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Degree Theory for Variational Inequalities in Complementary Systems

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Abstract. Working out degree theory for the investigation of finite-dimensional variational inequalities with continuous mappings in the usual way. All properties typical for a topologiocal degree are proved. Then the K-degree is generalized by the Galerkin procedure for some class $S_A(X)$ of monotone-like operators in complementary systems. On the basis of our theory some new results concerning solvability of variational inequalities in complementary systems are proved. These results make it possible to obtain new facts on solvability of variational inequalities as well as operator equations with stongly nonlinear differential operators.

Keywords: *Topological degree, variational inequalities, Leray-Schauder lemma, complementary systems, ρ-convergence, Galerkin approximation*

AMS subject classification: 47 H 11, 47 H 19, 49 J 40

0. Introduction

Degree theory is one of the most useful tools of modern nonlinear analysis applicable to the investigation of operator equations

$$Ty = h. \tag{0.1}$$

The theory was created for continuous mappings on finite-dimensional spaces in [2]. Then, because of the objective impossibility to develop a unique degree theory for all kinds of operators on infinite-dimensional spaces, there appeared a lot of variants of degree theory for various classes of operators on various types of infinite-dimensional spaces, such as [11, 13] for compact perturbations of the identity map on arbitrary Banach spaces, [3, 17] for operators of the class $(S)_+$ on reflexive spaces, [9, 19] for some classes of operators of pseudomonotone type on complementary systems. It should be noted that another topological characteristic has been used in most Russian works. This characteristic is called rotation and is equivalent to degree.

The main purpose of the present paper is to develop a new topological characteristic of the degree type, that will be applicable to the investigation of variational inequalities

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in the usual way. Such an idea, in terms of rotation, was partly described in [8] for a class of operators similar to $(S)_+$ on reflexive spaces and in [10] for a class of operators similar to pseudomonotone quasibounded operators on complementary systems. Section 1 is concerned with continuous operators on finite-dimensional spaces and Section 2 with operators of the class $S_A(X)$ on complementary systems.

Let Y and Z be Banach spaces in duality with respect to the pairing $\langle \cdot, \cdot \rangle$, $K \subset Y$ be a convex closed set, $h \in Z$ be some fixed element and $T : Y \to Z$ be some operator. Then

$$\langle v - y, Ty \rangle \geq \langle v - y, h \rangle \qquad \forall v \in K \tag{0.2}$$

is called a ***variational inequality*** [14], and $y \in K$ which satisfies (0.2) is called a ***solution*** of this variational inequality, or a (K, h)-*critical point* of T. If some set contains none of the (K, h)-critical points of T, we will say that the operator T is (K, h)-*non-degenerate* on this set.

It is known that in the case of $K = Y$ the variational inequality (0.2) is equivalent to equation (0.1). We will refer to a solution of this equation as to a (Y, h)-*critical point* of T.

1. Finite-dimensional K-degree

Let V be a real Euclidean space equipped with a scalar product $(\cdot, \cdot)$ and a norm $|\cdot|$. We shall remind of some definitions and main properties of the well-known finite-dimensional degree theory. The details can be found, for example, in [12, 15, 16, 18] or some other works.

Let us consider a bounded domain $\Omega \subset V$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. Let $h \in V$ and a continuous mapping $T : \overline{\Omega} \to V$ be (V, h)-non-degenerate on $\partial\Omega$. An integer-valued function is called a *degree of T at h relative to Ω* and denoted as

$$\deg(T, \Omega, h)$$

if it satisfies the following statements:

(1) *Additivity*: If Ω_l $(l = 1, \ldots, m)$ are open mutually disjont subsets of Ω and a mapping T is (V, h)-non-degenerate on $\overline{\Omega} \setminus \cup_{l=1}^{m} \Omega_l$, then

$$\deg(T, \Omega, h) = \sum_{l=1}^{m} \deg(T, \Omega_l, h).$$

(2) *Homotopy invariance*: If $T_t : \overline{\Omega} \to V$ is a family of continuous mappings, continuously dependent on $t \in [0, 1]$ and (V, h)-non-degenerate on $\partial\Omega$, then

$$\deg(T_0, \Omega, h) = \deg(T_1, \Omega, h).$$

(3) *Normalization*: If $Iy \equiv y$ and $h \in \Omega$, then

$$\deg(I, \Omega, h) = 1.$$

A one-parameter family of continuous mappings from statement (2) is called a *homotopy at h relative to Ω, connecting T_0 with T_1*, and such mappings are said to be *homotope at h relative to Ω*. A simple sufficient condition of homotopy, usually referred to the Poincaré-Bohl theorem, is stated below.

Proposition 1. *Continuous mappings T_0 and T_1 are homotope at 0 relative to Ω if there is no such $y \in \partial\Omega$ that the vectors T_0y and T_1y are directed in opposite ways.*

Indeed, in this case

$$T_ty \equiv (1-t)T_0y + tT_1y \neq 0 \qquad \forall\, y \in \partial\Omega, t \in [0,1].$$

Statements (1) - (3) completely define the degree, and can be treated as its axioms.

The following statements are directly derived from these statements:

(4) If $(T-h)y \equiv Ty - h$, then $\deg(T-h,\Omega,0) = \deg(T,\Omega,h)$.

(5) If $\Omega_* \subset \Omega$ and T is (V,h)-non-degenerate on $\overline{\Omega_*}$, then $\deg(T,\Omega,h) = \deg(T,\Omega \setminus \overline{\Omega_*}, h)$.

(6) If $T_0y = T_1y$ for all $y \in \partial\Omega$, then $\deg(T_0,\Omega,h) = \deg(T_1,\Omega,h)$.

(7) If T is (V,h)-non-degenerate on $\overline{\Omega}$, then $\deg(T,\Omega,h) = 0$.

(8) If $\deg(T,\Omega,h) \neq 0$, then there exists a $y_0 \in \Omega$ such that $Ty_0 = h$.

(9) If $\deg(T_0,\Omega,h) \neq \deg(T_1,\Omega,h)$, then there exist $y_0 \in \partial\Omega$ and $\lambda_0 \in (0,1)$ such that $(1-\lambda_0)T_0y_0 + \lambda_0T_1y_0 = h$.

A (V,h)-critical point $y_0 \in \Omega$ of T is called *isolated* if there exists an $r_0 > 0$ such that the ball

$$B_{r_0}(y_0) = \{y \in V : |y - y_0| < r_0\} \subset \Omega$$

includes no other (V,h)-critical point. It follows from statement (5) that

$$\deg(T, B_r(y_0), h) = \text{const} \qquad (0 < r < r_0).$$

This integer independent of r is called an *index* of an isolated (V,h)-critical point y_0 of T and denoted as

$$\operatorname{ind}(T, y_0, h).$$

We can formulate the next statement on the basis of this notation.

(10) If Ω includes only isolated (V,h)-critical points y_l of T, then the number of such points is finite and $\deg(T,\Omega,h) = \sum_{l=1}^{m} \operatorname{ind}(T,y_l,h)$.

Keeping in mind the investigation of operator equations (0.1), we can say that (8) and (10) are the key statements of the degree.

Finally, we will formulate the last statement, which is well-known as Leray-Schauder lemma. This proposition allows, on one hand, to reduce the dimension for evaluation of the degree, and, on the other hand, to construct finite-dimensional approximations for the generalization of degree theory on some classes of operators on infinite-dimensional spaces.

Proposition 2. *Let $V^0 \subset V$ be a proper subspace, let $\pi_0 : V \to V^0$ be an orthogonal projector, let $\Omega \subset V$ be a bounded domain such that $\Omega^0 = \Omega \cap V^0$ is non-empty, let $F : \overline{\Omega} \to V^0$ be a continuous mapping, let a mapping*

$$\Phi(y) = y - F(y) \tag{1.1}$$

be $(V, 0)$-non-degerate on $\partial\Omega$ and $\Phi^0(y) = \pi_0 y - F(y)$. Then

$$\deg(\Phi, \Omega, 0) = \deg(\Phi^0, \Omega^0, 0).$$

Let $K \subset V$ be a convex closed set with non-empty interior $\overset{\circ}{K}$. Let $P_K : V \to K$ be the projector, which transforms any $y \in V$ into $P_K(y) = z \in K$ with

$$|y - z| = \inf_{v \in K} |y - v|.$$

Because K is closed and convex, the projector P_K is well-defined on V, single-valued and continuous. Furthermore, it acts as identity on K. It is known [14] that a (K, h)-critical point of the mapping T is the same as a fixed point of the mapping

$$F(y) = P_K(y + h - Ty). \tag{1.2}$$

So, if the mapping Φ is defined by formula (1.1), then the variational inequality

$$(v - y, Ty) \geq (v - y, h) \qquad \forall v \in K \tag{1.3}$$

is equivalent to the operator equation

$$\Phi(y) = 0. \tag{1.4}$$

For a bounded domain Ω, let us consider a relatively open set $\omega = \Omega \cap K$ with a relative boundary $\partial_K\omega = \partial\Omega \cap K$, a whole boundary $\partial\omega$, a closure $\overline{\omega}$ and an interior $\overset{\circ}{\omega}$. Let $h \in V$ and a continuous mapping $T : \overline{\omega} \to V$ be (K, h)-non-degenerate on $\partial_K\omega$. In this case the mapping F, defined by formula (1.2), has no fixed point on $\partial_K\omega$. If the set of (K, h)-critical points of T or, which is the same, the set of fixed points of F from $\partial\omega \setminus \partial_K\omega$ is non-empty, we can shift it into $\overset{\circ}{\omega}$. Such an approach was suggested in [1]. So, we put

$$\delta(y) = \inf_{v \in K \setminus N} |y - v| \tag{1.5}$$

for some sufficiently small neighborhood N of this set and for any $y \in \overline{\omega}$, and define the mapping

$$\tilde{F}(y) = F(y) + \delta(y)(\tilde{y} - F(y)) \tag{1.6}$$

for some fixed $\tilde{y} \in \overset{\circ}{K}$. This mapping is obviously continuous on $\overline{\omega}$, coincides with the mapping F on $\overline{\omega} \setminus N$, has no fixed point on $\partial\omega$, and $\tilde{F}(y) \in \overset{\circ}{K}$ for all $y \in \partial\omega \cap N$. Hence, the mapping

$$\tilde{\Phi}(y) = y - \tilde{F}(y) \tag{1.7}$$

is continuous on $\overline{\omega}$ and $(V, 0)$-non-degenerate on $\partial\omega$. It is clear that the mappings $\tilde{F}$ and $\tilde{\Phi}$ depend on $\tilde{y}$. But the corresponding mapping $\tilde{\tilde{F}}$ has the same properties for another $\tilde{\tilde{y}} \in \overset{\circ}{K}$, and $\tilde{\tilde{F}}(y) \in \overset{\circ}{K}$ for all $y \in \partial\omega \cap N$. Since K is convex, the vectors $\tilde{\Phi}(y)$ and $\tilde{\tilde{\Phi}}(y)$ are directed in opposite ways for none of $y \in \partial\omega \cap N$, and $\tilde{\Phi}(y) \equiv \tilde{\tilde{\Phi}}(y)$ for all $y \in \partial\omega \setminus N$. Then, according to Proposition 1, the mappings $\tilde{\Phi}$ and $\tilde{\tilde{\Phi}}$ are homotope at 0 relative to $\overset{\circ}{\omega}$, and by statement (2) of the degree $\deg(\tilde{\Phi}, \overset{\circ}{\omega}, 0) = \deg(\tilde{\tilde{\Phi}}, \overset{\circ}{\omega}, 0)$. So, this integer is independent from $\tilde{y} \in \overset{\circ}{K}$. Its independence from a sufficiently small neighbourhood N is obvious. This shows that the following definition is legitimate.

Definition 1. Let $h \in V$ and a continuous mapping $T : \overline{\omega} \to V$ be (K,h)-non-degenerate on $\partial_K \omega$. Then we can set

$$\deg_K(T, \omega, h) = \deg(\tilde{\Phi}, \mathring{\omega}, 0)$$

and name this integer the *K-degree of T at h relative to ω.*

It will be shown in the next part of this section that the K-degree has typical degree properties and may be used for the investigation of the finite-dimensional variational inequalities (1.3).

Definition 2. Continuous mappings $T_0, T_1 : \overline{\omega} \to V$ are said to be *K-homotope at h relative to ω* if there exists a family of continuous mappings $T_t : \overline{\omega} \to V$, continuously dependent on $t \in [0,1]$ and (K,h)-non-degenerate on $\partial_K \omega$, which connects T_0 with T_1.

Definition 3. A (K,h)-critical point $y_0 \in \omega$ of T is said to be *isolated* if there exists an $r_0 > 0$ such that the set $B_{r_0}(y_0) \cap K \subset \omega$ includes no other (K,h)-critical points of T.

Definition 4. We set

$$\operatorname{ind}_K(T, y_0, h) = \deg_K(T, B_r(y_0) \cap K, h) \qquad (0 < r < r_0)$$

and name this integer *index of an isolated (K,h)-critical point y_0 of T.*

Theorem 1. *Let $K \subset V$ be a convex closed set, let $\omega \subset K$ be a non-empty bounded relatively open set, let $h \in V$, let a continuous mapping $T : \overline{\omega} \to V$ be (K,h)-non-degenerate on $\partial_K \omega$. Then the K-degree of T at h relative to ω has the following properties:*

(1) *If ω_l $(l = 1, \dots, m)$ are relatively open, mutually disjoint subsets of ω and a continuous mapping T is (K,h)-non-degenerate on $\overline{\omega} \setminus \cup_{l=1}^m \omega_l$, then*

$$\deg_K(T, \omega, h) = \sum_{l=1}^{m} \deg_K(T, \omega_l, h).$$

(2) *If continuous mappings T_0 and T_1 are K-homotope at h relative to ω, then*

$$\deg_K(T_0, \omega, h) = \deg_K(T_1, \omega, h).$$

(3) *If there exists a $y_0 \in \omega$ with*

$$(y - y_0, Ty) \geq (y - y_0, h) \qquad \forall y \in \partial_K \omega, \tag{1.8}$$

then $\deg_K(T, \omega, h) = 1$.

(4) $\deg_K(T - h, \omega, 0) = \deg_K(T, \omega, h)$.

(5) *If $\omega_* \subset \omega$ and T is (K,h)-non-degenerate on $\overline{\omega_*}$, then*

$$\deg_K(T, \omega, h) = \deg_K(T, \omega \setminus \overline{\omega_*}, h).$$

(6) *If* $T_0 y = T_1 y$ *for all* $y \in \partial_K \omega$, *then* $\deg_K(T_0, \omega, h) = \deg_K(T_1, \omega, h)$.

(7) *If* T *is* (K, h)*-non-degenerate on* $\overline{\omega}$, *then* $\deg_K(T, \omega, h) = 0$.

(8) *If* $\deg_K(T, \omega, h) \neq 0$, *then there exists a* $y_0 \in \omega$ *such that*

$$(v - y_0, Ty_0) \geq (v - y_0, h) \qquad \forall v \in K.$$

(9) *If* $\deg_K(T_0, \omega, h) \neq \deg_K(T_1, \omega, h)$, *then there exist* $y_0 \in \partial_K \omega$ *and* $\lambda_0 \in (0,1)$ *such that*

$$(v - y_0, (1 - \lambda_0)T_0 y_0 + \lambda_0 T_1 y_0) \geq (v - y_0, h) \qquad \forall v \in K. \tag{1.9}$$

(10) *If* ω *includes only isolated* (K, h)*-critical points* y_l *of* T, *then the number of such points is finite and*

$$\deg_K(T, \omega, h) = \sum_{l=1}^{m} \operatorname{ind}_K(T, y_l, h). \tag{1.10}$$

Proof. (1) Since $\overline{\omega} \setminus \cup_{l=1}^{m} \omega_l$ contains none of the (K, h)-critical points of T, then it contains none of the $(V, 0)$-critical points of Φ and, moreover, of $\tilde{\Phi}$. Therefore it is sufficient to use Definition 1 and statement (1) of the degree.

(2) Since the continuous mappings T_0 and T_1 are K-homotope at h relative to ω, there exists a family of continuous mappings $T_t : \overline{\omega} \to V$, continuously dependent on $t \in [0, 1]$ and (K, h)-non-degenerate on $\partial_K \omega$, which connects T_0 with T_1. We define a family of mappings as

$$F_t(y) = P_K(y + h - T_t y) \tag{1.11}$$

and denote by N some sufficiently small neighbourhood of the set

$$\Big\{ y \in \partial\omega \setminus \partial_K \omega : \ F_t(y) = y \ \text{ for some } \ t \in [0, 1] \Big\}. \tag{1.12}$$

By analogy with (1.6), we set

$$\tilde{F}_t(y) = F_t(y) + \delta(y)(\tilde{y} - F_t(y)) \tag{1.13}$$

for $\delta(y)$ from (1.5) and some fixed $\tilde{y} \in \mathring{K}$. By construction, the family of mappings $\tilde{F}_t$ is continuous on $\overline{\omega}$, continuously dependent on $t \in [0, 1]$ and has no fixed point on $\partial\omega$. Then

$$\tilde{\Phi}_t(y) = y - \tilde{F}_t(y) \tag{1.14}$$

is a homotopy at 0 relative to $\mathring{\omega}$, and according to Definition 1 and statement (2) of the degree

$$\deg_K(T_0, \omega, h) = \deg(\tilde{\Phi}_0, \mathring{\omega}, 0) = \deg(\tilde{\Phi}_1, \mathring{\omega}, 0) = \deg_K(T_1, \omega, h).$$

(3) First of all, we shall show that the family of mappings

$$T_t y = t(y - y_0) + (1 - t)(Ty - h)$$

is a K-homotopy at 0 relative to ω. Indeed, if there exist $y \in \partial_K \omega$ and $t \in (0, 1)$ with $(v - y, T_t y) \geq 0$ for all $v \in K$ or, which is the same,

$$t(v - y, y - y_0) + (1 - t)(v - y, Ty - h) \geq 0 \qquad \forall v \in K,$$

then for $v = y_0$ we obtain $t|y - y_0|^2 + (1-t)(y - y_0, Ty - h) \le 0$, which contradicts (1.8). It is obvious that the family of mappings T_t is continuous and continuously dependent on $t \in [0,1]$. Thus, according to property (4), proved below, and (2) of the K-degree we have

$$\deg_K(T, \omega, h) = \deg_K(T - h, \omega, 0) = \deg_K(I - y_0, \omega, 0) = \deg_K(I, \omega, y_0).$$

To evaluate this integer, we construct the following mappings, according to (1.2) and (1.1) for $T \equiv I$ and $h \equiv y_0$ (because of $y_0 \in \omega \subset K$):

$$F(y) = P_K(y + y_0 - y) = P_K(y_0) = y_0 \qquad \text{and} \qquad \Phi(y) = y - y_0.$$

If in addition $y_0 \in \mathring{\omega}$, then Φ is $(V,0)$-non-degenerate on $\partial\omega$ and we can set $\tilde{\Phi}(y) = \Phi(y)$. Otherwise $y_0 \in \partial\omega \setminus \partial_K \omega$ and we must construct $\tilde{\Phi}$ according to the scheme (1.5) - (1.7). It is obvious that for any fixed $0 < \epsilon < 1$ we have $y_0 + \epsilon(\tilde{y} - y_0) = y_* \in \mathring{\omega}$, and vectors $\tilde{\Phi}(y)$ and $\Phi_*(y) = y - y_*$ are directed in the opposite ways for none of $y \in \partial\omega$. Thus, according to Proposition 1, $\tilde{\Phi}$ and Φ_* are homotope at 0 relative to $\mathring{\omega}$, and according to Definition 1 and statements (2) - (4) of the degree

$$\deg_K(I, \omega, y_0) = \deg(\tilde{\Phi}, \mathring{\omega}, 0) = \deg(\Phi_*, \mathring{\omega}, 0) = \deg(I, \mathring{\omega}, y_*) = 1.$$

(4) It is evident that one and the same mapping F, described by formula (1.2), corresponds to both variational inequalities (1.3) and $(v - y, Ty - h) \ge 0$ $(v \in K)$. So, this property follows from Definition 1.

(5) (K,h)-non-degeneracy of T on $\overline{\omega_*}$ implies $(V,0)$-non-degeneracy of Φ on this set. So, this property follows from Definition 1 and statement (5) of the degree.

(6) Since $T_0 y = T_1 y = (1-t)T_0 y + tT_1 y$ for all $y \in \partial_K \omega$, then the continuous (K,h)-non-degenerate on $\partial_K \omega$ mappings T_0 and T_1 are K-homotope at h relative to ω. So, this property follows from property (2) of the K-degree.

(7) (K,h)-non-degeneracy of T on $\overline{\omega}$ implies $(V,0)$-non-degeneracy of Φ on this set. So, by putting $\tilde{\Phi} \equiv \Phi$, we have, according to Definition 1 and statement (7) of the degree $\deg_K(T, \omega, h) = \deg(\tilde{\Phi}, \mathring{\omega}, 0) = 0$.

(8) Suppose that T is (K,h)-non-degenerate on ω. By property (7) of the K-degree we get $\deg_K(T, \omega, h) = 0$, which is a contradiction.

(9) Assuming K-homotopy of T_0 and T_1 at h relative to ω implies a contradiction with property (2) of the K-degree. So, there exist $y_0 \in \partial_K \omega$ and $\lambda_0 \in (0,1)$ satisfying (1.9).

(10) Property (5) of the K-degree legitimate Definition 4. Boundedness of ω implies finitness of the set of isolated (K,h)-critical points of T in ω – otherwise this set contains a limit point, which is non-isolated. So, this set consists of y_l $(l = 1, \dots, m)$, and there exists an $r > 0$ such that the balls $B_r(y_l)$ $(l = 1, \dots, m)$ are mutually disjoint. It is evident that T is (K,h)-non-degenerate on $\omega \setminus \cup_{l=1}^m \overline{B_r(y_l)}$. Then, according to property (1) of the K-degree and Definition 4, equality (1.10) holds ∎

The next theorem is a K-degree variant of the Leray-Schauder lemma.

Theorem 2. *Let $V^0 \subset V$ be a proper subspace, let $\pi_0 : V \to V^0$ be an orthogonal projector, let $K \subset V$ be a closed convex set with $K^0 = K \cap V^0$ and $\mathring{K}^0 = \mathring{K} \cap V^0 \neq \emptyset$, let $\omega \subset K$ be a relatively open bounded set with $\omega^0 = \omega \cap V^0$ and $\mathring{\omega}^0 = \mathring{\omega} \cap V^0 \neq \emptyset$, let $T : \overline{\omega} \to V$ be a continuous mapping and for some $h \in V$*

$$\left\{ y \in \partial_K \omega : (v - y, Ty) \geq (v - y, h) \quad \forall v \in K^0 \right\} = \emptyset. \tag{1.15}$$

Then

$$\deg_K(T, \omega, h) = \deg_{K^0}(\pi_0 T, \omega^0, \pi_0 h). \tag{1.16}$$

Remark. Formula (1.15) implies (K, h)-non-degeneracy of T on $\partial_K \omega$ as well as $(K^0, \pi_0 h)$-non-degeneracy of $\pi_0 T$ on $\partial_{K^0} \omega^0$. Thus, both sides of equality (1.16) are well-defined.

Proof of Theorem 2. We introduce a one-parameter family of closed convex subsets of K

$$K^s = \left\{ y \in K : \inf_{v \in V^0} |y - v| \leq s \right\} \qquad (0 \leq s < \infty)$$

and a corresponding family of projectors $P_s : V \to K^s$. It is obvious that

$$P_0(y) = P_{K^0}(\pi_0 y) \qquad \forall y \in V, \tag{1.17}$$

where $P_{K^0} : V^0 \to K^0$ is a projector. Because the set ω is bounded and the mapping T is continuous on $\overline{\omega}$, the set $U = \cup_{y \in \overline{\omega}}(y + h - Ty)$ is bounded too; its projection on K is bounded all the more. Thus, there exists an $r > 0$ such that $P_K(U) \subset K^r$. Hence

$$P_r(y + h - Ty) = P_K(y + h - Ty) \qquad \forall y \in \omega. \tag{1.18}$$

For $t \in [0, 1]$ we set

$$F_t(y) = P_{tr}(y + h - Ty) \tag{1.19}$$

and

$$\Phi_t(y) = y - F_t(y). \tag{1.20}$$

Then the equation $\Phi_t(y) = 0$ is equivalent to the variational inequality $(v - y, Ty) \geq (v - y, h)$ for all $v \in K^{tr}$ (compare with (1.1) - (1.4)). Because of (1.15) and $K^0 \subset K^{tr}$ the family of continuous mappings Φ_t, continuously dependent on $t \in [0, 1]$, is (K, h)-non-degenerate on $\partial_K \omega$. By analogy with (1.11) - (1.14), we can construct a homotopy $\tilde{\Phi}_t$ at 0 relative to $\mathring{\omega}$. By Definition 1 and formulas (1.18) - (1.20)

$$\deg_K(T, \omega, h) = \deg(\tilde{\Phi}_1, \mathring{\omega}, 0). \tag{1.21}$$

According to statement (2) of the degree,

$$\deg(\tilde{\Phi}_1, \mathring{\omega}, 0) = \deg(\tilde{\Phi}_0, \mathring{\omega}, 0). \tag{1.22}$$

But the set $\mathring{\omega}$ and the mapping $\tilde{\Phi}_0$ satisfy the conditions of Proposition 2. So, for the mapping $\tilde{\Phi}^0(y) = \pi_0 y - \tilde{F}_0(y)$ we obtain

$$\deg(\tilde{\Phi}_0, \mathring{\omega}, 0) = \deg(\tilde{\Phi}^0, \mathring{\omega}^0, 0). \tag{1.23}$$

It follows from (1.17) and the linearity of the projector π_0 that

$$P_0(y + h - Ty) = P_{K^0}(y + \pi_0 h - \pi_0 Ty) \qquad \forall y \in \omega^0.$$

Specifying Definition 1 for the space V^0, we have

$$\deg(\tilde{\Phi}^0, \mathring{\omega}^0, 0) = \deg_{K^0}(\pi_0 T, \omega^0, \pi_0 h). \tag{1.24}$$

Finally, equalities (1.21) - (1.24) imply (1.16) ∎

It should be noted that in terms of rotation an analogous proposition is contained in [8] and [10].

2. K-degree in complementary systems

Let Y and Z be Banach spaces in duality with respect to the pairing $\langle\cdot,\cdot\rangle$, Y_0 and Z_0 be closed subspaces of Y and Z, respectively, such that the dual of Y_0 can be identified with Z and the dual of Z_0 can be identified with Y by means of the pairing $\langle\cdot,\cdot\rangle$. Then the quadruple $(Y, Y_0; Z, Z_0)$ is called *complementary system* [4]. In particular, it is a natural generalization of a dual pair (X, X^*), which is useful for a reflexive space X, to $(X^{**}, X; X^*, X^*)$. Principal examples are complementary systems formed of Orlicz-Sobolev spaces [4].

For further considerations, we assume that the subspaces Y_0 and Z_0 are separable, and that $K \subset Y$ is a σ-set (see the next definition).

Definition 5. A set $K \subset Y$ with $K_0 = K \cap Y_0$ so that

$$\sigma(Y, Z_0)\,\mathrm{cl}K = K \tag{2.1}$$

$$\sigma(Y, Z)\,\mathrm{cl}K_0 = K \tag{2.2}$$

will be called *σ-set*.

Clearly, a σ-set is convex and closed. It is known [4] that a space Y itself and a closed ball

$$\overline{B_r(y_0)} = \{y \in Y : \|y - y_0\| \leq r\}$$

with $y_0 \in Y_0$ are σ-sets. Provided that the support of an Orlicz-Sobolev space has some natural properties (for example, its boundary has the segment property), not only the space itself and corresponding balls [4], but also the cone of functions that are non-negative together with their lower derivatives, is a σ-set [10].

Let X be a convex set with $Y_0 \subset X \subset Y$. Let us equip X with a metric ρ such that a norm convergent sequence from Y_0 is ρ-convergent, and a ρ-bounded set from Y_0 is norm bounded. The ρ-closure of Y_0 will be denoted by X_0. Obviously, $Y_0 \subset X_0 \subset X$.

Such set X and metric ρ were defined, studied and used for construction of a rotation theory and a K-rotation theory in complementary systems formed of Orlicz-Sobolev spaces in [9] and [10], respectively. It was shown that the set of functions, belonging to the Orlicz class together with their derivatives, may be taken as X, and the corresponding integral metric may be taken as ρ. In this case ρ-convergence is situated between strong convergence and modular convergence, which was applied later in [19] for the construction of a degree theory in Orlicz-Sobolev spaces. Note that for reflexive Orlicz-Sobolev spaces modular convergence is equivalent to norm convergence, hence, to ρ-convergence.

Definition 6. We will say that an operator $T : X \to Z$ belongs to the class $S_A(X)$ if it satisfies the following conditions:

(T_1) For any finite-dimensional subspace $V \subset Y_0$, the function $f(u,v) = \langle v, Tu\rangle$ is continuous on $V \times V$.

(T_2) For any sequence $y_n \in Y_0$ such that $y_n \to y \in Y$ for $\sigma(Y, Z_0)$, $Ty_n \to z \in Z$ for $\sigma(Z, Y_0)$ and $\limsup_{n\to\infty}\langle y_n, Ty_n\rangle \le \langle y, z\rangle$ there follows that

$$\lim_{n\to\infty} \rho(y_n, y) = 0, \qquad Ty = z, \qquad \lim_{n\to\infty}\langle y_n, Ty_n\rangle = \langle y, z\rangle.$$

(T_3) For any $v_0 \in Y_0$ and constants $c_1 > 0$, $c_2 > 0$ there exists such constant $k(c_1, c_2, v_0) > 0$ with

$$\left.\begin{aligned} \|y\| &\le c_1 \\ \langle y - v_0, Ty\rangle &\le c_2 \end{aligned}\right\} \quad\Longrightarrow\quad \|Ty\| \le k(c_1, c_2, v_0).$$

It is clear that an operator T belongs to the class $S_A(X)$ simultaneously with the operator $T - h$, where $(T-h)y = Ty - h$ with some fixed $h \in Z_0$.

The conditions $(T_1) - (T_3)$ are similar to analogous conditions from [4] and subsequent works about complementary systems. In fact, the only difference is the additional claim of ρ-convergence of the sequence y_n in (T_2).

The class of operators $S_A(X)$ was defined, studied and used for a complementary system formed of Orlicz-Sobolev spaces in [9, 10]. It was shown that elliptic differential operators with natural conditions on their coefficients such as those stated in [4, 5] belong to this class. Note that a similar class of operators was used in [19] for constructing a degree theory in Orlicz-Sobolev spaces.

Let us consider a sequence of finite-dimensional subspaces $V_n \subset Y_0$ satisfying

$$V_n \subset V_{n+1} \quad \forall n \qquad \text{and} \qquad \overline{\bigcup_{n=1}^{\infty} V_n} = Y_0, \tag{2.3}$$

which exists because of the separability of Y_0. For a closed convex set K_0, the sets $K_n = K_0 \cap V_n$ are obviously closed, convex and

$$K_n \subset K_{n+1} \quad \forall n \qquad \text{and} \qquad \overline{\bigcup_{n=1}^{\infty} K_n} = K_0. \tag{2.4}$$

Lemma 1. *Let K be a σ-set, let $M \subset K \cap X$ be a ρ-bounded and ρ-closed set and $M_0 = M \cap Y_0$, let $h \in Z_0$ and an operator T of the class $S_A(X)$ be (K,h)-non-degenerate on $M \cap X_0$, that is*

$$\Big\{y \in M \cap X_0 : \langle v - y, Ty\rangle \ge \langle v - y, h\rangle \quad \forall v \in K\Big\} = \emptyset. \tag{2.5}$$

Then there exists finite-dimensional subspace $V^0 \subset Y_0$ with $K^0 = K \cap V^0$ such that

$$\Big\{y \in M_0 : \langle v - y, Ty\rangle \ge \langle v - y, h\rangle \quad \forall v \in K^0\Big\} = \emptyset.$$

Proof. Suppose the contrary. Then for any index n

$$\Big\{y \in M_0 : \langle v-y, Ty\rangle \geq \langle v-y, h\rangle \quad \forall v \in K_n\Big\} \neq \emptyset, \tag{2.6}$$

that is there exist $y_n \in M_0$ such that

$$\langle v-y_n, Ty_n\rangle \geq \langle v-y_n, h\rangle \qquad \forall v \in K_n. \tag{2.7}$$

The ρ-boundedness of M implies norm boundedness of M_0. So, there exists a $c_1 > 0$ such that $\|y_n\| \leq c_1$ for all n. Then for some $v = v_0 \in V_1$ we have

$$\langle y_n - v_0, Ty_n\rangle \leq \langle y_n - v_0, h\rangle \leq \|y_n - v_0\|\,\|h\| \leq c_2 \qquad \forall n, \tag{2.8}$$

and by condition (T_3) there exists a $k(c_1, c_2, v_0)$ such that $\|Ty_n\| \leq k(c_1, c_2, v_0)$. Without loss of generality, we may assume that $y_n \to y \in Y$ for $\sigma(Y, Z_0)$ and $Ty_n \to z \in Z$ for $\sigma(Z, Y_0)$. Besides, from (2.7) $\langle y_n, Ty_n\rangle \leq \langle v, Ty_n\rangle + \langle y_n - v, h\rangle$ for all $v \in K_n$. Taking the limit we obtain

$$\limsup_{n\to\infty}\langle y_n, Ty_n\rangle \leq \langle v, z\rangle + \langle y - v, h\rangle \qquad \forall\, v \in \overline{\bigcup_{n=1}^{\infty} K_n}.$$

According to (2.4), that is true for any $v \in K_0$, and according to (2.2) for any $v \in K$. Because of (2.1), $y \in K$. By setting $v = y$ we obtain $\limsup_{n\to\infty}\langle y_n, Ty_n\rangle \leq \langle y, z\rangle$, and by condition (T_2)

$$\lim_{n\to\infty}\rho(y_n, y) = 0, \qquad Ty = z, \qquad \lim_{n\to\infty}\langle y_n, Ty_n\rangle = \langle y, z\rangle.$$

Taking the limit in inequality (2.7) we obtain

$$\langle v-y, Ty\rangle \geq \langle v-y, h\rangle \qquad \forall\, v \in \overline{\bigcup_{n=1}^{\infty} K_n}.$$

According to (2.4), that is true for any $v \in K_0$, and according to (2.2) for any $v \in K$. Since the ρ-convergent sequence y_n belongs to M_0, its limit y belongs to $\rho\,\mathrm{cl}M_0 \subset M \cap X_0$. But this contradicts to (2.5). Hence there exists an index n_0 for which relation (2.6) is violated. Finally, we put $V^0 = V_{n_0}$ ∎

Corollary 1. *If some finite-dimensional subspace $V^0 \subset Y_0$ satisfies Lemma* 1, *then any finite-dimensional subspace $V^1 \subset Y_0$ with $V^0 \subset V^1$ satisfies it as well.*

Let us consider a finite-dimensional subspace $V^0 \subset Y_0$. Let $j_0 : V^0 \to Y_0$ be the identity and $j_0^* : Z \to V^{0*}$ be the dual projection. We identify the dual space V^{0*} with V^0 and equip it with a scalar product $(\cdot\,,\cdot)$, coordinated with the pairing $\langle\cdot\,,\cdot\rangle$ by the relation $\langle j_0 v, h\rangle = (v, j_0^* h)$ for all $v \in V^0$ and $h \in Z$. Then $j_0^* T j_0 : V^0 \to V^0$ is the corresponding Galerkin approximation for the operator T.

Let $V^1 \subset Y_0$ be another finite-dimensional subspace so that $V^0 \subset V^1$. We associate the corresponding operators $j_1 : V^1 \to Y_0$, $j_1^* : Z \to V^{1*} \equiv V^1$ and $j_1^* T j_1 : V^1 \to V^1$

with V^1, by analogy, with V^0. Let $\pi_0 : V^1 \to V^0$ be an orthogonal projector. It is obvious that $\pi_0 j_1^* = j_0^*$. In particular,

$$\pi_0 j_1^* h = j_0^* h \quad \forall h \in Z_0 \qquad \text{and} \qquad \pi_0 j_1^* T j_1 y = j_0^* T j_0 y \quad \forall y \in V^0. \tag{2.9}$$

Definition 7. A set ω will be called K-*admissible* if there exists such a ρ-open and ρ-bounded set $\Omega \subset X$ with $\omega = \Omega \cap K$, $\omega_0 = \omega \cap Y_0$ and $\mathring{\omega}_0 = \mathring{\omega} \cap Y_0 \neq \emptyset$, where $\mathring{\omega}$ is the ρ-interior of ω.

We denote the closure and the relative boundary of the K-admissible set ω with respect to the ρ-topology of $K \cap X$ as $\rho\,\mathrm{cl}\omega$ and $\rho\,\partial_K\omega$, respectively.

Lemma 2. *Let ω be a K-admissible set, $h \in Z_0$ and an operator T of the class $S_A(X)$ be (K,h)-non-degenerate on $\rho\,\partial_K\omega$. Then there exists a finite-dimensional subspace $V^0 \subset Y_0$ such that the Galerkin approximation $j_0^* T j_0$ is $(K^0, j_0^* h)$-non-degenerate on $\partial_{K^0}\omega^0$ with $\omega^0 = \omega \cap V^0$, and for any finite-dimensional subspace $V^1 \subset Y_0$ with $V^0 \subseteq V^1$, the K^1-degree of $j_1^* T j_1$ at $j_1^* h$ relative to $\omega^1 = \omega \cap V^1$ is well-defined and*

$$\deg_{K^1}(j_1^* T j_1, \omega^1, j_1^* h) = \deg_{K^0}(j_0^* T j_0, \omega^0, j_0^* h). \tag{2.10}$$

Proof. It is obvious that the set K^0 is closed and convex for any finite-dimensional subspace $V^0 \subset Y_0$, while the Galerkin approximation $j_0^* T j_0$, according to condition (T_1), is continuous. The existence of a finite-dimensional subspace $V^0 \subset Y_0$ such that the Galerkin approximation $j_0^* T j_0$ is $(K^0, j_0^* h)$-non-degenerate on $\partial_{K^0}\omega^0$ follows from Lemma 1 with $M = \rho\,\partial_K\omega$. The $(K^1, j_1^* h)$-non-degeneracy of the Galerkin approximation $j_1^* T j_1$ on $\partial_{K^1}\omega^1$ follows from Corollary 1. Without loss of generality, we may assume that $\mathring{\omega}^0 \neq \emptyset$. Thus both sides of equality (2.10) are well-defined and the equality itself is valid, that follows from Theorem 2, according to (2.9) ∎

Corollary 2. *If $V^{01} \subset Y_0$ and $V^{02} \subset Y_0$ are two different finite-dimensional subspaces such as V^0 in Lemma 2, then*

$$\deg_{K^{01}}(j_{01}^* T j_{01}, \omega^{01}, j_{01}^* h) = \deg_{K^{02}}(j_{02}^* T j_{02}, \omega^{02}, j_{02}^* h). \tag{2.11}$$

To prove this formula we consider a finite-dimensional subspace $V^1 \subset Y_0$ with $V^{01} \cup V^{02} \subset V^1$. According to (2.10), the corresponding K^1-degree equals to each side of the equality (2.11). Hence it is valid ∎

This shows that the following definition is legitimate.

Definition 8. Let $(Y, Y_0; Z, Z_0)$ be a complementary system, K be a σ-set, ω be a K-admissible set, $h \in Z_0$, an operator T of the class $S_A(X)$ be (K,h)-non-degenerate on $\rho\,\partial_K\omega \cap X_0$ and a finite-dimensional subspace $V^0 \subset Y_0$ corresponding to Lemma 2. Then we set

$$\deg_K(T, \omega, h) = \deg_{K^0}(j_0^* T j_0, \omega^0, j_0^* h)$$

and name this integer the *K-degree of T at h relative to ω*.

We will show that the K-degree of operators of the class $S_A(X)$ in complementary systems has the same properties as the finite-dimensional K-degree.

Definition 9. We will say that a family of operators $T_t : X \to Z$ belongs to the *class* $S'_A(X)$ if it satisfies the following conditions:

(T'_1) For any finite-dimensional subspace $V \subset Y_0$, the function $f(u,v,t) = \langle v, T_t u\rangle$ is continuous on $V \times V \times [0,1]$.

(T'_2) For any sequences $y_n \in Y_0$, $t_n \in [0,1]$ such that $y_n \to y \in Y$ for $\sigma(Y, Z_0)$, $t_n \to t \in [0,1]$, $T_{t_n} y_n \to z \in Z$ for $\sigma(Z, Y_0)$ and $\limsup_{n\to\infty}\langle y_n, T_{t_n} y_n\rangle \le \langle y, z\rangle$ there follows that

$$\lim_{n\to\infty} \rho(y_n, y) = 0, \qquad T_t y = z, \qquad \lim_{n\to\infty}\langle y_n, T_{t_n} y_n\rangle = \langle y, z\rangle.$$

(T'_3) For any $v_0 \in Y_0$, $c_1 > 0$, $c_2 > 0$ there exists a $k(c_1, c_2, v_0) > 0$ such that

$$\left.\begin{array}{r} \|y\| \le c_1 \\ \langle y - v_0, T_t y\rangle \le c_2 \end{array}\right\} \implies \|T_t y\| \le k(c_1, c_2, v_0).$$

It is obvious that the class $S'_A(X)$ is a one-parameter analogue of the class $S_A(X)$. Moreover, for any operators T_0 and T_1 of the class $S_A(X)$, a family of operators $T_t = (1-t)T_0 + tT_1$ belongs to the class $S'_A(X)$ [6, 10].

Definition 10. Operators T_0, T_1 of the class $S_A(X)$ are said to be *K-homotope* at $h \in Z_0$ relative to a K-admissible set ω if there exists a (K,h)-non-degenerate on $\rho\,\partial_K\omega \cap X_0$ family of operators T_t of the class $S'_A(X)$, which connects T_0 with T_1.

Definition 11. A (K,h)-critical point $y_0 \in K \cap X_0$ of an operator T of the class $S_A(X)$ is said to be *isolated* if there exists an $r_0 > 0$ such that the set

$$\mathcal{B}_{r_0}(y_0) = \{y \in K \cap X_0 : \rho(y, y_0) < r_0\}$$

includes no other (K,h)-critical points of T.

Note that this set is obviously K-admissible.

Definition 12. We set

$$\mathrm{ind}_K(T, y_0, h) = \deg_K(T, \mathcal{B}_r(y_0), h) \qquad (0 < r < r_0)$$

and name this integer the *index* of an isolated (K,h)-critical point y_0 of an operator T of the class $S_A(X)$.

Theorem 3. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system, K be a σ-set, ω be a K-admissible set, $h \in Z_0$ and an operator T of the class $S_A(X)$ be (K,h)-non-degenerate on $\rho\,\partial_K\omega \cap X_0$. Then the K-degree of T at h relative to ω has the following properties:*

(1) *If ω_l $(l = 1, \ldots, m)$ are mutually disjoint K-admissible subsets of ω and an operator T is (K,h)-non-degenerate on $\rho\,\mathrm{cl}(\omega \setminus \cup_{l=1}^m \omega_l) \cap X_0$, then*

$$\deg_K(T, \omega, h) = \sum_{l=1}^m \deg_K(T, \omega_l, h).$$

(2) *If operators T_0 and T_1 are K-homotope at h relative to ω, then* $\deg_K(T_0,\omega,h) = \deg_K(T_1,\omega,h)$.

(3) *If there exists a $y_0 \in \omega_0$ with $\langle y-y_0, Ty\rangle \geq \langle y-y_0, h\rangle$ for all $y \in \rho\,\partial_K\omega \cap X_0$, then* $\deg_K(T,\omega,h) = 1$.

(4) $\deg_K(T-h,\omega,0) = \deg_K(T,\omega,h)$.

(5) *If $\omega_* \subset \omega$ is such that the set $\omega \setminus \rho\,\mathrm{cl}\omega_*$ is K-admissible and an operator T is (K,h)-non-degenerate on $\rho\,\mathrm{cl}\omega_* \cap X_0$, then*

$$\deg_K(T,\omega,h) = \deg_K(T,\omega \setminus \rho\,\mathrm{cl}\omega_*, h).$$

(6) *If $T_0 y = T_1 y$ for all $y \in \rho\,\partial_K\omega \cap X_0$, then* $\deg_K(T_0,\omega,h) = \deg_K(T_1,\omega,h)$.

(7) *If T is (K,h)-non-degenerate on $\rho\,\mathrm{cl}\omega \cap X_0$, then* $\deg_K(T,\omega,h) = 0$.

(8) *If* $\deg_K(T,\omega,h) \neq 0$, *then there exists a $y_0 \in \omega \cap X_0$ such that $\langle y-y_0, Ty_0\rangle \geq \langle v-y_0, h\rangle$ for all $v \in K$.*

(9) *If* $\deg_K(T_0,\omega,h) \neq \deg_K(T_1,\omega,h)$, *then there exist $y_0 \in \rho\,\partial_K\omega \cap X_0$ and $\lambda_0 \in (0,1)$ such that $\langle v-y_0, (1-\lambda_0)T_0y_0 + \lambda_0 T_1 y_0\rangle \geq \langle v-y_0, h\rangle$ for all $v \in K$.*

(10) *If $\omega \cap X_0$ includes only isolated (K,h)-critical points y_l of T, then the number of such points is finite and*

$$\deg_K(T,\omega,h) = \sum_{l=1}^{m} \mathrm{ind}_K(T,y_l,h).$$

Proof. (1) Applying Lemma 1 to the set $M = \rho\,\mathrm{cl}(\omega \setminus \cup_{l=1}^m \omega_l)$, we obtain the existence of a finite-dimensional subspace $V^0 \subset Y_0$ such that $\mathring{\omega}^0 \neq \emptyset$, $\mathring{\omega}_l^0 \neq \emptyset$ $(l = 1,\ldots,m)$, and the Galerkin approximation $j_0^* T j_0$ is $(K^0, j_0^* h)$-non-degenerate on

$$\left(\rho\,\mathrm{cl}\left(\omega \setminus \bigcup_{l=1}^m \omega_l\right)\right)^0 = \rho\,\mathrm{cl}\left(\omega \setminus \bigcup_{l=1}^m \omega_l\right) \cap V^0 = \overline{\omega^0} \setminus \bigcup_{l=1}^m \omega_l^0.$$

So, it is sufficient to use Definition 8 and property (1) of the finite-dimensional K-degree.

(2) According to Definition 10, there exists a (K,h)-non-degenerate (on $\rho\,\partial_K\omega \cap X_0$) family of operators T_t of the class $S_A'(X)$, which connects T_0 with T_1. Applying parameter-variants of Lemma 1 (which are not stated because of the total analogy) to the set $M = \rho\,\partial_K\omega$, we obtain the existence of a finite-dimensional subspace $V^0 \subset Y_0$ such that $\mathring{\omega}^0 \neq \emptyset$ and the Galerkin approximation $j_0^* T_t j_0$ is a family of continuous mappings, continuously dependent on $t \in [0,1]$ and $(K^0, j_0^* h)$-non-degenerate on $\partial_{K^0}\omega^0$. So, it is sufficient to use Definition 8 and property (2) of the finite-dimensional K-degree.

(3) Applying Lemma 1 to the set $M = \rho\,\partial_K\omega$, we obtain the existence of a finite-dimensional subspace $V^0 \subset Y_0$ such that $\mathring{\omega}^0 \neq \emptyset$ and $y_0 \in \omega^0$, and the Galerkin approximation $j_0^* T j_0$ is $(K^0, j_0^* h)$-non-degenerate on $\rho\,\partial_{K^0}\omega^0$. But then

$$(y-y_0, j_0^* T j_0 y) \geq (y-y_0, j_0^* h) \qquad \forall y \in \partial_{K^0}\omega^0,$$

and it is sufficient to use Definition 8 and property (3) of the finite-dimensional K-degree.

(4) Because the variational inequality $\langle v - y, Ty - h\rangle \geq 0$ $(v \in K)$ is equivalent to (0.2), this property follows from Lemma 1, applied to the set $M = \rho\,\partial_K\omega$, Definition 8 and property (4) of the finite-dimensional K-degree.

(5) Applying Lemma 1 to the set $M = \rho\,\partial_K\omega \cup \rho\,\mathrm{cl}\omega_*$, we obtain the existence of a finite-dimensional subspace $V^0 \subset Y_0$ such that $\mathring{\omega}^0 \neq \emptyset$ and the Galerkin approximation $j_0^* T j_0$ is $(K^0, j_0^* h)$-non-degenerate on

$$(\rho\,\partial_K\omega \cup \rho\,\mathrm{cl}\omega_*)^0 = (\rho\,\partial_K\omega \cup \rho\,\mathrm{cl}\omega_*) \cap V^0 = \partial_{K^0}\omega^0 \cup \overline{\omega_*^0}.$$

So, it is sufficient to use Definition 8 and property (5) of the finite-dimensional K-degree.

(6) As in the finite-dimensional case, this follows immediately from property (2).

(7) Applying Lemma 1 to the set $M = \rho\,\mathrm{cl}\omega$, we obtain the existence of a finite-dimensional subspace $V^0 \subset Y_0$ such that $\mathring{\omega}^0 \neq \emptyset$ and the Galerkin approximation $j_0^* T j_0$ is $(K^0, j_0^* h)$-non-degenerate on $\overline{\omega^0}$. So, it is sufficient to use Definition 8 and property (7) of the finite-dimensional K-degree.

(8) As in the finite-dimensional case, this follows immediately from the previous property.

(9) As in the finite-dimensional case, this follows immediately from property (2).

(10) As in the finite-dimensional case, the legitimacy of Definition 12 follows from property (5) above. To verify finitness of the set of isolated (K, h)-critical points of T from $\omega \cap X_0$, suppose the contrary. K-admissibility of ω implies its ρ-boundedness. Therefore $\omega \cap X_0$ is strongly bounded and contains a sequence y_n of isolated (K, h)-critical points of T, satisfying (2.7) and (2.8). By condition (T_3), the sequence Ty_n is strongly bounded as well. Without loss of generality, we may assume that

$$y_n \rightarrow y \in Y \text{ for } \sigma(Y, Z_0) \qquad \text{and} \qquad Ty_n \rightarrow z \in Z \text{ for } \sigma(Z, Y_0).$$

Repeating the arguments from the proof of Lemma 1, we obtain that y is a non-isolated (K, h)-critical point of T from $\omega \cap X_0$, which contradicts our assumption. Thus, the set of (K, h)-critical points of T from $\omega \cap X_0$ consists of y_l $(l = 1, \ldots, m)$, and there exists an $r > 0$ such that the sets $\mathcal{B}_r(y_l)$ $(l = 1, \ldots, m)$ are mutually disjoint. To complete the proof, we use property (1) above and Definition 12 ∎

To illustrate the possibilities of the K-degree theory for operators of the class $S_A(X)$, we will formulate several theorems concerning solvability of variational inequalities (0.2) in complementary systems.

Theorem 4. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system, let K be a σ-set, let $T_t : X \rightarrow Z$ be a family of operators of the class $S'_A(X)$, let $h \in Z_0$ and an estimate $\rho(0, y) \leq R_0$ being valid for all solutions $y \in K \cap X_0$ of the variational inequality $\langle v - y, T_t y\rangle \geq \langle v - y, h\rangle$ $(v \in K)$ uniformly for $t \in [0, 1]$. Finally, let*

$$\deg_K(T_1, \mathcal{B}_R(0), h) \neq 0 \tag{2.12}$$

for some $R > R_0$. Then the operator T_0 has at least one (K,h)-critical point belonging to $K \cap X_0$.

The proof of this theorem follows easily from properties (2) and (8) of the K-degree. Thus, to prove solvability of the variational inequality (0.2) with an operator T_0 of the class $S_A(X)$, we may construct a homotopy of the class $S'_A(X)$ with T_1 satisfying (2.12). Note that this theorem is a variant of the Leray-Schauder theorem [13] for variational inequalities in complementary systems.

Theorem 5. *The previous theorem is still valid if (2.12) is substituted by the next condition: there exists a $y_0 \in K_0$ such that*

$$\langle y - y_0, T_1 y\rangle \geq \langle y - y_0, h\rangle \qquad \forall y \in \rho\, \partial_K \mathcal{B}_R(0). \tag{2.13}$$

The proof of this theorem is obvious, because inequality (2.13) implies inequality (2.12), according to property (3) of the K-degree. Note that this theorem is a special, but more convenient variant of the previous.

Theorem 6. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system, let K be a σ-set, let ω be a K-admissible set, let $h \in Z_0$, let operators T_0 and T_1 belong to the class $S_A(X)$, T_0 be (K,h)-non-degenerate on $\rho\,\mathrm{cl}\omega \cap X_0$, T_1 be (K,h)-non-degenerate on $\rho\, \partial_K \omega \cap X_0$ and for some $y_0 \in \omega_0$*

$$\langle y - y_0, T_1 y\rangle \geq \langle y - y_0, h\rangle \qquad \forall y \in \rho\, \partial_K \omega \cap X_0.$$

Then for some $\lambda > 0$ the set $\rho\, \partial_K \omega \cap X_0$ contains at least one (K,h)-critical point of the operator $T_0 + \lambda T_1$, or a solution of the eigenvalue problem for the variational inequality

$$\langle v - y, T_0 y\rangle + \lambda \langle v - y, T_1 y\rangle \geq \langle v - y, h\rangle \qquad \forall v \in K.$$

The proof of this theorem follows immediately from properties (7), (3) and (9) of the K-degree.

Theorem 7. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system, let K be a σ-set, let ω be a K-admissible set, let $h \in Z_0$, let an operator T of the class $S_A(X)$ be (K,h)-non-degenerate on $\rho\, \partial_K \omega \cap X_0$ and let $y_l \in \omega \cap X_0$ $(l = 1, \ldots, m)$ be isolated (K,h)-critical points of T, but*

$$\deg_K(T, \omega, h) \neq \sum_{l=1}^{m} \mathrm{ind}_K(T, y_l, h).$$

Then the set $\omega \cap X_0$ contains at least $(m+1)$ (K,h)-critical points of the operator T.

The proof of this theorem follows immediately from property (10) of the K-degree.

In the case of a variational inequality without right side

$$\langle v - y, Ty\rangle \geq 0 \qquad \forall v \in K \tag{2.14}$$

the existence of the trivial solution is often obvious. If 0 is an isolated (K,h)-critical point of T with a computable index, then the next specialization of Theorem 7 is very useful.

Theorem 8. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system, let K be a σ-set, let ω be a K-admissible set, let an operator T of the class $S_A(X)$ be $(K, 0)$-non-degenerate on $\rho\, \partial_K \omega \cap X_0$ and let $0 \in \omega$ be an isolated $(K, 0)$-critical point of T, but*

$$\deg_K(T, \omega, 0) \neq \mathrm{ind}_K(T, 0, 0).$$

Then the set $\omega \cap X_0$ contains at least one non-trivial solution of the variational inequality (2.14).

The next theorem allows to find approximate solutions of a variational inequality.

Theorem 9. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system, let K be a σ-set, let ω be a K-admissible set, let $h \in Z_0$, let an operator T of the class $S_A(X)$ be (K, h)-non-degenerate on $\rho\, \partial_K \omega \cap X_0$, let $y_0 \in \omega \cap X_0$ be a unique (K, h)-critical point of T with*

$$\mathrm{ind}_K(T, y_0, h) \neq 0 \tag{2.15}$$

and let a sequence of finite-dimensional subspaces $V_n \subset Y_0$ satisfies (2.3). *Then there exists an index n_0 such that for all $n \geq n_0$ the corresponding approximation of the variational inequality* (0.2)

$$(v - y, j_n^* T j_n y) \geq (v - y, j_n^* h) \qquad \forall y \in K_n \tag{2.16}$$

has a solution $y_n \in \omega_n = \omega \cap V_n$ and

$$\lim_{n \to \infty} \rho(y_n, y_0) = 0. \tag{2.17}$$

Proof. Applying Lemma 1 to the set $M = \rho\, \partial_K \omega$, we find an index n_0 such that for the finite-dimensional subspace $V_{n_0} = V^0 \subset Y_0$, there follows $\mathring{\omega}^0 \neq \emptyset$ and that the Galerkin approximation $j_0^* T j_0$ is $(K^0, j_0^* h)$-non-degenerate on $\rho\, \partial_{K^0} \omega^0$. Then according to Lemma 2, Definition 8, property (10) of the K-degree and (2.15) for any $n \geq n_0$ we have

$$\deg_{K_n}(j_n^* T j_n, \omega_n, j_n^* h) = \deg_K(T, \omega, h) = \mathrm{ind}_K(T, y_0, h) \neq 0.$$

Thus, by property (8) of the finite-dimensional K-degree there exists a $y_n \in \omega_n$, which satisfies (2.16).

To prove (2.17), we fix an $\epsilon > 0$ such that $\mathcal{B}_\epsilon(y_0) \subset \omega$. The operator T is obviously (K, h)-non-degenerate on the set $M = \rho\, \mathrm{cl}(\omega \backslash \mathcal{B}_\epsilon(y_0))$. Applying Lemma 1 and Corollary 1 to this set, we find an index n_ϵ such that for any $n \geq n_\epsilon$

$$\deg_{K_n}(j_n^* T j_n, \mathcal{B}_\epsilon(y_0) \cap V_n, j_n^* h) = \deg_K(T, \mathcal{B}_\epsilon(y_0), h) = \mathrm{ind}_K(T, y_0, h) \neq 0.$$

Thus, by property (8) of the finite-dimensional K-degree there exists a $y_n \in \mathcal{B}_\epsilon(y_0) \cap V_n$ which satisfies (2.16). Hence $\lim_{n \to \infty} \rho(y_n, y_0) \leq \epsilon$ with arbitrary $\epsilon > 0$, or $\lim_{n \to \infty} \rho(y_n, y_0) = 0$ ∎

Note that, in terms of rotation, variants of Theorems 4, 5 and 7 - 9 are contained in [10], and a variant of Theorem 6 in [6].

Keep in mind that the effectiveness of using the K-degree theory, as well as degree theory itself, depends on whether one can calculate the index of an isolated (K, h)-critical point. The evaluation of the index considered to be a hard problem. The majority of works dealing with approximative generalizations of the degree theory completely avoids considering this problem. As for the K-degree, the techniques to calculate the index of isolated (K, h)-critical points have not been worked out even for the finite-dimensional case (except [7]).

References

[1] Borisovich, Yu. G.: *On relative rotation of compact vector fields in linear spaces* (in Russian). Trudy Sem. Funct. Anal. Voronezh Gos. Univ. 12 (1969), 3 - 27.

[2] Brouwer, L.: *Über Abbildung von Mannigfaltigkeiten.* Math. Annal. 71 (1912), 97 - 115.

[3] Browder, F. E. and W. V. Petryshyn: *Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces.* J. Funct. Anal. 3 (1969), 217 - 245.

[4] Gossez, J.-P.: *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients.* Trans. Amer. Math. Soc. 190 (1974), 163 - 205.

[5] Gossez, J.-P. and V. Mustonen: *Variational inequalities in Orlicz-Sobolev spaces.* Nonlin. Anal. 11 (1987), 379 - 392.

[6] Khidirov, Yu. E.: *Structure of the class $S_A(X)$ of operators and eigenvalue problem for variational inequalities in complementary systems* (in Russian). Kachestv. Met. Issled. Operator. Uravn. Yaroslavl. Gos. Univ. 1988, pp. 27 - 34.

[7] Khidirov, Yu. E.: *K-rotation of a vector field and index of a K-critical point for n-dimensional half-space* (in Russian). Actual. Probl. Estestv. Human. Nauk. Mat. Inform. Yaroslavl. Gos. Univ. 1995, pp. 89 - 92.

[8] Klimov, V. S.: *On the theory of variational inequalities* (in Russian). Kachestv. Pribl. Met. Issled. Operator. Uravn. Yaroslavl. Gos. Univ. 1982, pp. 109 - 119.

[9] Klimov, V. S. and Yu. E. Khidirov: *Topological characteristics of nonlinear operators in complementary systems* (in Russian). Kachestv. Pribl. Met. Issled. Operator. Uravn. Yaroslavl. Gos. Univ. 1982, pp. 120 - 134.

[10] Klimov, V. S. and Yu. E. Khidirov: *On variational inequalities in Orlicz-Sobolev spaces* (in Russian). VINITI Inform. Center (Moscow) 1982, pp. 1 - 38.

[11] Krasnoselskij, M. A.: *Topological Methods in the Theory of Nonlinear Integral Equations* (in Russian). Moscow: Gostechizdat 1956; English transl.: New-York: Macmillan 1964.

[12] Krasnoselskij, M. A. and P. P. Zabreiko: *Geometric Methods of Nonlinear Analysis* (in Russian). Moscow: Nauka 1975.

[13] Leray, J. and J. Schauder: *Topologie et equations fonctionelles.* Ann. Sci. Ec. Norm. Sup. 51 (1934), 45 - 78.

[14] Lions, J. L.: *Quelques Mèthodes de Rèsolution des Problèmes aux Limites.* Paris: Dunod 1969.

[15] Lloyd, N. G.: *Degree Theory* (Cambridge Tracts in Mathematics: Vol. 73). Cambridge: Univ. Press 1978.

[16] Schwartz, J.: *Nonlinear Functional Analysis.* New York: Gordon and Breach 1969.

[17] Skrypnik, I. V.: *Quasilinear Elliptic Equations of Higher Order* (in Russian). Donetsk: Gos. Univ. 1971.

[18] Skrypnik, I. V.: *Methods of Investigation of Nonlinear Elliptic Boundary Value Problems* (in Russian). Moscow: Nauka 1990.

[19] Tienāry, M.: *A Degree Theory for a Class of Mappings of Monotone Type in Orlicz-Sobolev Spaces.* Theses. Ann. Acad. Sci. Fenn. Ser. A1. Math. Diss. 97 (1994), pp. 68.

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