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Strong Duality for Transportation Flow Problems

R. Klötzler

Abstract. This paper is a supplement and correction to the author's article "Optimal transportation flows" [2]. By new methods the existence of optimal transportation flows and the strong duality to deposit problems is proved.

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1. Introduction

In conformity with [2] we consider the following *transportation flow problem*:

$$K(\mu) := \int_\Omega r(x, d\mu(x)) \longrightarrow \min \qquad \text{on } Y \tag{1}$$

where

$$Y := \left\{ \mu \in L_\infty^{mn}(\Omega)^* \;\middle|\; \langle \nabla\sigma, \mu \rangle = K_D(\sigma) \quad \forall\, \sigma \in W_\infty^{1,n}(\Omega) \right\} \tag{2}$$

and

$$K_D(\sigma) := \int_\Omega \sigma(x)^\top d\alpha(x) \qquad \text{on } W_\infty^{1,n}(\Omega)\,. \tag{3}$$

We assume, Ω is a bounded strongly Lipschitz domain of E^m, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a given vector of finite Borel measures α_k on the σ-algebra $\mathfrak{B}$ of all Lebesgue-measurable subsets of $\mathfrak{B}$ which satisfy the assumption

$$\int_\Omega d\alpha_k = 0 \qquad (k = 1, \ldots, n)\,; \tag{4}$$

r is a given local cost rate on $\Omega \times E^{mn}$ with the following basic properties:

$$\left.\begin{array}{l} r(\cdot, v) \text{ is summable on } \Omega \\ r(x, \cdot) \text{ is positive homogeneous of degree one and convex on } E^{mn}\ \forall\, x \in \Omega \\ \gamma_1|v| \le r(x,v) \le \gamma_2|v| \ \ (v \in E^{mn},\ x \in \Omega) \text{ for some constants } \gamma_1, \gamma_2 > 0. \end{array}\right\} \tag{5}$$

R. Klötzler: Tauchaer Str. 202, D – 04349 Leipzig

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The objective functional of (1) is defined by

$$\int_{\Omega} r(x, d\mu(x)) := \sup_{u} \left\{ \langle u, \mu \rangle \,\middle|\, u \in L_{\infty}^{mn}(\Omega),\; u^{\top}(x)v \leq r(x,v)\; \forall\, v \in E^{mn} \right\}. \qquad (6)$$

Every element $\mu = (\mu_1, \ldots, \mu_n) \in Y$ is said to be a *feasible flow* and μ_k the *flow of the k-th tranportation good.*

Referring to [2], between the transportation flow problem (1) and the *deposit problem*

$$K_D(S) = \int_{\Omega} S(x)^{\top} d\alpha(x) \longrightarrow \max \qquad \text{on } \mathfrak{S}' \qquad (7)$$

there exists *duality*, i.e.

$$K(\mu) \geq K_D(S) \qquad \forall\, \mu \in Y,\; S \in \mathfrak{S}', \qquad (8)$$

if we define $\mathfrak{S}'$ by

$$\mathfrak{S}' := \left\{ S \in W_{\infty}^{1,n}(\Omega) \,\middle|\, \nabla S(x) \in \mathfrak{F}(x) \text{ for a.e. } x \in \Omega \right\} \qquad (9)$$

with

$$\mathfrak{F}(x) := \left\{ z \in E^{mn} \,\middle|\, z^{\top} v \leq r(x,v)\; \forall\, v \in E^{mn} \right\}. \qquad (10)$$

The restrictions of (9) characterize slope restrictions in the sense that $\nabla S(x)$ belongs to the convex *figuratrix set* $\mathfrak{F}(x)$ for a.e. $x \in \Omega$.

Since (4), the linear functional K_D has the property $K_D(S) = K_D(S + C)$ for any constant vektor $C \in E^n$. Therefore, without loss of generality we can reduce the deposit problem (7) on the restricted class $\mathfrak{S} := \{S \in \mathfrak{S}' \,|\, S(\hat{x}) = 0\}$ where $\hat{x}$ is an arbitrary fixed point in $\bar{\Omega}$.

We know from [2] the following theorem.

Theorem 1. *The deposit problem* (7) *has an optimal solution* S_0.

2. The existence of optimal flows

In $L_{\infty}^{mn}(\Omega)^*$ the standardized norm is defined by

$$\|\mu\| := \sup_{u} \left\{ \langle u, \mu \rangle \,\middle|\, u \in L_{\infty}^{mn}(\Omega),\; |u(x)| \leq 1 \text{ a.e. on } \Omega \right\}. \qquad (11)$$

We introduce in this Banach space an equivalent norm by

$$\|\mu\|^* := \sup_{u} \left\{ \langle u, \mu \rangle \,\middle|\, u \in L_{\infty}^{mn}(\Omega),\; u(x) \in \mathfrak{F}(x) \text{ a.e. on } \Omega \right\}. \qquad (12)$$

The equivalence of both norms is obvious under consideration of the third property of assumption (5):

$$\sup_u \left\{ \langle u, \mu \rangle \,\middle|\, u \in L_\infty^{mn}(\Omega),\ u(x)^\top v \le \gamma_1 |v|\ \forall v \in E^{mn},\ \text{a.e. on } \Omega \right\}$$
$$\le \sup_u \left\{ \langle u, \mu \rangle \,\middle|\, u \in L_\infty^{mn}(\Omega),\ u(x)^\top v \le r(x,v)\ \forall v \in E^{mn},\ \text{a.e. on } \Omega \right\}$$
$$\le \sup_u \left\{ \langle u, \mu \rangle \,\middle|\, u \in L_\infty^{mn}(\Omega),\ u(x)^\top v \le \gamma_2 |v|\ \forall v \in E^{mn},\ \text{a.e. on } \Omega \right\},$$

and this means $\gamma_1 \|\mu\| \le \|\mu\|^* \le \gamma_2 \|\mu\|$ thus equivalence of both norms.

Now, let $\mathfrak{S}_0 = \{\sigma \in W_\infty^{1,n}(\Omega) | \sigma(\hat{x}) = 0\}$ and U be a subspace of $L_\infty^{mn}(\Omega)$, characterized by

$$U := \left\{ u \in L_\infty^{mn}(\Omega) \,\middle|\, u = \nabla\sigma,\ \sigma \in \mathfrak{S}_0 \right\}. \tag{13}$$

In virtue of Sóbolev's embedding theorems [3: p. 60], the mapping $f : U \to \mathbb{R}$ is a linear continuous functional μ_0 on U, if we define $f(\nabla\sigma) := K_D(\sigma)$ for all $\sigma \in \mathfrak{S}_0$. Namely, there is a constant $M > 0$ such that for every σ of this type

$$\|\sigma\|_{C^n(\bar{\Omega})} \le M \operatorname*{ess\,sup}_{\Omega} |\nabla\sigma|$$

holds and therefore

$$|f(\nabla\sigma)| = |K_D(\sigma)| \le M \int_\Omega d|\alpha|\, \|\nabla\sigma\|_{L_\infty^{mn}(\Omega)}. \tag{14}$$

The linearity of f is obvious. Together with the boundedness (14) of f it follows that f is a linear continuous functional μ_0 on U. By the Hahn-Banach extension theorem [1: p. 109] we can extend μ_0 as a continuous linear functional on the whole space $L_\infty^{mn}(\Omega)$ with the same norm. That means, for each $u \in U$ there is uniquely a $\sigma \in \mathfrak{S}_0$ such that $u = \nabla\sigma$,

$$f(u) = K_D(\sigma) = \langle \nabla\sigma, \mu_0 \rangle, \tag{15}$$

and, with (12),

$$\|\mu_0\|^* = \sup \left\{ \langle \nabla\sigma, \mu_0 \rangle \,\middle|\, \sigma \in \mathfrak{S} \right\} = \sup_{\mathfrak{S}} K_D = K_D(S_0) \tag{16}$$

hold.

After the extension of μ_0 on the totality of $L_\infty^{mn}(\Omega)$, it holds again, according to (12),

$$\|\mu_0\|^* = \sup_u \left\{ \langle u, \mu_0 \rangle \,\middle|\, u \in L_\infty^{mn}(\Omega),\ u(x) \in \mathfrak{F}(x) \text{ a.e. on } \Omega \right\}$$

and since (6), (10) and (16)

$$\|\mu_0\|^* = K(\mu_0) = K_D(S_0). \tag{17}$$

From (15) $\mu_0 \in Y$ follows such that (8) and (17) lead to the optimality of μ_0 with respect to problem (1). So we can summarize:

Theorem 2. *The transportation flow problem* (1) *has an optimal solution* μ_0.

3. Conclusions and generalizations

The existence of optimal solutions S_0 of the deposit problem (7) and μ_0 of the transportation flow problem (1) has in connection with (8) and (17) the following consequence.

Theorem 3. *Between the dual problems* (1) *and* (7) *there exists strong duality in the sense that* $\min_Y K = \max K_D$.

From this theorem we obtain under consideration of (3), (12), (15) and (17)

$$K_D(S_0) = \int_\Omega S_0(x)^\top d\alpha(x) = K(\mu_0) = \langle \nabla S_0, \mu_0 \rangle \geq \langle u, \mu_0 \rangle$$

for all $u \in L_\infty^{mn}(\Omega)$, $u(x) \in \mathfrak{F}(x)$ a.e. This leads to the following conclusion.

Theorem 4. *An element* $S_0 \in \mathfrak{S}$ *is an optimal solution of the deposit problem* (7) *if and only if there is a vectorial set function* $\mu_0 \in L_\infty^{mn}(\Omega)^*$ *which satisfies the continuity equation*

$$\langle \nabla \sigma, \mu_0 \rangle = \int_\Omega \sigma(x)^\top d\alpha(x) \qquad \forall\, \sigma \in W_\infty^{1,n}(\Omega) \tag{18}$$

and the maximum condition

$$\langle \nabla S_0, \mu_0 \rangle \geq \langle u, \mu_0 \rangle \qquad \forall\, u \in L_\infty^{mn}(\Omega),\ u(x) \in \mathfrak{F}(x) \text{ a.e. on } \Omega. \tag{19}$$

Remark. Theorems 3 and 4 coincide essentially with Theorems 4 and 3 from [2]. However, unfortunately the proof of Theorem 3 in that paper was not correct because of a mistake in identifying weak*-compactness and sequentially weak*-compactness by the application of Alaoglu's theorem. Finally, we mention that all results proved here hold also for the case in which $W_\infty^{1,n}(\Omega)$ in (2) and (9) is replaced by $\mathring{W}_\infty^{1,n}(\Omega)$. Then we can omit even assumption (4).

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