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Hausdorff and Fractal Dimension Estimates for Invariant Sets of Non-Injective Maps

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Abstract. In this paper we are concerned with upper bounds for the Hausdorff and fractal dimensions of negatively invariant sets of maps on Riemannian manifolds. We consider a special class of non-injective maps, for which we introduce a factor describing the "degree of non-injectivity". This factor can be included in the Hausdorff dimension estimates of Douady-Oesterlé type [2, 7, 10] and in fractal dimension estimates [5, 13, 15] in order to weaken the condition to the singular values of the tangent map. In a number of cases we get better upper dimension estimates.

Keywords: *Hausdorff dimension estimates, fractal dimension estimates, non-injective maps, tangent map, singular values*

AMS subject classification: Primary 58 F 12, secondary 58 F 08

1. Introduction

In [2] Hausdorff dimension estimates for compact sets $K \subset \mathbb{R}^n$ that are invariant under C^1-maps φ are given. The main idea consists in showing that for a number $j \in \mathbb{N}$ the Hausdorff outer measure of $\varphi^j(K)$ is by a certain factor smaller than the outer measure of K, i.e. the iterated map is contracting with respect to the Hausdorff outer measure on K. The contraction constant can be estimated by means of a singular-value function of the tangent map, i.e. if the singular-value function is less than 1, then the map is contracting. In [7, 8] the condition for the contraction of the Hausdorff outer measure in $\mathbb{R}^n$ is weakened using Lyapunov-type functions. The latter results are generalized in [10] to maps on Riemannian manifolds (see also [6]). Using a technique similar to that of Douady and Oesterlé , Temam gave in [13] (see also [14]) upper bounds for the Hausdorff and fractal dimensions of semiflow invariant sets in a Hilbert space. Analogously fractal dimension estimates are derived in [5] for semiflows on Riemannian manifolds.

In practice the maps describing concrete physical or technical systems are often non-injective (see, for instance, [1]). For such non-injective maps it may be possible to use

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information about the "degree of non-injectivity" in order to get Hausdorff and fractal outer measure and dimension estimates under weakened conditions compared with the theorems mentioned above. For the first time such Douady-Oesterlé-type Hausdorff dimension estimates using the "degree of non-injectivity" are considered in [9]. There a class of $k-1$-endomorphisms is described, where the given invariant set can be split into k compact subsets and where each of those subsets is mapped onto the whole invariant set. The factor $\frac{1}{k}$ can be used to compensate the missing contraction property for the Hausdorff outer measure.

In the present paper we consider a class of maps satisfying even a weaker non-injectivity condition than the $k-1$-property. In general, such a class may be described as follows. Let φ be a C^1-map on a smooth (for simplicity C^∞) n-dimensional Riemannian manifold (M, g) and $K \subset M$ a compact set. (A class of maps that are only piecewise C^1 is considered in [11]. For these maps many results of this paper are also true.) Suppose that for a given outer measure $m(\cdot, d)$ on M (d-dimensional Hausdorff or fractal outer measure of the given set or of a covering class of this set) there exist a number $0 < a < 1$ and a family $\{K_j\}_{j \geq j_0}$ of subsets of K such that $m(\varphi^j(K_j), d) = m(\varphi^j(K), d)$ and $m(K_j, d) \leq a^j m(K, d)$ for all $j \geq j_0$. A map φ with such properties can be considered as piecewise $m(\cdot, d)$-expansive on K ($\frac{1}{a}$ is the expansion parameter and also describes the "degree of non-injectivity"). It follows that for such a map and any set $A \subset K$ there exists a $j \geq j_0$ such that $m(K_j, d) \leq a^j m(K, d) \leq m(A, d)$ and $m(\varphi^j(K_j), d) = m(\varphi^j(K), d)$, i.e. the semidynamical system $\{\varphi^j\}_{j \geq 0}$ generated by a piecewise $m(\cdot, d)$-expansive map has a certain transitive Markov-type property on K. It will be shown that for negatively invariant sets K of piecewise $m(\cdot, d)$-expansive maps, where $m(\cdot, d)$ is the d-dimensional Hausdorff or fractal outer measure, the parameter d is an upper bound of the associated dimension.

2. Hausdorff dimension estimates

First let us recall some notations of the Hausdorff outer measure and Hausdorff dimension for compact subsets K of a metric space (X, ϱ). Let $\varepsilon > 0$ and $d \geq 0$ be arbitrary real numbers. For a fixed cover $\{B_{r_i}\}_{i \in I}$ of K by a finite number of balls of radii r_i the value $\sum_{i \in I} r_i^d$ is the d-dimensional Hausdorff outer measure of the cover. Considering all possible finite covers of K by balls of radii at most ε we get the d-dimensional Hausdorff outer measure for the covering class of K by balls of radii at most ε by

$$\mu_H(K, d, \varepsilon) = \inf \textstyle\sum_i r_i^d.$$

For fixed d and K the d-dimensional Hausdorff outer measure of K is the limit of the monotone decreasing function $\mu_H(K, d, \cdot)$

$$\mu_H(K, d) = \lim_{\varepsilon \to 0+0} \mu_H(K, d, \varepsilon).$$

For every compact set K the uniquely defined critical number $d^* \geq 0$ with

$$\mu_H(K, d) = \begin{cases} \infty & \text{for } 0 \leq d < d^* \\ 0 & \text{for } d > d^* \end{cases}$$

is the Hausdorff dimension of K and is denoted by $\dim_H(K)$.

To describe how a ball in an Euclidean space is transformed under a linear map, the singular values of this map are used. Let E and E' be two n-dimensional Euclidean spaces with the scalar products $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_{E'}$, respectively, and let $L : E \to E'$ be a linear map. By L^* we denote the adjoint operator, i.e. the unique linear operator $L^* : E' \to E$ satisfying $\langle L^*u, v\rangle_E = \langle u, Lv\rangle_{E'}$ for all $u \in E'$ and $v \in E$. The singular values $\alpha_1(L) \geq \ldots \geq \alpha_n(L)$ of L are defined to be the eigenvalues of the positive semidefinite operator $\sqrt{L^*L}$. They are ordered with respect to their size and algebraic multiplicity. For an arbitrary integer $k \in \{0, 1, \ldots, n\}$ we define

$$\omega_k(L) = \begin{cases} \alpha_1(L) \cdots \alpha_k(L) & \text{for } k > 0 \\ 1 & \text{for } k = 0. \end{cases}$$

Furthermore, for an arbitrary number $d \in (0, n]$, further on written in the form $d = d_0 + s$ with $d_0 \in \{0, \ldots, n-1\}$ and $s \in (0, 1]$, we define a singular-value function by

$$\omega_d(L) = \omega_{d_0}^{1-s}(L)\,\omega_{d_0+1}^{s}(L).$$

For two linear maps $L : E \to E'$ and $L' : E' \to E''$ between n-dimensional Euclidean spaces there holds the relation (see [6, 10])

$$\omega_d(LL') \leq \omega_d(L')\,\omega_d(L) \qquad \text{for } d \in [0, n]. \tag{1}$$

Remark 1. If we restrict ourselves to the case $\alpha_i(L) > 0 \quad (i = 1, \ldots, n)$, then the geometrical interpretation of the singular values is as follows. Let $u_1, \ldots, u_n$ be an orthonormal basis of E such that u_i is an eigenvector of $\sqrt{L^*L}$ corresponding to the eigenvalue $\alpha_i(L) \quad (i = 1, \ldots, n)$. Then there exists an orthonormal basis $v_1, \ldots, v_n$ in E' such that $v_i = \frac{1}{\alpha_i(L)} L u_i$ for any $i = 1, \ldots, n$. The image of the unit ball

$$B_1(0) = \left\{ a_1 u_1 + \ldots + a_n u_n \in E \,\middle|\, (a_1, \ldots, a_n) \in \mathbb{R}^n, a_1^2 + \ldots + a_n^2 \leq 1 \right\}$$

in E under the map L is the set

$$\left\{ \sum_{1 \leq i \leq n} b_i v_i \in E' \,\middle|\, (b_1, \ldots, b_n) \in \mathbb{R}^n, \sum_{1 \leq i \leq n} \left(\frac{b_i}{\alpha_i(L)} \right)^2 \leq 1 \right\},$$

i.e. an ellipsoid in E' where the lengths of the semiaxes are $\alpha_1(L), \ldots, \alpha_n(L)$, respectively.

Therefore a similar concept for ellipsoids is introduced. Let $\mathcal{E}$ be an ellipsoid in an n-dimensional Euclidean space E, and let $\alpha_1(\mathcal{E}) \geq \ldots \geq \alpha_n(\mathcal{E}) \geq 0$ denote the lengths of its semiaxes (not necessarily positive). This means there exists an orthonormal basis $u_1, \ldots, u_n$ in E such that

$$\mathcal{E} = \left\{ \sum_{1 \leq i \leq n, \alpha_i(\mathcal{E}) \neq 0} a_i u_i \in E \,\middle|\, a_i \in \mathbb{R}, \sum_{1 \leq i \leq n, \alpha_i(\mathcal{E}) \neq 0} \left(\frac{a_i}{\alpha_i(\mathcal{E})} \right)^2 \leq 1 \right\}.$$

For an arbitrary integer $k \in \{0, 1, \ldots, n\}$ we define

$$\omega_k(\mathcal{E}) = \begin{cases} \alpha_1(\mathcal{E}) \cdots \alpha_k(\mathcal{E}) & \text{for } k > 0 \\ 1 & \text{for } k = 0 \end{cases}$$

and for an arbitrary number $d \in (0, n]$, $d = d_0 + s$ with $d_0 \in \{0, \ldots, n-1\}$ and $s \in (0, 1]$ we denote the d-dimensional ellipsoid volume by

$$\omega_d(\mathcal{E}) = \omega_{d_0}^{1-s}(\mathcal{E})\, \omega_{d_0+1}^{s}(\mathcal{E}). \tag{2}$$

Now let (M, g) be a smooth n-dimensional Riemannian manifold, $U \subset M$ an open set, $\varphi : U \to M$ a C^1-map and let $d_u\varphi$ denote the tangent map of φ at the point $u \in U$. Then for an arbitrary set $K \subset U$ and a number $d \in [0, n]$ we define

$$\omega_{d,K}(\varphi) = \sup_{u \in K} (\omega_d(d_u\varphi)) \tag{3}$$

which is an upper bound for the growth rate of the d-dimensional ellipsoid volume of small balls in the tangent bundle over K under the tangent map.

The following theorem describes the asymptotic behaviour of the d-dimensional Hausdorff outer measure of a compact set under a piecewise $\mu_H(\cdot, d)$-expansive C^1-map.

Theorem 1. *Let (M, g) be a smooth n-dimensional Riemannian manifold, $U \subset M$ be an open set and $\varphi : U \to M$ be a C^1-map. Suppose K and $\widetilde{K}$ are compact sets satisfying the relations $K \subset \widetilde{K} \subset U$ and $\varphi^j(K) \subset \widetilde{K}$ for any $j \in \mathbb{N}$. Suppose that for some numbers $d \in (0, n]$ and $a > 0$ the following conditions are satisfied:*

1) $\omega_{d,\widetilde{K}}(\varphi) < \frac{1}{a}$.

2) *There is a number $j_0 \in \mathbb{N}$ such that for any natural number $j \geq j_0$ there exists a set $K_j \subset K$ such that $\mu_H(\varphi^j(K_j), d) = \mu_H(\varphi^j(K), d)$ and $\mu_H(K_j, d) \leq a^j \mu_H(K, d)$.*

3) $\mu_H(K, d) < \infty$.

Then $\lim_{j \to \infty} \mu_H(\varphi^j(K), d) = 0$.

Proof. It follows from (1) that the singular value function satisfies the relation

$$\omega_{d,K}(\varphi^j) \leq \omega^j_{d,\widetilde{K}}(\varphi)$$

for any $j \in \mathbb{N}$. Further, for any number $\delta > 0$ using condition 1) we find a number $j_\delta > j_0$ so that for d written as $d = d_0 + s$ with $d_0 \in \{0, \ldots, n-1\}$ and $s \in (0, 1]$ the inequality

$$2^{d_0}(d_0 + 1)^{\frac{d}{2}} (\omega_{d,\widetilde{K}}(\varphi) \cdot a)^j \leq \delta$$

will be true for any $j > j_\delta$. Using additionally condition 2) and [10: Lemma 2.3] we get for any $j > j_\delta$ the relations

$$\begin{aligned} \mu_H(\varphi^j(K), d) &= \mu_H\left(\varphi^j(K_j), d\right) \\ &\leq 2^{d_0}(d_0 + 1)^{\frac{d}{2}} \omega^j_{d,\widetilde{K}}(\varphi) \mu_H(K_j, d) \\ &\leq 2^{d_0}(d_0 + 1)^{\frac{d}{2}} (\omega_{d,\widetilde{K}}(\varphi) \cdot a)^j \mu_H(K, d) \\ &\leq \delta \mu_H(K, d). \end{aligned}$$

Since δ can be chosen arbitrarily small, by condition 3) we get $\lim_{j \to \infty} \mu_H(\varphi^j(K), d) = 0$ ∎

Corollary 1. *If the conditions* 1) *and* 2) *of Theorem* 1 *are satisfied for certain numbers* $a > 0$ *and* $d \in (0, n]$ *and furthermore* $\varphi(K) \supset K$ *holds, then either* $\mu_H(K, d) = 0$, *or* $\mu_H(K, d) = \infty$.

Corollary 2. *Let the conditions* 2) *and* 3) *of Theorem* 1 *be satisfied for certain numbers* $a > 0$ *and* $d \in (0, n]$. *Furthermore, let* $p : \widetilde{K} \to \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ *be a continuous function such that the condition*

$$\sup_{u \in \widetilde{K}} \left(\frac{p(\varphi(u))}{p(u)} \omega_d(d_u \varphi) \right) < \frac{1}{a} \tag{4}$$

is satisfied. Then $\lim_{j \to \infty} \mu_H(\varphi^j(K), d) = 0$.

Proof. The proof is based on a technique considering p as Lyapunov-type function, similarly to [8] for $M = \mathbb{R}^n$. It follows from condition (4) that there is a number $0 < \nu < 1$ with $a \frac{p(\varphi(u))}{p(u)} \omega_d(d_u \varphi) < \nu$ for any $u \in \widetilde{K}$. Therefore by the chain rule and by applying (1) we get for any $u \in \widetilde{K}$ and arbitrary $j \in \mathbb{N}$

$$\begin{aligned} a^j \omega_d(d_u \varphi^j) &\le a^j \omega_d(d_{\varphi^{j-1}(u)} \varphi) \cdots \omega_d(d_u \varphi) \\ &\le a^j \frac{\nu}{a} \frac{p(\varphi^{j-1}(u))}{p(\varphi^j(u))} \cdots \frac{\nu}{a} \frac{p(u)}{p(\varphi(u))} \\ &= \nu^j \frac{p(u)}{p(\varphi^j(u))} \\ &\le \nu^j \frac{\sup_{u \in \widetilde{K}} p(u)}{\inf_{u \in \widetilde{K}} p(u)}. \end{aligned}$$

For any $\delta > 0$ we find a number $j_\delta > j_0$ such that for d ($d = d_0 + s$ with $d_0 \in \{0, \ldots, n-1\}$ and $s \in (0, 1]$) the relation

$$2^{d_0}(d_0 + 1)^{\frac{d}{2}} a^j \omega_{d, \widetilde{K}}(\varphi^j) \le \delta$$

will be true for any $j > j_\delta$. For these numbers j we get, similarly as in the proof of Theorem 1,

$$\begin{aligned} \mu_H(\varphi^j(K), d) &= \mu_H\left(\varphi^j(K_j), d\right) \\ &\le 2^{d_0}(d_0 + 1)^{\frac{d}{2}} \omega_{d, \widetilde{K}}(\varphi^j) \mu_H(K_j, d) \\ &\le 2^{d_0}(d_0 + 1)^{\frac{d}{2}} \omega_{d, \widetilde{K}}(\varphi^j) a^j \mu_H(K, d) \\ &\le \delta \mu_H(K, d) \end{aligned}$$

and therefore, $\lim_{j \to \infty} \mu_H(\varphi^j(K), d) = 0$ ∎

Example 1. Consider a class of modified horseshoe maps φ defined on the unit n-cube $Q = [0, 1] \times \ldots \times [0, 1] \subset \mathbb{R}^n$. The map φ contracts Q with a factor $\alpha < \frac{1}{2}$ in the first $(n-1)$ coordinate directions $x_1, \ldots, x_{n-1}$ and stretches it with the factor $\beta_1 > 1$ if $x_n < h$ and with the factor $\beta_2 > \beta_1$ if $x_n > h$ $(0 < h < 1)$ in the remaining direction x_n. Then the resulting parallelepiped is folded along the hyperplane $H =$

$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = h\}$ seperating the regions with different stretching factors, and finally it is formed to an n-dimensional horseshoe. Figure 1 illustrates such a map in the two-dimensional case.

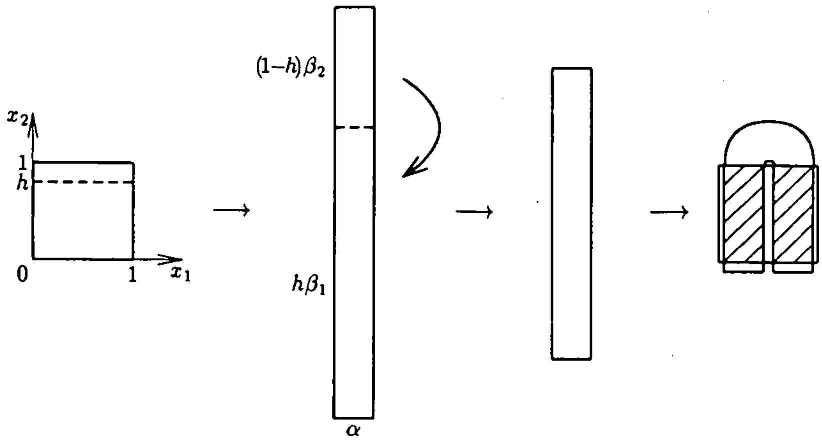

Figure 1: Modified horseshoe map

Further we assume that the map can be extended continuously differentiable to an open neighbourhood U of $K = \bigcap_{i=-\infty}^{\infty} \varphi^i(Q)$, where $\varphi^i(\cdot)$ for negative numbers i means the preimage under the map φ^{-i}. Let us consider the case $\alpha^{n-1}\beta_2 \geq 1$, where the Theorem of Douady and Oesterlé (see [2]) is not applicable.

The set K is invariant under the map. For $\widetilde{K} = K$ the condition $\varphi^j(K) \subset \widetilde{K}$ is satisfied for each $j \in \mathbb{N}$. Choose K_1 as that part of K where the stretching factor of φ is β_1. Then $\varphi(K_1) = K$. Iteratively define $K_j = K_{j-1} \cap \varphi^{-1}(K_{j-1})$ $(j \geq 2)$. By induction there is $\varphi(K_j) = K_{j-1}$ and therefore, $\varphi^j(K_j) = K = \varphi^j(K)$. That means the first part of condition 2) of Theorem 1 holds. Furthermore, in the two-dimensional case illustrated in Figure 1 it can be shown that K_j consists of 4^j linear copies of K obtained by horizontal contraction with factor α^j and vertical contraction with factor $(\frac{1}{\beta_1})^j$. The 4^j parts of K_j are disjoint and compact, and therefore have a certain distance $2\varepsilon_j > 0$. In a cover of K by balls of radius smaller than ε_j every ball can contain points of only one of the 4^j parts of K_j. Therefore, taking $\varepsilon < \varepsilon_j$ and $d \in [0, 2]$ we have

$$\mu_H(K_j, d, \varepsilon) = (4\gamma^d)^j \mu_H(K, d, \gamma^{-j}\varepsilon), \tag{5}$$

where $\gamma = \min\{\alpha, \frac{1}{\beta_1}\}$. Passing to the limit for $\varepsilon \to 0+0$ we get the second part of condition 2) of Theorem 1 with $a = 4\gamma^d$.

It remains to check condition 1). Because of $\alpha\beta_2 \geq 1$ we can restrict ourselves to $d \in [1, 2]$. Now the singular-value function has the form $\omega_{d,K}(\varphi) = \beta_2\alpha^{d-1}$, thus we are looking for a number $d \in [1, 2]$ satisfying

$$4\gamma^d\beta_2\alpha^{d-1} < 1. \tag{6}$$

This is equivalent to $d > \frac{\ln\alpha - \ln 4\beta_2}{\ln\alpha + \ln\gamma}$. For instance, for the parameters $\alpha = \frac{1}{3}$, $\beta_1 = 3$, $\beta_2 = 5$ we get $d > \frac{1}{2}\left(\frac{\ln 20}{\ln 3} + 1\right) \approx 1.863$. For such numbers d the conditions 1) and 2) of Theorem 1 are satisfied, and Corollary 1 yields $\mu_H(K,d) = 0$ or $\mu_H(K,d) = \infty$ ∎

Now we want to use the same method as above to find an upper estimate of the Hausdorff dimension of the set K considered in Theorem 1. Let us additionally assume that K is negatively invariant under φ, i.e. $K \subset \varphi(K)$ as in Corollary 1. In order to find an upper bound for the Hausdorff dimension of K we can not assume $\mu_H(K,d) < \infty$ as in Theorem 1. However it is possible to consider the Hausdorff outer measure of the class of finite covers of K by balls of radii at most ε instead of the Hausdorff outer measure of K itself, because the outer measure of a finite cover is always finite.

Theorem 2. *Let (M,g) be a smooth n-dimensional Riemannian manifold, $U \subset M$ be an open set and $\varphi : U \to M$ be a C^1-map. Suppose K and $\widetilde{K}$ are compact sets satisfying the relation $K \subset \varphi^j(K) \subset \widetilde{K} \subset U$ for any $j \in \mathbb{N}$. Suppose that for some numbers $a > 0$ and $d \in (0,n]$ of the form $d = d_0 + s$ with $d_0 \in \{0,\ldots,n-1\}$ and $s \in (0,1]$ the following conditions are satisfied:*

1) $\omega_{d,\widetilde{K}}(\varphi) < \frac{1}{a}$.

2) *There are numbers l with $\omega_{d,\widetilde{K}}(\varphi) < l < \frac{1}{a}$ and $j_0 \in \mathbb{N}$ such that for any natural number $j > j_0$ there exist a set $K_j \subset K$ and a number $\varepsilon_j > 0$ such that*

$$\mu_H(\varphi^j(K_j), d, \varepsilon) = \mu_H(\varphi^j(K), d, \varepsilon)$$
$$\mu_H\left(K_j, d, (d_0+1)^{-\frac{1}{2}} l^{-\frac{j}{d}} \varepsilon\right) \le a^j \mu_H(K, d, \varepsilon)$$

holds for any $\varepsilon \in (0, \varepsilon_j]$.

Then $\dim_H(K) \le d$.

Proof. Like in the proof of Theorem 1 we have $\omega_{d,K}(\varphi^j) \le \omega^j_{d,\widetilde{K}}(\varphi) < l^j$ for any $j \in \mathbb{N}$. For any $\delta > 0$ we find an integer $j_\delta > j_0$ so that the relation $2^{d_0}(d_0+1)^{\frac{d}{2}}(al)^j < \delta$ will be true for any $j > j_\delta$. Now let $j > j_\delta$ be fixed and consider $0 < \varepsilon \le \min\{\varepsilon_j, (d_0+1)^{-\frac{1}{2}} l^{-\frac{j}{d}} \varepsilon_0(l^j)\}$, where ε_0 is defined in [10: Lemma 2.3]. Then condition 2) and [10: Lemma 2.3] result in the following inequalities:

$$\begin{aligned}
\mu_H(K,d,\varepsilon) &\le \mu_H(\varphi^j(K),d,\varepsilon) \\
&= \mu_H(\varphi^j(K_j),d,\varepsilon) \\
&\le 2^{d_0}(d_0+1)^{\frac{d}{2}} l^j \mu_H(K_j, d, (d_0+1)^{-\frac{1}{2}} l^{-\frac{j}{d}} \varepsilon) \\
&\le 2^{d_0}(d_0+1)^{\frac{d}{2}} (al)^j \mu_H(K,d,\varepsilon) \\
&\le \delta \mu_H(K,d,\varepsilon).
\end{aligned}$$

Since the number δ can be choosen arbitrarily small and $\mu_H(K,d,\varepsilon)$ is finite, this means $\mu_H(K,d,\varepsilon) = 0$ for any $\varepsilon \in \left(0, \min\{\varepsilon_j, (d_0+1)^{-\frac{1}{2}} l^{-\frac{j}{d}} \varepsilon_0(l^j)\}\right)$ and therefore, $\mu_H(K,d) = 0$. Hence we get $\dim_H(K) \le d$ ∎

Using now a Lyapunov-type function we get a corollary of this theorem analogous to Corollary 2.

Corollary 3. *Let (M,g), U, K, $\widetilde{K}$ and φ be defined as in Theorem 2, and let $p : \widetilde{K} \to \mathbb{R}_+$ be a continuous function, such that for some numbers $a > 0$ and $d \in (0,n]$, $d = d_0 + s$ with $d_0 \in \{0, \ldots, n-1\}$ and $s \in (0,1]$ the following conditions are satisfied:*

1) $\sup_{u\in\widetilde{K}} \left(\frac{p(\varphi(u))}{p(u)}\omega_d(d_u\varphi)\right) < \frac{1}{a}$.

2) *There are numbers l with $\sup_{u\in\widetilde{K}} \left(\frac{p(\varphi(u))}{p(u)}\omega_d(d_u\varphi)\right) < l < \frac{1}{a}$ and $j_0 \in \mathbb{N}$ such that for any natural number $j > j_0$ there exist a set $K_j \subset K$ and a number $\varepsilon_j > 0$ with*

$$\mu_H(\varphi^j(K_j), d, \varepsilon) = \mu_H(\varphi^j(K), d, \varepsilon)$$

$$\mu_H\left(K_j, d, (d_0+1)^{-\frac{1}{2}} \left(l^j \frac{\sup_{u\in\widetilde{K}} p(u)}{\inf_{u\in\widetilde{K}} p(u)}\right)^{-\frac{1}{d}} \varepsilon\right) \le a^j \mu_H(K, d, \varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_j]$.

Then $\dim_H(K) \le d$.

Example 2 (Example 1 continued). For the modified horseshoe map described before, in the two-dimensional case the first part of condition 2) of Theorem 2 is satisfied for arbitrary numbers $\varepsilon > 0$ and $d \in [1,2]$. Furthermore, we can show the existence of a number l with $\omega_{d,K}(\varphi) < l < \frac{1}{a}$ satisfying $(d_0+1)^{-\frac{1}{2}} l^{-\frac{j}{d}} \ge \gamma^j$. Together with (5) this yields the second part of condition 2). Thus we get $\dim_H(K) \le d$ for any number $d > \frac{\ln\alpha - \ln 4\beta_2}{\ln\alpha + \ln\gamma}$. In the limit this yields $\dim_H(K) \le \frac{\ln\alpha - \ln 4\beta_2}{\ln\alpha + \ln\gamma}$. For the parameters $\alpha = \frac{1}{3}$, $\beta_1 = 3$ and $\beta_2 = 5$ we get $\dim_H(K) \le 1.863$.

If $\varphi(K \setminus K_1) \subset K_1$ holds, then the dimension estimate can be improved by means of Corollary 3. Using an appropriate Lyapunov-type function the condition 1) of Theorem 2 can be replaced by condition 1) of Corollary 3. Since here the singular-value function is constant on K_1 and $K \setminus K_1$, respectively, the simplest type of Lyapunov function is of the same kind, i.e. $p(u) = 1$ for $u \in K_1$ and $p(u) = P > 0$ for $u \in K \setminus K_1$. Since the distance between the sets K_1 and $K \setminus K_1$ is positive, such a function is continuous on $\widetilde{K} = K$. The constant P has to be chosen in such a way that $\sup_{u\in K} \left(\frac{p(\varphi(u))}{p(u)}\omega_d(d_u\varphi)\right)$ becomes minimal. Because of

$$\frac{p(\varphi(u))}{p(u)}\omega_d(d_u\varphi) = \begin{cases} \frac{1}{P}\beta_2\alpha^{d-1} & \text{for } u \in K \setminus K_1 \\ P\beta_1\alpha^{d-1} & \text{for } u \in K_1, \varphi(u) \in K \setminus K_1 \\ \beta_1\alpha^{d-1} & \text{for } u \in K_1, \varphi(u) \in K_1 \end{cases}$$

we have to choose P such that $\frac{1}{P}\beta_2 = P\beta_1$, i.e. $P = \sqrt{\beta_2/\beta_1}$. Thus we get the Lyapunov-type function

$$p(u) = \begin{cases} 1 & \text{for } u \in K_1 \\ \sqrt{\beta_2/\beta_1} & \text{for } u \in K \setminus K_1. \end{cases}$$

With this function p for $d \in [1,2]$ we get

$$\sup_{u\in K}\left(\frac{p(\varphi(u))}{p(u)}\omega_d(d_u\varphi)\right) = \sqrt{\beta_1\beta_2}\alpha^{d-1}$$

which is less than $\omega_{d,K}(\varphi) = \beta_2\alpha^{d-1}$. For any number $d > \frac{\ln\alpha - \ln 4\sqrt{\beta_1\beta_2}}{\ln\alpha + \ln\gamma}$ the conditions of Corollary 3 are satisfied, and we get the improved estimate $\dim_H(K) \le \frac{\ln\alpha-\ln 4\sqrt{\beta_1\beta_2}}{\ln\alpha+\ln\gamma}$. For the parameters $\alpha = \frac{1}{3}$, $\beta_1 = 3$ and $\beta_2 = 5$ this means $\dim_H(K) \le 1.747$ ∎

Remark 2. In Example 2 we would have got the same improved result if we had changed the standard metric on $\mathbb{R}^2$ by multiplying the metric tensor with the Lyapunov-type function p.

Since condition 2) of Theorem 2 is not easy to check, especially if the map is not piecewise linear, we now give some stronger conditions which can be checked more easily.

Corollary 4. *Let (M,g) be a smooth n-dimensional Riemannian manifold, $U \subset M$ an open set, $\varphi: U \to M$ a C^1-map and $K \subset U$ a compact φ-invariant set. Suppose that for some numbers $a > 0$ and $d \in (0,n]$ of the form $d = d_0 + s$ with $d_0 \in \{0,\ldots,n-1\}$ and $s \in (0,1]$ the following conditions are satisfied:*

1) $\omega_{d,K}(\varphi) < \frac{1}{a}$.

2) *There is a number $j_0 \in \mathbb{N}$ such that for any natural number $j \ge j_0$ there exist a set $K_j \subset K$ with $\varphi^j(K_j) = K$, a natural number N_j, a number l_0 and C^1-maps $f_{i,j}: U \to M$ $(i = 1,\ldots,N_j)$ with*

$$K_j = \bigcup_{i=1}^{N_j} f_{i,j}(K), \qquad \max_{i=1,\ldots,N_j}\omega_{d,K}(f_{i,j}) < l_0^j, \qquad N_j \le 2^{-d_0}(d_0+1)^{-\frac{d}{2}}a^j l_0^{-j}.$$

Then $\dim_H(K) \le d$.

Proof. Using [10: Lemma 2.3], for any $j \in \mathbb{N}$ with $j > j_0$ there exists a number ε_j such that

$$\begin{aligned}\mu_H(K_j, d, \sqrt{d_0+1}\,l_0^{\frac{j}{d}}\varepsilon) &\le \sum_{i=1}^{N_j} 2^{d_0}(d_0+1)^{\frac{d}{2}} l_0^j \mu_H(K,d,\varepsilon)\\ &= N_j 2^{d_0}(d_0+1)^{\frac{d}{2}} l_0^j \mu_H(K,d,\varepsilon)\\ &\le a^j \mu_H(K,d,\varepsilon)\end{aligned}$$

holds for any $\varepsilon \in (0,\varepsilon_j]$. Because of $N_j \ge 1$ and of condition 2) we have $2^{-d_0}(d_0 + 1)^{-\frac{d}{2}}a^j l_0^{-j} \ge 1$ and therefore, $2^{d_0}(d_0+1)^{\frac{d}{2}} l_0^j \omega_{d,K}^j(\varphi) < 1$ for any $j \ge j_0$. This means $l_0\,\omega_{d,K}(\varphi) < 1$, and because of condition 1), there are numbers $l \in \mathbb{R}$ and $j_0 \in \mathbb{N}$ such that $\omega_{d,K}(\varphi) < l < \frac{1}{a}$ and $(l_0 l)^{\frac{j}{d}} < \frac{1}{d_0+1}$ for any $j > j_0$ are satisfied. For these numbers j and all $\varepsilon \in (0,\varepsilon_j]$ we have

$$\mu_H(K_j, d, (d_0+1)^{-\frac{1}{2}} l^{-\frac{j}{d}}\varepsilon) \le \mu_H(K_j, d, \sqrt{d_0+1}\,l_0^{\frac{j}{d}}\varepsilon) \le a^j \mu_H(K,d,\varepsilon).$$

Applying Theorem 2 we get $\dim_H(K) \le d$ ∎

Example 3 (Example 1 continued). Since in our example of the modified horseshoe map (two-dimensional) the set K_j consists of 4^j linear copies of K we define $f_{i,j}$ to be the linear map of K onto the ith piece of K_j $(i = 1, \dots, 4^j)$. Then for $N_j = 4^j$ and $a > 4l_0 > 4\gamma^d$ condition 2) of Corollary 4 is satisfied. Condition 1) results in $a\beta_2\alpha^{d-1} < 1$, and the limit for $a \to 4\gamma^d$ yields (6). In this way we get the same result as before without a Lyapunov-type function, but we could reach it with less expense ∎

3. Fractal dimension estimates

First let us recall the notation of the fractal outer measure and fractal dimension for compact subsets K of a metric space (X, ϱ). For a given number $\varepsilon > 0$ let $N_\varepsilon(K)$ denote the smallest number of balls of radius ε needed to cover K. Then for $d \geq 0$ the d-dimensional fractal outer measure of the covering class of K by balls of radius ε is the number

$$\mu_F(K, d, \varepsilon) = N_\varepsilon(K)\varepsilon^d.$$

The upper limit

$$\mu_F(K, d) = \limsup_{\varepsilon \to 0+0} \mu_F(K, d, \varepsilon)$$

is the d-dimensional fractal outer measure of K. The uniquely defined critical number $d^* \geq 0$ satisfying

$$\mu_F(K, d) = \begin{cases} \infty & \text{for } 0 \leq d < d^* \\ 0 & \text{for } d > d^* \end{cases}$$

is called the upper capacity dimension [6], upper box-counting dimension [4] or fractal dimension [3, 13 - 15] of the set K. It can be shown that $\dim_H(K) \leq \dim_F(K)$. In order to estimate the fractal dimension of a negatively invariant set of a C^1-map on a Riemannian manifold we need two lemmata, which are formulated and proved analogously to assertions of [2, 10].

Lemma 1. *Let $(E, \langle \cdot, \cdot \rangle_E)$ be an n-dimensional Euclidean space, $u_1, \dots, u_n$ an orthonormal basis and*

$$\mathcal{E} = \left\{ a_1u_1 + \dots + a_nu_n \in E \,\middle|\, (a_1, \dots, a_n) \in \mathbb{R}^n, \left(\frac{a_1}{\alpha_1(\mathcal{E})}\right)^2 + \dots + \left(\frac{a_n}{\alpha_n(\mathcal{E})}\right)^2 \leq 1 \right\}$$

an ellipsoid with $\alpha_1(\mathcal{E}) \geq \dots \geq \alpha_n(\mathcal{E}) > 0$. Then for any $\eta > 0$, the set $\mathcal{E} + B_\eta(0)$, where $B_\eta(0)$ denotes the ball with radius η centered at the origin, is contained in the ellipsoid $\mathcal{E}' = \left(1 + \frac{\eta}{\alpha_n(\mathcal{E})}\right)\mathcal{E}$.

Proof. We have

$$B_{\alpha_n(\mathcal{E})}(0) = \left\{ a_1u_1 + \dots + a_nu_n \in E \,\middle|\, \left(\frac{a_1}{\alpha_n(\mathcal{E})}\right)^2 + \dots + \left(\frac{a_n}{\alpha_n(\mathcal{E})}\right)^2 \leq 1 \right\} \subset \mathcal{E}$$

and therefore,

$$\mathcal{E} + B_\eta(0) = \mathcal{E} + \frac{\eta}{\alpha_n(\mathcal{E})} B_{\alpha_n(\mathcal{E})}(0) \subset \mathcal{E} + \frac{\eta}{\alpha_n(\mathcal{E})}\mathcal{E} = \left(1 + \frac{\eta}{\alpha_n(\mathcal{E})}\right)\mathcal{E},$$

i.e. the statement of the lemma ∎

Lemma 2. *Let $(E,\langle\cdot,\cdot\rangle_E)$ be an n-dimensional Euclidean space, $u_1,\dots,u_n$ an orthonormal basis,*

$$\mathcal{E}=\left\{a_1u_1+\ldots+a_nu_n\in E\;\middle|\;\left(\frac{a_1}{\alpha_1(\mathcal{E})}\right)^2+\ldots+\left(\frac{a_n}{\alpha_n(\mathcal{E})}\right)^2\le 1\right\}$$

an ellipsoid with $\alpha_1(\mathcal{E})\ge\ldots\ge\alpha_n(\mathcal{E})>0$ and $0<r<\alpha_n(\mathcal{E})$. Then the relation $N_{\sqrt{n}r}(\mathcal{E})\le\frac{2^n\omega_n(\mathcal{E})}{r^n}$ holds, where $\omega_n(\mathcal{E})$ is defined in (2).

Proof. The ellipsoid $\mathcal{E}$ is contained in the parallelepiped

$$P=\left\{a_1u_1+\ldots+a_nu_n\in E\;\middle|\;|a_1|\le\alpha_1(\mathcal{E}),\dots,|a_n|\le\alpha_n(\mathcal{E})\right\},$$

whereas the edges have the lengths $2\alpha_1(\mathcal{E}),\dots,2\alpha_n(\mathcal{E})$, respectively. This parallelepiped can be covered by

$$\prod_{j=1}^{n}\left(\left[\frac{\alpha_j(\mathcal{E})}{r}\right]+1\right)\le 2^n\prod_{j=1}^{n}\frac{\alpha_j(\mathcal{E})}{r}=\frac{2^n\omega_n(\mathcal{E})}{r^n}$$

cubes with edges of the length $2r$ parallel to the directions u_i $(i=1,\dots,n)$ where $[\cdot]$ denotes the integer part. Each of the cubes is contained in a ball of radius $\sqrt{n}r$ ∎

The first theorem in this section provides an upper bound for the fractal dimension of a negatively invariant set if no information about the "degree of non-injectivity" is known.

Theorem 3. *Let (M,g) be a smooth n-dimensional Riemannian manifold, $U\subset M$ be an open set and $\varphi:U\to M$ be a C^1-map. Suppose $K\subset U$ is a compact set satisfying the relation $K\subset\varphi(K)\subset U$. Assume that*

$$0<\rho_K(\varphi):=\min_{u\in K}\alpha_n(d_u\varphi)<n^{-\frac{1}{2}} \tag{7}$$

and there exists a number $d\in(0,n]$ such that

$$\omega_{n,K}(\varphi)\rho_K^{d-n}(\varphi)\le 8^{-n}n^{-\frac{d}{2}}. \tag{8}$$

Then $\dim_F(K)\le d$.

Proof. Let $\eta\in(0,\rho_K(\varphi))$ be an arbitrary number and $r_1>0$ be so small that there exists an open set $V\subset M$ containing K which itself lies inside a compact subset of U such that

$$\|\tau_{\varphi(v)}^{\varphi(u)}d_v\varphi\tau_u^v-d_u\varphi\|\le\eta \tag{9}$$

for any $u,v\in V$ with $\varrho(u,v)\le r_1$ is satisfied, where $\|\cdot\|$ here denotes the operator norm. By $\varrho(\cdot,\cdot)$ we mean the geodesic distance between the points of M and by τ_u^v we denote the isometry between the tangent spaces T_uM and T_vM defined by parallel transport. Let $\exp_u:T_uM\to M$ denote the exponential map at an arbitrary point $u\in M$. Since $\exp_u$ is a smooth map satisfying $\|d_{O_u}\exp_u\|=1$ for any point $u\in M$ we find a number $r_u > 0$ such that $\|d_v \exp_u\| \leq 2$ for any $v \in B_{r_u}(O_u)$, where O_u denotes the origin of the tangent space T_uM. Since V is contained in a compact set there is a number $r_2 > 0$ such that $\|d_v \exp_u\| \leq 2$ is satisfied for any $u \in V$ and any $v \in B_{r_2}(O_u)$. Furthermore, there is a number $\alpha > 0$ such that $\alpha_1(d_u\varphi) < \alpha$ is satisfied for any $u \in V$.

Now we can find a number $r_0 \leq \min\{r_1, \frac{r_2}{2+\alpha+\eta}\}$ such that any ball $B_{r_0}(u)$ containing points of K is entirely contained in V. Let $r \in (0, r_0)$ be fixed. Since K is compact there is a finite number of points $u_j \in V$ $(j = 1, \dots, N_r(K))$ such that $K = \bigcup_{j=1}^{N_r(K)} B_r(u_j) \cap K$ and therefore,

$$\varphi(K) = \bigcup_{j=1}^{N_r(K)} \varphi(B_r(u_j) \cap K)$$

is satisfied. The Taylor formula for the differentiable map φ guarantees the relation

$$\begin{aligned} &\left\| \exp^{-1}_{\varphi(u_j)} \varphi(v) - d_{u_j}\varphi(\exp^{-1}_{u_j}(v)) \right\| \\ &\quad \leq \sup_{w \in B_r(u_j)} \left\| \tau^{\varphi(u_j)}_{\varphi(w)} d_w \varphi \tau^{w}_{u_j} - d_{u_j}\varphi \right\| \cdot \left\| \exp^{-1}_{u_j}(w) \right\| \end{aligned} \tag{10}$$

for every $v \in B_r(u_j)$. Thus, using (9) and (10), the image of every ball $B_r(u_j)$ under φ satisfies the inclusion

$$\varphi(B_r(u_j)) \subset \exp_{\varphi(u_j)}(d_{u_j}\varphi(B_r(O_{u_j})) + B_{\eta r}(O_{\varphi(u_j)})).$$

Since $\mathcal{E}_j := d_{u_j}\varphi(B_1(O_{\varphi(u_j)})))$ is an ellipsoid in $E = T_{\varphi(u_j)}M$ we get for this $\mathcal{E}_j$ and $\mathcal{E}'_j = \left(1 + \frac{\eta}{\alpha_n(\mathcal{E}_j)}\right)\mathcal{E}_j$ with Lemma 1

$$\varphi(B_r(u_j)) \subset \exp_{\varphi(u_j)}\left(r(\mathcal{E}_j + B_\eta(O_{\varphi(u_j)}))\right) \subset \exp_{\varphi(u_j)}\left(r\mathcal{E}'_j\right).$$

With

$$\sigma := \sqrt{n}\rho_K(\varphi) \tag{11}$$

we have

$$N_{\sigma r}(\varphi(K)) \leq N_r(K) \max_{j=1,\dots,N_r(K)} N_{\sigma r}(\exp_{\varphi(u_j)}(r\mathcal{E}'_j))$$

and therefore,

$$\mu_F(\varphi(K), d, \sigma r) \leq \left(\sigma^d \max_{j=1,\dots,N_r(K)} N_{\sigma r}(\exp_{\varphi(u_j)}(r\mathcal{E}'_j))\right)\mu_F(K, d, r). \tag{12}$$

Every ball $B_{\sigma r}(v)$ $(v \in M)$ containing points of $\exp_{\varphi(u_j)}(r\mathcal{E}'_j)$ is contained in the ball $B_{(2+\alpha_1(\mathcal{E}'_j))r}(u_j) \subset B_{r_2}(u_j)$, and so we have $B_{\sigma r}(v) \supset \exp_{\varphi(u_j)}(B_{\frac{1}{2}\sigma r}(\exp^{-1}_{\varphi(u_j)} v))$. This means

$$N_{\sigma r}(\exp_{\varphi(u_j)}(r\mathcal{E}'_j)) \leq N_{\frac{1}{2}\sigma r}(r\mathcal{E}'_j).$$

Since $\rho_K(\varphi) \leq \alpha_n(d_{u_j}\varphi) = \alpha_n(\mathcal{E}_j) \leq \alpha_n(\mathcal{E}'_j)$ is satisfied, Lemma 2 yields

$$N_{\frac{1}{2}\sigma r}(r\mathcal{E}'_j) \leq \frac{2^n\omega_n(r\mathcal{E}'_j)}{(\frac{1}{2}r\rho_K(\varphi))^n} = \frac{4^n\omega_n(\mathcal{E}'_j)}{\rho^n_K(\varphi)} \leq \frac{4^n\left(1 + \frac{\eta}{\rho_K(\varphi)}\right)^n\omega_n(\mathcal{E}_j)}{\rho^n_K(\varphi)} \leq \frac{8^n\omega_n(d_{u_j}\varphi)}{\rho^n_K(\varphi)}.$$

Using (11), (12) and assumption (8) we get

$$\begin{aligned}\mu_F(K,d,\sigma r) &\le \mu_F(\varphi(K),d,\sigma r)\\ &\le \sigma^d \frac{8^n\omega_{n,K}(\varphi)}{\rho_K^n(\varphi)}\mu_F(K,d,r)\\ &= n^{\frac{d}{2}}8^n\omega_{n,K}(\varphi)\rho_K^{d-n}(\varphi)\mu_F(K,d,r)\\ &< \mu_F(K,d,r).\end{aligned}$$

Because of (8) we have $\sigma < 1$. Therefore, for any $\varepsilon \in (0,r_0)$ we can find a number $l \in \mathbb{N}_0$ such that $\sigma^{l+1}r_0 \le \varepsilon < \sigma^l r_0$ is satisfied. Finally we get

$$\mu_F(K,d,\varepsilon) < \mu_F(K,d,\sigma^{-l}\varepsilon) < N_{\sigma^{-l}\varepsilon}(K)r_0^d \le N_{\sigma r_0}(K)r_0^d = \sigma^{-d}\mu_F(K,d,\sigma r_0),$$

which yields $\mu_F(K,d) < \infty$ and thus $\dim_F(K) \le d$ ∎

Corollary 5. *Let (M,g), U and φ be as in Theorem 3, K and $\widetilde{K}$ be compact sets satisfying the relation $K \subset \varphi^j(K) \subset \widetilde{K} \subset U$ and $\rho_{\widetilde{K}}(\varphi) > 0$. Suppose that there exists a number $d \in (0,n]$ with*

$$\omega_{n,\widetilde{K}}(\varphi)\rho_{\widetilde{K}}^{d-n}(\varphi) < 1. \tag{13}$$

Then $\dim_F(K) \le d$.

Proof. From $K \subset \varphi^j(K) \subset \widetilde{K} \subset U$ we get $K \subset \varphi^i(K) \subset U$ for any $i \in \mathbb{N}$. With respect to (1) the iterates of φ satisfy the relations

$$\omega_{n,K}(\varphi^i) \le \omega^i_{n,\widetilde{K}}(\varphi) \qquad \text{and} \qquad \rho_K(\varphi^i) \ge \rho^i_{\widetilde{K}}(\varphi) \tag{14}$$

and therefore,

$$\omega_{n,K}(\varphi^i)\rho_K^{d-n}(\varphi^i) \le \left(\omega_{n,\widetilde{K}}(\varphi)\rho_{\widetilde{K}}^{d-n}(\varphi)\right)^i.$$

Furthermore, we have from the definition (3)

$$\omega_{n,K}(\varphi^i)\rho_K^{d-n}(\varphi^i) \ge \rho_K^n(\varphi^i)\rho_K^{d-n}(\varphi^i) = \rho_K^d(\varphi^i). \tag{15}$$

By using (13) - (15), without loss of generality we can assume

$$\omega_{n,K}(\varphi)\rho_K^{d-n}(\varphi) \le 8^{-n}n^{-\frac{d}{2}} \qquad \text{and} \qquad \rho_K(\varphi) < n^{-\frac{1}{2}}.$$

In the opposite case consider the map φ^i with sufficiently large i. With Theorem 3 we get $\dim_F(K) \le d$ ∎

Corollary 6. *Let (M,g), U, K and φ be defined as in Theorem 3 and $p: U \to \mathbb{R}_+$ be a continuous function. Suppose that the following conditions are satisfied:*

1) $\omega_n(d_u\varphi) = \text{const} \neq 0$ *for all* $u \in K$.

2) *There exists a number* $s \in (0,1]$ *such that* $\frac{p(\varphi(u))}{p(u)}\omega_{n-1+s}(d_u\varphi) < 1$ *for all* $u \in K$.

Then $\dim_F(K) \le n-1+s$.

Proof. According to condition 2) there exists a positive number $\nu < 1$ with

$$\frac{p(\varphi(u))}{p(u)}\,\omega_{n-1+s}(d_u\varphi) \le \nu$$

for any $u \in K$. Therefore, by the chain rule and by (1) we have

$$\begin{aligned}\omega_{n-1+s,K}(\varphi^i) &\le \max_{u\in K} \frac{p(u)}{p(\varphi^i(u))}\frac{p(\varphi^i(u))}{p(\varphi^{i-1}(u))}\omega_{n-1+s}(d_{\varphi^{i-1}(u)}\varphi)\ \dots\ \frac{p(\varphi(u))}{p(u)}\omega_{n-1+s}(d_u\varphi)\\ &\le \frac{\max_{u\in K} p(u)}{\min_{u\in K} p(u)}\nu^i.\end{aligned}$$

Furthermore, the relation $\omega_{n-1+s,K}(\varphi^i) \ge \rho_K^{n-1+s}(\varphi^i)$ holds. Therefore, without loss of generality we can assume that $\omega_{n-1+s,K}(\varphi) < 8^{-n}n^{\frac{n-1+s}{2}}$ and $\rho_K(\varphi) < n^{-\frac{1}{2}}$ is satisfied. In the opposite case consider φ^i with sufficiently large i. We take $u_0 \in K$ such that $\alpha_n(d_{u_0}\varphi) = \rho_K(\varphi)$. Resulting from (3) and condition 1) we obtain

$$\omega_{n,K}(\varphi)\rho_K^{s-1}(\varphi) = \omega_n(d_{u_0}\varphi)\alpha_n^{s-1}(d_{u_0}\varphi) = \omega_{n-1+s}(d_{u_0}\varphi) < 8^{-n}n^{\frac{n-1+s}{2}}.$$

With Theorem 3 we get $\dim_F(K) \le n-1+s$ ∎

Remark 3. Conditions analogous to 1) of Corollary 6 are considered in [15] for invertible maps as the Hénon system. In contrast to our results the fractal dimension estimates in [15] are given in terms of Lyapunov exponents and without use of a Lyapunov-type function p.

Now we want to include the "degree of non-injectivity" in the method of estimating the fractal dimension developed in Theorem 3.

Theorem 4. *Let (M,g) be a smooth n-dimensional Riemannian manifold, $U \subset M$ be an open set and $\varphi : U \to M$ be a C^1-map. Suppose $K \subset U$ is a compact set satisfying the relation $K \subset \varphi(K) \subset U$. Suppose $\rho_K(\varphi) > 0$, and let $a, b > 0$ be numbers such that the following conditions are satisfied:*

1) *There exists a number $d \in (0,n]$ with*

$$\rho_K(\varphi) < a^{-\frac{1}{n}}b^{\frac{d-n}{n}}n^{-\frac{1}{2}}$$
$$\omega_{n,K}(\varphi)\rho_K^{d-n}(\varphi) \le a^{-\frac{d}{n}}b^{\frac{d}{n}(d-n)}8^{-n}n^{-\frac{d}{2}}.$$

2) *For any $j \in \mathbb{N}$ there are a compact set $K_j \subset K$ and a number $\varepsilon_j > 0$ such that*

$$\mu_F(\varphi^j(K_j), d, \varepsilon) = \mu_F(\varphi^j(K), d, \varepsilon)$$
$$\mu_F(K_j, d, b^j\varepsilon) \le a^j\mu_F(K, d, \varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_j]$ are satisfied.

Then $\dim_F(K) \le d$.

Proof. Analogous to the proof of Theorem 3 let $\eta \in (0, \rho_K(\varphi))$ be an arbitrary number. Let $r_1, r_2 > 0$ be so small that there exists an open set $V \subset M$ containing K and that V is a contained in a compact subset of U such that the inequalities $\|\tau_{\varphi(v)}^{\varphi(u)} d_v\varphi\tau_u^v - d_u\varphi\| \leq \eta$ for any $u, v \in V$ with $\varrho(u,v) \leq r_1$ and $\|d_v \exp_u\| \leq 2$ for any $u \in V$ and any $v \in B_{r_2}(O_u)$ are satisfied. With α defined in the proof of Theorem 3 we can find a number $r_0 \leq \min\{r_1, \frac{r_2}{2+\alpha+\eta}, \varepsilon_1\}$ such that any ball $B_{r_0}(u)$ containing points of K is entirely contained in V. Let $r \in (0, r_0)$ be fixed. Since K_1 is compact, there is a finite number of points $u_j \in V$ $(j = 1, \ldots, N_r(K_1))$ such that $K_1 = \bigcup_{j=1}^{N_r(K_1)} B_r(u_j) \cap K_1$ and therefore,

$$\varphi(K_1) = \bigcup_{j=1}^{N_r(K_1)} \varphi(B_r(u_j) \cap K_1)$$

is satisfied. Using the Taylor formula we get that the image of every ball $B_r(u_j)$ under φ satisfies the inclusion

$$\varphi(B_r(u_j)) \subset \exp_{\varphi(u_j)}(d_{u_j}\varphi(B_r(O_{u_j})) + B_{\eta r}(O_{\varphi(u_j)})).$$

With $\mathcal{E}_j := d_{u_j}\varphi(B_1(O_{\varphi(u_j)}))$ and $\mathcal{E}'_j = \left(1 + \frac{\eta}{\alpha_n(\mathcal{E}_j)}\right)\mathcal{E}_j$ we get by means of Lemma 1

$$\varphi(B_r(u_j)) \subset \exp_{\varphi(u_j)}\left(r(\mathcal{E}_j + B_\eta(O_{\varphi(u_j)}))\right) \subset \exp_{\varphi(u_j)}\left(r\mathcal{E}'_j\right).$$

With $\sigma := a^{\frac{1}{n}} b^{\frac{n-d}{n}} \sqrt{n}\rho_K(\varphi)$ we obtain

$$N_{\sigma r}(\varphi(K_1)) \leq N_{br}(K_1) \max_{j=1,\ldots,N_r(K_1)} N_{\sigma r}(\exp_{\varphi(u_j)}(br\mathcal{E}'_j))$$

and therefore,

$$\mu_F(\varphi(K_1), d, \sigma r) \leq \left(\sigma^d b^{-d} \max_{j=1,\ldots,N_r(K_1)} N_{\sigma r}(\exp_{\varphi(u_j)}(br\mathcal{E}'_j))\right)\mu_F(K, d, r).$$

Analogous to the proof of Theorem 3 we get $N_{\sigma r}(\exp_{\varphi(u_j)}(br\mathcal{E}'_j)) \leq N_{\frac{1}{2}\sigma r}(br\mathcal{E}'_j)$. Since $\rho_K(\varphi) \leq \alpha_n(d_{u_j}\varphi) = \alpha_n(\mathcal{E}_j) \leq \alpha_n(\mathcal{E}'_j)$ is satisfied, Lemma 2 yields

$$\begin{aligned} N_{\frac{1}{2}\sigma r}(br\mathcal{E}'_j) &\leq \frac{2^n\omega_n(br\mathcal{E}'_j)}{(\frac{1}{2}a^{\frac{1}{n}}b^{\frac{n-d}{n}}r\rho_K(\varphi))^n} = \frac{4^n b^d \omega_n(\mathcal{E}'_j)}{a\rho_K^n(\varphi)} \leq \frac{4^n b^d \left(1 + \frac{\eta}{\rho_K(\varphi)}\right)^n \omega_n(\mathcal{E}_j)}{a\rho_K^n(\varphi)} \\ &\leq \frac{8^n b^d \omega_n(d_{u_j}\varphi)}{a\rho_K^n(\varphi)}. \end{aligned}$$

Thus we have

$$\begin{aligned} \mu_F(K, d, \sigma r) &\leq \mu_F(\varphi(K), d, \sigma r) \\ &= \mu_F(\varphi(K_1), d, \sigma r) \\ &\leq \sigma^d b^{-d} \frac{8^n b^d \omega_n(d_{u_j}\varphi)}{a\rho_K^n(\varphi)} \mu_F(K_1, d, br) \\ &\leq a^{\frac{d}{n}} b^{\frac{d}{n}(n-d)} n^{\frac{d}{2}} \frac{8^n \omega_n(d_{u_j}\varphi)}{a\rho_K^n(\varphi)} \mu_F(K_1, d, br) \\ &\leq \mu_F(K, d, r). \end{aligned}$$

Since $\sigma < 1$, analogous to the end of the proof of Theorem 3, this yields $\dim_F(K) \leq d$ ∎

Corollary 7. *Let (M,g), U and φ be as in Theorem 4, K and $\widetilde{K}$ be compact sets satisfying the relation $K \subset \varphi^j(K) \subset \widetilde{K} \subset U$ and $\rho_{\widetilde{K}}(\varphi) > 0$. Let condition 2) of Theorem 4 be satisfied and assume that there exists a number $d \in (0,n]$ with*

$$\omega_{n,\widetilde{K}}(\varphi)\rho_{\widetilde{K}}^{d-n}(\varphi) < a^{-\frac{d}{n}} b^{\frac{d}{n}(d-n)}. \tag{16}$$

Then $\dim_F(K) \leq d$.

Proof. From $K \subset \varphi^j(K) \subset \widetilde{K} \subset U$ we get $K \subset \varphi^i(K) \subset U$ for any $i \in \mathbb{N}$. The iterates of φ satisfy the relations

$$\omega_{n,K}(\varphi^i) \leq \omega^i_{n,\widetilde{K}}(\varphi) \qquad \text{and} \qquad \rho_K(\varphi^i) \geq \rho^i_{\widetilde{K}}(\varphi)$$

and therefore

$$a^{\frac{di}{n}} b^{\frac{di}{n}(n-d)} \omega_{n,K}(\varphi^i)\rho_K^{d-n}(\varphi^i) \leq \left(a^{\frac{d}{n}} b^{\frac{d}{n}(n-d)} \omega_{n,\widetilde{K}}(\varphi)\rho_{\widetilde{K}}^{d-n}(\varphi)\right)^i.$$

Furthermore,

$$a^{\frac{d}{n}} b^{\frac{d}{n}(n-d)} \omega_{n,K}(\varphi^i)\rho_K^{d-n}(\varphi^i) \geq a^{\frac{d}{n}} b^{\frac{d}{n}(n-d)} \rho_K^n(\varphi^i)\rho_K^{d-n}(\varphi^i) = \left(a^{\frac{1}{n}} b^{\frac{n-d}{n}} \rho_K(\varphi^i)\right)^d$$

holds. Thus without loss of generality we can assume

$$a^{\frac{d}{n}} b^{\frac{d}{n}(n-d)} \omega_{n,K}(\varphi)\rho_K^{d-n}(\varphi) \leq 8^{-n} n^{-\frac{d}{2}} \qquad \text{and} \qquad a^{\frac{1}{n}} b^{\frac{n-d}{n}} \rho_K(\varphi) < n^{-\frac{1}{2}}.$$

Otherwise consider the map φ^i with sufficiently large i and substitute a by a^i and b by b^i. With Theorem 4 we get $\dim_F(K) \leq d$ ∎

Example 4 (Example 1 continued). Let us again consider the modified horseshoe map in two dimensions. For the sets K_j defined before, analogous to (5) we have $N_\varepsilon(K_j) \leq 4^j N_{\frac{\varepsilon}{\gamma^j}}(K)$ for sufficiently small $\varepsilon > 0$, i.e. with $a = 4\gamma^d$ and $b = \gamma$ condition 2) of Theorem 4 is satisfied. Then condition (16) results in

$$4^{\frac{d}{2}} \gamma^d \beta_2 \alpha^{d-1} < 1 \tag{17}$$

which is equivalent to $d > \frac{\ln \alpha - \ln \beta_2}{\ln 2 + \ln \alpha + \ln \gamma}$. Corollary 7 can be applied for any such d and shows that $\dim_F(K) \leq \frac{\ln \alpha - \ln \beta_2}{\ln 2 + \ln \alpha + \ln \gamma}$. For the parameters $\alpha = \frac{1}{3}$, $\beta_1 = 3$ and $\beta_2 = 5$ we get $\dim_F(K) \leq 1.800$.

By changing the metric with the Lyapunov-type function p used in Example 2 we alter the form of the balls covering $K \setminus K_1$. However, again we have $N_{\gamma^j\varepsilon}(K_j) \leq 4^j N_\varepsilon(K)$, i.e. condition 2) holds with $a = 4\gamma^d$ and $b = \gamma$. Condition (16) now results in $4^{\frac{d}{2}} \gamma^d \sqrt{\beta_1\beta_2}\alpha^{d-1} < 1$, which means $\dim_F(K) \leq \frac{\ln \alpha - \frac{1}{2}\ln \beta_1\beta_2}{\ln 2 + \ln \alpha + \ln \gamma}$. For $\alpha = \frac{1}{3}$, $\beta_1 = 3$ and $\beta_2 = 5$ we get $\dim_F(K) \leq 1.631$.

Remark 4. For two-dimensional horseshoe maps the upper bound for the fractal dimension of an invariant set obtained by Corollary 7 is always smaller than the bound for the Hausdorff dimension by Theorem 2, because for $d < 2$ condition (17) is weaker than condition (6). For $d > 2$ this relation is reversed, i.e. for the considered horseshoe maps in more than two dimensions the estimates of Section 2 really will be useful.

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