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Fixed Point Theory for Weakly Contractive Maps with Applications to Operator Inclusions in Banach Spaces Relative to the Weak Topology

D. O'Regan

Abstract. A variety of fixed point results is presented for weakly contractive multi-valued maps. Our theory will then be applied to establish general existence principles for operator inclusions in Banach spaces relative to the weak topology.

Keywords: *Fixed points, weakly contractive maps, operator inclusions*

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1. Introduction

This paper presents a new fixed point theory for weakly contractive multi-valued maps between Banach spaces. This theory is then used to establish new results for differential and integral inclusions in a Banach space relative to the weak topology.

In 1971, Szep [24] discussed the abstract Cauchy problem

$$\left.\begin{aligned} y' &= f(t,y) \quad \text{on } [0,T] \\ y(0) &= y_0 \end{aligned}\right\} \tag{1.1}$$

where $f: [0,T] \times E \to E$ is a weakly-weakly continuous function and E is a reflexive Banach space. More recently [3, 4, 17, 18, 20, 21] the non-reflexive case was examined, the weakly-weakly continuity of f was relaxed and also the more general integral equation case was discussed. However, the inclusion analogue of (1.1) has received very little attention; we refer the reader to [3, 5]. In this paper we will discuss in detail operator inclusions in a Banach space relative to the weak topology. Our general theory will include as particular cases differential and integral inclusions.

For the remainder of this section we gather together some notation and preliminary facts. Let Ω_E be the bounded subsets of a Banach space E and let K^w be the family of all weakly compact subsets of E. Also, let B be the closed unit ball of E. The DeBlasi

D. O'Regan: National University of Ireland, Department of Mathematics, Galway, Ireland

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measure of weak non-compactness ([7]; see also [11]) is the map $w : \Omega_E \to [0,\infty)$ defined by

$$w(X) = \inf\Big\{t > 0 : \text{There exists } Y \in K^w \text{ with } X \subseteq Y + t\,B\Big\} \qquad (X \in \Omega_E).$$

For convenience we recall some properties of w. For this let $X_1, X_2 \in \Omega_E$. Then:

(i) $X_1 \subseteq X_2$ implies $w(X_1) \leq w(X_2)$.

(ii) $w(X_1) = 0$ if and only if $\overline{X_1^w} \in K^w$; here $\overline{X_1^w}$ is the weak closure of X_1 in E.

(iii) $w(\overline{X_1^w}) = w(X_1)$.

(iv) $w(X_1 \cup X_2) = \max\{w(X_1), w(X_2)\}$.

(v) $w(r\,X_1) = r\,w(X_1)$ for all $r > 0$.

(vi) $w(co(X_1)) = w(X_1)$.

(vii) $w(X_1 + X_2) \leq w(X_1) + w(X_2)$.

(viii) If $\emptyset \neq X_n \subset E$ $(n \in \mathbb{N})$, X_1 bounded, are weakly subsets with $X_n \downarrow$ and $\lim_{n\to\infty} w(X_n) = 0$, then $\cap_{n=1}^{\infty} X_n \neq \emptyset$ and weakly compact.

Suppose $F : Z \subseteq E \to 2^E$ (here 2^E denotes the family of non-empty subsets of E) maps bounded sets into bounded sets. We call F an α *w-contractive map* if $0 \leq \alpha < 1$ and $w(F(X)) \leq \alpha\,w(X)$ for all bounded sets $X \subseteq Z$. We say $F : E_1 \to 2^{E_2}$ (here E_1 and E_2 are Banach spaces) is *weakly upper semicontinuous* if the set $F^{-1}(A)$ is weakly closed in E_1 for any weakly closed set A in E_2.

We now state a theorem of Ambrosetti type (see [17, 21] and [22: pp. 86 - 88]).

Theorem 1.1.

(a) *Let H be a bounded subset of $C([0,T],E)$ (here E is a Banach space). Then*

$$\sup_{t\in[0,T]} w(H(t)) \leq w(H)$$

where $H(t) = \{\phi(t) : \phi \in H\}$.

(b) *Let $H \subseteq C([0,T],E)$ be bounded and equicontinuous. Then*

$$w(H) = \sup_{t\in[0,T]} w(H(t)) = W(H[0,T])$$

where $H[0,T] = \cup_{t\in[0,T]} \{\phi(t) : \phi \in H\}$.

Next we state two results (the second one follows immediately from the first) which will be used frequently in Section 2 (see [23] for definitions and proofs).

Theorem 1.2. *Let X and Y be topological spaces and $F : X \to 2^Y$ be an upper semicontinuous, point-compact multifunction. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in F(x_\alpha)$ for each α, then there is a $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of the net $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Theorem 1.3. *Let X and Y be topological spaces and $F : X \to 2^Y$ be a upper semicontinuous, point-compact multifunction. Suppose $\{x_\alpha\}$ is a net in X and $y_\alpha \in F(x_\alpha)$ for each α. If $x_\alpha \to x_0$ and $y_\alpha \to y_0$, then $y_0 \in F(x_0)$.*

Throughout Section 2, E will be a Banach space with norm $|\cdot|$. E^* will denote the dual of E. We will let E_w denote the space E when endowed with the weak topology generated by the continuous linear functionals on E (the family of seminorms $\{\rho_h : h \in E^*\}$ is defined by $\rho_h(x) = |h(x)|$ for all $x \in E$). A function g from a measure space $(\Omega, \mathcal{M})$ to E is said to be *scalarly measurable* if for any $\phi \in E^*$ the function $\phi(g)$ is measurable on $(\Omega, \mathcal{M})$. Two scalarly measurable functions $g, h : \Omega \to E$ are said to be *weakly equivalent* if for all $\phi \in E^*$ we have $\phi(g) = \phi(h)$ a.e. Let y be a function from $[a,b]$ into E. Then y is said to be weakly continuous at $t_0 \in [a,b]$ if for every $\phi \in E^*$ we have $\phi(y(\cdot))$ continuous at t_0. Denote by $C([a,b], E_w)$ (or $C_w([a,b], E)$) the space of weakly continuous functions on $[a,b]$ with the topology of weak uniform convergence (the family of seminorms $\{\eta_h\}$ is defined by $\eta_h(g) = \sup_{x\in[a,b]} \rho_h(g(x))$ for all $g \in C([a,b], E_w)$). This topology is of course determined by the basis

$$V_u(\phi_1, \ldots, \phi_m; \varepsilon) = \bigcap_{k=1}^{m} \left\{ g \in C([a,b], E_w) : \sup_{[a,b]} |\phi_k(g(t)) - u(t)| < \epsilon \right\},$$

where $u \in C([a,b], E_w)$, $\phi_1, \ldots, \phi_m \in E^*$, $\varepsilon > 0$ and $m \in \mathbb{N}$.

2. Theory and applications

We begin by establishing some new fixed point results which will be useful when we are discussing abstract operator inclusions. We first state a fixed point result due to Arino, Gautier and Penot [3]; the proof follows easily from Himmelberg's fixed point result [13].

Theorem 2.1. *Let E be a metrizable locally convex linear topological space and let C be a weakly compact, convex subset of E. Then any weakly sequentially upper semicontinuous map $F : C \to C(C)$ has a fixed point; here $C(C)$ denotes the family of non-empty, closed, convex subsets of C.*

Remark 2.1. $F : C \to C(C)$ is weakly sequentially upper semicontinuous if for any weakly closed set A of C, $F^{-1}(A)$ is sequentially closed for the weak topology on C.

Remark 2.2. The proof of Theorem 2.1 follows immediately from [13] once we note $F : C \to C(C)$ is weakly upper semicontinuous (see the argument in Theorem 2.3) and that a convex subset of a locally convex space is closed if and only if it is weakly closed.

Our next result replaces the weak compactness of the space C with a weak compactness type assumption on the operator F.

Theorem 2.2. *Let Q be a non-empty, bounded, convex, closed set in a Banach space E. Assume $F : Q \to C(Q)$ is weakly sequentially upper semicontinuous and α w-contractive (as defined in Section 1; here $0 \le \alpha < 1$). Then F has a fixed point.*

Proof. Let

$$\left.\begin{aligned} S_1 &= Q \\ S_{n+1} &= \overline{co}(F(S_n)) \quad (n \geq 1). \end{aligned}\right\}$$

It is easy to see that

$$S_{n+1} \subseteq S_n \quad \text{and} \quad w(S_{n+1}) \leq \alpha^n\, w(S_1) \qquad \text{for } n \geq 1.$$

Since $w(S_n) \to 0$ as $n \to \infty$ we have that $\cap_{n=1}^{\infty} S_n = S_\infty$ is non-empty. In addition, S_∞ is weakly closed and convex since each S_n is; in fact S_∞ is weakly compact since $w(S_\infty) = 0$. Also, since

$$F(S_n) \subseteq F(S_{n-1}) \subseteq \overline{co}(F(S_{n-1})) = S_n \qquad \text{for all } n$$

we have $F : S_\infty \to C(S_\infty)$. Theorem 2.1 implies that F has a fixed point in $S_\infty \subseteq Q$ ∎

We now use Theorem 2.2 to obtain a nonlinear alternative of Leray-Schauder type.

Theorem 2.3. *Let Q and C be closed, bounded, convex subsets of a Banach space E with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $0 \in U$, $\overline{U^w}$ a weakly compact subset of Q and $F : \overline{U^w} \to CK(C)$ a weakly sequentially upper semicontinuous, α w-contractive (here $0 \leq \alpha < 1$) map; here $CK(C)$ denotes the family of non-empty, convex , weakly compact subsets of C. Then either*

(A1) *F has a fixed point*

or

(A2) *there is a point $u \in \partial_Q U$ (the weak boundary of U in Q) and $\lambda \in (0,1)$ with $u \in \lambda Fu$.*

Proof. Suppose (A2) does not hold and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished, i.e. (A1) occurs). Let us look at $E = (E, w)$ (the space E endowed with the weak topology); note E is a locally convex Hausdorff linear topological space. Let

$$H = \left\{ x \in \overline{U^w} :\ x \in \lambda F(x) \ \text{ for some } \ \lambda \in [0,1] \right\}.$$

Now $0 \in H$. Also, H is closed in (E, w). To see this we claim $F : \overline{U^w} \to CK(C)$ is weakly upper semicontinuous (i.e. upper semicontinuous in (E, w)). Suppose the claim is true for the moment. To see that H is closed in (E, w) let (x_α) be a net in H (i.e. $x_\alpha \in \lambda_\alpha F(x_\alpha)$ for some $\lambda_\alpha \in [0,1]$) with $x_\alpha \to x_0 \in \overline{U^w}$ in (E, w) (i.e. x_α converges weakly to x_0). We must show $x_0 \in H$. Without loss of generality assume $\lambda_\alpha \to \lambda_0 \in [0,1]$. Let

$$N(x, \lambda) = \lambda\, F(x) \quad \text{and note} \quad N : \overline{U^w} \times [0,1] \to CK(C).$$

We first show N is weakly upper semicontinuous. Let Ω be a weakly closed subset of C (i.e. Ω is a closed set in (E, w)), (y_α, t_α) is a net in $\overline{U^w} \times [0,1]$, $y_\alpha \to y_0$ in (E, w), $t_\alpha \to t_0$ and $t_\alpha\, F(y_\alpha) \cap \Omega \neq \emptyset$. Suppose $w_\alpha \in F(y_\alpha)$ with $t_\alpha\, w_\alpha \in \Omega$. Now since F is weakly upper semicontinuous (i.e. upper semicontinuous in (E,w)), then there exist (Theorem 1.2) $w_0 \in F(y_0)$ and a subnet (w_β) of (w_α) with $w_\beta \to w_0$ in (E,w). Since Ω is weakly closed we have $t_0\, w_0 \in \Omega$. Consequently, $t_0\, F(y_0) \cap \Omega \neq \emptyset$, so $N : \overline{U^w} \times [0,1] \to CK(C)$ is weakly upper semicontinuous. Then since $(x_\alpha, \lambda_\alpha)$ is a net in $\overline{U^w} \times [0,1]$ with $x_\alpha \to x_0$ in (E,w), $\lambda_\alpha \to \lambda_0$ and $x_\alpha \in N(x_\alpha, \lambda_\alpha)$ we have (from Theorem 1.3) that $x_0 \in N(x_0, \lambda_0)$, so $x_0 \in H$. Thus H is closed in (E,w) if our claim is true.

To show $F : \overline{U^w} \to CK(C)$ is upper semicontinuous in (E,w) let A be a weakly closed subset of C.

Remark 2.3. $F^{-1}(A)$ is sequentially closed in E (with respect to the strong topology); recall a subset M is sequentially closed in E (with respect to the strong topology) if whenever $x_n \in M$ for all $n \in \mathbb{N}$ and $x_n \to x$ (with respect to the strong topology), then $x \in M$. Let $y_n \in F^{-1}(A)$ and $y_n \to y$ (with respect to the strong topology). Then y_n converges to y in (E,w). Now since $F : \overline{U^w} \to CK(C)$ is sequentially upper semicontinuous in (E,w) (i.e. $F^{-1}(A)$ is sequentially closed in (E,w)) we have $y \in F^{-1}(A)$. Consequently, if A is a weakly closed subset of C, we have $F^{-1}(A)$ sequentially closed in E (of course, by definition also weakly sequentially closed).

Now since $\overline{U^w}$ is weakly compact we have $\overline{F^{-1}(A)^w}$ weakly compact. Let $x \in \overline{F^{-1}(A)^w}$. The Eberlein-Šmulian theorem [10: p. 549] implies there exists a sequence $x_n \in F^{-1}(A)$ with $x_n \to x$ in (E,w). Since $F^{-1}(A)$ is weakly sequentially closed, we have $x \in F^{-1}(A)$. Thus $\overline{F^{-1}(A)^w} = F^{-1}(A)$, so $F^{-1}(A)$ is weakly closed. Consequently, $F : \overline{U^w} \to CK(C)$ is upper semicontinuous in (E,w).

Next we show H is compact in (E,w). To see this notice $H \subseteq co(F(H) \cup \{0\})$ and so

$$w(H) \le w(F(H)) \le \alpha\, w(H).$$

Consequently, $w(H) = 0$, so H is weakly compact. Now (A2) does not hold and F does not have a fixed point on $\partial_Q U$, so $H \cap \partial_Q U = \emptyset$. Also, (E,w) is Tychonoff, so there exists a continuous (continuous in (E,w)) $\mu : \overline{U^w} \to [0,1]$ with $\mu(H) = 1$ and $\mu(\partial_Q U) = 0$. Let

$$N(x) = \begin{cases} \mu(x)\, F(x) & \text{for } x \in \overline{U^w} \\ \{0\} & \text{for } x \in C \setminus \overline{U^w}. \end{cases}$$

Consider E with the norm topology. It is easy to see since $F : \overline{U^w} \to CK(C)$ is weakly upper semicontinuous that $N : C \to CK(C)$ is weakly upper semicontinuous. Also, $N : C \to CK(C)$ is α w-contractive. To see this let $X \subseteq C$ and notice

$$N(X) \subseteq co(F(X \cap U) \cup \{0\}).$$

Thus

$$w(N(X)) \le w(F(X \cap U)) \le w(F(X)) \le \alpha\, w(X).$$

Now Theorem 2.2 implies that there exists $x \in C$ with $x \in N(x)$. Now $x \in U$ since $0 \in U$. Consequently, $x \in \lambda\, F(x)$ with $0 \le \lambda = \mu(x) \le 1$. Consequently, $x \in H$, which implies $\mu(x) = 1$ and so $x \in F(x)$ ∎

Remark 2.4. The condition that $\overline{U^w}$ is weakly compact can be removed in Theorem 2.3 if we assume $F : \overline{U^w} \to CK(C)$ is weakly upper semicontinuous.

Theorem 2.3 can now be used to establish a new fixed point result for weakly sequentially upper semicontinuous maps in separable, reflexive Banach spaces.

Theorem 2.4. *Let $E = (E, \|\cdot\|)$ be a separable and reflexive Banach space, C and Q are closed, bounded, convex subsets of E with $Q \subseteq C$ and $0 \in Q$. Also, assume $F : Q \to CK(C)$ is a weakly sequentially upper semicontinuous (and weakly compact) map. In addition suppose the following:*

$$\left.\begin{array}{l}\textit{For any } \Omega_\varepsilon = \{x \in E : d(x,Q) \le \varepsilon\}\ (\varepsilon > 0),\ \textit{if } \{(x_j,\lambda_j)\}_{j=1}^\infty \textit{ is a sequence} \\ \textit{in } Q \times [0,1] \textit{ with } x_j \rightharpoonup x \in \partial_{\Omega_\varepsilon} Q \textit{ and } \lambda_j \to \lambda \textit{ and if } x \in \lambda F(x),\ 0 \le \lambda < 1, \\ \textit{then } \{\lambda_j F(x_j)\} \subseteq Q \textit{ for } j \textit{ sufficiently large } (\partial_{\Omega_\varepsilon} Q \textit{ is the weak boundary} \\ \textit{of } Q \textit{ relative to } \Omega_\epsilon,\ d(x,y) = |x-y| \textit{ and } \rightharpoonup \textit{ denotes weak convergence).}\end{array}\right\} \quad (2.1)$$

Then F has a fixed point in Q.

Remark 2.5. A special case of (2.1) (this is all we need for applications) is the following condition:

$$\left.\begin{array}{l}\text{If } \{(x_j,\lambda_j)\}_{j=1}^\infty \text{ is a sequence in } Q \times [0,1] \text{ with } x_j \rightharpoonup x \text{ and } \lambda_j \to \lambda \text{ and} \\ \text{if } x \in \lambda F(x) \text{ with } 0 \le \lambda < 1, \text{ then } \{\lambda_j F(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.}\end{array}\right\} \quad (2.2)$$

Proof of Theorem 2.4. Let $r : E \to Q$ be a weakly continuous retraction guaranteed from [4] (see also [19, 20]) and consider

$$B = \{x \in E : x \in F r(x)\}.$$

We first show $B \ne \emptyset$. Notice the argument in Theorem 2.3 (note Q is weakly compact since a subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology) implies that $F : Q \to CK(C)$ is weakly upper semicontinuous and so $F r : E \to CK(C)$ is weakly upper semicontinuous.

Remark 2.6. Notice $F : Q \to CK(C)$ is a weakly compact map is redundant in the statement of Theorem 2.4 (this is why we use brackets) since $F : Q \to CK(C)$ is weakly upper semicontinuous and Q is weakly compact (see [1: p. 464]).

Now since F is a weakly compact map, we have $F r(E)$ weakly compact. Theorem 2.2 implies that $B \ne \emptyset$. Also, B is weakly closed (see the argument in Theorem 2.3); in fact B is weakly compact since $B \subseteq F(Q)$.

We now show $B \cap Q \ne \emptyset$. Suppose $B \cap Q = \emptyset$. Then since Q is weakly compact and B is weakly closed we have from [12: p. 65] that $d(B,Q) = \inf\{\|x-y\| : x \in B,\, y \in Q\} > 0$. Thus there exists $\varepsilon > 0$ with $\Omega_\varepsilon \cap B = \emptyset$; here Ω_ε is as described in (2.1). Note Ω_ε is weakly compact (since Ω_ϵ is closed and convex (so weakly closed) and bounded). Also, since E is separable, we know from [8] that weak topology on Ω_ε is metrizable; let d^* denote the metric. For $i \in \mathbb{N}$ let

$$U_i = \left\{x \in \Omega_\epsilon : d^*(x,Q) < \frac{\epsilon}{i}\right\}.$$

Fix $i \in \mathbb{N}$. Now U_i is $d^\star$-open in Ω_ε, so U_i is weakly open in Ω_ε. Also,

$$\overline{U_i^w} = \overline{U_i^{d^\star}} = \left\{x \in \Omega_\varepsilon : d^\star(x, Q) \leq \frac{\varepsilon}{i}\right\} \quad \text{and} \quad \partial_{\Omega_\varepsilon} U_i = \left\{x \in \Omega_\varepsilon : d^\star(x, Q) = \frac{\varepsilon}{i}\right\}.$$

Now Theorem 2.3 implies (since $\Omega_\varepsilon \cap B = \emptyset$) that there exists $y_i \in \partial_{\Omega_\varepsilon} U_i$ and $\lambda_i \in (0,1)$ with $y_i \in \lambda_i F r(y_i)$.

Remark 2.7. Note: (i) $\overline{U_i^w}$ is weakly compact since $U_i \subseteq \Omega_\varepsilon$ and (ii) $F r : \overline{U_i^w} \to CK(C)$ is a weakly upper semicontinuous map.

Consequently, for each $j \in \mathbb{N}$ there exists $(y_j, \lambda_j) \in \partial_{\Omega_\varepsilon} U_j \times (0,1)$ with $y_j \in \lambda_j F r(y_j)$. Notice in particular since $y_j \in \partial_{\Omega_\varepsilon} U_j$ that

$$\{\lambda_j F r(y_j)\} \not\subseteq Q \qquad \text{for } j \in \mathbb{N}. \tag{2.3}$$

We now look at

$$D = \left\{x \in E : x \in \lambda F r(x) \text{ for some } \lambda \in [0,1]\right\}.$$

Clearly, D is weakly closed since $Fr : E \to CK(C)$ is weakly upper semicontinuous. Also, since $D \subseteq co(F(Q) \cup \{0\})$ we have that D is weakly compact (so weakly sequentially compact by the Eberlein-Šmulian theorem). This together with $d^\star(y_j, Q) = \frac{\varepsilon}{j}$, $|\lambda_j| \leq 1$ $(j \in \mathbb{N})$ implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \rightharpoonup y^\star \in \partial_{\Omega_\varepsilon} Q$. Also, since $y_j \in \lambda_j F r(y_j)$, we have $y^\star \in \lambda^\star F r(y^\star)$ (note $H : \overline{U_1^w} \times [0,1] \to CK(C)$ given by $H(u, \lambda) = \lambda F r(u)$ is weakly upper semicontinuous (see the argument in Theorem 2.3)). Thus $y^\star \in \lambda^\star F(y^\star)$. Now $\lambda^\star \neq 1$ since $B \cap Q = \emptyset$ and so we may assume $0 \leq \lambda^\star < 1$. But in this case, (2.1) with $x_j = r(y_j)$, $x = y^\star = r(y^\star)$ implies $\{\lambda_j F r(y_j)\} \subseteq Q$ for j sufficiently large. This contradicts (2.3). Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x \in F r(x)$, i.e. $x \in F(x)$ ∎

We now use Theorems 2.2 and 2.4 to establish some general existence principles for the nonlinear abstract operator inclusion

$$y(t) \in F y(t) \quad \text{on} \quad [0, T]. \tag{2.4}$$

Remark 2.8. Notice (2.4) is understood for a.e. $t \in [0,T]$ if we are looking for solutions in $L^p([0,T], E)$.

Theorem 2.5. *Let E_1 be a Banach space and let E be either $C([0,T], E_1)$ or $L^p([0,T], E_1)$, $1 \leq p < \infty$. Let Q be a non-empty, bounded, convex, closed subset of E and assume $F : Q \to C(Q)$ is a weakly sequentially upper semicontinuous and α w-contractive $(0 \leq \alpha < 1)$ map. Then (2.4) has a solution in Q.*

Proof. The result follows immediately from Theorem 2.2 ∎

Theorem 2.6. *Let E_1 be a separable and reflexive Banach space and let Q and C be closed, bounded, convex subsets of $L^p([0,T], E_1)$, $1 \leq p < \infty$, with $Q \subseteq C$ and $0 \in Q$. Assume $F : Q \to CK(C)$ is a weakly sequentially upper semicontinuous map such that (2.1) (or (2.2)) holds. Then (2.4) has a solution in Q.*

Remark 2.9. Note [8: Chapters III and IV] implies $L^p([0,T], E_1)$ is reflexive and separable.

Proof of Theorem 2.6. The result follows immediately from Theorem 2.4 ∎

It is also of interest (see [3, 4 - 6, 16 - 18, 20 - 22, 24]) to find solutions to (2.4) in $C([0,T],E_w)$; here E is a Banach space. To establish existence in this case we use Himmelberg's fixed point theorem [13]. For convenience we state it here.

Theorem 2.7. *Let Q be a non-empty, convex, closed subset of a locally convex Hausdorff linear topological space B. Assume that $F: Q \to C(Q)$ is upper semicontinuous and $F(Q)$ is relatively compact in B. Then F has a fixed point.*

Theorem 2.7 immediately yields the following existence principle for (2.4).

Theorem 2.8. *Let E be a Banach space with Q a non-empty, closed, convex subset of $C([0,T],E_w)$. Also, assume Q is a subset of $C([0,T],E)$ and $F: Q \to Cc(Q)$ is w-upper semicontinuous (i.e. for any closed set B of $C([0,T],E_w)$, $F^{-1}(B)$ is closed in $C([0,T],E_w)$). In addition suppose the family $F(Q)$ is weakly equicontinuous and $F(Q(t))$ is weakly relatively compact in E, for each $t \in [0,T]$. Then (2.4) has a solution in Q.*

Remark 2.10. $Cc(Q)$ denotes the family of non-empty, convex (subset of $C([0,T],E)$), closed (in $C([0,T],E)$) subsets of Q.

Proof of Theorem 2.8. The result follows from Theorem 2.7 (there the locally convex Hausdorff space B is $C([0,T],E_w)$) if we show $F(Q)$ is relatively compact in $C([0,T],E_w)$ and if $F: Q \to Cw(Q)$ (here $Cw(Q)$ denotes the family of non-empty, convex, closed (in $C([0,T],E_w)$) subsets of Q).

To see that $F(Q)$ is relatively compact in $C([0,T],E_w)$ we apply the Arzela-Ascoli theorem [15: Theorem 7.17/p. 233]. We must show (which were assumed)

(i) for each $t \in [0,T]$, $F\,Q(t) = \{F\,y(t) : y \in Q\}$ is weakly relatively compact in E

(ii) $F(Q)$ is weakly equicontinuous.

To show $F: Q \to Cw(Q)$ we need to check that Fy is closed in $C([0,T],E_w)$ for each $y \in Q$. Fix $y \in Q$ and look at Fy. Now Fy has closed, convex values in $C([0,T],E)$. Also note, if $\phi \in E^\star$ and if we define the point functional ψ by $\psi(u) = \phi(u(0))$ (here $u \in C([0,T],E)$), then $\psi \in (C([0,T],E))^\star$. Suppose (x_β) is a net in Fy with $x_\beta \to x_0$ in $C([0,T],E_w)$. Take $\phi \in E^\star$ and define the point functional ψ as above. Now $\psi(x_\beta) \to \psi(x_0)$ together with the fact that Fy is weakly closed (since Fy is a closed, convex subset of $C([0,T],E)$) implies $x_0 \in F(y)$. Thus Fy is closed in $C([0,T],E_w)$ for each $y \in Q$ ∎

Remark 2.11. Notice $F: Q \to Cc(Q)$ could be replaced by $F: Q \to Cw(Q)$ in the statement of Theorem 2.8. Also, the condition Q is a subset of $C([0,T],E)$ is not needed in the proof if we wish to guarantee a solution in $C([0,T],E_w)$ (but not necessarily in $C([0,T],E)$).

A more general version of Theorem 2.8 is the following result.

Theorem 2.9. *Let E be a Banach space with Q a non-empty, closed, convex subset of $C([0,T],E_w)$. Also, assume Q is a closed, bounded subset of $C([0,T],E)$, $F: Q \to Cc(Q)$ is w-upper semicontinuous, and there exists α, $0 \le \alpha < 1$, with $w(F(X)) \le \alpha\, w(X)$ for all bounded subsets $X \subseteq Q$. In addition suppose the family $F(Q)$ is weakly equicontinuous. Then (2.4) has a solution in Q.*

Proof. Let

$$\left.\begin{aligned} S_1 &= Q \\ S_{n+1} &= \overline{co}(F(S_n)) \quad (n \geq 1). \end{aligned}\right\}$$

It is easy to see that

$$S_{n+1} \subseteq S_n \quad \text{and} \quad w(S_{n+1}) \leq \alpha^n w(S_1) \qquad \text{for } n \geq 1.$$

Since $0 \leq \alpha < 1$ we have $w(S_n) \to 0$ as $n \to \infty$. Also, since S_n is a weakly closed subset of $C([0,T],E)$ for each n we have that $S_\infty = \cap_{n=1}^{\infty} S_n$ is non-empty. In addition, S_∞ is weakly closed and convex. Also, an easy argument (see the ideas in Theorem 2.8 using point functionals) implies that S_∞ is closed in $C([0,T], E_w)$. In addition, since

$$F(S_n) \subseteq F(S_{n-1}) \subseteq \overline{co}(F(S_{n-1})) = S_n \qquad \text{for all } n$$

we have $F : S_\infty \to Cc(S_\infty)$; in fact $F : S_\infty \to Cw(S_\infty)$ as in Theorem 2.8. The result follows immediately from Theorem 2.7 if we show $F(S_\infty)$ is relatively compact in $C([0,T], E_w)$. Apply the Arzela-Ascoli theorem [15: p. 233]. We have immediately, by assumption, that $F(S_\infty)$ is weakly equicontinuous so it remains for us to show for each $t \in [0,T]$ that the set $F S_\infty(t) = \{F y(t) : y \in S_\infty\}$ is weakly relatively compact in E. To see this notice since $w(S_\infty) = 0$ and $F(S_\infty) \subseteq S_\infty$ so we have $w(F(S_\infty)) = 0$. This together with Theorem 1.1/(a) implies $w(F S_\infty(t)) = 0$ for each $t \in [0,T]$. Thus for each $t \in [0,T]$ we have that $F S_\infty(t)$ is weakly relatively compact in E ∎

To illustrate the theory derived we now consider the Volterra integral inclusion

$$y(t) \in h(t) + \int_0^t k(t,s)\, F(s, y(s))\, ds \qquad \text{on } [0,T]. \tag{2.5}$$

When we are discussing (2.5) we will assume the following conditions hold:

$$F : [0,T] \times E \to E \text{ has non-empty, compact, convex values.} \tag{2.6}$$

$$\left.\begin{aligned} &\text{For each continuous } y : [0,T] \to E \text{ there exists a scalarly} \\ &\text{measurable } z : [0,T] \to E \text{ with } z(t) \in F(t, y(t)) \text{ a.e. on } [0,T]. \end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned} &\text{For any } r > 0 \text{ there exists a constant } M_r > 0 \text{ with} \\ &|F(t,y)| \leq M_r \text{ for all } t \in [0,T] \text{ and } y \in E \text{ with } |y| \leq r. \end{aligned}\right\} \tag{2.8}$$

$$\left.\begin{aligned} &\text{For each continuous } y : [0,T] \to E \text{ there exists a } z \text{ (as in (2.7))} \\ &\text{with either } z\,[0,T] \text{ relatively weakly compact or } z \text{ is Pettis} \\ &\text{integrable and } \overline{co}(z\,[0,T]) \text{ has the Radon-Nikodym property.} \end{aligned}\right\} \tag{2.9}$$

$$h : [0,T] \to E \quad \text{is a continuous single-valued function.} \tag{2.10}$$

$$\left.\begin{aligned} &k_t(s) \in L^1([0,t], \mathbb{R}) \text{ for each } t \in [0,T] \text{ (here } k_t(s) = k(t,s)) \\ &\text{and there exists } v \in L^1[0,T] \text{ and constants } \alpha, \beta > 0 \text{ such that} \\ &\text{for } x,\, t \in [0,T],\, x < t \text{ we have } \textstyle\int_x^t |k(t,s)|\, ds \leq \beta\left(\int_x^t v(s)\, ds\right)^{\alpha}. \end{aligned}\right\} \tag{2.11}$$

$$\int_0^{t^*} |k_t(s) - k_{t'}(s)|\, ds \to 0 \quad \text{as} \quad t \to t', \quad \text{where} \quad t^* = \min\{t, t'\}. \tag{2.12}$$

Assign a multi-valued operator

$$N : C([0,T], E_w) \cap C([0,T], E) \quad (= C([0,T], E)) \;\to\; Cc(C([0,T], E)) \tag{2.13}$$

by letting

$$N\, y(t) = \left\{ h(t) + \int_0^t k(t,s)\, v(s)\, ds \;\middle|\; \begin{array}{l} v : [0,T] \to E \text{ is scalarly measurable} \\ \text{with } v(t) \in F(t, y(t)) \text{ a.e. } t \in [0,T] \end{array} \right\}. \tag{2.14}$$

We first show (2.13) makes sense. Suppose $y : [0,T] \to E$ is continuous. Then there exists a scalarly measurable $z : [0,T] \to E$ with $z(t) \in F(t, y(t))$ a.e. $t \in [0,T]$. From (2.9) and [9: p. 671] we have that z is weakly equivalent to a strongly Bochner-measurable mapping g. Also, $|g|$ is bounded since $|\phi(z)|$ is bounded (see (2.8)) for all $\phi \in E^*$ with $|\phi| = 1$. Let $r(t) = h(t) + \int_0^t k(t,s)\, g(s)\, ds$. Then for each $\phi \in E^*$ we have

$$\phi\left(\int_0^t k(t,s)\, g(s)\, ds \right) = \int_0^t k(t,s)\, \phi(g(s))\, ds = \int_0^t k(t,s)\, z(s)\, ds$$

and so $\phi(r) = \phi\big(h(t) + \int_0^t k(t,s)\, z(s)\, ds\big)$.

Next we show

$$u(t) = h(t) + \int_0^t k(t,s)\, z(s)\, ds \in C([0,T], E_w) \cap C([0,T], E).$$

Now $u \in C([0,T], E_w)$ immediately since

$$\phi(u(t)) = \phi(h(t) + \int_0^t k(t,s)\, z(s)\, ds) = \phi(r(t)).$$

To see that u is continuous first notice that there exists $r > 0$ with $|y|_0 = \sup_{[0,T]} |y(t)| \le r$ and so (2.8) implies that there exists a constant M_r with

$$|F(t, y(t))| \le M_r \qquad \text{for all } \; t \in [0,T].$$

Let $t,\, x \in [0,T]$ with $t > x$. Without loss of generality assume $u(t) - u(x) \neq 0$. Then there exists (consequence of the Hahn Banach theorem) $\phi \in E^*$ with $|\phi| = 1$ and $|u(t) - u(x)| = \phi(u(t) - u(x))$. Thus

$$\begin{aligned} |u(t) - u(x)| &= \phi\left(h(t) - h(x) + \int_0^x [k(t,s) - k(x,s)]\, z(s)\, ds + \int_x^t k(t,s)\, z(s)\, ds \right) \\ &\le |h(t) - h(x)| + M_r \int_0^x |k(t,s) - k(x,s)|\, ds + M_r \int_x^t |k(t,s)|\, ds, \end{aligned}$$

so $u \in C([0,T],E)$. Consequently,

$$N : C([0,T],E_w) \cap C([0,T],E) \quad (= C([0,T],E)) \rightarrow 2^{C([0,T],E)}.$$

To show (2.13) it just remains for us to show that N has closed (in $C([0,T],E)$) values (note N has automatically convex values since (2.6) is true). Let $y : [0,T] \rightarrow E$ be continuous and look at the set $N y$. Suppose $w_n \in N y$ $(n \geq 1)$. Then there exists $z_n : [0,T] \rightarrow E$ $(n \geq 1)$, scalarly measurable with $z_n(s) \in F(s, y(s))$ for a.e. $s \in [0,T]$. Suppose

$$w_n(t) \rightarrow h(t) + \int_0^t k(t,s)\, z(s)\, ds = w(t) \qquad \text{in } C([0,T],E).$$

Fix $t \in (0,T]$ and $\phi \in E^*$. Then $\phi(k(t,\cdot)\, z_n) \rightarrow \phi(k(t,\cdot)\, z)$ in $L^1[0,t]$, so $\phi(k(t,\cdot)\, z_n) \rightarrow \phi(k(t,\cdot)\, z)$ in measure. Thus there exists a subsequence S of integers with

$$\phi(k(t,s)\, z_n(s)) \rightarrow \phi(k(t,s)\, z(s)) \qquad \text{for a.e. } s \in [0,t] \text{ (as } n \rightarrow \infty \text{ in } S).$$

Now since $k(t,s)\, z_n(s) \in k(t,s)\, F(s, y(s))$ for a.e. $s \in [0,t]$ and since the values of F are closed and convex (so weakly closed), we have $k(t,s)\, z(s) \in k(t,s)\, F(s,y(s))$ for a.e. $s \in [0,t]$. Thus $w \in N y$ and so N has closed (in $C([0,T],E)$) values ∎

Theorem 2.10. *Let E be a Banach space and Q a non-empty, closed, convex subset of $C([0,T],E_w)$ with Q a bounded subset of $C([0,T],E)$. Also, assume (2.6) – (2.12) are satisfied. In addition, suppose the following conditions hold:*

$$\left.\begin{array}{l} N : C([0,T],E_w) \cap C([0,T],E) \ \ (= C([0,T],E)) \rightarrow Cc(C([0,T],E)) \\ \textit{is } w - \textit{upper semicontinuous (here } N \textit{ is as defined in (2.14)).} \end{array}\right\} \tag{2.15}$$

$$\left.\begin{array}{l} K(\{t\} \times [0,t] \times Q[0,t]) \textit{ is weakly relatively compact} \\ \textit{in } E \textit{ for each } t \in [0,T]; \ \textit{ here } K(t,s,u) = k(t,s)\, F(s,u). \end{array}\right\} \tag{2.16}$$

$$N : Q \rightarrow Cc(Q). \tag{2.17}$$

Then (2.5) has a solution in Q.

Remark 2.12. Condition (2.15) is our replacement for conditions of Type A in [4, 5]. Certainly, if $F(p)$ is point-valued for every $p \in [0,T] \times E$ and $F : [0,T] \times E \rightarrow E$ is weakly-weakly continuous, then (2.15) is satisfied; in fact, F is a w-continuous [21] single-valued map.

Proof of Theorem 2.10. The result follows from Theorem 2.8 once we show:

(i) $N\, Q(t) = \{N\, y(t) : y \in Q\}$ is weakly relatively compact in E for each $t \in [0,T]$.

(ii) $N(Q)$ is weakly equicontinuous.

To see (i) fix $t \in [0,T]$ and take $y \in Q$. Let $u \in N y$. Then there exists a scalarly measurable $z : [0,T] \rightarrow E$ with $z(t) \in F(t, y(t))$ a.e. and

$$u(t) = h(t) + \int_0^t k(t,s)\, z(s)\, ds.$$

By [16] we have

$$\int_0^t k(t,s)\,z(s)\,ds \in t\,\overline{co}\left\{k(t,s)\,z(s),\ s \in [0,t]\right\},$$

so

$$\begin{aligned} w(N\,Q\,(t)) &\leq w\big(t\ \overline{co}\{K(t,s,y(s)):\ y \in Q,\ s \in [0,t]\}\big) \\ &= T\ w\big(K(\{t\} \times [0,t] \times Q\,[0,t])\big) \\ &= 0. \end{aligned}$$

This implies $N\,Q\,(t)$ is weakly relatively compact in E for each $t \in [0,T]$.

To see (ii) notice since Q is bounded that there exists $r > 0$ with $|u|_0 \leq r$ for all $u \in Q$ and so from (2.8) there exists $M > 0$ with

$$|F(t,y(t))| \leq M \qquad \text{for all } t \in [0,T] \text{ and all } y \in Q.$$

Let $u \in N\,Q$ and $t,\,x \in [0,T]$ with $t > x$. Without loss of generality assume $u(t) - u(x) \neq 0$. Then there exists $\phi \in E^*$ with $|\phi| = 1$ and $|u(t) - u(x)| = \phi(u(t) - u(x))$. Thus

$$\begin{aligned} |u(t) - u(x)| &= \phi(u(t) - u(x)) \\ &\leq |h(t) - h(x)| + M \int_0^x |k(t,s) - k(x,s)|\,ds + M \int_x^t |k(t,s)|\,ds. \end{aligned}$$

Thus $N\,Q$ is weakly equicontinuous (of course, $N\,Q$ is also strongly equicontinuous) ∎

Remark 2.13. Since $N\,Q$ is relatively compact in $C([0,T],E_w)$, then to check (2.15) we need to show [1: p. 465] the following property holds: if (y_α) (here (y_α) is a net in $C([0,T],E_w))$ converges to y in $C([0,T],E_w)$ and $v_\alpha \in N(y_\alpha)$ is such that (v_α) converges to v in $C([0,T],E_w)$, then $v \in N\,y$ (i.e. the graph of N is a closed subset of $C([0,T],E_w) \times C([0,T],E_w)$).

A more general version of Theorem 2.10 is the following result.

Theorem 2.11. *Let E be a Banach space and Q a non-empty, closed, convex subset of $C([0,T],E_w)$ with Q a bounded subset of $C([0,T],E)$. Also, assume* (2.6) – (2.12), (2.15) *and* (2.17) *hold. In addition, suppose*

$$N\,Q\,(t) \ \textit{ is weakly relatively compact in } E \textit{ for each } t \in [0,T]. \tag{2.18}$$

Then (2.5) *has a solution in* Q.

Remark 2.14. If E is reflexive, then (2.18) holds since a subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology (now use (2.8) and the fact that Q is a bounded subset of $C([0,T],E)$).

Remark 2.15. We could also use Theorem 2.9 to obtain another existence result for (2.5). We could replace (2.16) in Theorem 2.10 by

$$\left.\begin{array}{l}\text{there exists } \gamma \geq 0 \text{ with } \gamma T < 1 \text{ and } w(K(\{t\} \times [0,t] \times \Omega)) \leq \gamma\, w(\Omega) \\ \text{for } t \in [0,T] \text{ and for any bounded subset } \Omega \text{ of } Q\end{array}\right\}$$

provided Q is a equicontinuous subset of $C([0,T],E)$ (the idea is to use Theorem 1.1(b)).

Remark 2.16. It is also possible to discuss the Hammerstein inclusion

$$y(t) \in h(t) + \int_0^T k(t,s)\, F(s, y(s))\, ds \qquad \text{for } t \in [0,T].$$

Here (2.11) and (2.12) are replaced by

$$\left.\begin{array}{l}k_t(s) \in L^1([0,T],\mathbb{R}) \text{ for each } t \in [0,T] \text{ and the map} \\ t \mapsto k_t(s) \text{ is continuous from } [0,T] \text{ to } L^1([0,T],\mathbb{R}).\end{array}\right\}$$

To conclude this section we discuss "approximation type methods" for (2.4). A set

$$\mathcal{K} = \big\{T_\alpha : \alpha \in J \text{ (some index set)}\big\},$$

where

$$T_\alpha : C([0,T],E_w) \cap C([0,T],E) \;\; (= C([0,T],E)) \;\rightarrow\; Cc(C([0,T],E))$$

for each $\alpha \in J$, is *collectively compact* in $C([0,T],E_w)$ if for each bounded set $\Omega \subseteq C([0,T],E)$ the set $\mathcal{K}\,\Omega$ is relatively compact in $C([0,T],E_w)$. To discuss (2.4) we consider for each $n \in \mathbb{N}$ the equations (think of these as corresponding numerical approximations)

$$x(t) \in F_n\, x(t) \qquad \text{on } [0,T]. \tag{2.19^n}$$

Theorem 2.12. *Let E be a Banach space and Q a non-empty, closed, convex subset of $C([0,T],E_w)$ with Q a bounded subset of $C([0,T],E)$. Suppose the following conditions are satisfied:*

$$\text{For each } n \in \mathbb{N}, \;\; F_n : Q \rightarrow Cc(Q) \;\; \text{is } w-\text{upper semicontinuous} \tag{2.20}$$

$$\mathcal{K} = \{F_n : n \in \mathbb{N}\} \quad \text{is collectively compact in } C([0,T],E_w). \tag{2.21}$$

$$\left.\begin{array}{l}\text{The sequence } \{F_n\}_{n=1}^{\infty} \text{ has the following closure property:} \\ \text{For any compact subset (in } C([0,T],E_w)) \;\Omega \text{ of } Q, \\ \text{if } z_n \text{ in } \Omega \text{ with } z_n \in F_n\, z_n \;\; (n \geq 1) \text{ and there exists} \\ z_0 \in C([0,T],E_w) \text{ with } z_n \rightarrow z_0 \text{ in } C([0,T],E_w), \text{ then } z_0 \in F\, z_0.\end{array}\right\} \tag{2.22}$$

Then there exists a subsequence S of $\mathbb{N}$ and a sequence $\{x_n\}$ of solutions of $(2.19)^n$ $(n \in S)$ *with* $x_n \to x_0$ *(as $n \to \infty$ in S) in* $C([0,T], E_w)$ *and x_0 is a solution of* (2.4).

Proof. For each $n \in \mathbb{N}$, F_n has a fixed point by Theorem 2.8 since $F_n Q$ is relatively compact in $C([0,T], E_w)$. Thus there exists $x_n \in Q$ with $x_n \in F_n x_n$. Let

$$\Omega = \overline{\{F_n y : y \in Q,\ n \geq 1\}} \qquad \text{(closure in } C([0,T], E_w)).$$

Now Ω is a compact subset of $C([0,T], E_w)$. Also, $x_n \in \Omega$ for each $n \in \mathbb{N}$. Thus (x_n) has a subsequence which converges weakly uniformly on $[0,T]$ to a weakly continuous function $x_0 \in \Omega$. Now (2.22) implies $x_0 \in F x_0$ ∎

A set

$$\mathcal{K} = \{T_\alpha : \alpha \in J \text{ (some index set)}\},$$

where $T_\alpha : Z \to C(Z)$ (here $Z = C([0,T], E)$ or $Z = L^p([0,T], E)$ $(1 \leq p < \infty)$, or Z are subsets of such spaces) for each $\alpha \in J$ is *weakly collectively compact* with respect to Z if for each bounded set Ω of Z, the set $\mathcal{K}\,\Omega$ is weakly relatively compact in Z.

Theorem 2.13. *Let E be a Banach space and let Z be either $C([0,T], E)$ or $L^p([0,T], E)$ $(1 \leq p < \infty)$. Also, assume Q is a non-empty, bounded, convex, closed subset of Z. Suppose the following conditions are satisfied:*

$$\forall\, n \in \mathbb{N},\ F_n : Q \to C(Q) \text{ is weakly sequentially upper semicontinuous.} \tag{2.23}$$

$$\mathcal{K} = \{F_n : n \in \mathbb{N}\} \text{ is weakly collectively compact with respect to } Z. \tag{2.24}$$

$$\left.\begin{array}{l} \text{For any weakly compact subset } \Omega \text{ of } Z, \text{ if } z_n \in \Omega \text{ with } z_n \in F_n z_n \\ (n \geq 1) \text{ and there exists } z_0 \in Z \text{ with } z_n \rightharpoonup z_0 \text{ in } Z, \text{ then } z_0 \in F z_0. \end{array}\right\} \tag{2.25}$$

Then there exists a subsequence S of $\mathbb{N}$ and a sequence $\{x_n\}$ of solutions of $(2.19)^n$ $(n \in S)$ *with* $x_n \rightharpoonup x_0$ *(as $n \to \infty$ in S) in Z and x_0 is a solution of* (2.4).

Proof. Let $\Omega = \overline{\{F_n y : y \in Q,\, n \geq 1\}}^{w}$ (weak closure in Z) ∎

A collection

$$\mathcal{K} = \{T_\alpha : \alpha \in J \text{ (some index set)}\},$$

where $T_\alpha : L^p([0,T], E) \to CK(L^p([0,T], E))$ for each $\alpha \in J$, is *collectively bounded* in X $(X \subseteq L^p([0,T], E))$ if for each bounded set Ω of X, the set $\mathcal{K}\,\Omega$ is bounded in $L^p([0,T], E)$ $(1 < p < \infty)$.

Theorem 2.14. *Let E be a separable and reflexive Banach space and let Q and C be closed, bounded, convex subsets of $L^p([0,T], E)$ $(1 < p < \infty)$, with $Q \subseteq C$ and $0 \in Q$. Suppose the following conditions are satisfied:*

$$\forall\, n \in \mathbb{N},\ F_n : Q \to CK(C) \text{ is weakly sequentially upper semicontinuous.} \tag{2.26}$$

$$\left.\begin{array}{l} \text{For each } n \in \mathbb{N}, \text{ if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } Q \times [0,1] \\ \text{with } x_j \rightharpoonup x \text{ and } \lambda_j \to \lambda, \text{ and if } x \in \lambda F_n(x) \text{ with } 0 \leq \lambda < 1, \\ \text{then } \{\lambda_j F_n(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{array}\right\} \tag{2.27}$$

$$\left.\begin{array}{l} \textit{For any weakly compact subset } \Omega \textit{ of } L^p([0,T],E), \\ \textit{if } z_n \in \Omega \textit{ with } z_n \in F_n\, z_n \ \ (n \geq 1) \textit{ and there exists} \\ z_0 \in L^p([0,T],E) \textit{ with } z_n \rightharpoonup z_0 \textit{ in } L^p([0,T],E), \textit{ then } z_0 \in F\, z_0. \end{array}\right\} \qquad (2.28)$$

Then there exists a subsequence S *of* $\mathbb{N}$ *and a sequence* $\{x_n\}$ *of solutions of* $(2.19)^{n'}$ $(n \in S)$ *with* $x_n \rightharpoonup x_0$ $(as\ n \to \infty\ in\ S)$ *in* $L^p([0,T],E)$ *and* x_0 *is a solution of* (2.4).

Proof. For each $n \in \mathbb{N}$, F_n has a fixed point by Theorem 2.6.

Remark 2.17. Note $\mathcal{K} = \{F_n : n \in \mathbb{N}\}$, where $F_n : Q \to CK(C)$, is collectively bounded in Q since C is a bounded subset of $L^p([0,T],E)$.

Let $\Omega = \overline{\{F_n\, y : y \in Q,\, n \geq 1\}}^w$ ∎

References

[1] Aliprantis, C. D. and K. C. Border: *Infinite Dimensional Analysis.* Berlin: Springer-Verlag 1985.

[2] Anselone, P. M.: *Collectively Compact Operator Approximation Theory.* Engelwood Cliffs (New Jersey): Prentice Hall 1971.

[3] Arino, O., Gautier, S. and J. P. Penot: *A fixed point theorem for sequentially continuous mappings with applications to ordinary differential equations.* Funkc. Ekvac. 27 (1984), 273 - 279.

[4] Chow, S. N. and J. D. Schuur: *An existence theorem for ordinary differential equations in Banach spaces.* Bull. Amer. Math. Soc. 77 (1971), 1018 - 1020.

[5] Chow, S. N. and J. D. Schuur: *Fundamental theory of contingent differential equations in Banach spaces.* Trans. Amer. Math. Soc. 179 (1973), 133 - 144.

[6] Cramer, E., Lakshmikantham, V. and A. R. Mitchell: *On the existence of weak solutions of differential equations in nonreflexive Banach spaces.* Nonlinear Anal. 2 (1978), 169 - 177.

[7] De Blasi, F. S.: *On a property of the unit sphere in Banach spaces.* Bull. Math. Soc. Sci. Math. Roum. 21 (1977), 259 - 262.

[8] Dunford, N. and J. T.Schwartz: *Linear Operators.* Part I: *General Theory.* New York - London: Intersci. Publ. 1958.

[9] Edgar, G. A.: *Measurability in a Banach space.* Indiana Univ. Math. J. 26 (1977), 663 - 678.

[10] Edwards, R. E.: *Functional Analysis. Theory and Applications.* New York et al.: Holt, Rinehart and Winston 1965.

[11] Emmanuele, G.: *Measures of weak noncompactness and fixed point theorems.* Bull. Math. Soc. Sci. Math. Roum. 25 (1981), 353 - 358.

[12] Floret, K.: *Weakly Compact Sets.* Lect. Notes Math. 801 (1980), 1 - 123.

[13] Himmelberg, J. C.: *Fixed points of compact multifunctions.* J. Math. Anal. Appl. 38 (1972), 205 - 207.

[14] Jameson, G. J. O.: *An elementary proof of the Arens and Borsuk extension theorems.* J. London Math. Soc. 14 (1976), 364 - 368.

[15] Kelley, J.: *General Topology.* Toronto: D. Van Nostrand Comp. 1955.

[16] Knight, W. J.: *Solutions of differential equations in B-spaces.* Duke Math. J. 41 (1974), 437 - 442.

[17] Kubiaczyk, I. and S. Szufla: *Kneser's theorem for weak solutions of ordinary differential equations in Banach spaces.* Publ. Inst. Math. (Beograd) 46 (1982), 99 - 103.

[18] Lakshmikantham, V. and S. Leela: *Nonlinear Differential Equations in Abstract Spaces.* Oxford: Pergamon Press 1981.

[19] O'Regan, D.: *A continuation method for weakly condensing operators.* Z. Anal. Anw. 15 (1996), 565 - 578.

[20] O'Regan, D.: *Fixed point theory for weakly sequentially continuous mappings.* Mathl. Comput. Modelling (to appear).

[21] O'Regan, D.: *Operator equations in Banach spaces relative to the weak topology.* (to appear).

[22] Smith, S. K. L.: *Measure of Nonconvergence and Noncompactness.* Ph.D Thesis. Arlington: University of Texas 1978.

[23] Su, C. H. and V. M. Sehgal: *Some fixed point theorems for condensingmultifunctions in locally convex spaces.* Proc. Amer. Math. Soc. 50 (1975), 150 - 154.

[24] Szep, A.: *Existence theorems for weak solutions of ordinary differential equations in reflexive Banach spaces.* Studia Sci. Math. Hungar. 6 (1971), 197 - 203.

[25] Zeidler, E.: *Nonlinear Functional Analysis and its Applications.* Volume I: *Fixed-Point Theorems.* New York: Springer 1986.

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