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Essential Properties of L^∞-functions

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Abstract. The paper deals with local characteristics of L^∞-elements given as equivalence classes of measurable, essentially bounded functions $f : \mathbb{R}^m \to \mathbb{R}$. Besides of essential lower and upper limit functions we introduce a new set-valued map carrying the information on a class, the essential limit set at a point, and analyze their main properties. Criteria for qualifying the continuity of function representatives are appended. The results can be applied e.g. in control theory to interprete "almost everywhere" conditions.

Keywords: L^∞ *function space, upper and lower limit functions, set-valued maps, integrability and continuity criteria*

AMS subject classification: 26 B 15, 26 B 40, 28 A 20, 46 E 30, 49 N 65

1. Introduction

In recent time the spaces of essentially bounded, measurable functions find application to a growing extent in different fields of mathematics such as nonsmooth analysis (occuring there as generalized derivatives of Lipschitz functions) or in differential inclusions. In many cases, problems of optimal control under inequality constraints may be adequately described only by use of L^∞-spaces. Typical examples are problems involving state functions from the Sobolev space $W^{1,p}$ $(p < \infty)$ under box constraints leading to appropriate essential control bounds. Another reason may be seen in the fact that in contrast to spaces L^p $(1 \leqslant p < \infty)$, in L^∞ the cone of essentially nonnegative functions has a nonempty interior, i.e. the feasible set for the related optimal control problem has desirable regularity properties.

In connection with these applications usually two types of questions arise:

1) Often one has to assess the local ("pointwise") behavior of the considered L^∞-functions. As far as their values are determined only "up to sets of zero measure" in fact one has to consider not "individual", proper functions but related equivalence classes. Therefore, tools are needed which allow to characterize local properties *valid for all representatives* of a given L^∞-class.

2) For many problems it is of importance whether a given equivalence class includes representatives with specific analytical properties like continuity in a certain point or area, semicontinuity, Riemann integrability etc.

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In order to contribute answering these questions we propose the following concept. In a first step (Section 2) we introduce *essential upper* and *lower limit functions* (Definition 2.1) which turn out to be measurable and semicontinuous (Theorem 2.5). The pair of these essential limit functions yield a complete characterization of an L^∞-element (Theorem 2.3). Secondly, in Section 3 we define *essential limit sets* (Definition 3.2) as objects carrying the whole information on the local behavior common for all functions belonging to a given equivalence class. We show that they are compact sets (Theorem 3.8) containing the function value of a particular representative at almost every point (Theorem 3.6). Assembling them to a set-valued mapping on the whole range of definition, we again obtain an one-to-one correspondence to the underlying L^∞-class (Theorem 3.10). In addition, this mapping is upper semicontinuous (Theorem 3.11). Section 4 is dedicated to answer the second kind of questions, e.g. to the formulation of conditions under which a given equivalence class in L^∞ contains continuous (Theorem 4.5) or Riemann integrable representatives (Theorem 4.6). Finally, we consider inequalities involving L^∞-arguments which are valid only almost everywhere. We prove for them an insertion rule extending the validity to the whole domain (Theorem 4.8).

Notations. Suppose there is given a range of definition Ω representing the closure of a bounded domain in $\mathbb{R}^m$. Denote by $\mathrm{K}(t_0,\delta)$ the intersection of Ω and the open ball with centre t_0 and radius δ. Further, let λ be the m-dimensional Lebesgue measure, and $\mathfrak{B}$ the σ-algebra of Borel subsets of Ω extended by λ-null sets. Instead of $\lambda(\mathrm{A})$, we write shortly $|\mathrm{A}|$. Concerning the boundary $\partial\Omega$ let us assume that for any $t \in \partial\Omega$ and arbitrary $\delta > 0$ the relation $|\mathrm{K}(t,\delta)| > 0$ is satisfied. For example, this is true for Lipschitz domains (in strong sense) or domains having the "cone property" [1: p. 66]. We define $L^\infty(\Omega, \mathfrak{B}, \lambda)$ as the Banach space of equivalence classes of functions $f: \Omega \to \mathbb{R}$ which are λ-essentially bounded and measurable. Two functions f_1 and f_2 fall into the same equivalence class if and only if the set $\{ t \in \Omega : f_1(t) \neq f_2(t) \}$ is a λ-null set. In the forthcoming, we will distinguish between an individual essentially bounded, measurable function f and the equivalence class $\tilde{f}$ as element of $L^\infty(\Omega, \mathfrak{B}, \lambda)$ containing the individual function f. The norm in $L^\infty(\Omega, \mathfrak{B}, \lambda)$ is given in the usual way by $\| \tilde{f} \| = \operatorname{ess\,sup} \{ |f(t)| : t \in \Omega \}$ with f chosen arbitrarily from $\tilde{f}$.

2. Essential limit functions

Let $f: \Omega \to \mathbb{R}$ be essentially bounded and measurable. For arbitrary given $\delta > 0$ denote

$$G_f(t_0,\delta) = \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta)} f(t) \qquad \text{and} \qquad g_f(t_0,\delta) = \operatorname*{ess\,inf}_{t \in \mathrm{K}(t_0,\delta)} f(t).$$

As a result of our assumptions on Ω, these expressions are well defined for any $t_0 \in \Omega$. For fixed t_0 and a monotonically decreasing sequence $\delta_n \downarrow 0$ the corresponding sequences $\{ G_f(t_0,\delta_n) \}$ and $\{ g_f(t_0,\delta_n) \}$ are monotone and bounded.

Definition 2.1. Let the function f be essentially bounded and measurable on Ω. Then the real-valued functions

$$\Phi(t) = \lim_{\delta \to 0+0} G_f(t,\delta) \qquad \text{and} \qquad \phi(t) = \lim_{\delta \to 0+0} g_f(t,\delta)$$

are called the *essential upper limit function* and *essential lower limit function of* f, respectively.

If the functions f_1 and f_2 belong to the same equivalence class $\tilde{f} \in L^\infty(\Omega, \mathfrak{B}, \lambda)$ then the values $G_{f_1}(t,\delta)$ and $G_{f_2}(t,\delta)$ coincide for all $t \in \Omega$ and arbitrary $\delta > 0$, and the same is true for $g_{f_1}(t,\delta)$ and $g_{f_2}(t,\delta)$. Consequently, all individual functions in an equivalence class have the same essential limit functions. To avoid confusion, we will write $\Phi_f(t)$ and $\phi_f(t)$ to underline the relation to the original function whenever this is necessary.

Lemma 2.2. *Let* $\mathrm{A}' \subseteq \Omega$ *have positive measure. Then a subset* $\mathrm{A} \subseteq \mathrm{A}'$ *of equal measure exists with the property*

($*$) *For all* $t \in \mathrm{A}$ *and arbitrary* $\delta > 0$ *the set* $\mathrm{A} \cap \mathrm{K}(t,\delta)$ *has positive measure.*

Proof. Consider the set B of all points $t \in \mathrm{A}'$ such that a radius $\delta(t) > 0$ exists with $|\mathrm{A}' \cap K(t,\delta(t))| = 0$. Obviously, $|\mathrm{B} \cap \mathrm{K}(t,\delta(t))| = 0$ holds for all $t \in \mathrm{B}$. On the other hand, the balls $\mathrm{K}(t,\delta(t))$ form an open covering of the set B. Since Ω as a metric subspace of $\mathbb{R}^m$ is separable, we may apply Lindelöf's theorem [3: p. 46] and choose from $\{\mathrm{K}(t,\delta(t))\}$ an at most countable subcovering for B. If t_i denote the centers of the balls used herein, we obtain $|\mathrm{B}| \leqslant \sum_{i=1}^{\infty} |\mathrm{B} \cap \mathrm{K}(t_i,\delta(t_i))| = 0$. Taking $\mathrm{A} = \mathrm{A}' \setminus \mathrm{B}$, we conclude that for all $t \in \mathrm{A}$ and arbitrary $\delta > 0$ the relation $|\mathrm{A}' \cap \mathrm{K}(t,\delta)| > 0$ holds. But then from $|\mathrm{B}| = 0$ we get $|\mathrm{A} \cap \mathrm{K}(t,\delta)| > 0$ so that A is a set with the desired property ($*$) ∎

Remark that under our assumptions Ω itself is a set with property ($*$).

Theorem 2.3. *Two measurable, essentially bounded functions* f_1 *and* f_2 *belong to different classes* $\tilde{f}_1 \neq \tilde{f}_2$ *if and only if there exists a point* $t_0 \in \Omega$ *where at least one of the relations* $\Phi_{f_1}(t_0) \neq \Phi_{f_2}(t_0)$ *or* $\phi_{f_1}(t_0) \neq \phi_{f_2}(t_0)$ *holds.*

Proof. Assume (without loss of generality) that the set $\mathrm{A}' = \{t \in \Omega : f_1(t) < f_2(t)\}$ has positive measure. At least one of the sets $\mathrm{A}'(\varepsilon) = \{t \in \Omega : f_2(t) - f_1(t) \geqslant \varepsilon\}$ (which are measurable since the functions f_1 and f_2 are such) has positive measure then. Denote this set by $\mathrm{A}'(\varepsilon_0)$. Due to Lemma 2.2, $\mathrm{A}'(\varepsilon_0)$ contains a subset $\mathrm{A}(\varepsilon_0)$ of equal measure but with property ($*$). For arbitrary $t_0 \in \mathrm{A}(\varepsilon_0)$ the estimate

$$G_{f_2}(t_0,\delta) = \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta)} f_2(t) \geqslant \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta)} f_1(t) + \varepsilon_0 = G_{f_1}(t_0,\delta) + \varepsilon_0$$

holds, so that $\Phi_{f_2}(t_0) - \Phi_{f_1}(t_0) \geqslant \varepsilon_0$. Analogously, in the case that $\{t \in \Omega : f_1(t) > f_2(t)\} = \mathrm{B}'$ is a set of positive measure one can find a $t_0 \in \Omega$ where $\phi_{f_1}(t_0) - \phi_{f_2}(t_0) \geqslant \varepsilon_0$.

On the contrary, let (without loss of generality) $\Phi_{f_1}(t_0) < \Phi_{f_2}(t_0)$ in a point $t_0 \in \Omega$. Then choose a sequence $\delta_n \downarrow 0$. From the monotonicity of G_f we deduce the existence of a neighborhood $\mathrm{K}(t_0,\delta_n)$ of t_0 where

$$G_{f_1}(t_0,\delta_n) = \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta_n)} f_1(t) < \Phi_{f_2}(t_0) \leqslant \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta_n)} f_2(t).$$

Consequently, the functions f_1 and f_2 must be distinct on some subset of $\mathrm{K}(t_0,\delta_n)$ having nonzero measure so that they cannot belong to the same equivalence class in $L^\infty(\Omega, \mathfrak{B}, \lambda)$. The proof for the case that $\phi_{f_1}(t_0) \neq \phi_{f_2}(t_0)$ is analogous ∎

Lemma 2.4.

(1) *If two measurable, essentially bounded functions f_1 and f_2 satisfy the relation $f_1(t) \leqslant f_2(t)$ almost everywhere on Ω then for all $t \in \Omega$ the inequalities $\Phi_{f_1}(t) \leqslant \Phi_{f_2}(t)$ and $\phi_{f_1}(t) \leqslant \phi_{f_2}(t)$ hold.*

(2) *For all $t \in \Omega$ and arbitrary $\delta > 0$ the following estimates hold:*

$$-\|\tilde{f}\| \leqslant \operatorname*{ess\,inf}_{t \in \Omega} f(t) \leqslant g_f(t,\delta) \leqslant \phi(t) \leqslant \Phi(t) \leqslant G_f(t,\delta) \leqslant \operatorname*{ess\,sup}_{t \in \Omega} f(t) \leqslant \|\tilde{f}\|.$$

Proof. Both assertions follow directly from Definition 2.1 ∎

Theorem 2.5. *For any sequence $t_n \to t_0$,*

$$\limsup_{n\to\infty} \Phi(t_n) \leqslant \Phi(t_0) \qquad \text{as well as} \qquad \liminf_{n\to\infty} \phi(t_n) \geqslant \phi(t_0).$$

Therefore, the functions Φ and ϕ are upper resp. lower semicontinuous, measurable and bounded functions on Ω.

Proof. Suppose the contrary, i.e. the existence of a sequence $t_n \to t_0$ such that $\limsup_{n\to\infty} \Phi(t_n) - \Phi(t_0) \geqslant \varepsilon > 0$. Then $\{t_n\}$ must contain some subsequence $\{t_i\}$ with $\Phi(t_i) \geqslant \Phi(t_0) + \varepsilon/2$. Consequently, for arbitrary given $\delta > 0$ there exists an index i such that, at first, $t_i \in \mathrm{K}(t_0,\delta)$, and secondly,

$$G_f(t_0,\delta) = \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta)} f(t) \geqslant \Phi(t_i) = \lim_{\gamma \to 0+0} \operatorname*{ess\,sup}_{\mathrm{K}(t_i,\gamma) \subseteq \mathrm{K}(t_0,\delta)} f(t) \geqslant \Phi(t_0) + \varepsilon/2.$$

This leads to the contradiction

$$\Phi(t_0) = \lim_{\delta \to 0+0} G_f(t_0,\delta) \geqslant \liminf_{i\to\infty} \Phi(t_i) > \Phi(t_0).$$

In analogous manner the respective relation for ϕ may be proved. Thus the functions Φ and ϕ are semicontinuous and, consequently, belong to the first Baire's class [3: p. 403 / Theorem 3] what particularly yields that Φ and ϕ are measurable. The boundedness follows from Lemma 2.4 ∎

Lemma 2.6. *For any given $t_0 \in \Omega$ there exist a sequence $t_n \to t_0$ with the property $\lim_{n\to\infty} \Phi(t_n) = \Phi(t_0)$ and a sequence $t_n' \to t_0$ with the property $\lim_{n\to\infty} \phi(t_n') = \phi(t_0)$.*

Proof. Suppose the contrary, i.e. $\limsup_{n\to\infty} \Phi(t_n) < \Phi(t_0)$ for any sequence $t_n \to t_0$ what means the existence of some $\varepsilon > 0$ with $\Phi(t_0) > \Phi(t) + \varepsilon$ for all $t \in \mathrm{K}(t_0,\delta)$ ($\delta > 0$). Then for every $t \in \mathrm{K}(t_0,\delta)$ we find a ball $\mathrm{K}(t,\gamma(t))$ with the property $G_f(t,\gamma(t)) < \Phi(t_0) - \varepsilon/2$. From this family of balls we select a countable subfamily (with centres t_i) covering the original set $\mathrm{K}(t_0,\delta)$. Then we obtain a contradiction by

$$\begin{aligned}
\Phi(t_0) &\leqslant G_f(t_0,\delta) = \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta)} f(t) \\
&\leqslant \sup_i \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_i,\gamma(t_i))} f(t) = \sup_i G_f(t_i,\gamma(t_i)) \\
&\leqslant \Phi(t_0) - \varepsilon/2.
\end{aligned}$$

Analogously one can show the relation for ϕ, and hence the lemma ∎

The essential limit functions Φ and ϕ and the usual limit functions

$$\Psi(t_0) = \lim_{\delta \to 0+0} \sup_{t \in \mathrm{K}(t_0,\delta)} f(t) \qquad \text{and} \qquad \psi(t_0) = \lim_{\delta \to 0+0} \inf_{t \in \mathrm{K}(t_0,\delta)} f(t),$$

interpreted as functions with values in $\mathbb{R} \cup \{-\infty, +\infty\}$, are connected by the inequalities

$$\psi(t) \leqslant \phi(t) \qquad \text{and} \qquad \Phi(t) \leqslant \Psi(t)$$

for all $t \in \Omega$. Indeed, the relations

$$G_f(t_0,\delta) = \operatorname*{ess\,sup}_{t \in \mathrm{K}(t_0,\delta)} f(t) \leqslant \sup_{t \in \mathrm{K}(t_0,\delta)} f(t)$$

and

$$\inf_{t \in \mathrm{K}(t_0,\delta)} f(t) \leqslant \operatorname*{ess\,inf}_{t \in \mathrm{K}(t_0,\delta)} f(t) = g_f(t_0,\delta)$$

remain to be valid in the limit for $\delta \downarrow 0$. The example of the function

$$f(x) = \begin{cases} (-2)^n & \text{for } x = (2k+1)/2^n \quad (n \in \mathbb{N},\ k \in \mathbb{Z}) \\ 0 & \text{elsewhere} \end{cases}$$

given on $\Omega = [-1\,,\,1]$ shows that usual and essential limit functions may differ on the whole domain Ω. Indeed, $\phi_f(t) = \Phi_f(t) \equiv 0$ but $\psi_f(t) \equiv -\infty$ and $\Psi_f(t) \equiv +\infty$.

Remark that a function h is upper semicontinuous at an accumulation point t_0 of its range of definition if and only if $h(t_0) = \Psi_h(t_0)$, and lower semicontinuous in the case $h(t_0) = \psi_h(t_0)$ (see [3: p. 127]).

Theorem 2.7. *The inequalities*

$$\phi(t) \leqslant f(t) \leqslant \Phi(t) \qquad \textit{and} \qquad g_f(t,\delta) \leqslant f(t) \leqslant G_f(t,\delta)$$

are valid for almost every $t \in \Omega$ with arbitrary positive δ.

Proof. The proof is based on the following theorem [3: p. 410 / Theorem 7]:

Assume that the set A *is measurable and of nonzero measure. Let f be a measurable function on* A *which is equivalent to some finite function, and ε any positive constant. Then a perfect set* $\mathrm{B}' \subseteq \mathrm{A}$ *with* $|\mathrm{A}|/2 \leqslant |\mathrm{B}'| < |\mathrm{A}|$ *exists such that the restriction* $f|\mathrm{B}'$ *is finite and continuous.*

We have to show that the set A$'$ of the points with $f(t) > \Phi_f(t)$ has zero measure. Supposing the contrary, Lemma 2.2 yields the existence of a subset $\mathrm{A} \subseteq \mathrm{A}'$ having equal measure and satisfying condition $(*)$. Consider now the restriction $f|\mathrm{A}$ which is equivalent (on A) to some bounded, measurable function. The above cited theorem allows to find a subset $\mathrm{B}' \subseteq \mathrm{A}$ with $|\mathrm{A}|/2 \leqslant |\mathrm{B}'| < |\mathrm{A}|$ where $f|\mathrm{B}'$ is continuous. According to Lemma 2.2 this set also contains a subset of equal measure with property $(*)$, say B. Restricting f to B we see that

$$\lim_{\delta \to 0+0} \inf_{t \in \mathrm{K}(t_0,\delta) \cap \mathrm{B}} f(t) = \psi_{f|\mathrm{B}}(t) = f|\mathrm{B}(t) = \Psi_{f|\mathrm{B}}(t) = \lim_{\delta \to 0+0} \sup_{t \in \mathrm{K}(t_0,\delta) \cap \mathrm{B}} f(t).$$

Since B has the property $(*)$, the values $\Phi_{f|B}(t)$ and $\phi_{f|B}(t)$ are well-defined for arbitrary $t \in B$. From $\psi_{f|B}(t) \leqslant \phi_{f|B}(t) \leqslant \Phi_{f|B}(t) \leqslant \Psi_{f|B}(t)$ we deduce $\phi_{f|B}(t) = f|B(t) = \Phi_{f|B}(t)$ for all $t \in B$. Finally, we conclude

$$\Phi_{f|B}(t) = \lim_{\delta\to 0+0} \operatorname*{ess\,sup}_{t\in K(t_0,\delta)\cap B} f(t) \leqslant \lim_{\delta\to 0+0} \operatorname*{ess\,sup}_{t\in K(t_0,\delta)} f(t) = \Phi_f(t_0)$$

for any $t_0 \in B$.

Summing up, we see that A' must have a subset B of nonzero measure where $f(t) = f|B(t) = \Phi_{f|B}(t) \leqslant \Phi_f(t)$ holds in contradiction to the definition of A'. Thus A' is a λ-null set. Repeating the arguments with $\phi(t)$ we see that the set where $f(t) < \phi(t)$ is of zero measure too, and hence the first statement of the theorem is valid. Taking into account now Lemma 2.4, we arrive at the second inclusion $g_f(t,\delta) \leqslant f(t) \leqslant G_f(t,\delta)$ ∎

Lemma 2.8.

(1) *There exist points $t_0 \in \Omega$ with $\Phi(t_0) = \operatorname{ess\,sup}_{t\in\Omega} f(t)$ and $s_0 \in \Omega$ with $\phi(s_0) = \operatorname{ess\,inf}_{t\in\Omega} f(t)$. Thus,*

$$\max_{t\in\Omega} \Phi(t) = \operatorname*{ess\,sup}_{t\in\Omega} f(t) \qquad \text{and} \qquad \min_{t\in\Omega} \phi(t) = \operatorname*{ess\,inf}_{t\in\Omega} f(t).$$

(2) *In addition, the essential limit functions satisfy*

$$\max_{t\in\Omega} \Phi(t) = \operatorname*{ess\,sup}_{t\in\Omega} \Phi(t) \qquad \text{and} \qquad \min_{t\in\Omega} \phi(t) = \operatorname*{ess\,inf}_{t\in\Omega} \phi(t).$$

So the norm of $\tilde{f}$ may be expressed by

$$\| \tilde{f} \| = \max\left(\left|\max_{t\in\Omega} \Phi_f(t)\right|, \left|\min_{t\in\Omega} \phi_f(t)\right| \right) = \max\left(\| \tilde{\Phi}_f \|, \| \tilde{\phi}_f \| \right).$$

Proof. (1) Denote for shortness $\operatorname{ess\,sup}_{t\in\Omega} f(t) = c$. The set $B = \{ t \in \Omega : c < f(t) \}$ then has measure zero. On the other hand, for arbitrary $\varepsilon > 0$ there exists a set $A'(\varepsilon)$ of nonzero measure where $c - \varepsilon < f(t)$. Lemma 2.2 guarantees the existence of a subset $A(\varepsilon) \subseteq A'(\varepsilon) \setminus B$ with equal measure and property $(*)$. For all $t \in A(\varepsilon)$ it follows immediately that $c - \varepsilon \leqslant G_f(t,\delta) \leqslant c$ for all $\delta > 0$, and thus $c - \varepsilon \leqslant \Phi_f(t) \leqslant c$. Given any sequence $\varepsilon_n \downarrow 0$ and choosing $t_n \in A(\varepsilon_n)$ arbitrarily, we obtain $\lim_{n\to\infty} \Phi_f(t_n) = c$. The compactness of Ω allows to find a subsequence $\{ t_i \}$ converging to a point $t_0 \in \Omega$, and the upper semicontinuity of Φ then yields the relation

$$\Phi(t_0) \leqslant \operatorname*{ess\,sup}_{t\in\Omega} f(t) = \lim_{i\to\infty} \Phi(t_i) \leqslant \Phi(t_0).$$

Therefore we have $\Phi(t_0) = \operatorname{ess\,sup}_{t\in\Omega} f(t)$. Analogous arguments lead to the result on ϕ.

(2) Since Theorem 2.7 we have $f(t) \leqslant \Phi(t) \leqslant \max_{t\in\Omega} \Phi(t) = \operatorname{ess\,sup}_{t\in\Omega} f(t)$ almost everywhere on Ω, so it follows that

$$\operatorname*{ess\,sup}_{t\in\Omega} f(t) \leqslant \operatorname*{ess\,sup}_{t\in\Omega} \Phi(t) \leqslant \operatorname*{ess\,sup}_{t\in\Omega} f(t).$$

Analogously,

$$\operatorname*{ess\,inf}_{t\in\Omega} f(t) = \min_{t\in\Omega} \phi(t) \leqslant \phi(t) \leqslant f(t)$$

holds on Ω a.e., so it follows

$$\operatorname*{ess\,inf}_{t\in\Omega} f(t) \leqslant \operatorname*{ess\,inf}_{t\in\Omega} \phi(t) \leqslant \operatorname*{ess\,inf}_{t\in\Omega} f(t).$$

This result together with the inequalities

$$\operatorname*{ess\,inf}_{t\in\Omega} \phi(t) \leqslant \operatorname*{ess\,inf}_{t\in\Omega} \Phi(t) \qquad \text{and} \qquad \operatorname*{ess\,sup}_{t\in\Omega} \phi(t) \leqslant \operatorname*{ess\,sup}_{t\in\Omega} \Phi(t)$$

and the equation

$$\|\tilde f\| = \max\Big(\big|\operatorname*{ess\,inf}_{t\in\Omega} f(t)\big|, \big|\operatorname*{ess\,sup}_{t\in\Omega} f(t)\big|\Big)$$

gives our last assertion ∎

3. The essential limit set

In the present section we consider a set-valued characteristic for L^∞-classes which generalizes in appropriate manner the usual limit of a function in a point $t_0 \in \Omega$.

Definition 3.1. As before, let $f : \Omega \to \mathbb{R}$ be a given essentially bounded and measurable function. The value $v \in \mathbb{R}$ is called *essential accumulation value of f in $t_0 \in \Omega$* if there exists a set $\mathrm{M}_v \subseteq \Omega$ with both the following properties:

(a) For arbitrary $\delta > 0$ the set $\mathrm{M}_v \cap \mathrm{K}(t_0, \delta)$ is of positive measure.

(b) For all $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{ess\,sup}_{t\in \mathrm{M}_v\cap\mathrm{K}(t_0,\delta)} |f(t) - v| \leqslant \varepsilon$

(cf. [5: Definition 1]).

Remark that the set M_v may be always replaced by a set M'_v differing from M_v by a null set only. The same holds for $\mathrm{M}_v \cap \mathrm{K}(t_0, \delta)$.

Definition 3.2. The set $\mathrm{E}_f(t_0)$ of all essential accumulation values of f in t_0 is called the *essential limit set of f in t_0* (cf. [5: Definition 2]).

If two functions f_1 and f_2 are equivalent, then they coincide almost everywhere on arbitrary sets of positive measure what is particularly true for the sets M_v with property (a) from Definition 3.1. Hence, $\mathrm{E}_{f_1}(t) = \mathrm{E}_{f_2}(t)$ for all $t \in \Omega$.

Lemma 3.3. *For all $t_0 \in \Omega$ the values $\phi_f(t_0)$ and $\Phi_f(t_0)$ are enclosed in $\mathrm{E}_f(t_0)$.*

Proof. The definition $G_f(t_0,\delta) = \operatorname{ess\,sup}_{t\in\mathrm{K}(t_0,\delta)} f(t)$ guarantees the existence of a subset $\mathrm{K}_\delta \subseteq \mathrm{K}(t_0,\delta) \subseteq \Omega$ of positive measure such that $G_f(t,\delta) - \delta \leqslant f(t) \leqslant G_f(t,\delta)$ almost everywhere on K_δ or, equivalently, $\operatorname{ess\,sup}_{t\in\mathrm{K}_\delta} |f(t) - G_f(t,\delta)| \leqslant \delta$. Setting $\mathrm{M} = \cup_{0<\delta\leqslant 1} \mathrm{K}_\delta$, we have $|\mathrm{M} \cap \mathrm{K}(t_0,\delta_0)| = |\cup_{0<\delta\leqslant\delta_0} \mathrm{K}_\delta| > 0$ for all $\delta_0 > 0$. Given now some $\varepsilon > 0$, we choose $\delta \leqslant \varepsilon/2$ according to $|G_f(t_0,\delta) - \Phi(t_0)| \leqslant \varepsilon/2$ and conclude that

$$\begin{aligned} &\operatorname*{ess\,sup}_{t \in M \cap K(t_0,\delta)} |f(t) - \Phi(t_0)| \\ &\leqslant \operatorname*{ess\,sup}_{t \in M \cap K(t_0,\delta)} |f(t) - G_f(t_0,\delta)| + |G_f(t_0,\delta) - \Phi(t_0)| \leqslant \varepsilon. \end{aligned}$$

Thus we obtain $\Phi(t_0) \in \mathrm{E}_f(t_0)$. The relation $\phi(t_0) \in \mathrm{E}_f(t_0)$ follows analogously ∎

Lemma 3.4. *The inclusion* $\mathrm{E}_f(t_0) \subseteq [\phi(t_0), \Phi(t_0)] \subseteq [-\|\tilde f\|, \|\tilde f\|]$ *is valid for all* $t_0 \in \Omega$.

Proof. In view of Lemma 2.4, we have to prove only the first inclusion. Let v be an element of $\mathrm{E}_f(t_0)$ satisfying $v - \Phi(t_0) = \varepsilon > 0$. Then we find a set M_v such that $|\mathrm{M}_v \cap \mathrm{K}(t_0,\delta)| > 0$ for all $\delta > 0$ and $f(t) \geqslant \Phi(t_0) + \varepsilon/2$ for almost all $t \in \mathrm{M}_v$. But this last relation leads to a contradiction: we get

$$\Phi(t_0) = \lim_{\delta \to 0+0} G_f(t_0,\delta) = \lim_{\delta \to 0+0} \operatorname*{ess\,sup}_{t \in K(t_0,\delta)} f(t) \geqslant \Phi(t_0) + \varepsilon/2.$$

It is checked similiarly that $\mathrm{E}_f(t_0)$ cannot contain any element v with $\phi(t_0) - v > 0$ what completes the proof ∎

Lemma 3.5. *Given* $t_0 \in \Omega$ *and a set* $\mathrm{K} \subseteq \Omega$ *with property* (a), *i.e.* $|\mathrm{K} \cap \mathrm{K}(t_0,\delta)| > 0$ *for arbitrary* $\delta > 0$. *Let* $f|\mathrm{K}$ *denote the restriction of* f *to* K. *Then* $\mathrm{E}_{f|\mathrm{K}}(t_0)$ *is a subset of* $\mathrm{E}_f(t_0)$.

Proof. For any $v \in \mathrm{E}_{f|\mathrm{K}}(t_0)$ there exists a set $\mathrm{M}_v \subseteq \mathrm{K} \subseteq \Omega$ with the properties

(a) $|\mathrm{M}_v \cap \mathrm{K}(t_0,\delta)| > 0$ for all $\delta > 0$ and

(b) For all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname*{ess\,sup}_{t \in M_v \cap K(t_0,\delta)} |f|\mathrm{K}(t) - v| = \operatorname*{ess\,sup}_{t \in (M_v \cap K) \cap K(t_0,\delta)} |f(t) - v| \leqslant \varepsilon.$$

Consequently, $v \in \mathrm{E}_f(t_0)$ ∎

Theorem 3.6. *For almost every* $t \in \Omega$ *the relation* $f(t) \in \mathrm{E}_f(t)$ *is true.*

Proof. The proof is similiar to that of Theorem 2.7. Assuming that the set A′ of all points where $f(t) \notin \mathrm{E}_f(t)$ has positive measure, we can construct a subset $\mathrm{B} \subset \mathrm{A}'$ with property (*) and positive measure such that $f|\mathrm{B}(t) = \psi_{f|\mathrm{B}}(t) = \phi_{f|\mathrm{B}}(t) = \Phi_{f|\mathrm{B}}(t) = \Psi_{f|\mathrm{B}}(t)$ hold on B. Consequently, $f|\mathrm{B}(t) = \Phi_{f|\mathrm{B}}(t) \in \mathrm{E}_{f|\mathrm{B}}(t)$. Since B has property (*), we may apply Lemma 3.5 and for arbitrary $t \in \mathrm{B}$ obtain $\mathrm{E}_{f|\mathrm{B}}(t) \subseteq \mathrm{E}_f(t)$. Thus, $f(t) \in \mathrm{E}_f(t)$ on B what stands in direct contradiction to the definition of A′ ∎

Lemma 3.7.

(1) *Let* $\mathrm{E}_f(t_0)$ *contain at least two elements* $v_1 \neq v_2$ *with corresponding sets* M_{v_1} *and* M_{v_2}. *There exists a* $\delta_0 > 0$ *such that* $|(\mathrm{M}_{v_1} \cap \mathrm{M}_{v_2}) \cap \mathrm{K}(t_0,\delta_0)| = 0$. *In particular, the sets* M_{v_i} *may be chosen such that their intersection is empty or equal to* $\{t_0\}$.

(2) *If* $\mathrm{E}_f(t_0)$ *contains* n *different values* $v_1,\dots,v_n$, *then the corresponding sets* $\mathrm{M}_{v_1},\dots,\mathrm{M}_{v_n}$ *may be chosen disjunct or having the intersection* $\{t_0\}$ *only.*

Proof. (1) Assume that for arbitrary $\delta > 0$ the set $(M_{v_1} \cap M_{v_2}) \cap K(t_0, \delta)$ has positive measure. Choose $\varepsilon < (v_1 + v_2)/2$. Then we find a $\delta_1 > 0$ with $\operatorname{ess\,sup}_{t \in M_{v_1} \cap K(t_0,\delta_1)} |f(t) - v_1| \leqslant \varepsilon$, and also a $\delta_2 > 0$ such that $\operatorname{ess\,sup}_{t \in M_{v_2} \cap K(t_0,\delta_2)} |f(t) - v_2| \leqslant \varepsilon$. Taking $\delta_3 = \min(\delta_1, \delta_2)$, we get

$$\operatorname*{ess\,sup}_{t \in (M_{v_1} \cap M_{v_2}) \cap K(t_0,\delta_3)} |f(t) - v_1| \leqslant \operatorname*{ess\,sup}_{t \in M_{v_1} \cap K(t_0,\delta_1)} |f(t) - v_1| \leqslant \varepsilon$$

$$\operatorname*{ess\,sup}_{t \in (M_{v_1} \cap M_{v_2}) \cap K(t_0,\delta_3)} |f(t) - v_2| \leqslant \operatorname*{ess\,sup}_{t \in M_{v_2} \cap K(t_0,\delta_2)} |f(t) - v_2| \leqslant \varepsilon.$$

Since the relations $|f(t) - v_1| \leqslant (v_1 + v_2)/2$ and $|f(t) - v_2| \leqslant (v_1 + v_2)/2$ should be true almost everywhere on $(M_{v_1} \cap M_{v_2}) \cap K(t_0, \delta_3)$, we arrived at a contradiction. Consequently, there exists a $\delta_0 > 0$ with $|(M_{v_1} \cap M_{v_2}) \cap K(t_0, \delta_0)| = 0$, and further we can replace M_{v_1} by $(M_{v_1} \setminus (M_{v_1} \cap M_{v_2})) \cap K(t_0, \delta_0)$ resp. $((M_{v_1} \setminus (M_{v_1} \cap M_{v_2})) \cup \{t_0\}) \cap K(t_0, \delta_0)$. Analogous relations hold with M_{v_2} then. Repeating this construction n times respectively, we arrive at assertion (2) ∎

Theorem 3.8. *For all $t \in \Omega$ the essential limit set $E_f(t)$ is compact.*

Proof. Consider a sequence of elements $v_n \in E_f(t_0)$ converging to $v \in \mathbb{R}$. Without loss of generality we will assume that $\{v_n\}$ is monotone. Every v_n corresponds to some set $M_{v_n} \subseteq \Omega$ with properties

(a) $|M_{v_n} \cap K(t_0, \delta)| > 0$ for all $\delta > 0$ and

(b) for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{ess\,sup}_{t \in K(t_0,\delta)} |f(t) - v_n| \leqslant \varepsilon/2$

from Definition 3.1. Let us fix the values δ_n related to $\varepsilon = 2|v_n - v|$. Since

$$|M_{v_n} \cap K(t_0, \delta_n)| = \sum_{k=0}^{\infty} |M_{v_n} \cap (K(t_0, \delta_n/2^k) \setminus K(t_0, \delta_n/2^{k+1}))| > 0,$$

in relation to each $n \in \mathbb{N}$ there exists a minimal $k_n \in \mathbb{N}$ for which $M'_{v_n} = M_{v_n} \cap (K(t_0, \delta_n/2^{k_n}) \setminus K(t_0, \delta_n/2^{k_n+1}))$ is of positive measure. Now we choose from $\{\delta_n\}$ a subsequence $\delta_i \downarrow 0$ satisfying $\delta_{i+1} \leqslant \delta_i/2^{k_i+1}$ and define $M = \cup_i M'_{v_i}$. Obviously, this set has property (a). Further, for $\delta = \delta_i$ corresponding to $\varepsilon \geqslant 2|v_i - v|$ we have

$$\operatorname*{ess\,sup}_{t \in M \cap K(t_0,\delta_i)} |f(t) - v| \leqslant \operatorname*{ess\,sup}_{t \in M \cap K(t_0,\delta_i)} |f(t) - v_i| + |v_i - v| \leqslant \varepsilon,$$

that is, (b) holds too, and v consequently lies in $E_f(t_0)$. Hence the essential limit set is closed. While Lemma 3.4 yields the boundedness of $E_f(t_0) \subseteq \mathbb{R}$, it turns out to be compact ∎

Example 3.9. The function

$$f(t) = \begin{cases} \sin(1/t) & \text{for } 0 < |t| \leqslant \pi \\ 0 & \text{for } t = 0 \end{cases}$$

in $t_0 = 0$ has the non-denumerable essential limit set $E_f(0) = [-1, 1]$. To see this, for given $v \in [-1, 1]$ we arrange the (countably many) isolated points t_n with $\sin(1/t_n) =$

v into a sequence $t_n \to t_0$, and then we find for every t_n a neighborhood where $v - 1/n < \sin(1/t) < v + 1/n$ holds. Joining these neighborhoods we obtain a set M_v satisfying conditions (a) and (b) from the definition with respect to the point $t_0 = 0$.

Theorem 3.10. *Two given measurable, essentially bounded functions f_1 and f_2 belong to different equivalence classes in $L^\infty(\Omega, \mathfrak{B}, \lambda)$ if and only if there exists a point $t_0 \in \Omega$ where their essential limit sets do not coincide.*

Proof. If $\tilde{f}_1 \neq \tilde{f}_2$, then one can find a point t_0 with $\Phi_{f_1}(t_0) \neq \Phi_{f_2}(t_0)$ or $\phi_{f_1}(t_0) \neq \phi_{f_2}(t_0)$ according to Theorem 2.3. By Lemma 3.3 together with Lemma 3.4, the essential limit sets $\mathrm{E}_{f_1}(t_0)$ and $\mathrm{E}_{f_2}(t_0)$ then must differ at least by one of these values.

On the contrary, let us fix some $t_0 \in \Omega$ and assume that some element $v_1 \in \mathrm{E}_{f_1}(t_0)$ does not belong to $\mathrm{E}_{f_2}(t_0)$. Remember that (due to Theorem 3.8) the essential limit sets are closed; so this means the existence of a closed interval $[v_1 - \varepsilon, v_1 + \varepsilon] \subset \mathbb{R}$ which is disjoint to $\mathrm{E}_{f_2}(t_0)$. Further, a set M_{v_1} with property (a) is related to v_1. Let us consider the restriction $f_2|\mathrm{M}_{v_1}$. By Lemma 3.5, $\mathrm{E}_{f_2|\mathrm{M}_{v_1}}(t_0) \subseteq \mathrm{E}_{f_2}(t_0)$ is also disjoint to $[v_1 - \varepsilon, v_1 + \varepsilon]$. Denote by v_2 the element in $\mathrm{E}_{f_2|\mathrm{M}_{v_1}}(t_0)$ having minimal distance to v_1, and let $\mathrm{M}_{v_2} \subseteq \mathrm{M}_{v_1}$ be the corresponding set with property (a). There exist a $\delta_1 > 0$ then with $\operatorname{ess\,sup}_{t \in \mathrm{M}_{v_1} \cap \mathrm{K}(t_0,\delta_1)} |f_1(t) - v_1| < \varepsilon/2$ and, respectively, a $\delta_2 > 0$ with $\operatorname{ess\,sup}_{t \in \mathrm{M}_{v_2} \cap \mathrm{K}(t_0,\delta_2)} \big| f_2|\mathrm{M}_{v_1}(t) - v_2 \big| < \varepsilon/2$. Setting $\delta_3 = \min(\delta_1, \delta_2)$ the values of f_1 and $f_2|\mathrm{M}_{v_1}$ turn out to be distinct almost everywhere on $\mathrm{M}_{v_2} \cap \mathrm{K}(t_0, \delta_3)$, i.e. on a set of positive measure. It follows that f_1 and f_2 cannot fall into the same equivalence class ∎

Our next concern is to analyze some properties of the set-valued map $\mathrm{E}_f(t) \colon \Omega \to \mathfrak{P}(\mathbb{R})$. We call a set-valued map $\Gamma \colon \Omega \to \mathfrak{P}(\mathbb{R})$ *upper semicontinuous* (in the sense of Bouligand, Kuratowski, Wilson) at a point $t_0 \in \Omega$ if for every open set S with $\Gamma(t_0) \subseteq \mathrm{S} \subseteq \Omega$ a neighborhood $\mathrm{K}(t_0, \delta) \subseteq \Omega$ exists such that $\Gamma(t) \subseteq \mathrm{S}$ is valid for all $t \in \mathrm{K}(t_0, \delta)$. The *lower semicontinuity* at $t_0 \in \Omega$ is described then as follows: For any $v_0 \in \Gamma(t_0)$ and any sequence $t_n \to t_0$ there exists a sequence of elements $v_n \in \Gamma(t_n)$ converging to v_0 [2: p. 38 f./Definitions 1.4.1 and 1.4.2].

Theorem 3.11. *For any measurable, essentially bounded function f the set-valued map $\mathrm{E}_f(t) : \Omega \to \mathfrak{P}(\mathbb{R})$ is upper semicontinuous at every $t \in \Omega$.*

Proof. Let the upper semicontinuity of the map $t \mapsto \mathrm{E}_f(t)$ fail at some point $t_0 \in \Omega$. As defined above, we find then some open set $\mathrm{S} \subset \mathbb{R}$ with $\mathrm{E}_f(t_0) \subseteq \mathrm{S}$ and a sequence $t_n \to t_0$ such that every set $\mathrm{E}_f(t_n)$ contains at least one point $v_n \notin \mathrm{S}$. Due to Lemma 3.4 the sequence $\{v_n\}$ is bounded by $\|\tilde{f}\|$, consequently, it has a convergent subsequence $v_i \to v_0 \in [-\|\tilde{f}\|, \|\tilde{f}\|]$. While the complement of S is closed we conclude $v_0 \notin \mathrm{S}$. To obtain a contradiction, we show that $v_0 \in \mathrm{E}_f(t_0)$.

First extract a subsequence of $\{t_i\}$ for which the distances $|t_j - t_0|$ are monotonically decreasing. Find further a monotone sequence $\delta_j \downarrow 0$ such that t_j is an inner point of $\mathrm{K}(t_0, \delta_j) \setminus \mathrm{K}(t_0, \delta_{j+1})$ for the respective index j. The choice of v_j and of t_j then guarantees the existence of sets M_{v_j} (having nonzero measure) and of balls $\mathrm{K}(t_j, \delta(t_j)) \subset \mathrm{K}(t_0, \delta_j) \setminus \mathrm{K}(t_0, \delta_{j+1})$ for which $\operatorname{ess\,sup}_{t \in \mathrm{M}_{v_j} \cap \mathrm{K}(t_j, \delta(t_j))} |f(t) - v_j| \leqslant 1/j$. Now define $\mathrm{M}_{v_0} = \cup_j (\mathrm{M}_{v_j} \cap \mathrm{K}(t_j, \delta(t_j)))$. By construction, the set M_{v_0} has properties (a) and (b) with respect to t_0 and v_0 what implies $v_0 \in \mathrm{E}_f(t_0)$ ∎

According to Theorem 3.6, the set-valued map $t \mapsto \mathrm{E}_f(t)$ has the bounded, measurable selector function $h(t)$ defined by $h(t) = f(t)$ in all points t where $f(t) \in \mathrm{E}_f(t)$ and $h(t) = \Phi_f(t)$ elsewhere.

Lemma 3.12. *Suppose there exists a set of positive measure where the essential limit sets* $\mathrm{E}_f(t)$ *contain at least two elements. Then the set-valued map* $\mathrm{E}_f(t)\colon \Omega \to \mathfrak{P}(\mathbb{R})$ *has at least two bounded, measurable selector functions which belong to different equivalence classes in* $L^\infty(\Omega, \mathfrak{B}, \lambda)$.

Proof. From $\mathrm{E}_f(t)$ we can always choose the measurable selections $\phi(t)$ and $\Phi(t)$ (Lemma 3.3). If the essential limit sets have more than one element on a set $\mathrm{K} \subseteq \Omega$ of positive measure, then $\phi(t) < \Phi(t)$ on K, and thus $\tilde{\phi} \neq \tilde{\Phi}$ ∎

Example 3.13. The construction of the classical ternary Cantor's set on $\Omega = [0, 1]$ allows the following modification: In the first step, delete from Ω the centered open interval of length $1/4$, then in the second step an open interval of length $1/4^2$ from the entre of every remaining subinterval and so on, i.e. step k consists in the deleting of 2^{k-1} subintervals of length $1/4^k$ every. In the limit for $k \to \infty$ we get a set A with analogous topological properties as Cantor's discontinuum, i.e. a perfect, totally disconnected and thus nowhere dense subset of Ω (see [3: p. 291 / No. 280]). The measure of A may be expressed by

$$|\mathrm{A}| = 1 - \sum_{k=1}^{\infty} 2^{k-1}/4^k = 1/2.$$

If $t_0 \in \mathrm{A}$, then any neighborhood $\mathrm{K}(t_0, \delta)$ $(\delta > 0)$ contains infinitely many open intervals not belonging to A. On the other hand, for symmetry reasons in the interval between the centres of arbitrary two "gaps" of A we find a subset of A with positive measure. This leads to the conclusion that for any $t_0 \in \mathrm{A}$ and $\delta > 0$ both sets $\mathrm{K}(t_0, \delta) \cap \mathrm{A}$ as well as $\mathrm{K}(t_0, \delta) \cap (\Omega \setminus \mathrm{A})$ have positive measure.

To illustrate the statement of Lemma 3.12, let us consider the characteristic function $f(t) = \chi_\mathrm{A}(t)$ of the set A. This is a bounded, measurable function fulfilling the assumptions of this lemma. Indeed, $\mathrm{E}_f(t_0)$ is equal to $\{0, 1\}$ in every point $t_0 \in \mathrm{A}$ with $\mathrm{M}_0 = \Omega \setminus \mathrm{A}$ and $\mathrm{M}_1 = \mathrm{A}$, whereas $\mathrm{E}_f(t_0) = \{0\}$ holds for all $t_0 \in \Omega \setminus \mathrm{A}$. Choosing an arbitrary subset B of A with positive measure the functions $h_1(t) = \chi_\mathrm{B}(t)$ and $h_2 \equiv 0$ represent two measurable selections from $t \mapsto \mathrm{E}_f(t)$ which are in different L^∞ equivalence classes. Let us mention that the map $\mathrm{E}_f(t)\colon \Omega \to \mathfrak{P}(\mathbb{R})$ in no point $t_0 \in \mathrm{A}$ will be lower semicontinuous in the above defined sense. Since any neighborhood of $t_0 \in \mathrm{A}$ does contain some point $t \in \Omega \setminus \mathrm{A}$, we find a sequence of points $t_n \in \Omega \setminus \mathrm{A}$ converging to t_0 but there is no sequence of elements of $\mathrm{E}_f(t_n) = \{0\}$ converging to $1 \in \mathrm{E}_f(t_0)$. Therefore, we can find a set of positive measure where lower semicontinuity does not hold.

4. Essentially continuous functions

The concluding section is directed to the discussion of continuity and integrability criteria in terms of essential limit sets.

Definition 4.1. A measurable, essentially bounded function $f: \Omega \to \mathbb{R}$ is called *essentially continuous at a point* $t_0 \in \Omega$ if $\phi(t_0) = \Phi(t_0)$, and *essentially discontinuous* else. The value $S_f(t_0) = \Phi(t_0) - \phi(t_0)$ denotes the *essential oscillation of* f *at* t_0.

In [4] the "essential oscillation of f over a set $\mathrm{K} \subseteq \Omega$" of nonzero measure was defined by $O(f, \mathrm{K}) = \operatorname{ess\,sup}_{t \in \mathrm{K}} f(t) - \operatorname{ess\,inf}_{t \in \mathrm{K}} f(t)$. For null sets N, $O(f, \mathrm{N})$ was set to zero.

At a given point $t \in \Omega$, two functions belonging to the same class $\tilde{f} \in L^\infty(\Omega, \mathfrak{B}, \lambda)$ are either both essentially continuous, or essentially discontinuous both. Moreover, for every point $t \in \Omega$ the essential oscillations of these functions coincide.

Lemma 4.2. *A measurable, essentially bounded function f is essentially continuous at a point $t_0 \in \Omega$ if and only if the set $\mathrm{E}_f(t_0)$ is a singleton.*

Proof. In view of Lemma 3.4, it is sufficient to remark that $\mathrm{E}_f(t_0)$ contains one and only one element iff the relation $\phi(t_0) = \Phi(t_0)$ holds ∎

Lemma 4.3. *Let v be a given element of $\mathrm{E}_f(t_0)$ and M_v a set with properties* (a) *and* (b) *with respect to t_0 and v. Then the restriction $f|\mathrm{M}_v$ is essentially continuous in t_0.*

Proof. Obviously, $\mathrm{E}_{f|\mathrm{M}_v}(t_0) = \{ v \}$ is a singleton so that the result follows from the previous lemma ∎

Theorem 4.4. *A measurable, essentially bounded function f is essentially continuous in a point $t_0 \in \Omega$ if and only if the class $\tilde{f}$ contains a representative h which is continuous in t_0.*

Proof. Let the function f be essentially continuous in t_0. Given a sequence $\varepsilon_n \downarrow 0$ we choose δ_n with the property

$$\phi(t_0) - \varepsilon_n \leqslant g_f(t_0, \delta_n) \leqslant \phi(t_0) = \Phi(t_0) \leqslant G_f(t_0, \delta_n) \leqslant \Phi(t_0) + \varepsilon_n.$$

The subset K_n of points of $\mathrm{K}(t_0, \delta_n)$ where the function values lie outside of the interval $[\phi(t_0) - \varepsilon_n , \Phi(t_0) + \varepsilon_n]$ has zero measure then. To obtain the required function $h \in \tilde{f}$ we set $h(t) = \Phi(t)$ on the set $\mathrm{K} = \cup_n \mathrm{K}_n \cup \{ t_0 \}$ and $h(t) = f(t)$ for the remaining points of Ω. Then the function h is continuous at t_0. But the set K by construction is a countable union of null sets, so that the equivalence of h and f is evident.

Assume next that a representative $h \in \tilde{f}$ exists which is continuous at $t_0 \in \Omega$. This implies $\psi_h(t_0) = \Psi_h(t_0)$ so that also $\phi_h(t_0) = \Phi_h(t_0)$. The last relation is independent of the particular representative choice, i.e. it holds for f as well ∎

Theorem 4.5. *If a measurable, essentially bounded function f is essentially continuous in all points $t \in \Omega$, then $\tilde{f}$ contains an element which is continuous everywhere on Ω, namely $h(t) = \Phi(t)$. On the contrary, every function equivalent to some continuous function is essentially continuous on Ω.*

Proof. Let $\phi(t) = \Phi(t)$ be valid everywhere on Ω. Then it follows from Theorem 2.5 that the essential limit function Φ is continuous on Ω. Further, Theorem 2.7 yields $\Phi(t) = \phi(t) \leqslant f(t) \leqslant \Phi(t)$ for almost all $t \in \Omega$, i.e. the functions f and Φ are equivalent. If we assume on the other hand that a function h is continuous on Ω, then in analogy to the preceding proof we have $\psi_h(t) = \Psi_h(t)$ and $\phi_h(t) = \Phi_h(t)$ for arbitrary $t \in \Omega$. Therefore, any representative of $\tilde{h}$ must be essentially continuous on Ω ∎

The last result allows to conclude that any function which is essentially continuous on the whole set Ω can be at most discontinuous of first kind in the sense of [3: p. 412].

Theorem 4.6. *Suppose the set Ω to be outer squarable in the sense that $|\operatorname{cl}(\Omega) \setminus \Omega| = 0$. If the measurable, essentially bounded function f is essentially continuous almost everywhere on Ω, then $\tilde{f}$ contains at least one Riemann integrable on Ω representative, e.g. $h(t) = \Phi_f(t)$. Reversely, any function equivalent to some Riemann integrable on Ω function must be essentially continuous almost everywhere on Ω.*

Proof. A function f is Riemann integrable on $\Omega \subset \mathbb{R}^m$ if and only if f is bounded and almost everywhere continuous on Ω while Ω itself has to be bounded and outer squarable in the above described sense [3: p. 460 / No. 415, Theorem 1]. Suppose that f is essentially continuous outside of some null set N. Then $\phi(t) = \Phi(t)$ holds for every $t \in \Omega \setminus \mathrm{N}$ and, due to Theorem 2.7, $\phi(t) = f(t) = \Phi(t)$ holds for almost every $t \in \Omega \setminus \mathrm{N}$ what proves $\Phi \in \tilde{f}$. Given a sequence $t_n \to t_0 \in \Omega \setminus \mathrm{N}$, we have

$$\Phi(t_0) = \phi(t_0) \leqslant \liminf_{n\to\infty} \phi(t_n) \leqslant \liminf_{n\to\infty} \Phi(t_n) \leqslant \limsup_{n\to\infty} \Phi(t_n) \leqslant \Phi(t_0)$$

according to Theorem 2.5. Thus Φ is continuous on $\Omega \setminus \mathrm{N}$ so that Φ as representative of $\tilde{f}$ is Riemann integrable over Ω. Let now f be a Riemann integrable function and h an arbitrary representative of $\tilde{f}$. Then for almost all $t \in \Omega$ the identity $\psi_f(t) = \phi_f(t) = \Phi_f(t) = \Psi_f(t)$ holds. But then for the same points also $\phi_h(t) = \Phi_h(t)$ so that h is essentially continuous almost everywhere on Ω ∎

It follows from this theorem that a function which is essentially continuous almost everywhere on Ω is at most discontinuous of the second kind in the sense of [3: p. 412].

Theorem 4.7. *There exists an upper semicontinuous representative h in the class $\tilde{f}$ if and only if $f(t) = \Phi(t)$ holds almost everywhere on Ω. Analogously, $\tilde{f}$ contains a lower semicontinuous representative whenever $f(t) = \phi(t)$ for almost all $t \in \Omega$.*

Proof. If $f(t) = \Phi_f(t)$ is true almost everywhere on Ω, then we deduce from Theorem 2.5 that the upper semicontinuous function Φ_f belongs to $\tilde{f}$. On the other hand, if a function $h \in \tilde{f}$ is upper semicontinuous, then $h(t) = \Psi_h(t)$ for all $t \in \Omega$ so that almost everywhere on Ω we get from Theorem 2.7 the relation $\Psi_h(t) = h(t) \leqslant \Phi_h(t) \leqslant \Psi_h(t)$. So the equality $f(t) = h(t) = \Phi_h(t) = \Phi_f(t)$ holds for almost all t in Ω. The proof of the second statement is similiar ∎

Theorem 4.8. *Suppose $r: \Omega \times \mathbb{R} \to \mathbb{R}$ to be continuous with respect to all arguments. Let $r(t, f(t)) \geqslant 0$ hold for almost all $t \in \Omega$ using a given measurable, essentially bounded function $f: \Omega \to \mathbb{R}$. If h is an arbitrary selector function of the set-valued map $t \mapsto \mathrm{E}_f(t)$, i.e. a function satisfying $h(t) \in \mathrm{E}_f(t)$ for all $t \in \Omega$, then the inequality $r(t, h(t)) \geqslant 0$ is fulfilled even everywhere on Ω.*

Proof. For given $t_0 \in \Omega$ and $v_0 = h(t_0) \in \mathrm{E}_f(t_0)$, first find a set $\mathrm{M}_{v_0} \subset \Omega$ with the properties (a) and (b) from Definition 3.1. Since r is continuous, for arbitrary $\varepsilon > 0$ a positive δ exists such that $|t - t_0| + |v - v_0| \leqslant \delta$ guarantees that $|r(t,v) - r(t_0, v_0)| \leqslant \varepsilon$. Using (b), for $\varepsilon_1 = \delta/2 > 0$ we determine δ_1 with $0 < \delta_1 \leqslant \varepsilon_1$ such that

$$\operatorname*{ess\,sup}_{t \in \mathrm{M}_{v_0} \cap \mathrm{K}(t_0, \delta_1)} |f(t) - v_0| \leqslant \varepsilon_1.$$

Further, we denote by $\mathrm{K}(\varepsilon)$ the set of all points of $\mathrm{M}_{v_0} \cap \mathrm{K}(t_0, \delta_1)$ reduced by the null set of points where $|f(t) - v_0| > \varepsilon_1$ or $r(t, f(t)) < 0$. Then for arbitrary $t \in \mathrm{K}(\varepsilon)$ the estimate $|t - t_0| + |f(t) - v_0| \leqslant \delta_1 + \varepsilon_1 \leqslant \delta$ is valid leading finally to $r(t_0, v_0) \geqslant r(t, f(t)) - |r(t, f(t)) - r(t_0, v_0)| \geqslant -\varepsilon$ using arbitrary $t \in \mathrm{K}(\varepsilon)$. This construction may be performed for arbitrary positive ε so that $r(t_0, v_0) \geqslant 0$ immediately follows ∎

Under the assumptions of Theorem 4.8 in particular the bounded, measurable functions $r(t, \phi_f(t))$ and $r(t, \Phi_f(t))$ are nonnegative everywhere on Ω.

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