Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 17 (1998), No. 2, 271–280

Convergence of the Newton-Kantorovich Method under Vertgeim Conditions: a New Improvement

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Abstract. Let $f : B(x_0, R) \subset X \to Y$ be an operator from a closed ball of a Banach space X to a Banach space Y. We give new conditions to ensure the convergence of Newton-Kantorovich approximations toward a solution of the equation $f(x) = 0$, under the hypothesis that f' be Hölder continuous. The case of f' being Hölder continuous in a generalized sense is analyzed as well.

Keywords: *Newton-Kantorovich approximations, Hölder type conditions*

AMS subject classification: 65 J 15, 47 H 17

1. Introduction

Let X and Y be Banach spaces, $B(x_0, R)$ the closed ball in X with center x_0 and radius R. Assume that $f : B(x_0, R) \to Y$ is an operator satisfying the following conditions (usually called the *Vertgeim conditions* [12, 13]):

a) f is Frechét differentiable at interior points of $B(x_0, R)$.

b) f' satisfies a Hölder condition with an exponent $\theta \in (0, 1]$.

c) $f'(x_0)$ is invertible.

In the sequel we use the following notations:

$$a = \|f'(x_0)^{-1} f(x_0)\|, \qquad b = \|f'(x_0)^{-1}\|$$

$$k = \sup\left\{ \frac{\|f'(x_1) - f'(x_2)\|}{\|x_1 - x_2\|^\theta} \,\middle|\, x_1, x_2 \in \mathring{B}(x_0, R) \text{ with } x_1 \neq x_2 \right\}.$$

Our aim is to study the solvability of the equation

$$f(x) = 0 \tag{1}$$

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This work was performed under the auspices of CNR - Italia and MURST - Italia. The second author was also supported by the Belorussian Foundation of Fundamental Scientific Research.

ISSN 0232-2064 / $ 2.50

and the convergence of the classical Newton-Kantorovich approximations

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n) \qquad (n \geq 0) \tag{2}$$

to a solution x_* of equation (1).

In Section 1 we simply collect some results, more or less known in the literature, about the unique solvability of equation (1). The parameter $\xi = a^\theta bk$ is fundamental to formulate the results. We observe that ξ is large for a "flat" function f. The first basic results on the convergenge of Newton-Kantorovich approximations were obtained by B. A. Vertgeim [12, 13]. The Vertgeim results imply that the Newton-Kantorovich approximations are defined and converge to the solution of equation (1) for $\xi \leq \xi_{ver}$, where ξ_{ver} is the unique solution in $(0, 1]$ of the scalar equation $\frac{1}{(1+\theta)^\theta}\left(\frac{t}{1-t}\right)^{1+\theta} = t$. J. V. Lysenko [7] proved that the result of Vertgeim still holds if $\xi \leq 2^{\theta-1}\left(\frac{\theta}{1+\theta}\right)^\theta$.

Our main theorem in Section 3 improves the result by Lysenko: in fact the convergence of the Newton-Kantorovich approximations still holds for $\xi \leq \frac{1}{\nu(\theta)}\left(\frac{\theta}{1+\theta}\right)^\theta$, where $\nu = \nu(\theta)$ is a suitable function constructed in such a way that $\nu(\theta) < 2^{1-\theta}$ for $\theta \in (0, 1]$. In our case, the estimates needed in the proof are obtained in a different way than the Lysenko one.

In Section 4 we discuss the rate of convergence of the approximation (2) to a solution of equation (1).

The idea we use in Section 3 to prove the convergence result for the Newton-Kantorovich approximations is modelled on a general scheme we sketch in Section 5, in the case of f' satisfying a generalized Hölder condition:

$$\|f'(x_1) - f'(x_2)\| \leq \omega(\|x_1 - x_2\|) \qquad (x_1, x_2 \in B(x_0, R)) \tag{3}$$

with $\omega : [0, +\infty) \to [0, +\infty)$ being some increasing function with $\lim_{t\to 0} \omega(t) = 0$. (The classical Hölder condition is of course obtained when $\omega(t) = kt^\theta$ for some $\theta \in (0, 1]$.) We want to remark that throughout the paper, one of the main techniques used in the proofs is that of the majorizing sequences introduced by Rheinboldt [10].

Applications to nonlinear integral equations can be obtained modifying in a suitable way the examples given in [1, 2, 6].

2. Unique solvability

In this section we deal with (unique) solvability of equation (1). We need a preliminary lemma about the solutions $r \geq 0$ of the scalar equation

$$\psi(r) = 0 \tag{4}$$

where ψ is defined by $\psi(r) = \frac{bk}{1+\theta} r^{1+\theta} - r + a$.

Lemma 1. *Equation* (4) *has no solution, a unique solution or exactly two solutions* $r_* < r_{**}$ *if and only if*

$$a^\theta bk > \left(\frac{\theta}{1+\theta}\right)^\theta, \qquad a^\theta bk = \left(\frac{\theta}{1+\theta}\right)^\theta, \qquad a^\theta bk < \left(\frac{\theta}{1+\theta}\right)^\theta,$$

respectively.

Proof. The solutions of equation (4) coincide with the fixed points of the strictly increasing and strictly convex function $d(r) = r + \psi(r)$. The tangent to the graph of d at r is parallel to the bisectrix if and only if $d'(r) = 1$, i.e. $r = r_{crit} = (bk)^{-\frac{1}{\theta}}$. The position of the point $(r_{crit}, d(r_{crit}))$ with respect to the bisectrix gives the precise number of fixed points for d. Since $d(r_{crit}) = a + \frac{(bk)^{-\frac{1}{\theta}}}{1+\theta}$, the result follows ∎

Theorem 1. *Suppose that f satisfies the Vertgeim conditions and that*

$$a^\theta bk \leq \left(\frac{\theta}{1+\theta}\right)^\theta \qquad \text{and} \qquad r_* \leq R$$

where r_* and r_{**} are the roots of $\psi(r) = 0$. Then equation (1) *has a unique solution x_* in the ball $B(x_0, r_*)$. Moreover, this solution is unique in the bigger ball $B(x_0, R)$ if $r_* \leq R < r_{**}$ or, in other words, if r_* is the unique root of $\psi(r)$ on $[0, R]$ and $\psi(R) \leq 0$.*

Proof. Since $f'(x_0)$ is invertible, the equation $f(x) = 0$ is equivalent to the equation $x = Tx$, where $T : B(x_0, R) \to X$, defined by the equality $Tx = x - f'(x_0)^{-1} f(x)$, is usually called the Goursát operator. We have $\|Tx_0 - x_0\| = a = d(0)$ and

$$\|T'(x)\| = \|f'(x_0)^{-1}\left[f'(x_0) - f'(x)\right]\| \leq bkr^\theta = d'(r)$$

for $\|x - x_0\| \leq r \leq R$. Now it is easy to obtain the result using Theorems 1 and 2 in Chapter XVIII of [5] ∎

3. Convergence of Newton-Kantorovich approximations

Let f satisfy the Vertgeim conditions with a fixed $\theta \in (0,1)$. The function $\nu = \nu(\theta)$, introduced in the following lemma, is crucial to improve the results by Ju. V. Lysenko about the convergence of the approximations (2) [7].

Lemma 2. *Set*

$$\nu(\theta) = \sup_{0<t<+\infty} h(t) \qquad \text{where} \qquad h(t) = \frac{t^{1+\theta} + (1+\theta)t}{(1+t)^{1+\theta} - 1}.$$

Then the inequalities $1 < \nu(\theta) < 2^{1-\theta}$ hold for $0 < \theta < 1$.

Proof. First we note that $\lim_{t\to 0} h(t) = \lim_{t\to+\infty} h(t) = 1$ and so $1 \leq \nu(\theta)$. The derivative of h has the numerator

$$\left[(1+\theta)t^\theta + (1+\theta)\right]\left[(1+t)^{1+\theta} - 1\right] - \left[t^{1+\theta} + (1+\theta)t\right](1+\theta)(1+t)^\theta.$$

Since $\lim_{t\to 0} h'(t) = +\infty$, $\nu(\theta)$ is strictly bigger than 1. Moreover, if τ is a maximum point for h, τ cannot be 0 and so is a critical point for h. Now $h'(\tau) = 0$ if and only if

$$h(\tau) = \frac{\tau^{1+\theta} + (1+\theta)\tau}{(1+\tau)^{1+\theta} - 1} = \frac{\tau^\theta + 1}{(1+\tau)^\theta} \leq \max_{0<t<+\infty} g(t) = g(1) = 2^{1-\theta}$$

where $g(t) = \frac{t^\theta + 1}{(1+t)^\theta}$. Consequently, $\nu(\theta) \leq 2^{1-\theta}$. If $\nu(\theta) = 2^{1-\theta}$, we have $h(\tau) = g(1) = g(\tau)$. Since the equality $g(\tau) = 2^{1-\theta}$ implies $\tau = 1$ and consequently $h(1) = \frac{2+\theta}{2^{1+\theta}-1} = 2^{1-\theta}$, i.e. $2^{1+\theta} = 2 - \theta$. The last equality is impossible since $2^{1+\theta}$ is strictly convex in θ ∎

The following numerical table compares the magnitude of $\nu(\theta)$ with that of $2^{1-\theta}$:

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\nu(\theta)$	1.850	1.716	1.596	1.488	1.389	1.297	1.213	1.137	1.065
$2^{1-\theta}$	1.865	1.740	1.623	1.514	1.413	1.318	1.230	1.148	1.071

Now we consider the function

$$\tilde{\psi}(r) = a + \frac{\nu(\theta) bk}{1+\theta} r^{1+\theta} - r \tag{5}$$

which is pointwise bigger than ψ. We observe that the scalar equation $\tilde{\psi}(r) = 0$ has similar properties as those stated in Lemma 1.

The next theorem is our main result.

Theorem 2. *Suppose that f satisfies the Vertgeim conditions, that*

$$\nu(\theta) a^\theta bk \leq \left(\frac{\theta}{1+\theta}\right)^\theta$$

and that $r^* \leq R$, where r^* is the smallest root of the scalar equation $\tilde{\psi}(r) = 0$. Then the Newton-Kantorovich approximations are defined for all n, belong to $B(x_0, r^*)$ and converge to the unique solution x_* of equation (1).

Proof . From the equality

$$f'(x)^{-1} = \left[I + f'(x_0)^{-1}(f'(x) - f'(x_0))\right]^{-1} f'(x_0)^{-1}$$

it follows that

$$\|f'(x)^{-1}\| \leq \frac{b}{1 - bk\|x - x_0\|^\theta}.$$

The sequence of scalars defined by the recurrence formula

$$\left.\begin{aligned} r_0 &= 0 \\ r_{n+1} &= r_n - \frac{\tilde{\psi}(r_n)}{\tilde{\psi}'(r_n)} \quad (n \geq 0) \end{aligned}\right\}$$

is increasing and converges to r^*. Suppose that for $k \le n$ the x_k are well defined and $\|x_k - x_{k-1}\| \le r_k - r_{k-1}$. Then we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le \|f'(x_n)^{-1}\| \, \|f(x_n)\| \\
&= \|f'(x_n)^{-1}\| \, \|f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1})\| \\
&\le \frac{b}{1 - bk\|x_n - x_0\|^\theta} \int_0^1 \|f'((1-t)x_{n-1} + tx_n) - f'(x_{n-1})\| \, \|x_n - x_{n-1}\| \, dt \\
&\le \frac{b}{1 - bk\|x_n - x_0\|^\theta} \, \frac{k}{1+\theta} \|x_n - x_{n-1}\|^{1+\theta} \\
&\le \frac{bk(r_n - r_{n-1})^{1+\theta}}{(1+\theta)(1 - bkr_n^\theta)} \\
&= \frac{bkr_{n-1}^{1+\theta}\left[\frac{1}{1+\theta}\left(\frac{r_n}{r_{n-1}} - 1\right)^{1+\theta} + \left(\frac{r_n}{r_{n-1}} - 1\right)\right] - bkr_{n-1}^\theta(r_n - r_{n-1})}{1 - bkr_n^\theta}.
\end{aligned}$$

By Lemma 2 applied for $t = \frac{r_n}{r_{n-1}} - 1$, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \frac{\frac{\nu(\theta)bkr_{n-1}^{1+\theta}}{1+\theta}\left[\left(\frac{r_n}{r_{n-1}}\right)^{1+\theta} - 1\right] - bkr_{n-1}^\theta(r_n - r_{n-1})}{1 - bkr_n^\theta} \\
&= \frac{\frac{\nu(\theta)bk}{1+\theta}r_n^{1+\theta} - \frac{\nu(\theta)bk}{1+\theta}r_{n-1}^{1+\theta} - bkr_{n-1}^\theta(r_n - r_{n-1})}{1 - bkr_n^\theta} \\
&= \frac{\tilde{\psi}(r_n) - \tilde{\psi}(r_{n-1}) - \psi'(r_{n-1})(r_n - r_{n-1})}{-\psi'(r_n)} \\
&= \frac{\tilde{\psi}(r_n)}{\psi'(r_n)} \\
&= r_{n+1} - r_n.
\end{aligned}$$

Consequently, $\{x_n\}$ is a Cauchy sequence converging to a solution x_* of equation (1). Moreover, $x_* \in B(x_0, r^*)$ and the estimate $\|x_* - x_n\| \le r^* - r_n$ holds ∎

A final remark is needed to conclude the section. It was proved in [3], implementing the iteration procedure on a computer, that there is a "critical point" $\xi = \xi_{num}$ in the interval $[0, (\frac{\theta}{1+\theta})^\theta]$ (the existence interval for equation (1)). More exactly, for $\xi < \xi_{num}$, "numerical convergence" of iteration (2) holds (which, of course, does not mean that iterations (2) actually converge in some Banach space). Likewise, one can show that, for $\xi > \xi_{num}$, iterations (2) diverge "numerically".

We can summarize our results in the following inequalities:

$$0 < \xi_{ver} < 2^{\theta-1}\left(\frac{\theta}{\theta+1}\right)^\theta < \frac{1}{\nu(\theta)}\left(\frac{\theta}{\theta+1}\right)^\theta \le \xi_{num} < \left(\frac{\theta}{\theta+1}\right)^\theta < 1.$$

4. The rate of convergence

Under the condition of Theorem 2, the Newton-Kantorovich approximations satisfy the estimates $\|x_* - x_n\| \le r_* - r_n$. This gives a bound for the rate of convergence of the sequence $\{x_n\}$ to x_*. The above error estimates demand the computation of r_n, which in some cases may not be convenient. Below we present another estimate that requires the computation of r_* only.

Theorem 3. *Under the hypotheses of Theorem 2, if $r_* < r_{\text{crit}}$, then*

$$\limsup_{n\to\infty} \frac{\|x_* - x_{n+1}\|}{\|x_* - x_n\|^{1+\theta}} \le \frac{bk}{(1+\theta)(1-bkr_*^\theta)}.$$

Proof. The following inequalities hold:

$$\begin{aligned}
\|x_* - x_{n+1}\| &= \|x_* - x_n - f'(x_n)^{-1} f(x_n)\| \\
&= \|f'(x_n)^{-1} [f(x_*) - f(x_n) - f'(x_n)(x_* - x_n)]\| \\
&\le \|f'(x_n)^{-1}\| \cdot \left\| \int_0^1 [f'((1-t)x_n + tx_*) - f'(x_n)](x_* - x_n)\, dt \right\| \\
&\le \frac{b}{1-bkr_n^\theta} \int_0^1 kt^\theta \|x_* - x_n\|^{1+\theta} dt \\
&\le \frac{bk\|x_* - x_n\|^{1+\theta}}{(1+\theta)(1-bkr_n^\theta)}.
\end{aligned}$$

From these inequalities it follows (if $r_* < r_{crit}$) that

$$\limsup_{n\to\infty} \frac{\|x_* - x_{n+1}\|}{\|x_* - x_n\|^{1+\theta}} \le \frac{bk}{(1+\theta)(1-bkr_*^\theta)}$$

and the statement is proved ∎

From the inequalities in the proof of Theorem 3 we have, for $n \in N$,

$$\|x_* - x_n\| \le \left[\frac{bk}{(1+\theta)(1-bkr^*)^\theta}\right]^{\frac{(1+\theta)^n - 1}{\theta}} (r_*)^{(1+\theta)^n}.$$

We do not study in this paper the "limit" case $r_* = r_{crit} = (bk)^{-\frac{1}{\theta}}$.

We conclude remarking that the Kantorovich case ($\theta = 1$) is completely studied in [14] (see also [8, 9]). Similar error estimates in the case $0 < \theta < 1$, to our knowledge, are not known.

5. The generalized Hölder case

The idea we use in Section 3 to prove the convergence result for the Newton-Kantorovich approximations is modelled on a general scheme we sketch in the case of f' satisfying the generalized Hölder condition (3). We proceed quite heuristically.

For the problem of existence we refer to the paper [1]. We only recall that the unique solvability for equation (1) is controlled, in this case, by the scalar equation $\psi(r) = 0$, with ψ given by

$$\psi(r) = a + b\int_0^r \omega(t)dt - r. \tag{6}$$

(Of course, in the case $\omega(t) = kt^\theta$ the function (6) reduces to (4)). As we have seen in the foregoing sections, the convergence of the iterative process (2) cannot be controlled, in general, by the same equation.

We begin introducing the scalar equation $\psi(r) = 0$ to control the convergence process. Afterwards we look for a function $\tilde{\psi}$ in such a way that the sequence of scalars defined by

$$\left.\begin{aligned} r_0 &= 0 \\ r_{n+1} &= r_n - \frac{\tilde{\psi}(r_n)}{\psi'(r_n)} \quad (n \geq 0) \end{aligned}\right\}$$

is a majorizing sequence for the Newton-Kantorovich approximations:

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n \qquad (n \geq 0). \tag{7}$$

Usually, (7) is achieved by induction. If we suppose that (7) is verified for $k \leq n$ and that the function $\tilde{\psi}$ satisfies the inequality

$$b\int_0^{v-u} \omega(t)\,dt \leq \tilde{\psi}(v) - \tilde{\psi}(u) - \psi'(u)(v-u) \qquad (0 \leq u \leq v \leq R) \tag{8}$$

we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|f'(x_n)^{-1}\|\,\|f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1})\| \\ &\leq \frac{b}{1 - b\omega(r_n)}\int_0^1 \|f'((1-t)x_{n-1} + tx_n) - f'(x_{n-1})\|\,\|x_n - x_{n-1}\|\,dt \\ &\leq \frac{1}{1 - b\omega(r_n)}\int_0^1 \omega(t\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|\,dt \\ &= \frac{1}{1 - b\omega(r_n)}\int_0^{\|x_n - x_{n-1}\|} \omega(t)\,dt \end{aligned}$$

$$\leq \frac{b \int_0^{r_n - r_{n-1}} \omega(t)\, dt}{1 - b\omega(r_n)}$$

$$\leq \frac{\tilde{\psi}(r_n) - \tilde{\psi}(r_{n-1}) - \psi'(r_{n-1})(r_n - r_{n-1})}{1 - b\omega(r_n)}$$

$$= -\frac{\tilde{\psi}(r_n)}{\psi'(r_n)}$$

$$= r_{n+1} - r_n.$$

Inequality (8) is crucial in the inductive step. If we look for $\tilde{\psi}$ of the type

$$\tilde{\psi}(r) = a + b \int_0^r \tilde{\omega}(t)\, dt - r,$$

we have for the unknown function $\tilde{\omega}$ the inequality

$$\int_0^{v-u} \omega(t)\, dt + \omega(u)(v-u) \leq \int_u^v \tilde{\omega}(t)\, dt \qquad (0 \leq u \leq v \leq R) \tag{9}$$

which does not depend on a and b.

The problem of determining $\tilde{\omega}$ in inequality (9) is not easy in general. In Section 3 we have used a function $\tilde{\omega}$ of the type $\tilde{\omega}(r) = \nu\omega(r)$ with ν constant. In this case, (9) is satisfied if and only if

$$\nu = \sup_{0 \leq u \leq v \leq R} \frac{\int_0^{v-u} \omega(t)\, dt + \omega(u)(v-u)}{\int_u^v \omega(t)\, dt}.$$

In [4] the authors considered a function $\tilde{\omega}$ defined by means of the inf-convolution

$$\tilde{\omega}(r) = \sup_{0 \leq t \leq r} (\omega(t) + \omega(r-t)).$$

An attempt to construct $\tilde{\omega}$ in general can be made using the following indications.

The existence of solutions for equation (1) is controlled by the scalar equation (6); in general, there is a gap between existence conditions and convergence conditions for Newton-Kantorovich approximations. This gap can be partially filled if we choose, among all possible functions $\tilde{\omega}$, one which is closer to ω. An optimality condition, for example, can be written in the following way:

$$\int_0^r \tilde{\omega}(t)\, dt = \sup_{\tau} \sum_{j=1}^{s} \left(\int_0^{r_j - r_{j-1}} \omega(t)\, dt + \omega(r_{j-1})(r_j - r_{j-1}) \right) \tag{10}$$

where the supremum is computed over all finite subdivisions $\tau = (r_0, r_1, ..., r_s)$ of the interval $(0, r)$.

The second member in (10) is the variation on $[0, r]$ of the non-negative interval function defined by

$$\chi([v,u]) = \int_v^u \omega(t)\,dt + \omega(v)(u-v).$$

To make the construction of $\tilde{\omega}$ more explicit is not an easy task. We refer to the enormous literature on interval functions. Such literature (see, for example, [11]) is not easily applicable to χ, since χ is not subadditive.

We conclude this section remarking that error estimates, analogous to that obtained in Section 4, hold also in the generalized Hölder case. In fact, from $\|f'(x)^{-1}\| \leq \frac{1}{1-b\omega(r^*)}$ for $\|x - x_0\| \leq r_*$ it follows that

$$\begin{aligned} \|x_* - x_{n+1}\| &\leq \frac{b}{1-b\omega(r_*)} \int_0^1 \omega(t\|x_* - x_n\|)\|x_* - x_n\|\,dt \\ &= \frac{b}{1-b\omega(r_*)} \int_0^{\|x_* - x_n\|} \omega(t)\,dt. \end{aligned}$$

If we introduce the function

$$\Omega(r) = \frac{b}{1-b\omega(r_*)} \int_0^r \omega(t)\,dt,$$

we obtain in a standard way, for $n \in \mathbb{N}$,

$$\|x_* - x_n\| \leq \Omega^{(n)}(r_*),$$

where $\Omega^{(n)}$ is the n-th iterate of Ω.

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Received 30.06.1997; in revised form 07.01.1998