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On the Kernel of the Klein-Gordon Operator

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Abstract. In this work with the aid of technics of quaternionic analysis we show that any solution of the Klein-Gordon equation can be represented via two solutions of the Dirac equation with the same mass. Moreover, the two functions corresponding to each solution of the Klein-Gordon equation are unique.

Keywords: *Klein-Gordon operator, Dirac operator, quaternions*

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1. Introduction

In this work we study the relationship between two most important operators of particle physics, namely the Klein-Gordon operator and the Dirac operator. Of course, some relations between them were established in the moment when Dirac discovered his operator as a square root of the Klein-Gordon operator. Thus, it is clear that any solution of the Dirac equation solves also the Klein-Gordon equation. Here using the formalism and methods of quaternionic analysis we prove a more intimate connection between the solutions of both equations. Namely, we show that any solution of the Klein-Gordon equation can be represented via two solutions of the Dirac equation with the same mass that seems to be a natural extension of Dirac's theory. In other words, any function describing the behaviour of a free particle with integer spin can be completely determined by two functions describing free particles with spin $\frac{1}{2}$. Moreover, the two functions corresponding to each solution of the Klein-Gordon equation are unique.

The methods of quaternionic analysis used in this article were developed recently in [1, 4, 8]. Here in Section 2 we present only some necessary definitions and results referring the reader to corresponding works. Section 3 contains the main results of the article resumed in Proposition 3.

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2. Preliminaries

We denote by $\mathbb{H}(\mathbb{C})$ the algebra of complex quaternions (i.e. biquaternions). Each element $\alpha \in \mathbb{H}(\mathbb{C})$ is represented in the form $\alpha = \sum_{k=0}^{3} \alpha_k i_k$, where $\alpha_k \in \mathbb{C}$, i_0 is the unit and i_k are standard basic quaternions. We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition i commutes with all i_k. The quaternion $\overline{\alpha} = \alpha_0 - \sum_{k=1}^{3} \alpha_k i_k$ is called *conjugated* to α. For multiplication from the right-hand side by a quaternion α we use the notation M^α, i.e. by definition $M^\alpha f = f\alpha$.

Let us denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$. This is the set of all complex quaternions α satisfying the condition $\alpha\overline{\alpha} = 0$. As usual zero is not included into $\mathfrak{S}$. We will use the following idempotents generated by zero divisors:

$$P_k^{\pm} = \frac{1}{2} M^{(1 \pm i i_k)} \qquad (1 \le k \le 3).$$

Let us consider the Helmholtz operator with quaternionic wave number $\Delta + M^{\alpha^2}$ studied in [7] (see also [8]). Here Δ is the 3-dimensional Laplace operator. With the aid of quaternions it is possible (see [3]) to factorize the Helmholtz operator:

$$\Delta + M^{\alpha^2} = -(D + M^\alpha)(D - M^\alpha),$$

where $D = \sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}$ is the well-known Moisil-Theodoresco operator, which was studied for the first time in [9, 10]. Moreover, for $\alpha \notin \mathfrak{S} \cup \{0\}$,

$$\ker(\Delta + M^{\alpha^2}) = \ker(D + M^\alpha) \oplus \ker(D - M^\alpha). \tag{1}$$

For $\alpha \in \mathfrak{S}$ we have

$$\ker(\Delta + M^{\alpha^2}) = M^\alpha(\ker D_{2\alpha_0} \oplus \ker D_{-2\alpha_0}) \oplus M^{\overline{\alpha}}(\ker \Delta). \tag{2}$$

Here and in what follows we use the notation $D_\alpha = D + M^\alpha$. In its turn for the kernel of the Laplace operator we have

$$\ker \Delta = \ker D_{-\alpha} \oplus M^{i_1}(\ker D_\alpha),$$

where α is an arbitrary zero divisor, $\alpha_0 = 0$, and $\alpha_1 \neq 0$. These results were obtained in [6, 7] (see also [8: Section 2]) with the help of the operators

$$\Pi_\alpha = \begin{cases} -\dfrac{1}{2\alpha\overline{\alpha}} M^{\overline{\alpha}} D_{-\alpha} & \text{if } \alpha \notin \mathfrak{S} \\ -\dfrac{1}{8\alpha_0^2} M^\alpha D_{-\alpha} & \text{if } \alpha \in \mathfrak{S} \end{cases}$$

which are defined on $\ker(\Delta + M^{\alpha^2})$ and possess the following properties:

$$\Pi_\alpha^2 = \Pi_\alpha \qquad \text{for all } \alpha \in \mathbb{H}(\mathbb{C}) \tag{3}$$

$$\Pi_\alpha \Pi_{-\alpha} = \Pi_{-\alpha}\Pi_\alpha = 0 \qquad \text{for all } \alpha \in \mathbb{H}(\mathbb{C}) \tag{4}$$

$$\Pi_\alpha + \Pi_{-\alpha} = I \qquad \text{for all } \alpha \notin \mathfrak{S} \tag{5}$$

$$\Pi_\alpha + \Pi_{-\alpha} + \frac{1}{2\alpha_0} M^{\overline{\alpha}} = I \qquad \text{for all } \alpha \in \mathfrak{S}. \tag{6}$$

For any $\alpha \in \mathbb{H}(\mathbb{C})$ we have $\Pi_\alpha : \ker(\Delta + M^{\alpha^2}) \longrightarrow \ker D_\alpha$.

We will consider $\mathbb{H}(\mathbb{C})$-valued functions given in a domain $\Omega \subset \mathbb{R}^4$. On the set $C^1(\Omega; \mathbb{H}(\mathbb{C}))$ the following operator is defined [5]:

$$\mathbb{D} = P_1{}^+(i\partial_t + D) + P_1^-(-i\partial_t + D) - mM^{i_2},$$

where $\partial_t = \frac{\partial}{\partial t}$. The operator $\mathbb{D}$ is equivalent to the classic Dirac operator

$$\mathfrak{D} = \gamma_0\partial_t - \sum_{k=1}^{3} \gamma_k \frac{\partial}{\partial x_k} + imI$$

in the sense that $\mathbb{D}$ may be obtained from $\mathfrak{D}$ by a simple invertible matrix transformation (see [5] and [8: Section 12]). We denote by F the Fourier transform with respect to the variable t:

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\beta t}dt.$$

3. Decomposition of the kernel of the Klein-Gordon operator

Let us denote by A the Klein-Gordon operator which describes free particles with integer spin:

$$A = \partial_t^2 - \Delta + m^2.$$

We will consider the operator A on the set $C^2(\Omega)$, where $\Omega = (-\infty, \infty) \times G$, $(-\infty, \infty)$ is the axis corresponding to the time variable t and G is a domain in $\mathbb{R}^3$. Let us introduce the operator

$$\overline{\mathbb{D}} = P_1{}^+(-i\partial_t + D) + P_1{}^-(i\partial_t + D) + mM^{i_2}$$

and consider the product

$$\mathbb{D}\overline{\mathbb{D}} = P_1{}^+(\partial_t{}^2 - \Delta) + P_1{}^-(\partial_t{}^2 - \Delta) + m^2 = \partial_t{}^2 - \Delta + m^2 = A.$$

Thus, $A = \mathbb{D}\overline{\mathbb{D}}$. We will use this fact for analysis of $\ker A$.

Let us rewrite the operators A, $\mathbb{D}$ and $\overline{\mathbb{D}}$ in the form

$$\begin{aligned} A &= F^{-1}(-\Delta + m^2 - \beta^2)F \\ \mathbb{D} &= F^{-1}(D + M^{(i\beta i_1 - m i_2)})F \\ \overline{\mathbb{D}} &= F^{-1}(D - M^{(i\beta i_1 - m i_2)})F. \end{aligned} \tag{7}$$

Let us denote $\alpha = -(i\beta i_1 - m i_2)$. Then for the Helmholtz operator from the brackets in (7) we have

$$-\Delta + m^2 - \beta^2 = D_\alpha D_{-\alpha}.$$

Using the corresponding projection operators Π_α and $\Pi_{-\alpha}$ we introduce the operators

$$Q_{\pm\alpha} = F^{-1}\Pi_{\pm\alpha}F.$$

The following proposition is a simple corrollary of (3) - (6).

Proposition 1. *On the set* $\ker A$ *the following equalities are true:*

1. $Q_\alpha^2 = Q_\alpha$.
2. $Q_\alpha Q_{-\alpha} = Q_{-\alpha} Q_\alpha = 0$.
3. $Q_\alpha + Q_{-\alpha} = I$.

Note that $\ker Q_\alpha = \ker \mathbb{D}$ and $\ker Q_{-\alpha} = \ker \overline{\mathbb{D}}$. The prove, for example, of the first equality is a corrollary of the fact that both inclusions $f \in \ker Q_\alpha$ and $f \in \ker \mathbb{D}$ are equivalent to the equality $D_{-\alpha} F f = 0$. Then we obtain the following

Proposition 2. *The kernel of the Klein-Gordon operator A can be represented as direct sum of the kernels of Dirac's operators* $\mathbb{D}$ *and* $\overline{\mathbb{D}}$, *i.e.* $\ker A = \ker \mathbb{D} \oplus \ker \overline{\mathbb{D}}$.

This proposition is a generalization of the following result obtained in [2]: $\mathcal{M} = \ker(-i\partial_t + D) \oplus \ker(i\partial_t + D)$, where $\mathcal{M}$ denotes all null-solutions of the wave operator $\Delta - \partial_t^2$ which depend on time.

Note that if $f \in \ker \mathbb{D}$, then $f i_3 \in \ker \overline{\mathbb{D}}$. In other words: $\ker \overline{\mathbb{D}} = M^{i_3}(\ker \mathbb{D})$ and we obtain the following principal result of this article:

Proposition 3. *The relation* $\ker A = \ker \mathbb{D} \oplus M^{i_3}(\ker \mathbb{D})$ *is true.*

A curious point in this result is the fact that due to the anticommutativity of the imaginary quaternionic units the kernel of the operator of second order A is represented via the kernel of the operator of first order $\mathbb{D}$. Consequently, any solution of the Klein-Gordon equation may be represented as a sum of two solutions of the Dirac equation one of which is multiplied by i_3. This result conforms, e.g., with the quark model of mesons. Each meson is constituted by one quark and one anti-quark. The multiplication by i_3 in Proposition 3 represents the transformation of a particle into an anti-particle.

Note that decomposition (1) of the kernel of Helmholtz's operator is not unique and can be rewritten in a form similar to Proposition 3. Let us consider the following Helmholtz operator $\Delta + \alpha^2 I$, where $\alpha \in \mathbb{C}$. Then we have

$$\Delta + \alpha^2 I = -(D + M^{\alpha i_1})(D - M^{\alpha i_1})$$

and with the aid of the projection operators Π_α and $\Pi_{-\alpha}$ we obtain

$$\ker(\Delta + \alpha^2 I) = \ker(D + M^{\alpha i_1}) \oplus \ker(D - M^{\alpha i_1}).$$

But $\ker(D - M^{\alpha i_1}) = M^{i_2}(\ker(D + M^{\alpha i_1}))$, consequently

$$\ker(\Delta + \alpha^2 I) = \ker(D + M^{\alpha i_1}) \oplus M^{i_2}(\ker(D + M^{\alpha i_1})).$$

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