{2\mu+1})}}{p/(ba^{2\mu+1})}.
$$
  

$$
-1) \text{ and } \alpha a^{2}/(a^{2} - 1) = b, \beta a^{2}/(a^{2} - 1) = ab, \text{ the for}
$$

$$
\Phi(p, a) = \Phi\left(\frac{p}{\alpha}, a^{2}\right) \Phi\left(\frac{p}{\beta}, a^{2}\right).
$$
  
nediately implies (2.8) by means of the inverse Laplace transform  
to the well known property  $\mathcal{L}\{\gamma\phi(\gamma t)\}$ 

In view of  $b = a/(a-1)$  and  $\alpha a^2/(a^2-1) = b$ ,  $\beta a^2/(a^2-1) = ab$ , the foregoing equation can be written as  $\Phi(p, a) = \Phi\left(\frac{p}{\alpha}, a^2\right) \Phi\left(\frac{p}{\beta}, a^2\right)$ . (2.9) can be written as

$$
\Phi(p,a) = \Phi\left(\frac{p}{\alpha},a^2\right) \Phi\left(\frac{p}{\beta},a^2\right). \tag{2.9}
$$

This equation immediately implies (2.8) by means of the inverse Laplace transform, using the convolution theorem and the well known property  $\mathcal{L}\{\gamma\phi(\gamma t)\} = \Phi(p/\gamma)$  for arbitrary  $\gamma > 0$ 

Let us mention that the function  $\phi(\alpha t)$  has the support  $[0, 1/\alpha]$  and  $\phi(\beta t)$  the support  $[0, 1/\beta]$ , so that in view of  $1/\alpha + 1/\beta = 1$ , the convolution (2.8) has in fact the support  $[0,1]$  (cf. the first three propositions in Section 4). mention that the function  $\phi(\alpha t)$  has the support  $[0, 1/\alpha]$  an  $/\beta$ , so that in view of  $1/\alpha + 1/\beta = 1$ , the convolution (2.8) hat  $[$ (cf. the first three propositions in Section 4).<br>
the convolution as usual by means of

Writing the convolution as usual by means of a star, we can generalize (2.8) by

$$
\phi(t,a) = \alpha_1 \alpha_2 \cdots \alpha_n \phi(\alpha_1 t, a^n) * \phi(\alpha_2 t, a^n) * \dots * \phi(\alpha_n t, a^n)
$$

where  $\alpha_{\nu} = a^{\nu}(1 - a^{-n})/(a - 1)$  for  $\nu = 1, ..., n$  and  $n \in \mathbb{N}$ .

Though we are actually only interested in the case  $a > 1$ , the power series (2.7) is also convergent in the case of  $0 \le a \le 1$  for  $|p| < 2\pi/(1 - a)$ . From (2.7), we get for  $\nu = 1, ..., n$  and  $n \in \mathbb{N}$ .<br>
interested in the case  $a > 1$ , the power series (2.7) is<br>  $\frac{a}{a} \le 1$  for  $|p| < 2\pi/(1 - a)$ . From (2.7), we get<br>  $\sum_{n=1}^{\infty} \frac{B_n}{n!n} \frac{(1 - a)^n}{1 - a^n} p^n = -\ln \Phi(-p, a)$ <br>
nus, for these p, we h

$$
a_1 = \alpha_1 \alpha_2 \cdots \alpha_n \phi(\alpha_1 t, a^n) * \phi(\alpha_2 t, a^n) * \cdots * \phi(\alpha_n
$$
  
\n
$$
- a^{-n})/(a - 1) \text{ for } \nu = 1, ..., n \text{ and } n \in \mathbb{N}.
$$
  
\ne actually only interested in the case  $a > 1$ , the pc  
\nthe case of  $0 \le a \le 1$  for  $|p| < 2\pi/(1 - a)$ . From (1)  
\n
$$
\ln \Phi(p, 1/a) = \sum_{n=1}^{\infty} \frac{B_n}{n! n} \frac{(1 - a)^n}{1 - a^n} p^n = -\ln \Phi(-p, a)
$$
  
\nuse of  $a > 1$ . Thus, for these  $p$ , we have  $\Phi(p, a)\Phi(-q)$   
\nequation  
\n
$$
\Phi(p, a) \Phi(p, 1/a) = e^{-p}.
$$
  
\nfind  
\n
$$
0) = \frac{p}{e^p - 1}, \qquad \Phi(p, 1) = e^{-p/2}, \qquad \Phi(p, \infty) = \frac{1}{2}
$$

for  $|p| < 2\pi b$  in case of  $a > 1$ . Thus, for these p, we have  $\Phi(p, a)\Phi(-p, 1/a) = 1$  and in view of (2.1) the equation

$$
\Phi(p,a)\,\Phi(p,1/a) = e^{-p}.\tag{2.10}
$$

In particular, we find

we are actually only interested in the case 
$$
a > 1
$$
, the power series (2.7) is  
ent in the case of  $0 \le a \le 1$  for  $|p| < 2\pi/(1 - a)$ . From (2.7), we get  

$$
\ln \Phi(p, 1/a) = \sum_{n=1}^{\infty} \frac{B_n}{n! n} \frac{(1 - a)^n}{1 - a^n} p^n = -\ln \Phi(-p, a)
$$
bin case of  $a > 1$ . Thus, for these  $p$ , we have  $\Phi(p, a)\Phi(-p, 1/a) = 1$  and in  
the equation  

$$
\Phi(p, a)\Phi(p, 1/a) = e^{-p}.
$$
 (2.10)  
or, we find  

$$
\Phi(p, 0) = \frac{p}{e^p - 1}, \qquad \Phi(p, 1) = e^{-p/2}, \qquad \Phi(p, \infty) = \frac{1 - e^{-p}}{p}
$$
 (2.11)

(cf. (1.1)) and  $\Phi(0, p) = 1$ . A product representation for  $\Phi(p, a)$  in case of  $0 < a < 1$ can be derived, writing (1.1) as

$$
\Phi(p,a) = \frac{ap/b}{1 - e^{-ap/b}} \, \Phi(ap,a)
$$

and iterating this equation. The result also follows from (2.10).

Later on, we need the coefficients of the expansion

$$
\Phi(p,a) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!} p^n \tag{2.12}
$$

These coefficients can be determined by means of the recursion formula

(0, p) = 1. A product representation for 
$$
\Psi(p, a)
$$
 in case of  $0 < a < 1$  \nWriting (1.1) as

\n
$$
\Phi(p, a) = \frac{ap/b}{1 - e^{-ap/b}} \Phi(ap, a)
$$
\nis equation. The result also follows from (2.10).

\nneed the coefficients of the expansion

\n
$$
\Phi(p, a) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!} p^n
$$
\n(2.12)

\nis can be determined by means of the recursion formula

\n
$$
\rho_0(a) = 1
$$
\n
$$
\rho_n(a) = \frac{1}{n} \sum_{k=1}^n {n \choose k} \rho_{n-k}(a) B_k \frac{(a-1)^k}{a^k-1} \quad \text{quad} \quad (2.13)
$$
\n(2.13)

\n(2.7)

\nand (2.12)

\nin view of  $\Phi' = \Phi(\ln \Phi)'$  by comparison of coefficients, and that the coefficients  $\rho_n(a)$  are rational functions with respect to  $a$ ,

which follows from (2.7) and (2.12) in view of  $\Phi' = \Phi(\ln \Phi)'$  by comparison of coefficients. From (2.13), we find that the coefficients  $\rho_n(a)$  are rational functions with respect to a, in particular  $\rho_1(a) = -\frac{1}{2}$ ,  $\rho_2(a) = \frac{2a+1}{6(a+1)}$ ,  $\rho_3(a) = \frac{-a}{4(a+1)}$ , in particular

Later on, we need the coefficients of the expansion  
\n
$$
\Phi(p, a) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!} p^n
$$
\nThese coefficients can be determined by means of the recursion formula  
\n
$$
\rho_0(a) = 1
$$
\n
$$
\rho_n(a) = \frac{1}{n} \sum_{k=1}^n {n \choose k} \rho_{n-k}(a) B_k \frac{(a-1)^k}{a^k - 1} \quad \text{quad}(n \ge 1)
$$
\nwhich follows from (2.7) and (2.12) in view of  $\Phi' = \Phi(\ln \Phi)'$  by comparison of coeffi  
\nFrom (2.13), we find that the coefficients  $\rho_n(a)$  are rational functions with respe  
\nin particular  
\n
$$
\rho_1(a) = -\frac{1}{2}, \qquad \rho_2(a) = \frac{2a+1}{6(a+1)}, \qquad \rho_3(a) = \frac{-a}{4(a+1)},
$$
\n
$$
\rho_4(a) = \frac{6a^4 + 3a^3 + 5a^2 + 2a - 1}{30(a+1)^2(a^2 + 1)}, \qquad \rho_5(a) = \frac{-2a^4 - 2a^2 + a}{12(a+1)^2(a^2 + 1)}, \qquad \text{According to (2.11), we have the special values}
$$
\n
$$
\rho_5(a) = \frac{(-1)^n}{(a+1)^2(a^2 + 1)}, \qquad (-1)^n
$$

$$
\rho_n(0) = B_n, \qquad \rho_n(1) = \frac{(-1)^n}{2^n}, \qquad \rho_n(\infty) = \frac{(-1)^n}{n+1},
$$
\n(2.14)

where  $B_n$  are the Bernoulli numbers (cf. (2.6)). Let us mention that (2.13), for  $a \to 0$ , turns over into a nonlinear recursion formula for the Bernoulli numbers.

For  $a > 1$ , the coefficients of the expansion (2.12) possess the representation

*pn (a) = (-1) I tnc(*<sup>t</sup> *, a) dt*  (2.15)

i.e.  $(-1)^n \rho_n(a)$  are the *n*-th moments of the solution  $\phi$  of (1), (2) (cf. [1: p. 87]). In view of (2.14) and the original functions of (2.11)  $\phi(t, 1) = \delta(t - 1/2)$  as well as  $\phi(t,\infty) = 1$  for  $0 < t < 1$  and  $\phi(t,\infty) = 0$  elsewhere, the formula (2.15) is also true both for  $a = 1$  and for  $a = \infty$ .  $b_n(a) = (-1)^n \int_0^1 t^n \phi(t, a)$ <br>
moments of the solution<br>
ginal functions of (2.11)<br>
d  $\phi(t, \infty) = 0$  elsewhere,<br>
bability density, the corr<br>
e variance  $(a - 1)/(12(a -$ <br>
in that for  $a > 1$  the *n*-th i<br>  $\frac{1}{2^n} \le (-1)^n \rho_n(a) \le \frac{1}{n+$ 

Considering  $\phi(t, a)$  as probability density, the corresponding random variable has the expected value 1/2 and the variance  $(a-1)/(12(a+1))$ .

Without proof, we mention that for  $a > 1$  the n-th moments  $(-1)^n \rho_n(a)$  satisfy the estimate

$$
\frac{1}{2^n} \leq (-1)^n \rho_n(a) \leq \frac{1}{n+1} ,
$$

so that in case of  $a \ge 1$  the power series (2.12) converges in fact for all  $p \in \mathbb{C}$ .

#### **3. Approximations**

 

The solution  $\phi$  of (1), (2),  $a > 1$ , can also be obtained by means of iteration.

**Theorem 3.1.** For every non-negative L-integrable function  $f_0$  on the interval  $[0,1]$ *with*  $f_0(t) = 0$  for  $t \notin [0, 1]$  *and the property* 

the Solution of an Integral-Functional Equation 165  
\nIn also be obtained by means of iteration.  
\nnegative L-integrable function 
$$
f_0
$$
 on the interval [0, 1]  
\nproperty  
\n
$$
\int_{0}^{1} f_0(t) dt = 1 , \qquad (3.1)
$$
\nconverges uniformly on [0, 1] to the unique solution

*the sequence*  $f_n = Lf_{n-1}$  $(n \geq 1)$  *converges uniformly on*  $[0,1]$  *to the unique solution*  $\phi$  of  $(1), (2)$ , which is also non-negative.

the sequence  $f_n = Lf_{n-1}$   $(n \ge 1)$  converges uniformly on [0, 1] to the unique solution  $\phi$  of (1), (2), which is also non-negative.<br>**Proof.** Obviously, the iterates  $f_n$   $(n \ge 1)$  are continuous, non-negative, and  $f_n(t) =$ **Proof.** Obviously, the iterates  $f_n$   $(n \geq 1)$  are continuous, non-negative, and i.e. we also have

Let the property

\n
$$
\int_{0}^{1} f_0(t) dt = 1,
$$
\n
$$
\int_{0}^{1} f_0(t) dt = 1,
$$
\nTherefore,  $f_n$   $(n \geq 1)$  are continuous, non-negative, and can easily be seen that (3.1) is invariant concerning  $L$ ,

\n
$$
\int_{0}^{1} f_n(t) dt = 1 \qquad (n \geq 1).
$$
\nSo, the sequence  $f_n$ , we wish to apply the selection theorem.

\nThus, the function  $f_n$  is a function of  $f_n$  and  $f_n$  is a function of  $f_n$ .

In order to prove convergence of the sequence  $f_n$ , we wish to apply the selection theorem due to Arzelà and Ascoli. For this reason, we have to show that the functions  $f_n$ are uniformly bounded on [0, 1] and equicontinuous. Since  $f_n(t)$  is non-negative, and satisfies  $f_n(t) = 0$  for  $t \notin (0, 1)$  as well as (3.2), we obtain for each  $t \in [0, 1]$ can easily be seen that<br>  $\int_{0}^{1} f_n(t) dt = 1$  ( $n \ge 0$ <br>
of the sequence  $f_n$ , we we<br>
for this reason, we have<br>  $\int_{0}^{1}$  and equicontinuous.<br>
as well as (3.2), we ob<br>  $\int_{a^{i-a+1}}^{a^{i}} f_{n-1}(\tau) d\tau \leq b \int_{0}^{a^{i}}$ <br>
quence. Th *a(t-fh) a(t+h)-a-l-I* 

$$
0 \leq f_n(t) = b \int_{a^1-a+1}^{a^1} f_{n-1}(\tau) d\tau \leq b \int_{0}^{1} f_{n-1}(\tau) d\tau = b,
$$

i.e. the houndedness of the sequence. The inequality

Let us prove convergence, the sequence 
$$
f_n
$$
, we then to apply the collection in a  $A$  is a  $A$ , where  $A$  is a  $A$  is a  $A$ , where  $A$  is a  $A$  is a  $A$ , where  $A$  is a  $A$  is a  $A$ , where  $A$  is a  $A$  is a  $A$ , where  $A$  is a  $A$  is a  $A$ , where  $A$  is a  $A$ ,  $A$  is a  $A$ , and  $A$  is a  $A$ , and  $A$ , and  $A$  is a  $A$ , and  $A$ , and  $A$  is a  $A$ , and  $A$ , and <

shows the equicontinuity. Hence, we can select a subsequence  $f_{k(n)}$  which is uniformly convergent to a non-negative function *f.* 

Next, we prove that this subsequence  $f_{k(n)}$  converges to  $\phi$ . For this reason we consider the (finite) Lapace transform  $F_n(p) = \int_0^1 e^{-pt} f_n(t) dt$ . Hence, from  $f_n = Lf_{n-1}$ and  $f_n(t) = 0$  for  $t \le 0$ , we get  $F_n(p) = \frac{b}{p}(1 - e^{-p/b}) F_{n-1}(\frac{p}{q})$   $(n \ge 1)$  and it follows that the transform  $F_n(p) = \int_0^1 e^{i\theta}$ <br>
we get  $F_n(p) = \frac{b}{p}(1 - e^{-i\theta})$ <br>  $F_n(p) = F_0(a^{-n}p) \prod_{k=0}^{n} \frac{1 - e^{-i\theta}}{p}$ 

$$
F_n(p) = F_0(a^{-n}p) \prod_{k=0}^n \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)}.
$$

Since  $a > 1$ , we obtain for  $n \to \infty$ 

$$
t \leq 0, \text{ we get } F_n(p) = \frac{v}{p} (1 - e^{-p/p}) F_{n-1} \left( \frac{r}{a} \right) \quad (n \geq 1) \text{ and it follow}
$$
\n
$$
F_n(p) = F_0(a^{-n}p) \prod_{k=0}^n \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)}.
$$
\nstation for  $n \to \infty$ 

\n
$$
F_0(a^{-n}p) = \int_0^1 e^{-a^{-n}p t} f_0(t) dt \to \int_0^1 f_0(t) dt = 1.
$$

In view of (1.2), we see that the sequence  $F_n$  converges to the Laplace transform of the solution  $\phi$ , and it follows that  $f = \phi$ .

Now, we can show that the sequence  $f_n$  is uniformly convergent to  $\phi$ . From above, we see that every subsequence of  $f_n$ , as sequence in  $C[0,1]$ , has a subsequence which converges in the maximum norm to  $\phi \in C[0,1]$ . Hence, by a convergence principle in Banach spaces (cf. Zeidler [9: p. 480]) the sequence  $f_n$  converges in this norm to  $\phi$ , i.e. the sequence  $f_n$  is uniformly convergent to  $\phi$ 

**Remark.** The condition  $f_0 \ge 0$  is not necessary. Namely, if  $f_0$  is an arbitrary L-integrable function on [0,1] with  $f_0(t) = 0$  for  $t \notin (0, 1)$ , then we can introduce a sequence  $g_n$  by  $g_0 = |f_0|$  and  $g_n = Lg_{n-1}$ , so that  $|f_n(t)| \leq g_n(t)$ , and the case is actually reduced to the foregoing one with some further modifications. The subsequence which<br>nvergence principle in<br>in this norm to  $\phi$ , i.e.<br>, if  $f_0$  is an arbitrary<br>en we can introduce a<br> $g_n(t)$ , and the case is<br>tions.<br>aat it also satisfies the<br> $\bar{f}$ ). (4.1)<br> $\bar{f}$ ). (4.1)

## **4. Symmetry and Monotony**

We know that the solution  $\phi$  of (1),  $a > 1$ , is differentiable so that it also satisfies the functional-differential equation **nd Monotony**<br>
plution  $\phi$  of (1),  $a > 1$ , is differentiable so the dependency of  $t$  =  $ab [\phi(at) - \phi(at - a + 1)]$   $(b = \frac{a}{a-1})$ 

$$
\phi'(t) = ab \left[ \phi(at) - \phi(at - a + 1) \right] \qquad (b = \frac{a}{a-1}). \tag{4.1}
$$

**Proposition 4.1.** The solution  $\phi$  of (1), (2) has the property  $\phi(t) = \phi(1-t)$ , *i.e. is symmetric (with respect to* 1/2).

**Proof.** It is easy to check that if a function f has the property  $f(t) = f(1-t)$ , then  $g = Lf$  has this property too. Thus, choosing  $f_0$  symmetric, this property transfers to all iterates  $f_n = Lf_{n-1}$  and also to the limit function  $\phi$ 

**Remark.** Note that the symmetry of  $\phi$  is also a consequence of (2.1).

**Proposition 4.2.** In case of  $1 < a < 2$ , the solution  $\phi$  of  $(1),(2)$  is strongly *increasing in* (0, **1/2).** 

**Proof.** Let  $1 < a < 2$ . At first, we show that  $\phi(t)$  is not decreasing for  $0 < t < 1/2$ . We choose  $f_0$  from Proposition 3.1 symmetric with respect to 1/2, continuous and strongly increasing in  $(0,1/2)$ . Then the iterates  $f_n$   $(n \geq 1)$  are also symmetric and differentiable, and we show by induction that they are also strongly increasing in  $(0,1/2)$ . By differentiation of  $f_{n+1} = Lf_n$ , we get

$$
f'_{n+1}(t) = ab[f_n(at) - f_n(at - a + 1)].
$$

Since  $f_n$  is strongly increasing in (0,1/2) and  $f_n(t) = 0$  for  $t \le 0$ , it follows

$$
f'_{n+1}(t) = ab f_n(at) > 0
$$

for  $0 < t < 1-1/a$ . Furthermore, each c with  $f'_{n+1}(c) = 0$  satisfies  $f_n(ac) = f_n(ac - a +$ 1). Since  $ac > ac - a + 1$ , it follows in view of  $f_n(1-t) = f_n(t)$  that  $1-ac = ac - a + 1$ and therefore  $c = 1/2$ . This means that  $f'_{n+1}(t) > 0$  for  $t \in (0, 1/2)$ . As  $n \to \infty$ , we obtain  $\phi'(t) \geq 0$ , so that  $\phi(t)$  is not decreasing for these t.

Next, we show that  $\phi(t)$  is strongly positive for  $t \in (0, 1)$ . Assume that  $\phi(t) = 0$  for  $0 \le t \le \varepsilon$  where  $\varepsilon$  is maximal. From (1) it follows that also  $\phi(t) = 0$  for  $t \le a\varepsilon$  and thus  $\varepsilon = 0$  in view of  $a > 1$ . The monotony of  $\phi$  implies  $\phi(t) > 0$  for  $t \in (0, 1/2]$ , and the symmetry transfers this property to the whole interval  $(0,1)$ .

Now, we are able to show that  $\phi$  is strongly increasing in  $(0,1/2)$ . In view of  $(4.1)$ , we have  $\phi'(t) = ab\phi(at) > 0$  for  $0 < t < 1 - 1/a$ , and in view of the symmetry, we have  $\phi'(1/2) = 0$ . Let c be the smallest point in  $[1 - 1/a, 1/2]$  with  $\phi'(c) = 0$  which exists in view of the continuity of  $\phi'$ . From (4.1) and the symmetry of  $\phi$  it follows that On the Solution of an Integral-Functional Equation 167<br>
ttrongly positive for  $t \in (0,1)$ . Assume that  $\phi(t) = 0$  for<br>
from (1) it follows that also  $\phi(t) = 0$  for  $t \le a\epsilon$  and thus<br>
conotony of  $\phi$  implies  $\phi(t) > 0$  for  $t$ 

$$
\phi(ac - a + 1) = \phi(ac) = \phi(1 - ac). \tag{4.2}
$$

With the notation  $d = \min(ac, 1 - ac)$  where  $d \leq 1/2$ , we have obviously  $ac - a + 1 \leq d$ , so that there arise two cases. In the case  $ac - a + 1 < d$  the monotony and (4.2) imply  $\phi(t)$  = const for  $ac-a+1 \leq t \leq d$  and hence  $\phi'(ac-a+1)=0$ . In view of  $0 < ac - a + 1 < c$ , this is a contradiction to the choice of *c*. It remains the case  $ac - a + 1 = d$ , which in view of  $a > 1$  is only possible for  $ac - a + 1 = ac$  and hence  $c = 1/2$ , i.e.  $\phi'(t) > 0$  for  $t \in (0, 1/2)$ 

**Proposition 4.3.** In case of  $a \geq 2$ , the solution  $\phi$  of (1),(2) has the following *properties:* 

- (i)  $\phi(t)$  *is strongly increasing in*  $(0, 1/a)$ .
- (ii)  $\phi(t) = b$  for  $t \in [1/a, 1-1/a]$ .

**Proof.** Let  $a \ge 2$ . In case of  $t \le 1/a$ , we have  $at - a + 1 \le 0$ . Hence, (4.1) implies that  $\phi'(t) = ab \phi(at) \ge 0$ , and  $\phi(t)$  is not decreasing for these t. Analogously to the foregoing proof it follows that  $\phi(t)$  is strongly positive for  $0 < t < 1$ . For  $0 < t < 1/a$ , we have  $0 < at < 1$  and  $at - a + 1 < 0$ , hence, (4.1) implies  $\phi'(t) > 0$  for these t.

On the other side, for  $1/a \le t \le 1 - 1/a$ , we have  $at - a + 1 \le 0$  and  $at \ge 1$ , hence  $\phi = L\phi$ ,  $\phi(\tau) = 0$  for  $\tau \notin (0,1)$  and (2) imply for these t that  $\phi(t) = b \int_0^1 \phi(\tau) d\tau = b$ and the assertion is proved  $\blacksquare$ 

### **5. Fourier expansions**

For the Laplace transform  $\Phi$  of the solution  $\phi$  of (1), (2),  $a > 1$ , we have in case of  $p = ix$ 

$$
\Phi(ix) = \int\limits_0^1 e^{-ixt}\phi(t) dt = \int\limits_0^1 \cos(xt)\phi(t) dt - i \int\limits_0^1 \sin(xt)\phi(t) dt.
$$

Comparision with (2.4) and separation in real and imaginary part yields

$$
\mathbf{n}\mathbf{s}
$$
\n
$$
\Phi \text{ of the solution } \phi \text{ of (1), (2), } a > 1, \text{ we have in case of}
$$
\n
$$
t^t \phi(t) dt = \int_0^1 \cos(xt) \phi(t) dt - i \int_0^1 \sin(xt) \phi(t) dt.
$$
\n
$$
\text{d separation in real and imaginary part yields}
$$
\n
$$
\int_0^1 \cos(xt) \phi(t) dt = \cos\left(\frac{x}{2}\right) P(x) \tag{5.1}
$$

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\nand  
\n
$$
\int_{0}^{1} \sin(xt)\phi(t) dt = \sin\left(\frac{x}{2}\right) P(x)
$$
\nwhere P is given by (2.5). These formulas can be used to e

where  $P$  is given by (2.5). These formulas can be used to expand  $\phi$  in a Fourier series. For this reason, let f be the 1-periodic function with  $f(t) = \phi(t)$  for  $0 \le t \le 1$ . In view of Proposition *4.1,* the function *f is* an even one and has the Fourier series

$$
f(t) = 1 + \sum_{n=1}^{\infty} a_n \cos(n\pi t)
$$

where in view of (5.1)

$$
\int_{0}^{1} (2.5)
$$
 These formulas can be used to expand  $\phi$  in a Fourier series.  
\n $f$  be the 1-periodic function with  $f(t) = \phi(t)$  for  $0 \le t \le 1$ . In view  
\nthe function  $f$  is an even one and has the Fourier series  
\n
$$
f(t) = 1 + \sum_{n=1}^{\infty} a_n \cos(n\pi t)
$$
\n1)  
\n1)  
\n
$$
a_n = 2 \int_{0}^{1} f(t) \cos(n\pi t) dt = 2 \cos\left(\frac{n\pi}{2}\right) P(n\pi),
$$
\n
$$
a_{2n} = 2(-1)^n P(2n\pi) \text{ for all } n \ge 1. \text{ Hence,}
$$
\n
$$
f(t) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n P(2n\pi) \cos(2n\pi t).
$$
\n
$$
a_n = \int_{0}^{\infty} f(t) \cos(n\pi t) e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}}.
$$
\n
$$
b_n = \int_{0}^{\infty} \cos\left(\frac{2n\pi t}{n}\right) e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}}.
$$
\n
$$
a_n = \int_{0}^{\infty} \cos\frac{\alpha}{2^n} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}}.
$$
\n
$$
a_n = \int_{0}^{\infty} \cos\frac{\alpha}{2^n} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}}.
$$
\n
$$
a_n = \int_{0}^{\infty} \cos\frac{\alpha}{2^n} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}} e^{-\frac{2\pi}{n}}.
$$
\n
$$
b_n = \int_{0}^{\infty} e^{-\
$$

i.e.  $a_{2n-1} = 0$  and  $a_{2n} = 2(-1)^n P(2n\pi)$  for all  $n \ge 1$ . Hence,

$$
f(t) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n P(2n\pi) \cos(2n\pi t) . \tag{5.3}
$$

The foregoing way to calculate the Fourier coefficients is simpler than the usual one via the theorem of residues (cf. Berg [1: P. *33])* because it saves us to check the premises of Jordan's Lemma. In view of  $f(t) \cos(n\pi t) dt$ <br>  $)^n P(2n\pi)$  for al<br>  $1 + 2 \sum_{n=1}^{\infty} (-1)^n I$ <br>  $e$ : the Fourier coe<br>  $\arg [1: p. 33])$  be<br>  $\frac{n\alpha}{\alpha} = \prod_{n=1}^{\infty} \cos \frac{\alpha}{2^n}$ <br>  $\frac{n\alpha}{\alpha}$  rther represents

$$
\frac{\sin \alpha}{\alpha} = \prod_{n=1}^{\infty} \cos \frac{\alpha}{2^n} \qquad (\alpha \neq 0)
$$
  
further representation  

$$
f(x) = \prod_{n=1}^{\infty} \prod_{n=1}^{\infty} \cos \left( \frac{a-1}{2^{n+1} \alpha^k} x \right)
$$

(cf. *[3: p. 109])* we have the further representation

 $\bar{.}$ 

$$
P(x) = \prod_{n=1}^{\infty} \prod_{k=1}^{\infty} \cos \left( \frac{a-1}{2^{n+1} a^k} x \right) .
$$

The last results in this section are only valid in the case of  $a = 2$ , where the foregoing formula simplifies to

$$
\frac{1}{\alpha} = \prod_{n=1}^{\infty} \cos \frac{n}{2^n} \quad (\alpha \neq 0)
$$
  
we the further representation  

$$
P(x) = \prod_{n=1}^{\infty} \prod_{k=1}^{\infty} \cos \left( \frac{a-1}{2^{n+1}a^k} x \right).
$$
  
this section are only valid in the case of  $a = 2$ , where the foregoing  

$$
P(x) = \prod_{k=1}^{\infty} \frac{\sin \frac{x}{2^{k+1}}}{\frac{x}{2^{k+1}}} = \prod_{\nu=1}^{\infty} \cos^{\nu} \left( \frac{x}{2^{\nu+2}} \right).
$$
(5.4)  
 $x = 0$  and therefore also  $a_{4n} = 0$  (cf. (5.2)). Hence, the Fourier

Consequently,  $P(4n\pi) = 0$  and therefore also  $a_{4n} = 0$  (cf. (5.2)). Hence, the Fourier series (5.3) simplifies to

0 and therefore also 
$$
a_{4n} = 0
$$
 (cf. (3)  
 $f(t) = 1 + \sum_{n=1}^{\infty} a_{4n-2} \cos((4n-2)\pi t)$ 

with  $a_{4n-2} = -2P((4n-2)\pi)$ , i.e.

On the Solution of an integral-Functional Equation 169  
\n
$$
2P((4n-2)\pi), i.e.
$$
\n
$$
a_{4n-2} = -2 \prod_{k=1}^{\infty} \frac{\sin \frac{(2n-1)\pi}{2^k}}{\frac{(2n-1)\pi}{2^k}} = -2 \prod_{\nu=1}^{\infty} \cos^{\nu} \left( \frac{2n-1}{2^{\nu+1}} \pi \right).
$$
\n(5.5)

\nmeans of a computer yield for the following first coefficients:

\n
$$
a_2 = -1.10754255179 \qquad a_6 = 0.09240459827
$$

Calculations by means of a computer yield for the following first coefficients:

On the Solution of an integral-Functional E<br>  $2P((4n-2)\pi)$ , i.e.<br>  $a_{4n-2} = -2 \prod_{k=1}^{\infty} \frac{\sin \frac{(2n-1)\pi}{2^k}}{\frac{(2n-1)\pi}{2^k}} = -2 \prod_{\nu=1}^{\infty} \cos^{\nu} \left( \frac{2n-1}{2^{\nu+1}} \pi \right)$ <br>
means of a computer yield for the following first coe On the Solution of an integral-Functional E<br>  $2P((4n - 2)\pi)$ , i.e.<br>  $a_{4n-2} = -2 \prod_{k=1}^{\infty} \frac{\sin \frac{(2n-1)\pi}{2^k}}{\frac{(2n-1)\pi}{2^k}} = -2 \prod_{\nu=1}^{\infty} \cos^{\nu} \left( \frac{2n-1}{2^{\nu+1}} \pi \right)$ <br>
means of a computer yield for the following first co On the Solution of an integral-Functional E<br>  $a_{4n-2} = -2 \prod_{k=1}^{\infty} \frac{\sin \frac{(2n-1)\pi}{2^k}}{\frac{(2n-1)\pi}{2^k}} = -2 \prod_{\nu=1}^{\infty} \cos^{\nu} \left( \frac{2n-1}{2^{\nu+1}} \pi \right)$ <br>
means of a computer yield for the following first coeffi<br>  $a_2 = -1.1075425$ On the Solution of an integral-Functional E<br>  $2P((4n-2)\pi)$ , i.e.<br>  $a_{4n-2} = -2 \prod_{k=1}^{\infty} \frac{\sin \frac{(2n-1)\pi}{2^k}}{\frac{(2n-1)\pi}{2^k}} = -2 \prod_{\nu=1}^{\infty} \cos^{\nu} \left( \frac{2n-1}{2^{\nu+1}} \pi \right)$ <br>
means of a computer yield for the following first coe 2 $P((4n-2)\pi)$ , i.e.<br>  $a_{4n-2} = -2 \prod_{k=1}^{\infty} \frac{\sin \frac{(2n-1)\pi}{2^k}}{(2n-1)\pi} = -2 \prod_{\nu=1}^{\infty} \cos^{\nu} \left( \frac{2n-1}{2^{\nu+1}} \pi \right)$ <br>
means of a computer yield for the following first coeffic<br>  $a_2 = -1.10754255179$   $a_6 = 0.09240459827$ <br>  $a$  $a_{14} = -0.0020$ <br>  $a_{24} = -0.0000$ <br>  $a_{32} = 0.00000$ <br>  $a_{40} = -0.00000$ <br>  $o(-1)^k$  for  $x \in (k, k + n - 2)\pi$  is equal to<br>  $\sum_{r=1}^{\infty} \left[\frac{2n-1}{2^k}\right]_{r=1}^{\infty} = (-1)^{k=0}$ 0.0001230572  $a_{24} = -0.00059673667$ <br>  $-0.00018280996$   $a_{32} = 0.00001311091$ <br>
0.00000676768  $a_{40} = -0.00002464224$ .<br>  $a_{4n-2} = -P((4n-2)\pi)$  is equal to<br>  $a_{4n-2} = -P((4n-2)\pi)$  is equal to<br>  $\sum_{k=1}^{\infty} \left[\frac{2n-1}{2^k}\right]_{k=0} \sum$ 

Since the sign from  $sin(\pi x)$  is equal to  $(-1)^k$  for  $x \in (k, k + 1)$ , it follows from (5.5) that the sign  $\varepsilon_{4n-2}$  of  $a_{4n-2} = -P((4n-2)\pi)$  is equal to

$$
\varepsilon_{4n-2} = -(-1)^{\sum_{k=1}^{\infty} \left[\frac{2n-1}{2^k}\right]} = (-1)^{\sum_{k=0}^{\infty} \left[\frac{2n-1}{2^k}\right]}.
$$
\n(5.6)

Assume that  $2n - 1$  has the dyadic representation  $2n - 1 = d_s \cdots d_1 d_0$  with  $d_s = 1$  and  $d_{\nu} \in \{0, 1\}$ . Since the number  $[(2n - 1)/2^{k}] = d_{s} \cdots d_{k}$  is odd if and only if  $d_{k} = 1$ , it follows from (5.6) that  $\varepsilon_{4n-2} = (-1)^{\nu(2n-1)}$ , where  $\nu(n)$  denotes the number of ones in the dyadic representation of *n*, i.e.  $\nu(n)$  is the binary sum-of-digits function (cf. [4: p. 293]). gn from sin( $\pi x$ ) is equal to  $(-1)^n$  for  $x \in (k, k + 1)$ , it<br>  $\lim_{\epsilon_4 n - 2}$  of  $a_{4n-2} = -P((4n - 2)\pi)$  is equal to<br>  $\sum_{\epsilon_1 = 2}^{\infty} \left[\frac{2n-1}{2k}\right]$ <br>  $\sum_{\epsilon_4 = 2}^{\infty} \left[\frac{2n-1}{2k}\right]$ <br>  $\sum_{\epsilon_1 = 2}^{\infty} \left[\frac{2n-1}{2k}\right]$ <br>  $\$ 

i.e.  $K = \left[\ln(n\pi)/\ln 2\right]$ . Then we get from (5.4) the inequality

\n dyadic representation of 
$$
n
$$
, i.e.  $\nu(n)$  is the binary sum-of-digits function (cf).\n

\n\n In order to obtain an estimate for  $|P(2n\pi)|$ , we choose  $K$  subject to  $\frac{n\pi}{2K+1} < 1$ .\n

\n\n $K = \left[\ln(n\pi)/\ln 2\right]$ . Then we get from (5.4) the inequality\n 
$$
|P(2n\pi)| \leq \prod_{k=1}^{K} \frac{2^k}{n\pi} = \frac{2^{K(K+1)2}}{n^{K}\pi^{K}} = \left(\frac{2}{n\pi} \frac{2^K}{n\pi}\right)^{K/2} \leq \left(\frac{2}{n\pi}\right)^{K/2},
$$
\n

\n\n sh matches to the asymptotic considerations in Section 9.\n

which matches to the asymptotic considerations in Section 9.

## 6. Spectral properties

Generalizing (1), we consider the integral equation

te for 
$$
|P(2n\pi)|
$$
, we choose K subject to  $\frac{n\pi}{2K+1} < 1 \le \frac{n\pi}{2K}$ ,  
get from (5.4) the inequality  

$$
= \frac{2^{K(K+1)2}}{n^K \pi^K} = \left(\frac{2}{n\pi} \frac{2^K}{n\pi}\right)^{K/2} \le \left(\frac{2}{n\pi}\right)^{K/2},
$$
  
ic considerations in Section 9.  
  
ne integral equation  

$$
\lambda \phi(t) = \int_{at-a+1}^{at} \phi(\tau) d\tau
$$
 (6.1)  
and the corresponding adjoint equation  

$$
(5+2\pi)^{1/2}
$$

with  $a > 1$  and supp  $\phi \subset [0, 1]$ , and the corresponding adjoint equation

tic considerations in Section 9.  
\nthe integral equation  
\n
$$
\lambda \phi(t) = \int_{at-a+1}^{at} \phi(\tau) d\tau
$$
\nand the corresponding adjoint equation  
\n
$$
\frac{(\tau+a-1)}{(\tau+a-1)} = \int_{\tau/a}^{\tau/a} \psi(t) dt
$$
\n(6.2)  
\n
$$
\phi(\tau) = \int_{\tau/a}^{\tau/a} \psi(t) dt
$$
\n(6.3)

for  $0 \leq \tau \leq 1$ . Both equations are eigenvalue problems for adjoint operators which are linear and compact. The compactness can be shown as in Section 2 by means of the selection theorem of Arzelà and Ascoli. Hence by the Fredhoim alternative, both equations must have the same non-zero eigenvalues with the same multiplicity (cf. 19: p. 789]). First, we consider equation (6.2).

**Proposition 6.1.**  $For \vert \lambda \vert > 1/b$ , the integral equation (6.2) with  $\tau \in [0, 1]$  has only **170** L. Berg and M. Krüppe<br> **Proposition 6.1.**  $For \, |$ <br> *the trivial solution*  $\psi(\tau) \equiv 0$ .

**Proof.** A solution  $\psi$  of (6.2) is continuous. Assume that  $K = \max |\psi(\tau)|$  on [0, 1]

**Proposition 6.1.** For 
$$
|\lambda| > 1/b
$$
, the integral equation (6.2) with the trivial solution  $\psi(\tau) \equiv 0$ . **Proof.** A solution  $\psi$  of (6.2) is continuous. Assume that  $K = n$  and  $d \in [0,1]$  with  $|\psi(d)| = K$ . From (6.2), we get\n
$$
(d+a-1)/a
$$
\n
$$
|\lambda|K = |\lambda| |\psi(d)| \leq \int_{d/a}^{(d+a-1)/a} |\psi(t)| dt \leq \frac{a-1}{a} K
$$
\nSince  $|\lambda| > 1/b = (a-1)/a$ , this implies that  $K = 0$ , i.e.  $\psi(\tau) \equiv 0$  **Proposition 6.2.** The integral equation (6.2) has exactly the no Proposition 6.2. The integral equation (6.2) has exactly the nonpolynomials\n
$$
\psi_n(\tau) = \sum_{\nu=0}^n {n \choose \nu} r_{n-\nu}(a) \tau^{\nu}
$$
,\nwhere  $r_0(a) = 1$  and the coefficients  $r_{\nu}(a)$  for  $\nu \geq 1$  are determined.

Since  $|\lambda| > 1/b = (a-1)/a$ , this implies that  $K = 0$ , i.e.  $\psi(\tau) \equiv 0$ 

**Proposition 6.2.** *The integral equation (6.2) has exactly the non-zero eigenvalues*   $\lambda_n = 1/(ba^n)$ ,  $n \in \mathbb{N}_0$ , which are all simple. Corresponding eigenfunctions are the *polynomials*  $d/a$ <br>  $=(a-1)/a$ , this implies that K<br>  $n \in \mathbb{N}_0$ , which are all simple.<br>  $\psi_n(\tau) = \sum_{\nu=0}^n {n \choose \nu} r_n$ <br>
and the coefficients  $r_{\nu}(a)$  for  $\nu \ge$ <br>  $r_{\nu}(a) = \frac{-1}{a^{\nu}-1} \frac{1}{\nu+1} \sum_{k=0}^{\nu-1} { \nu+1 \choose k}$ <br>  $o \lambda = 0$  is an eig *integral equation* (<br> *iich are all simple*<br>  $\psi_n(\tau) = \sum_{\nu=0}^n \binom{n}{\nu}$ <br> *fficients*  $r_{\nu}(a)$  *for*<br>  $\frac{1}{n-1} \frac{1}{\nu+1} \sum_{k=0}^{\nu-1} \binom{\nu}{k}$ <br> *an eigenvalue of* (  $|\psi(t)| dt \leq \frac{1}{a}$  *K*.<br>  $K = 0$ , i.e.  $\psi(\tau) \equiv 0$ <br> **(6.2)** has exactly the non-zero eigenvalues<br> *e.* Corresponding eigenfunctions are the<br>  $r_{n-\nu}(a) \tau^{\nu}$ , (6.3)<br>  $\nu \geq 1$  are determined recursively by<br>  $\begin{pmatrix} +1 \\ k \end{pmatrix$ 

$$
\psi_n(\tau) = \sum_{\nu=0}^n \binom{n}{\nu} r_{n-\nu}(a) \tau^{\nu} , \qquad (6.3)
$$

where  $r_0(a) = 1$  and the coefficients  $r_{\nu}(a)$  for  $\nu \ge 1$  are determined recursively by

$$
r_{\nu}(a) = \frac{-1}{a^{\nu} - 1} \frac{1}{\nu + 1} \sum_{k=0}^{\nu - 1} {\binom{\nu + 1}{k}} r_k(a) (a - 1)^{\nu - k} a^k
$$
 (6.4)

*Additionally, also*  $\lambda = 0$  *is an eigenvalue of* (6.2) and each integrable function  $\psi$  with *period 1 - 1/a and average 0, restricted to the interval [0, 11, is solution of (6.2) with*   $\lambda = 0$ . **Proof.** Assume that  $\psi$  is an eigenvalue of (6.2) and each integrable function  $\psi$  with  $\omega d$  1 - 1/a and average 0, restricted to the interval [0,1], is solution of (6.2) with 0.<br>**Proof.** Assume that  $\psi$  is a solutio

From (6.2), *we* find by differentiation **Proof.** Assume that  $\psi$  is a solution of (6.2) with  $\lambda \neq 0$ . Then  $\psi$  is differentiable.

$$
\lambda \psi'(\tau) = \frac{1}{a} \int\limits_{\tau/a}^{(\tau+a-1)/a} \psi'(t) dt ,
$$

i.e.  $\psi'$  is a solution of (6.2) with eigenvalue parameter  $a\lambda$  instead of  $\lambda$ . It follows that  $n \in \mathbb{N}$ , is a solution of (6.2) with eigenvalue parameter  $a^n \lambda$ . Since  $a > 1$ , for each i.e.  $\psi'$  is a solution of (6.2) with eigenvalue parameter  $a\lambda$  instead of  $\lambda$ . It follows that  $\psi^{(n)}$ ,  $n \in \mathbb{N}$ , is a solution of (6.2) with eigenvalue parameter  $a^n\lambda$ . Since  $a > 1$ , for each  $\lambda \neq 0$  there exi  $\lambda \neq 0$  there exists an integer *n* such that  $a^n|\lambda| > 1/b$ . Proposition 6.1 implies that  $\psi^{(n)} = 0$ , i.e.  $\psi$  must be a polynomial. If we make the ansatz ILET  $\int_{\tau/a}^{1} f(6.2)$  with eigenvalue particle in order (6.2) with eigenvalue particle in the apply point in the apply point of  $\psi(\tau) = \sum_{\nu=0}^{n} c_{n\nu} \tau^{\nu}$ , over into the equation

$$
\psi(\tau) = \sum_{\nu=0}^n c_{n\nu} \tau^{\nu} , \qquad c_{nn} = 1 ,
$$

equation (6.2) turns over into the equation

1 (T + *a -* 1) *<sup>t</sup>* <sup>+</sup> <sup>1</sup> - *A c,, r*  v *= c,,* ii+1 *u0 i,=0*

from which we obtain by comparison of the coefficients of  $\tau^n$  that  $\lambda = \frac{a-1}{a^{n+1}}$ , i.e. at most  $\lambda = 1/(ba^n)$  can be an eigenvalue of (6.2). With this value for  $\lambda$ , we obtain once more by comparison of coefficients recursively On the general comparison<br>  $\alpha$ : an eigenvalue officients recomposed<br>  $=\sum_{\nu=\mu}^{n}\frac{1}{\nu+1}$ <br>
0. Now, with Explore of an Integral-Functical of the coefficients of  $\tau^n$  time of (6.2). With this value<br>cursively<br> $\begin{pmatrix} \nu + 1 \\ \mu \end{pmatrix} c_{n\nu} (a - 1)^{\nu - \mu} a^{n-\nu}$ Frison of the coeff<br> *x* alue of (6.2). V<br> *x* s recursively<br>  $\frac{1}{k+1} \begin{pmatrix} \nu+1 \\ \mu \end{pmatrix} c_{n\nu}$ <br>
with the further<br>  $\sum_{k=0}^{n-\mu} \begin{pmatrix} n-\mu+1 \\ k \end{pmatrix}$ <br> *k* is equations with<br>  $k \geq 0$  be done in the origin

In be an eigenvalue of (6.2). With this value is  
of coefficients recursively  

$$
c_{n\mu} = \sum_{\nu=\mu}^{n} \frac{1}{\nu+1} {\nu+1 \choose \mu} c_{n\nu} (a-1)^{\nu-\mu} a^{n-\nu}
$$

 $c_{n\mu} = \sum_{\nu=\mu}^{n} \frac{1}{\nu+1} { \nu+1 \choose \mu} c_{n\nu}(a-1)^{\nu-\mu} a^{n-\nu}$ <br>for  $\mu = n-1, n-2, ..., 0$ . Now, with the further ansatz  $c_{n\mu} = {n \choose \mu} r_{n-\mu}(a)$  and the substitution  $\nu = n - k$ , we get

$$
a_n = \sum_{\nu=\mu}^n \frac{1}{\nu+1} \binom{\nu+1}{\mu} c_{n\nu} (a-1)^{\nu-\mu} a^{n-\nu}
$$
  
\n
$$
c_{n\mu} = \sum_{\nu=\mu}^n \frac{1}{\nu+1} \binom{\nu+1}{\mu} c_{n\nu} (a-1)^{\nu-\mu} a^{n-\nu}
$$
  
\n
$$
1, n-2, ..., 0. \text{ Now, with the further ansatz } c_{n\mu} = \binom{n}{\mu} r_{n-\mu}(a) \text{ and the}
$$
  
\n
$$
\nu = n-k, \text{ we get}
$$
  
\n
$$
r_{n-\mu}(a) = \frac{1}{n-\mu+1} \sum_{k=0}^{n-\mu} \binom{n-\mu+1}{k} r_k(a)(a-1)^{n-\mu-k} a^k.
$$
 (6.5)  
\n
$$
\mu = \nu \text{ and solving this equations with respect to } r_{\nu}(a), \text{ we immediately ob-}
$$

Writing  $n - \mu = \nu$  and solving this equations with respect to  $r_{\nu}(a)$ , we immediately obtain (6.4). Of course, all steps can be done in the opposite direction, so that  $\lambda = 1/(ba^n)$ is in fact an cigenvalue of (6.2) with the eigenfunction (6.3), (6.4). This eigenvalue is simple, since all coefficients of  $\psi_n$  are uniquely determined. The statement with respect to  $\lambda = 0$  is obvious, so that the proposition is proved  $\blacksquare$ respect to  $r_{\nu}(a)$ , we immediately ob-<br>
spect to  $r_{\nu}(a)$ , we immediately ob-<br>
simined. The statement with respect<br>
d<br>
dholm theory implies that equation<br>
mvalues  $\lambda_n = 1/(a^n b)$ ,  $n \in \mathbb{N}_0$ , but<br>
d by differentiation<br>

Since  $(6.1)$  is the adjoint equation of  $(6.2)$ , the Fredholm theory implies that equation (6.1) also possesses exactly the simple non-zero eigenvalues  $\lambda_n = 1/(a^n b)$ ,  $n \in \mathbb{N}_0$ , but  $\lambda = 0$  is not an eigenvalue of (6.1). From (1), we find by differentiation

$$
\phi^{(n)}(t) = a^n b \int_{at-a+1}^{at} \phi^{(n)}(\tau) d\tau
$$
\n(6.6)

so that  $\phi^{(n)}$  is an eigenfunction to  $\lambda_n$ . Hence, we have the

**Proposition 6.3.** *Equation* (6.1) with  $t \in [0,1]$  *has exactly the eigenvalues*  $\lambda_n =$  $1/(a^n b)$ ,  $n \in \mathbb{N}_0$ . *Corresponding eigenfunctions are*  $\phi^{(n)}$ , where  $\phi$  is the solution of *(1),(2).* 

Equation (6.2) has also in case of  $\tau \in \mathbb{R}$  only the eigenvalues  $\lambda_n = 1/(a^n b)$  with  $n \in \mathbb{N}_0$ , since the proof of Proposition 6.1 can be transferred to the case  $K = \max |\psi(\tau)|$ on  $[-M, M]$  with an arbitrary  $M > 1$ . This implies that the assertion of Proposition 6.2 is also true for arbitrary real *r,* and the polynomials (6.3) as well as periodic functions with period  $1 - 1/a$  are also corresponding eigenfunctions of  $(6.2)$  in this case.  $\in \mathbb{R}$  only the eigenvalues  $\lambda_n = 1/(a^n b)$  with<br>can be transferred to the case  $K = \max |\psi(\tau)|$ <br>is implies that the assertion of Proposition 6.2<br>olynomials (6.3) as well as periodic functions<br>ag eigenfunctions of (6.2) in t

However, if we admit also solutions of (6.1) with non-compact support, then there appear further eigenvalues, which are unbounded, i.e. then the corresponding operator cannot be compact, and the Frcdholm alternative cannot be applied. In order to show However, if we admit also solutions of (6.1) with non-compact support, then there<br>appear further eigenvalues, which are unbounded, i.e. then the corresponding operator<br>cannot be compact, and the Fredholm alternative canno  $\phi_n$ ,  $n \in \mathbb{N}_0$ , defined by  $\phi_0 = \phi$  from (1), (2) and

$$
\phi_{n+1}(t) = \int_{0}^{t} \phi_n(\tau) d\tau \tag{6.7}
$$

for  $n \geq 0$ .

**Proposition 6.4.** *Equation* (6.1) with  $a > 1$  has besides  $1/(a^n b)$  also the eigenval*ues*  $\lambda_{-n-1} = a^{n+1}/b, n \in \mathbb{N}_0$ , with corresponding eigenfunctions  $\psi_n(t, 1/a)$  and  $\phi_{n+1}(t)$ , *if we admit that the solutions have non-compact support.* 

**Proof.** In the proof of Proposition *6.2,* we never have used explicitely the assumption  $a > 1$ . Hence, the polynomials  $\psi_n(\tau, a)$  ( $n \in \mathbb{N}_0$ ) are also solutions of (6.2) with  $\lambda = 1/(ba^n)$  in case of  $0 < a < 1$ , if  $\tau$  is not restricted to [0,1]. This means with the substitution  $a = 1/\alpha$  for  $\alpha > 1$  that 1 has besides  $1/(a^n b)$  also the eigenval-<br> *eigenfunctions*  $\psi_n(t, 1/a)$  and  $\phi_{n+1}(t)$ ,<br> *support*.<br>
never have used explicitely the assump-<br>  $\iota \in \mathbb{N}_0$  are also solutions of (6.2) with<br>
stricted to [0,1]. This means tion  $a > 1$ . Hence, the polynomials  $\psi_n(\tau, a)$  ( $n \in \mathbb{N}_0$ ) are also solutions of (6.2) with  $\lambda = 1/(ba^n)$  in case of  $0 < a < 1$ , if  $\tau$  is not restricted to [0,1]. This means with the substitution  $a = 1/\alpha$  for  $\alpha > 1$  tha

ence, the polynomials 
$$
\psi_n(\tau, a)
$$
  $(n \in \mathbb{N}_0)$  are also solutions of (6.2) with  
\nn case of  $0 < a < 1$ , if  $\tau$  is not restricted to [0,1]. This means with the  
\n $= 1/\alpha$  for  $\alpha > 1$  that  
\n
$$
\alpha^n(1-\alpha)\psi_n(t,1/\alpha) = \int_{\alpha t}^{\alpha t-\alpha+1} \psi_n(\tau,1/\alpha) d\tau \qquad (n \in \mathbb{N}_0),
$$
\n(6.8)

i.e.  $\psi_n(t, 1/a)$  is an eigenfunction of equation (6.1) with  $t \in \mathbb{R}$  and  $a > 1$  to the eigenvalue  $\lambda_{-n-1} = a^n(a-1) = a^{n+1}/b, n \in \mathbb{N}_0$ . It is clear as in (6.6) that  $\phi_{n+1}$  is also an eigenfunction to the eigenvalue  $\lambda_{n-1}$ 

Later on in Section 9 we shall see that equation (6.1) with  $t \in \mathbb{R}$  and  $a > 1$  has even all non-negative numbers  $\lambda$  as multiple eigenvalues. Analogously as in Proposition 6.4, we can conclude by means of the substitution  $a = 1/\alpha$  that problem (1), (2) with  $\tau \in [0, 1]$  is unsolvable for  $0 < a < 1$ .

## 7. **Connections between eigenfunctions**

The next proposition shows a surprising connection between the eigenfunctions  $\psi_n$  of 7. **Connections between eigenfunctions**<br>The next proposition shows a surprising connection between the eigenfunctions  $\psi_n$  of (6.2) and the Laplace transform  $\Phi$  of the solution  $\phi$  of (1), (2), i.e. of the eigenfuncti (6.2) and the Laplace transform  $\Phi$  of the solution  $\phi$  of (1), (2), i.e. of the eigenfunction  $\phi$  of (6.1) with  $\lambda = 1/b$  and  $a > 1$ . The result, however, is also valid for  $0 \le a \le 1$  and all  $z \in \mathbb{C}$ . 1.<br>
eigenfunction<br>
prising connection<br>
of the solution  $\phi$ .<br>
The result, howe<br>
on  $g(\tau, z) = e^{(\tau - 1)}$ <br>
le have<br>  $\frac{e^{-1}}{z} = \sum_{n=0}^{\infty} \frac{\psi_n(\tau)}{n!}$ <br>
for all  $\tau \in \mathbb{R}$ , and tween the eigenfunctions  $\psi_n$  of<br>
1), (2), i.e. of the eigenfunction<br>
, is also valid for  $0 \le a \le 1$  and<br>  $\Phi(z)$  is the generating function<br>  $\Phi(z)$  is the generating function<br>
. (7.1)<br>  $< 2b\pi$ ,  $b = a/(a - 1)$ .

**Proposition 7.1.** The function  $g(\tau, z) = e^{(\tau-1)z}/\Phi(z)$  is the generating function *of the eigenfunctions*  $\psi_n(\tau)$ , *i.e. we have ()* =<br>
<sup>∞</sup> ±

$$
\frac{e^{(\tau - 1)z}}{\Phi(z)} = \sum_{n=0}^{\infty} \frac{\psi_n(\tau)}{n!} z^n , \qquad (7.1)
$$

*where the power series converges for all*  $r \in \mathbb{R}$ , and  $|z| < 2b\pi$ ,  $b = a/(a - 1)$ .<br> **Proof.** Introducing the generating function of the coefficients  $r_{\nu}(a)$ <br>  $g(z) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} r_{\nu}(a) z^{\nu}$ 

**Proof.** Introducing the generating function of the coefficients  $r_{\nu}(a)$ 

$$
\Phi(z) = \sum_{n=0}^{\infty} \frac{n!}{n!}
$$
\n*es for all*  $\tau \in \mathbb{R}$ , and  $|z|$  *generating function of the* 
$$
g(z) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} r_{\nu}(a) z^{\nu}
$$
\n*n* (6.5) with  $n - \mu = \nu$  *yi* 
$$
g(z) = \frac{e^{z(a-1)} - 1}{z(a-1)} g(az)
$$

as formal power series, equation (6.5) with  $n - \mu = \nu$  yields

$$
g(z) = \frac{e^{z(a-1)} - 1}{z(a-1)} g(az) .
$$

Hence, in view of  $g(0) = 1$ , (1.2) and (2.1), we obtain the equation

On the Solution of an Integral-Funct  
\n
$$
= 1, (1.2) \text{ and } (2.1), \text{ we obtain the equation}
$$
\n
$$
g(z) = \prod_{n=0}^{\infty} \frac{z/(ba^n)}{e^{z/(ba^n)} - 1} = \frac{1}{\Phi(-z)} = \frac{e^{-z}}{\Phi(z)},
$$

which implies also the convergence of (7.1) for  $|z| < 2b\pi$  (cf. Knopp [6: p. 451]). By multiplication with the power series of  $e^{rz}$ , we find

$$
e^{\tau z} \frac{e^{-z}}{\Phi(z)} = \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \frac{1}{\nu! (n-\nu)!} r_{n-\nu}(a) \tau^{\nu} z^{n} ,
$$

so that, in view of  $(6.3)$ , the proposition is proved  $\blacksquare$ 

According to (7.1), the eigenfunctions  $\psi_n$  are so-called *Appell polynomials* (cf. [2: p. 22]). By comparison of (2.12) and (7.1), we find in view of (2.10) and  $\psi_n(0, a) = r_n(a)$ the relation  $< 2b\pi$  (cf.<br>  $\mathbf{r}_{n-\nu}(a)\tau^{\nu}z$ <br>
called Appe<br>
view of (2.1<br>
Section 2)<br>
ms So that, in view of (0.3), the proposition is proved <br>According to (7.1), the eigenfunctions  $\psi_n$  are so-cal<br>22]). By comparison of (2.12) and (7.1), we find in vie<br>the relation<br> $r_n(a) = \rho_n \left(\frac{1}{a}\right)$ ,<br>so that according t 7.1), the  $\epsilon$ <br>
7.1), the  $\epsilon$ <br>
on of (2.1)<br>
to the firs<br>  $\psi_n(\tau, a) = \frac{n}{2} \tau^{n-1}$ 

$$
r_n(a) = \rho_n\left(\frac{1}{a}\right),\tag{7.2}
$$

so that according to the first coefficients  $\rho_n(a)$  (cf. Section 2) we obtain for the eigenfunctions  $\psi_n(\tau) = \psi_n(\tau, a)$  of (6.2) the representations<br>  $\psi_n(\tau) = \tau^n - \frac{n}{2}\tau^{n-1} + \frac{n(n-1)}{12}\frac{a+2}{a+1}\tau^{n-2} - \frac{n(n-1)(n-2)}{24(a+1)}\$ 

ons 
$$
\psi_n(\tau) = \psi_n(\tau, a)
$$
 of (6.2) the representations  
\n
$$
\psi_n(\tau) = \tau^n - \frac{n}{2} \tau^{n-1} + \frac{n(n-1)}{12} \frac{a+2}{a+1} \tau^{n-2} - \frac{n(n-1)(n-2)}{24(a+1)} \tau^{n-3} + \dots
$$
\nrticular, equation (7.1) implies in view of (2.11) that  
\n
$$
\psi_n(\tau, 0) = \frac{\tau^{n+1} - (\tau - 1)^{n+1}}{n+1}, \quad \psi_n(\tau, 1) = \left(\tau - \frac{1}{2}\right)^n, \quad \psi_n(\tau, \infty) = B_n(\tau)
$$

In particular, equation (7.1) implies in view of (2.11) that

at according to the first coefficients 
$$
\rho_n(a)
$$
 (cf. Section 2) we obtain for the  
lions  $\psi_n(\tau) = \psi_n(\tau, a)$  of (6.2) the representations  

$$
\psi_n(\tau) = \tau^n - \frac{n}{2} \tau^{n-1} + \frac{n(n-1)}{12} \frac{a+2}{a+1} \tau^{n-2} - \frac{n(n-1)(n-2)}{24(a+1)} \tau^{n-3} + \dots
$$
  
tricular, equation (7.1) implies in view of (2.11) that  

$$
\psi_n(\tau, 0) = \frac{\tau^{n+1} - (\tau - 1)^{n+1}}{n+1}, \quad \psi_n(\tau, 1) = \left(\tau - \frac{1}{2}\right)^n, \quad \psi_n(\tau, \infty) = B_n(\tau),
$$
  
the  $B_n$  are the Bernoulli polynomials. Moreover, the special values (2.14) and  
recursion formula (2.13) can easily be transferred by means of (7.2)

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where  $B_n$  are the Bernoulli polynomials. Moreover, the special values (2.14) and also<br>the recursion formula (2.13) can easily be transferred by means of (7.2).<br>The expansion (7.1) implies some properties of the eigenfunc polynomials, resp. Bernoulli numbers.

Proposition 7.2. The eigenfunctions  $\psi_n$  of equation (6.2) with  $a > 1$  have the *properties*

$$
(i) \psi'_n(\tau) = n\psi_{n-1}(\tau)
$$

$$
(ii) \ \psi_n(1-\tau) = (-1)^n \psi_n(\tau)
$$

and they form with the derivatives  $\phi^{(m)}$  a biorthogonal system in  $L^2[0,1]$ , i.e.

(iii)  $\int_{0}^{1} \phi^{(m)}(\tau)\psi_n(\tau) d\tau = (-1)^m m! \delta_{m,n}$   $(m, n \in \mathbb{N}_0).$ 

Proof. Property (i) is a general one for Appell polynomials. According to (2.1), we have the equation  $g(1-t, z) = g(t, -z)$ , which in view of (7.1) yields the symmetry

property (ii). For  $m \neq n$ , the functions  $\phi^{(m)}(\tau)$  and  $\psi_n(\tau)$  are orthogonal, since they are eigenfunctions of adjoint equations to different eigenvalues. Now, let be  $m = n$ .

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\nproperty (ii). For 
$$
m \neq n
$$
, the functions  $\phi^{(m)}(\tau)$  and  $\psi_n(\tau)$  are orthogonal, since t  
\nare eigenfunctions of adjoint equations to different eigenvalues. Now, let be  $m =$   
\nObviously, (iii) is clear for  $n = 0$  in view of (2) and  $\psi_0(\tau) = 1$ . Integration by parts  
\n
$$
\int_{0}^{1} \phi^{(n+1)}(\tau)\psi_{n+1}(\tau) d\tau = -\int_{0}^{1} \phi^{(n)}(\tau)\psi'_{n+1}(\tau) d\tau
$$
\nand (i) complete the proof by means of induction **ii**

and (i) complete the proof by means of induction  $\blacksquare$ 

(i) complete the proof by means of induction  $\blacksquare$ <br>(i) complete the proof by means of induction  $\blacksquare$ <br>**Remarks. 1.** We can easily check directly that  $\psi'_n$ <br> $\lambda = 1/(a^{n-1}b)$ . Since we know that all eigenval is also an eigenfunction of  $(6.2)$ with  $\lambda = 1/(a^{n-1}b)$ . Since we know that all eigenvalues are simple, it follows that  $\int_{0}^{\phi^{(n+1)}}(\tau)\psi_{n+1}(\tau)d\tau = -\int_{0}^{\phi^{(n)}}(\tau)\psi'_{n+1}(\tau)d\tau$ <br>
(i) complete the proof by means of induction **R**<br> **Remarks.** 1. We can easily check directly that  $\psi'_{n}$  is also an eigenfunction  $\lambda = 1/(a^{n-1}b)$ . Since we k

2. Since  $\psi_n(1 - \tau)$  is also an eigenfunction of (6.2) with the simple eigenvalue  $\lambda = 1/(a^n b)$ , it follows that  $\psi_n(1 - \tau) = c_n \psi_n(\tau)$  with  $c_n^2 = 1$ , and therefore  $c_n = 1$  or  $c_n = -1.$ 

3. Note that the properties (i) and (ii) are also valid for the polynomials  $\psi_n(t, 1/a)$ ,  $a>1$ .

4. Of course, also the periodic eigenfunctions of (6.2) with respect to the eigenvalue 0 are orthogonal to the eigenfunctions of (6.1).

Formula (6.3) with  $\tau = 1$  implies the relation

$$
r_n(a) = (-1)^n \sum_{\nu=0}^n \binom{n}{\nu} r_{\nu}(a) ,
$$

which can be used to check the already calculated  $r_{\nu}(a)$ . The last relation means that the functions  $r_{\nu}(a)$  are linearly dependent where more precisely

check the already calculated 
$$
r_{\nu}(a)
$$
. The last  
re linearly dependent where more precisely  

$$
\sum_{\nu=0}^{n} {n \choose \nu} \left(2 - \frac{\nu}{n}\right) r_{2n-1-\nu}(a) = 0 \qquad (n \ge 1).
$$

From (2.9), we can derive an addition theorem, namely

and to the eigenfunctions of (6.1).

\n6.3) with 
$$
\tau = 1
$$
 implies the relation

\n
$$
r_n(a) = (-1)^n \sum_{\nu=0}^n \binom{n}{\nu} r_{\nu}(a),
$$
\nused to check the already calculated  $r_{\nu}(a)$ . The last relation means that

\n
$$
r_{\nu}(a)
$$
 are linearly dependent where more precisely\n
$$
\sum_{\nu=0}^n \binom{n}{\nu} \left(2 - \frac{\nu}{n}\right) r_{2n-1-\nu}(a) = 0 \qquad (n \geq 1).
$$
\n), we can derive an addition theorem, namely

\n
$$
\sum_{\nu=0}^n \binom{n}{\nu} a^{\nu} \psi_{\nu}(\sigma, a^2) \psi_{n-\nu}(\tau, a^2) = (a+1)^n \psi_n \left(\frac{\sigma}{\alpha} + \frac{\tau}{\beta}, a\right), \qquad (7.3)
$$
\n
$$
\tau = 0 \text{ turns over into a nonlinear formula for } r_n(a). \text{ Moreover, from (7.1)}
$$
\nfollows that

which for  $\sigma = \tau = 0$  turns over into a nonlinear formula for  $r_n(a)$ . Moreover, from (7.1) and (2.10) it follows that

$$
\sum_{\nu=0}^n \binom{n}{\nu} \psi_{\nu}(\sigma, a) \psi_{n-\nu}(\tau, 1/a) = (\sigma + \tau - 1)^n,
$$

and the general formula for Appendix  

$$
\sum_{\nu=0}^{n} {n \choose \nu} \sigma^{\nu} \psi_{n-\nu}(\tau, a) = \psi_n(\sigma + \tau, a),
$$

which is equivalent to Proposition 7.2/(i) in view of Taylor's formula.

## 8. Polynomial relations

By means of certain relations between eigenfunctions, we can derive some polynomial relations for the solution  $\phi$  of the initial problem (1), (2).

**Proposition 8.1.** *The eigenfunctions*  $\phi_n$   $(n \in \mathbb{N})$  *of* (6.1) with  $a > 1$  have the *representations b* of the initial problem (1), (2).<br> *ne* eigenfunctions  $\phi_n$  ( $n \in \mathbb{N}$ ) of (6.1) with<br>  $a^{\frac{n(n-3)}{2}}(a-1)^n \phi(a^{-n}t, a)$  for  $0 \le t \le a-1$ 

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\nnial relations  
\nertain relations between eigenfunctions, we can derive some polynomial  
\ne solution 
$$
\phi
$$
 of the initial problem (1), (2).  
\non 8.1. The eigenfunctions  $\phi_n$   $(n \in \mathbb{N})$  of (6.1) with  $a > 1$  have the  
\n
$$
\phi_n(t) = \begin{cases}\na^{\frac{n(n-3)}{2}}(a-1)^n \phi(a^{-n}t, a) & \text{for } 0 \le t \le a-1\\ \n\frac{1}{(n-1)!} \psi_{n-1}(t, 1/a) & \text{for } 1 \le t.\n\end{cases}
$$
\n(e first equation in (8.1) can easily be proved by induction, since it is an  
\n= 0, and the induction step can be performed by means of (6.6) and  
\n
$$
\phi\left(\frac{t}{a}\right) = b \int_0^t \phi(\tau) d\tau \qquad (0 \le t \le a-1).
$$
\n(8.2)  
\nuation follows from  $\phi_n^{(n)}(t) = 0$  for  $t \ge 1$ , so that  $\phi_n$  for these  $t$  is

Proof. The first equation in (8.1) can easily be proved by induction, since it is an identity for  $n = 0$ , and the induction step can be performed by means of  $(6.6)$  and

$$
\phi\left(\frac{t}{a}\right) = b \int\limits_0^t \phi(\tau) d\tau \qquad (0 \le t \le a-1). \tag{8.2}
$$

The second equation follows from  $\phi_n^{(n)}(t) = 0$  for  $t \ge 1$ , so that  $\phi_n$  for these *t* is<br>
a polynomial of degree at most  $n - 1$ . This polynomial remains an eigenfunction of<br>
(6.1) to the eigenvalue  $a^n/b$  also under th a polynomial of degree at most  $n - 1$ . This polynomial remains an eigenfunction of (6.1) to the eigenvalue  $a^n/b$  also under the restriction  $t \geq 1$ , i.e. in view of (6.7)  $\phi_n(t) = k_n \psi_{n-1}(t, 1/a)$  since, as in the proof of Proposition 6.2, polynomials which are eigenfunctions of (6.1) are uniquely determined up to a constant factor. The factor is fixed by *for the action 6.2, polynomials which are*<br>  $f(t) = \int_{0}^{1} \phi(\tau) d\tau = 1$ <br> *for*  $\frac{1}{a^n} \le t \le \frac{a-1}{a^n}$  (8.3)

\n- \n
$$
-1(t, 1/a)
$$
 since, as in the proof of Proposition 6.2, polynomial,  $c$  of (6.1) are uniquely determined up to a constant factor.\n
\n- \n $k_n$  ( $n-1$ )! =  $k_n \psi_{n-1}^{(n-1)}(t, 1/a) = \phi_n^{(n-1)}(t) = \int_0^1 \phi(\tau) d\tau = 1$ \n
\n- \n (2)) \n
\n- \n**Ex** The equations of  $\phi_n$  in (8.1) imply the\n
\n- \n**Ex** For  $a \geq 2$ , the solution of (1), (2) has the representation\n
\n- \n $\phi(t) = \frac{\psi_{n-1}(a^n t, 1/a)}{(n-1)! a^{n(n-3)/2} (a-1)^n}$  for  $\frac{1}{a^n} \leq t \leq \frac{a-1}{a^n}$ \n
\n- \n**Ex** we know this already from Proposition 4.3/(ii) in view of\n
\n

for  $t \geq 1$  (cf. (2))

The two representations of  $\phi_n$  in (8.1) imply the

Corollary. For  $a \geq 2$ , the solution of  $(1), (2)$  has the representation

$$
\phi(t) = \frac{\psi_{n-1}(a^n t, 1/a)}{(n-1)! a^{n(n-3)/2} (a-1)^n} \qquad \text{for} \quad \frac{1}{a^n} \le t \le \frac{a-1}{a^n} \tag{8.3}
$$

*with*  $n \in \mathbb{N}$ .

For  $n = 1$ , we know this already from Proposition 4.3/(ii) in view of  $\psi_0(t) = 1$ . In case of  $a > 2$ , it can be shown by means of  $(8.2)$  and

$$
0 = \frac{\psi_{n-1}(a^{n}t, 1/a)}{(n-1)! a^{n(n-3)/2} (a-1)^{n}}
$$
 for  $\frac{1}{a^{n}} \le t \le \frac{a}{c}$   
the known this already from Proposition 4.3/(ii) in view  
can be shown by means of (8.2) and  

$$
\phi\left(\frac{t+a-1}{a}\right) = b \int_{t}^{1} \phi(\tau) d\tau
$$
 for  $2 - a \le t \le 1$   
minal of degree *n* in all  $2^{n}$  intervals which can be attain  
of the mappings  

$$
t \mapsto \frac{t}{a}
$$
 and 
$$
t \mapsto \frac{t+a-1}{a}
$$
  
where,  $n \in \mathbb{N}_{0}$ . The union of these intervals is an  
of the second representation in (8.1) we have

that  $\phi$  is a polynomial of degree *n* in all  $2^n$  intervals which can be attained from  $(\frac{1}{a}, 1-\frac{1}{a})$ by *n* applications of the mappings

$$
t \longmapsto \frac{t}{a}
$$
 and  $t \longmapsto \frac{t+a-1}{a}$ 

in an arbitrary order,  $n \in \mathbb{N}_0$ . The union of these intervals is an open Cantor set of Lebesgue measure 1. For  $a = 3$ , the last results are well-known from [8].

As generalization of the second representation in (8.1), we have

**Proposition 8.2.** *The eigenfunctions*  $\phi_n$  ( $n \in \mathbb{N}$ ) *of* (6.1) with  $a > 1$  possess the *property*

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\n2. The eigenfunctions 
$$
\phi_n
$$
  $(n \in \mathbb{N})$  of (6.1) with  $a > 1$  possess the  
\n
$$
\phi_n(t) - (-1)^n \phi_n(1-t) = \frac{1}{(n-1)!} \psi_{n-1}(t, 1/a)
$$
\n(8.4)

*for all*  $t \in \mathbb{R}$ .

**Proof.** First, we prove (8.4) for  $0 \le t \le 1$  by induction. For  $n = 1$  the equation is true in view of (6.7) and the symmetry property  $\phi(t) = \phi(1 - t)$  as well as (2) and  $\psi_0(t) = 1$ . According to (6.8) and Proposition 7.2/(i), we find by integration of (8.4) prove (8.4) for  $0 \le t \le 1$  by induction. For<br>
) and the symmetry property  $\phi(t) = \phi(1 - t)$ <br>
to (6.8) and Proposition 7.2/(i), we find by is<br>  $f(t) + (-1)^n \phi_{n+1}(1 - t) = \frac{1}{n!} \psi_n(t, 1/a) + c_0$  $\phi_n(t) - (-1)^n \phi_n(1-t) = \frac{1}{(n-1)!} \psi_{n-1}(t, 1/a)$  (8.4)<br>for all  $t \in \mathbb{R}$ .<br>Proof. First, we prove (8.4) for  $0 \le t \le 1$  by induction. For  $n = 1$  the equation<br>is true in view of (6.7) and the symmetry property  $\phi(t) = \phi(1-t)$  as

$$
\phi_{n+1}(t) + (-1)^n \phi_{n+1}(1-t) = \frac{1}{n!} \psi_n(t,1/a) + c_n
$$

 $\phi_{n+1}(t) + (-1)^n \phi_{n+1}(1-t) = \frac{1}{n!} \psi_n(t, 1/a) + c_n$ <br>with a certain constant  $c_n$ . For  $t = 1$  we get  $\phi_{n+1}(1) = \frac{1}{n!} \psi_n(1, 1/a) + c_n$  and, by<br>comparison with (8.1), we obtain  $c_n = 0$ . Thus equation (8.4) is proved for  $0 \le t \le$ by means of induction.  $\phi_{n+1}(t) + (-1)^n \phi_{n+1}(1-t) = \frac{1}{n!} \psi_n(t, 1/a) + c_n$ <br> **a** certain constant  $c_n$ . For  $t = 1$  we get  $\phi_{n+1}(1) = \frac{1}{n!} \psi_n(1, 1/a) + c_n$  and, by<br>
parison with (8.1), we obtain  $c_n = 0$ . Thus equation (8.4) is proved for  $0 \le t \le 1$ <br>

For  $t \ge 1$ , equation (8.4) is already known from Proposition 8.1 and  $\phi_n(1-t) = 0$ .<br>But (8.4) is also valid for  $t \le 0$ , in view of Proposition 7.2/(ii) **I** 

**Remark.** Though we have derived (8.4) by means of Proposition *7.2/(i), (ii)* and used that  $(8.1)$  is already known for  $t \geq 1$ , representation  $(8.4)$  implies conversely in view of definition  $(6.7)$  both properties (i), (ii) in Proposition 7.2, and for  $t > 1$  also the second formula in (8.1). 1, equation (8.4) is already known fr<br>also valid for  $t \le 0$ , in view of Propo<br>k. Though we have derived (8.4) by<br>8.1) is already known for  $t \ge 1$ , rep<br>nition (6.7) both properties (i), (ii) in<br>ula in (8.1).<br>**ry.** For  $a$ Proposition 8.1 and  $\phi_n(1-t)$ <br>
n 7.2/(ii) **s**<br>
ms of Proposition 7.2/(i), (ii)<br>
tation (8.4) implies converse<br>
position 7.2, and for  $t \ge 1$  also<br>
b has the property<br>  $\frac{\psi_{n-1}(1/2 + a^n \tau, 1/a)}{-1}!a^{n(n-3)/2}(a-1)^n$ 

Corollary. For  $a \geq 3/2$ , the solution  $\phi$  of  $(1)$ ,  $(2)$  has the property

$$
\begin{aligned}\n\text{Hungian (8.1)}\\
\text{llary. For } a \ge 3/2, \text{ the solution } \phi \text{ of (1), (2) has the property} \\
\phi \left( \frac{1}{2a^n} + \tau \right) + (-1)^{n-1} \phi \left( \frac{1}{2a^n} - \tau \right) &= \frac{\psi_{n-1}(1/2 + a^n \tau, 1/a)}{(n-1)! \, a^{n(n-3)/2} \, (a-1)^n} \end{aligned} \tag{8.5}
$$

*for*  $|\tau| \leq a^{-n}(a-3/2)$  *and*  $n \in \mathbb{N}$ .

This follows from (8.4) and the first equation in (8.1) with  $t = 1/2 \pm a^n \tau$ , where  $0 \le 1/2 \pm a^n \tau \le a-1$  implies that (8.5) is valid for  $|r| \le a^{-n}(a-3/2)$  and  $|r| \le 1/(2a^n)$ .<br>
But the last condition is unnecessary in view of (8.3).<br>
Generalizing the first equation in (8.1), we have for all t and  $n \in \mathbb{N}$ But the last condition is unnecessary in view of (8.3). *For*  $a \ge 3/2$ , the solution  $\phi$  of<br>  $\left(\frac{1}{2a^n} - \tau\right)$ <br>  $= 3/2$  and  $n \in \mathbb{N}$ .<br>
from (8.4) and the first equal  $a-1$  implies that (8.5) is valid<br>
dition is unnecessary in view<br>
the first equation in (8.1), we<br>  $= a^{n$ 

Generalizing the first equation in (8.1), we have for all  $t$  and  $n \in \mathbb{N}$  the representation

$$
\phi_n(t) = a^{n(n-3)/2} (a-1)^n \sum_{\nu_1, ..., \nu_n \ge 0} \phi \left( \frac{t}{a^n} - \frac{\nu_1}{a^{n-1} b} - ... - \frac{\nu_n}{b} \right), \tag{8.6}
$$

where the sum has only finitely many non-vanishing terms for fixed *t*. This equation<br>
can be proved by induction as the first equation in (8.1), if we use instead of (8.2) the<br>
relation<br>  $b \int_{0}^{t} \phi(\tau) d\tau = b \sum_{\nu \geq 0} \int$ can be proved by induction as the first equation in  $(8.1)$ , if we use instead of  $(8.2)$  the relation

$$
t) = a^{n(n-3)/2} (a-1)^n \sum_{\nu_1, \dots, \nu_n \ge 0} \phi \left( \frac{t}{a^n} - \frac{\nu_1}{a^{n-1}b} - \dots - \frac{\nu_n}{b} \right)
$$
  
\nIn has only finitely many non-vanishing terms for fixed  $t$ .  
\nby induction as the first equation in (8.1), if we use inset  
\n
$$
t - \nu(a-1)
$$
  
\n
$$
b \int_0^t \phi(\tau) d\tau = b \sum_{\nu \ge 0} \int_{t - (\nu+1)(a-1)}^t \phi(\tau) d\tau = \sum_{\nu \ge 0} \phi \left( \frac{t}{a} - \frac{\nu}{b} \right)
$$
  
\nfor  $t \ge 0$ .

which is valid for  $t > 0$ .

Representation (8.6) together with the second equation in (8.1) imply theببيتين

 $\mathcal{O}^{\bullet}$ 

 $\frac{1}{3}$  ,  $\frac{1}{3}$  ,  $\frac{1}{3}$  ,  $\frac{1}{3}$  ,  $\frac{1}{3}$ 

Corollary. For  $a > 1$ , the solution  $\phi$  of  $(1)$ ,  $(2)$  satisfies the polynomial relation

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\n
$$
\text{I. For } a > 1, \text{ the solution } \phi \text{ of } (1), (2) \text{ satisfies the polynomial relation}
$$
\n
$$
\sum_{\nu_1, \dots, \nu_n \ge 0} \phi \left( t - \frac{\nu_1}{a^{n-1}b} - \dots - \frac{\nu_n}{b} \right) = \frac{\psi_{n-1}(a^n t, 1/a)}{(n-1)! \ a^{\frac{n(n-3)}{2}} (a-1)^n} \tag{8.7}
$$
\n
$$
\text{I. } n \in \mathbb{N}.
$$

 

*for t*  $\geq 1$ *, the so*<br> $\sum_{\nu_1, ..., \nu_n \geq 0} \phi\left(t - \frac{\nu_1}{a^{n-1}t}\right)$ <br>*for t*  $\geq 1/a^n$ ,  $n \in \mathbb{N}$ .<br>For  $n = 1$ , we find in view of t<br> $a > 1$  satisfies the equation For  $n = 1$ , we find in view of  $\psi_0(t) = 1$  that the solution  $\phi$  of (1), (2) with arbitrary *a> 1* satisfies the equation

$$
\sum_{\nu=-\infty}^{+\infty} \phi\left(t-\frac{\nu}{b}\right) = b \tag{8.8}
$$

for all real *t*, which for  $1/a \le t \le 1$  follows directly from (8.7). Equation (8.8) can be used to check the orthogonality property mentioned in Remark 4 of the previous section.

The last results in this section concern some special properties of the polynomials  $\psi_n(t, 1/a)$  with  $a > 1$ .

**Proposition 8.3.** *The eigenfunctions*  $\psi_n(t,1/a)$  ( $n \in \mathbb{N}_0$ ) of (6.1) with  $a>1$  have *the representation*

$$
\sum_{\nu=-\infty}^{+\infty} \phi\left(t - \frac{\nu}{b}\right) = b \tag{8.8}
$$
\n
$$
\leq t \leq 1 \text{ follows directly from (8.7). Equation (8.8) can\nsonality property mentioned in Remark 4 of the previous\nsection concern some special properties of the polynomials\neigenfunctions  $\psi_n(t, 1/a)$  ( $n \in \mathbb{N}_0$ ) of (6.1) with  $a > 1$  have  
\n
$$
\psi_n(t, 1/a) = \int_0^1 (t - \tau)^n \phi(\tau) d\tau \tag{8.9}
$$
\nMoreover, the polynomials  $\psi_n(t, 1/a)$  ( $n \in \mathbb{N}$ ) are positive
$$

*w<sub>n</sub>*(*t*, 1/*a*) with *a* > 1.<br> **Proposition 8.3.** The eigenfunctions  $\psi_n(t, 1/a)$  ( $n \in \mathbb{N}_0$ ) of (6.1) with *a* > 1 have<br> *the representation*<br>  $\psi_n(t, 1/a) = \int_0^1 (t - \tau)^n \phi(\tau) d\tau$  (8.9)<br> *with the solution*  $\phi$  of (1) **Proposition 8.3.** The eigenfunctions  $\psi_n(t,1/a)$   $(n \in \mathbb{N}_0)$  of (6.1) with  $a > 1$  have<br>the representation<br> $\psi_n(t,1/a) = \int_0^1 (t-\tau)^n \phi(\tau) d\tau$  (8.9)<br>with the solution  $\phi$  of (1), (2). Moreover, the polynomials  $\psi_n(t,1/a)$   $(n$ **Proposition 8.3.** The eigenfunctions  $\psi_n(t,1/a)$  (<br>the representation<br> $\psi_n(t,1/a) = \int_0^1 (t-\tau)^n \phi(\tau)$ <br>with the solution  $\phi$  of (1), (2). Moreover, the polynomial<br>and strongly increasing for  $t > 1/2$ . For  $n \to +\infty$ , with<br>in

**Proof.** According to *(2.10)* and (7.1), we have

$$
e^{tz}\Phi(z)=\sum_{n=0}^{\infty}\frac{\psi_n(t,1/a)}{n!}z^n
$$

i.e.  $e^{iz}\Phi(z)$  is the generating function of the eigenfunctions  $\psi_n(t, 1/a)$ . On the other hand, we obtain

*treasing for t* > 1/2. For 
$$
n \to +\infty
$$
, we have  $\psi_n(t, 1/a)$   
\n $d |\psi_n(t, 1/a)| \to +\infty$  for each  $t \notin [0, 1]$ .  
\n $e^{tz}\Phi(z) = \sum_{n=0}^{\infty} \frac{\psi_n(t, 1/a)}{n!} z^n$ ,  
\nthe generating function of the eigenfunctions  $\psi_n(t, 1/a)$   
\n $e^{tz}\Phi(z) = \int_0^1 e^{(t-\tau)z} \phi(\tau) d\tau = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^1 (t-\tau)^n \phi(\tau) d\tau$   
\nson of coefficients yields the representation (8.9).  
\n>0 for  $0 < t < 1$ , equation (8.9) implies that for  $\epsilon$ 

hence, comparison of coefficients yields the representation (8.9).

Since  $\phi(t) > 0$  for  $0 < t < 1$ , equation (8.9) implies that for even *n*, we have  $\psi_n(t, 1/a) > 0$  for all *t.* In view of  $\psi_0(t) = 1$  and  $\psi_1(t) = t - 1/2$ , Proposition 7.2/(i) and (ii) yield the assertions concerning the monotony.  $f^n \phi(\tau) d\tau$ <br> *(.9)*.<br>
that for even *n*, w<br>  $-1/2$ , Proposition<br>  $f^n \phi(\tau) d\tau$ 

For  $0 \le t \le 1$ , we get from

$$
|\psi_n(t,1/a)|=\int\limits_0^t(t-\tau)^n\phi(\tau)\,d\tau+\int\limits_t^1(\tau-t)^n\phi(\tau)\,d\tau
$$

 $\label{eq:2} \begin{split} \mathcal{L}^{(1,0)}(t) = \frac{1}{2\pi\,t} \, . \end{split}$ 

in view of  $0 \leq \phi(t) \leq b$  (cf. Section 4) the estimate

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\nin view of 
$$
0 \le \phi(t) \le b
$$
 (cf. Section 4) the estimate  
\n
$$
|\psi_n(t,1/a)| \le \frac{t^{n+1} + (1-t)^{n+1}}{n+1}b \qquad (0 \le t \le 1)
$$
\nwith  $b = a/(a-1)$ . Hence,  $\psi_n(t,1/a)$  converges to 0 uniformly in [0,1]. Now let  $1 < t < 2$ , i.e.  $t = 1 + c$  with  $0 < c < 1$ . From (8.9), we obtain in view of the monotony

*b*  $(0 \le t \le 1)$  (8.10)<br>to 0 uniformly in [0,1]. Now let<br>we obtain in view of the monotony  $1 < t < 2$ , i.e.  $t = 1 + c$  with  $0 < c < 1$ . From  $(8.9)$ , we obtain in view of the monotony of  $\phi(t)$  for  $t \in [0, 1/2]$  that

$$
\leq \varphi(t) \leq b \text{ (cf. Section 4) the estimate}
$$
\n
$$
|\psi_n(t,1/a)| \leq \frac{t^{n+1} + (1-t)^{n+1}}{n+1}b \qquad (0 \leq t \leq 1)
$$
\n
$$
|(a-1). \text{ Hence, } \psi_n(t,1/a) \text{ converges to 0 uniformly in } [0,1/2] \text{ that}
$$
\n
$$
\text{Let } \psi_n \text{ is the probability of } \frac{1}{a} \text{ and }
$$

i.e.  $\psi_n(1+c) \to +\infty$  for  $n \to +\infty$ . This implies in view of the monotony of  $\psi_n(t, 1/a)$ for  $t > 1/2$  and Proposition 7.2/(ii) the divergence  $|\psi_n(t, 1/a)| \rightarrow +\infty$  for  $t \notin [0, 1]$ 

According to  $\psi_n(t,1) = (t - 1/2)^n$  and  $\phi(\tau,1) = \delta(\tau-1/2)$ , formula (8.9) is true also for  $a = 1$ , but not the assertion concerning the divergence for  $n \to +\infty$ . For  $t = 0$ , formula (8.9) reduced to (2.15).

## **9. The truncated equation**

It is useful to compare the solutions of the differentiated equation (6.1)  

$$
\lambda \phi'(t) = a(\phi(at) - \phi(at - a + 1))
$$
(9.1)

with the solutions of the truncated equation

$$
\lambda g'(t) = a g(at) \qquad (a > 1 , \lambda > 0). \tag{9.2}
$$

It is useful to compare the solutions of the differentiated equation (6.1)<br>  $\lambda \phi'(t) = a(\phi(at) - \phi(at - a + 1))$  (9.1)<br>
with the solutions of the truncated equation<br>  $\lambda g'(t) = a g(at)$  (a > 1,  $\lambda$  > 0). (9.2)<br>
We look for solutions with formation of (9.2) yields  $\lambda p G(p) = G(\frac{p}{q})$  with the general solution

**ncated equation**  
\ncompare the solutions of the differentiated equation (6.1)  
\n
$$
\lambda \phi'(t) = a(\phi(at) - \phi(at - a + 1))
$$
\n(9.1)  
\nons of the truncated equation  
\n
$$
\lambda g'(t) = a g(at) \quad (a > 1, \lambda > 0).
$$
\n(9.2)  
\nlutions with  $\text{supp} \subset [0, \infty)$ , which possess a Laplace transform *G*. Trans-  
\n9.2) yields  $\lambda p G(p) = G(\frac{p}{a})$  with the general solution  
\n
$$
G(p) = G_0(p) Q\left(\frac{\ln p}{\ln a}\right), \qquad G_0(p) = p^{\alpha} \exp(-\beta \ln^2 p),
$$
\n(9.3)  
\n
$$
\alpha = -\frac{1}{2} - \frac{\ln \lambda}{\ln a}, \qquad \beta = \frac{1}{2 \ln a}
$$
\n(9.4)  
\nis an arbitrary 1-periodic function. It can easily be seen that  $G_0$  is a  
\nform with the original function  
\n
$$
G(t) = \frac{1}{2} \log \frac{1}{\ln a}, \qquad \beta = \frac{1}{2 \ln a}
$$
\n(9.5)

where

$$
\alpha = -\frac{1}{2} - \frac{\ln \lambda}{\ln a}, \qquad \beta = \frac{1}{2 \ln a} \tag{9.4}
$$

and where  $Q$  is an arbitrary 1-periodic function. It can easily be seen that  $G_0$  is a Laplace transform with the original function

and where 
$$
Q
$$
 is an arbitrary 1-periodic function. It can easily be seen that  $G_0$  is a Laplace transform with the original function  
\n
$$
g_0(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(pt - \beta \ln^2 p) p^{\alpha} dp , \qquad (9.5)
$$
\n
$$
c > 0, g(t) = 0 \text{ for } t \leq 0, \text{ and that further solutions of (9.2) for } t \geq 0 \text{ are}
$$
\n
$$
g(t) = \int_0^t g'_0(t - \tau) P\left(\frac{\ln \tau}{\ln a}\right) d\tau
$$
\nwith an arbitrary (and for simplicity) continuous function  $P(t) = P(t + 1)$ .

$$
g(t) = \int_{0}^{t} g'_0(t-\tau) P\left(\frac{\ln \tau}{\ln a}\right) d\tau
$$

with an arbitrary (and for simplicity) continuous function  $P(t) = P(t + 1)$ .

Next, we want to study the asymptotic behaviour of  $g_0(t)$  for  $t \to +0$ .

**Proposition 9.1.** For  $t \to +0$ , the function (9.5) has the asymptotic behaviour

On the Solution of an Integral-Functional Equation 179  
position 9.1. For 
$$
t \to +0
$$
, the function (9.5) has the asymptotic behaviour  

$$
g_0(t) \sim \frac{(2\beta)^{\epsilon}}{\sqrt{2\pi}} t^{\gamma} (-\ln t)^{\delta} \exp\left(-\beta \ln^2\left(\frac{t}{-\ln t}\right)\right)
$$
(9.6)
$$
= -2\beta - \delta - \frac{1}{2} , \qquad \delta = \frac{1}{2} + \alpha - 2\beta \ln(2\beta) , \qquad \epsilon = \frac{1}{2} + \alpha - \beta \ln(2\beta) .
$$
of. According to L. Berg [1: Theorem 47.3], we have

*with*

$$
\gamma = -2\beta - \delta - \frac{1}{2} , \qquad \delta = \frac{1}{2} + \alpha - 2\beta \ln(2\beta) , \qquad \varepsilon = \frac{1}{2} + \alpha - \beta \ln(2\beta) .
$$

Proposition 9.1. For 
$$
t \to +0
$$
, the function (9.5) has the as  
\n
$$
g_0(t) \sim \frac{(2\beta)^{\epsilon}}{\sqrt{2\pi}} t^{\gamma} (-\ln t)^{\delta} \exp\left(-\beta \ln^2 \left(\frac{t}{-\ln t}\right)\right)
$$
\n
$$
\gamma = -2\beta - \delta - \frac{1}{2}, \qquad \delta = \frac{1}{2} + \alpha - 2\beta \ln(2\beta), \qquad \varepsilon = \frac{1}{2}.
$$
\nProof. According to L. Berg [1: Theorem 47.3], we have  
\n
$$
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[pt + f(p)] \, dp \sim \frac{\exp[xt + f(x)]}{\sqrt{2\pi}f''(x)}
$$
\nt \to +0, if  $x = x(t) \to \infty$  with  $t + f'(x) = o(\sqrt{f''(x)})$ ,  
\nmatrices are satisfied. Since the check of these assumptions

for  $t \to +0$ , if  $x = x(t) \to \infty$  with  $t + f'(x) = o(\sqrt{f''(x)})$ , and if some further assumptions are satisfied. Since the check of these assumptions is only a question of routine, we drop it and restrict ourselves to the necessary f assumptions are satisfied. Since the check of these assumptions is only a question of routine, we drop it and restrict ourselves to the necessary formal calculations. In view of

$$
f(p) = \alpha \ln p - \beta \ln^2 p , \quad f'(p) = \frac{1}{p} (\alpha - 2\beta \ln p) , \quad f''(p) = \frac{1}{p^2} (2\beta \ln p + 2\beta - \alpha)
$$

we choose  $x = -(2\beta/t) \ln t$ , so that  $\ln x \sim - \ln t$  for  $t \to +0$  and therefore  $f''(x)$  $t^2/(-2\beta \ln t)$ . Hence, the condition<br> $t + f'(x) = \frac{t}{\sqrt{2\pi}}$ 

$$
\beta \ln^2 p , \quad f'(p) = \frac{1}{p} (\alpha - 2\beta \ln p) , \quad f''(p) = \frac{1}{p^2} (2\beta)
$$
  

$$
(2\beta/t) \ln t, \text{ so that } \ln x \sim -\ln t \text{ for } t \to +0 \text{ and } t
$$
  
ace, the condition  

$$
t + f'(x) = \frac{t}{\ln t} \left[ \ln(-2\beta \ln t) - \frac{\alpha}{2\beta} \right] = o\left(\frac{t}{\sqrt{-\ln t}}\right)
$$
  
u view of

is satisfied, and in view of

choose 
$$
x = -(2\beta/t) \ln t
$$
, so that  $\ln x \sim -\ln t$  for  $t \to +0$  and therefore  $f''(x - 2\beta \ln t)$ . Hence, the condition  
\n
$$
t + f'(x) = \frac{t}{\ln t} \left[ \ln(-2\beta \ln t) - \frac{\alpha}{2\beta} \right] = o\left(\frac{t}{\sqrt{-\ln t}}\right)
$$
\n
$$
\text{t isfied, and in view of}
$$
\n
$$
xt + f(x) = -2\beta \ln t + \alpha \ln \left(\frac{-2\beta \ln t}{t}\right) - \beta \ln^2 \left(\frac{-2\beta \ln t}{t}\right)
$$
\n
$$
= -\beta \ln^2 \left(\frac{-t}{\ln t}\right) + (\gamma + 1) \ln t + \left(\delta - \frac{1}{2}\right) \ln(-\ln t) + \left(\varepsilon - \frac{1}{2}\right) \ln(2\beta)
$$
\n
$$
\sqrt{f''(x)} \sim t/\sqrt{-2\beta \ln t}, \text{ we obtain (9.6) after elementary calculations.}
$$

and  $\sqrt{f''(x)} \sim t/\sqrt{-2\beta \ln t}$ , we obtain (9.6) after elementary calculations **I** 

It can directly be checked that the right-hand side of (9.6) satisfies equation (9.2) asymptotically for  $t \rightarrow +0$ .

The results about the solutions of (9.2) can be used to obtain new informations concerning the solutions of (9.1). It is clear that all solutions of (9.2) with  $g(t) = 0$ for  $t \to +0$ .<br>The results about the solutions of (9.2) can be used to obtain new informations<br>concerning the solutions of (9.1). It is clear that all solutions of (9.2) with  $g(t) = 0$ <br>for  $t \le 0$  are  $C^{\infty}$ -functions, the The results about the solutions of (9.2) can be used to obtain new informations<br>concerning the solutions of (9.1). It is clear that all solutions of (9.2) with  $g(t) = 0$ <br>for  $t \le 0$  are  $C^{\infty}$ -functions, they satisfy  $g^{$ also (9.1) for  $t \le 1 - 1/a$ . Obviously, all these solutions of (9.2) can be continued to solutions of (9.1), defining  $\phi(t) = g(t)$  for  $t \le a - 1$  and  $\phi(t) = \phi(t - a + 1) + \frac{\lambda}{a} \phi'(\frac{t}{a})$ for  $t > a - 1$ . Since all these solutions are also solutions of (6.1) and in case of  $\lambda = 0$ also all  $(a - 1)$ -periodic integrable functions with average zero, we have proved the

**Proposition 9.2.** *Considering integral equation (6.1),* a *>* 1, *in the non-compact case, all*  $\lambda \geq 0$  *are multiple eigenvalues of it.* 

Finally, we return to equations  $(1)$ ,  $(2)$  and the Laplace transform  $(1.2)$  of the solution. This corresponds to the special case  $\lambda = 1/b$  in (9.1), so that  $\alpha = 1/2$  - $(\ln(a-1))/a$  in (9.4).

**Proposition 9.3.** *For positive p, we have the representation* 

Using integral equation (6.1), 
$$
a > 1
$$
, in the non-compact  
\ngenvvalues of it.

\nations (1), (2) and the Laplace transform (1.2) of the  
\nthe special case  $\lambda = 1/b$  in (9.1), so that  $\alpha = 1/2 -$ 

\nstitute *p*, we have the representation

\n
$$
Φ(p) = \frac{G(p)}{\prod_{k=1}^{\infty} (1 - e^{-pa^k/b})}
$$
\n(9.7)

\nfunction *Q* there and  $\alpha = 1/b$  in (9.4).

\nion 2, we write equation (1.1) for the (one-sided) Laplace

\n
$$
Φ(p) = \frac{ap}{b} \frac{Φ(ap)}{(1 - e^{-pa/b})}
$$
\n(9.8)

\n(b in (9.4) is the general solution of  $G(p) = \frac{ap}{b} G(ap)$ , the

with G from (9.3), a suitable function Q there and  $\alpha = 1/b$  in (9.4).

 $\mathcal{L}_{\text{max}}$  and  $\mathcal{L}_{\text{max}}$ 

**Proof.** As already in Section 2, we write equation (1.1) for the (one-sided) Laplace transform  $\Phi$  of  $\phi$  in the form

$$
\Phi(p) = \frac{ap}{b} \frac{\Phi(ap)}{(1 - e^{-pa/b})} \tag{9.8}
$$

Since G from (9.3) with  $\lambda = 1/b$  in (9.4) is the general solution of  $G(p) = \frac{ap}{b} G(ap)$ , the general solution of (9.8) reads (9.7), if we do not specify *Q.* Hence, also (1.2) has the form (9.7) with a special 1-periodic function *Q* in (9.3) which can be represented as

$$
Q(t) = a^{\frac{1}{2}t^2 - \alpha t} \prod_{k=0}^{\infty} \frac{1 - e^{-a^{t-k}/b}}{a^{t-k}/b} \prod_{l=1}^{\infty} (1 - e^{-a^{t+l}/b})
$$

and the statement is proved  $\blacksquare$ 

If we introduce the functions

$$
\Gamma_n(p) = \frac{G(p)}{\prod_{k=1}^n (1 - e^{-pa^k/b})}
$$

 $\Gamma_n(p) = \frac{G(p)}{\prod_{k=1}^n (1 - e^{-pa^k/b})}$ <br>with  $\Gamma_0(p) = G(p)$ , then for  $n \in \mathbb{N}$  we have  $\Gamma_n(p) = \frac{\Gamma_{n-1}(p)}{1 - e^{-pa^n/b}}$  and the corresponding<br>original functions original functions  $\vec{G}(p)$ <br>  $=\frac{\vec{\Pi}}{\prod_{k=1}^{n} (1 - e^{-pa^k/b})}$ <br>
we have  $\Gamma_n(p) = \frac{\Gamma_{n-1}(p)}{1 - e^{-pa^k/b}}}$  and the corr<br>  $=\sum_{k=0}^{\infty} \gamma_{n-1} \left(t - \frac{ka^n}{b}\right)$ <br>
ter finitely many steps.<br>  $\vec{F}$  we find from (9.7) that  $\Phi(n) \propto G(n)$ .

$$
\gamma_n(t) = \sum_{k=0}^{\infty} \gamma_{n-1} \left( t - \frac{ka^n}{b} \right)
$$

converge to  $\phi(t)$  for every fixed *t* after finitely many steps.

For  $\Re p \to \infty$  with  $|\arg p| \leq \vartheta < \frac{\pi}{2}$  we find from (9.7) that  $\Phi(p) \sim G(p)$  with (9.3). Hence, we expect that  $\phi$  has an asymptotic behavior as  $g_0$  in (9.6) up to a factor which is both bounded and bounded away from zero.

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