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On Persson's Theorem in Local Dirichlet Spaces

G. Grillo

Abstract. Given a strongly local, regular and irreducible Dirichlet form $\mathcal{E}$, we prove a version of Persson's theorem concerning the variational characterization of the bottom of the essential spectrum of the generator H of $\mathcal{E}$. Such a result is then used to prove L^p-exponential decay of the "small eigenfunctions" of H.

Keywords: *Local Dirichlet spaces, Persson's theorem, small eigenfunctions*

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1. Introduction

Given a positive second-order differential operator of elliptic type in divergence form

$$H_0 = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right)$$

on an open and connected domain $D \subset \mathbb{R}^n$, with Dirichlet boundary conditions and suitable assumptions on the matrix $(a_{i,j})$, one can consider the bilinear form

$$a(u,v) = (H_0^{1/2}u, H_0^{1/2}v)_{L^2(D)}$$

defined for $u,v \in D(a) = \mathrm{Dom}(H_0^{1/2})$. This bilinear form mirrors many of the features of the operator H and enjoices several interesting properties which can be considered in a more abstract setting as the defining properties of mathematical objects which are known as *Dirichlet forms*.

In fact, let us recall from [8] the following basic definitions:

Let X be a locally compact separable Hausdorff space, m a positive Radon measure on X with full support, and define H as the real Hilbert space $L^2(X,dm)$, whose scalar product is denoted by $(\cdot,\cdot)$. A Dirichlet form $\mathcal{E}$ defined on $D(\mathcal{E}) \subset H$ is a closed, symmetric, non-negative definite, bilinear form on $D(\mathcal{E})$ such that

$$u \in D(\mathcal{E}) \quad\Longrightarrow\quad v = (0 \vee u) \wedge 1 \in D(\mathcal{E}) \;\text{ and }\; \mathcal{E}(v,v) \le \mathcal{E}(u,u).$$

G. Grillo: Univ. di Udine, Dip. di Mat. e Inf., via delle Scienze 206 (loc. Rizzi), 33100 Udine, Italy. Present address: Univ. di Torino, Dip. di Mat., via Carlo Alberto 10, 10123 Torino, Italy. e-mail: grillo@dm.unito.it

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Definition. The Dirichlet form $\mathcal{E}$ is said to be

• *local* if, for all $u, v \in D(\mathcal{E})$ with disjoint supports, one has $\mathcal{E}(u,v) = 0$, where we define the support of f to be the support of the measure $f\,dm$;

• *strongly local* if, for all $u, v \in D(\mathcal{E})$ with compact support such that u is constant on a neighbourhood of supp(v), one has $\mathcal{E}(u,v) = 0$;

• *regular* if there exists a set $C \subset D(\mathcal{E}) \cap C_c(X)$ which is dense both in $C(X)$ with the uniform norm and in $D(\mathcal{E})$ with the norm induced by $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)$;

• *irreducible* if $u \in D_{\text{loc}}(\mathcal{E})$ is constant whenever $\mathcal{E}(u,u) = 0$, where $D_{\text{loc}}(\mathcal{E})$ is the set of the m-measurable functions u on X such that for all relatively compact open sets $A \subset X$ there exists $v \in D(\mathcal{E})$ which coincides with u on A.

In order that the present definition of $D_{\text{loc}}(\mathcal{E})$ is well posed, one usually requires that $\mathcal{E}$ is strongly local; however, it is also possible (and will be done later on) to consider $D_{\text{loc}}(\mathcal{E})$ even when $\mathcal{E}$ is the form sum of a strongly local Dirichlet form and of a bilinear form of the type $\int_X uv\,dV$ for a positive Radon measure V on X.

If X is a differentiable manifold, any strongly local Dirichlet form is associated, by the second Beurling-Deny formula [8], to a second-order differential elliptic operator in divergence form H_0 so that $\mathcal{E}(u,v) = (H_0^{1/2}u, H_0^{1/2}v)$ for all $u, v \in C_c^\infty(X)$. A local Dirichlet form is also associated in the above manner to a generalized Schrödinger operator $H = H_0 + V$, H_0 as above, V being a positive Radon measure on X.

Typical classes of differential operators which give rise to Dirichlet form include uniformly elliptic operators on manifolds, elliptic operators with weights in the Muckenhoupt class, subelliptic operators and Hörmander operators (cf. [2, 10] and references there).

In [5] several results are proved concerning the L^p exponential decay of solutions to equations of the form

$$\mathcal{E}(u,v) = \int_X fv\,dm$$

for all $v \in L_c^\infty(X,m) \cap D(\mathcal{E})$, where $f \in L^2(dm)$ is assigned and $\mathcal{E}$ is a local, regular and irreducible Dirichlet form. Those results, which are motivated by [1] and [4], rely upon general topological assumption to be described later on and, mainly, on the validity of a coerciveness assumption. Precisely, one must require that there exists a positive function λ defined on X such that

$$\int_X \lambda u^2\,dm \le \mathcal{E}(u,u) \qquad \text{for all } u \in C. \tag{1.1}$$

Notice that the classical Hardy inequality on (0,1) is a special case of (1.1), when setting

$$\mathcal{E}(u,u) = \int_0^1 (u')^2 dx, \quad \lambda(x) = \text{const}\,[x \wedge (1-x)]^{-2}, \quad C = C_c^\infty(0,1).$$

We prove here the analogue of a theorem of Persson [9] to obtain a version in the present context of his variational characterization of the bottom of the essential spectrum of elliptic differential operators in divergence form. Such a theorem and the results of [5] can be combined as in [1] to prove a suitable coercivity condition and, then, weighted L^p-bounds for those eigenfunctions of the self-adjoint operator associated to $\mathcal{E}$ whose eigenvalue lays below the essential spectrum of H ("small eigenfunctions"). This complements the results of [5], in which results of the same nature are proved for the solutions of the equation $Hu = f$ with f given.

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2. Persson's theorem and applications to eigenfunctions of generators of strongly local Dirichlet forms

To state our results, we need some more facts from the theory of Dirichlet form. First we recall that the *energy measures* of a regular, irreducible and strongly local Dirichlet form are the Radon measures $\mu(u,u)$ defined by

$$\int_X \phi \, d\mu(u,u) := \mathcal{E}(u, \phi u) - \frac{1}{2}\mathcal{E}(u^2, \phi)$$

for all $u \in D(\mathcal{E}) \cap L^\infty(X,m)$ and for all $0 \le \phi \in D(\mathcal{E}) \cap C_c(X)$. Truncation and monotone convergence allow to define $\mu(u,u)$ for all $u \in D(\mathcal{E})$, and by polarization one defines the signed Radon measure $\mu(u,v)$ for all $u,v \in D(\mathcal{E})$. It is known [2, 8, 10] that $\mathcal{E}(u,v) = \int_X d\mu(u,v)$ for all $u,v \in D(\mathcal{E})$. Besides, the energy measures $\mu(u,v)$ can be defined also for $u,v \in D_{\rm loc}(\mathcal{E})$ by setting

$$\mathbf{1}_A \mu(u,v) = \mathbf{1}_A \mu(u',v')$$

for any relatively compact open set $A \subset X$, where u' and v' are functions on $D(\mathcal{E})$ coinciding with u and v, respectively, on A. The definition is well posed by the strong locality of $\mathcal{E}$.

We shall need four properties of the energy measures [2, 8, 10]. In the sequel, $\tilde{u}$ and $\tilde{v}$ will denote the quasi-continuous modifications of u and v, respectively, which exist for the class of functions used below by the regularity of $\mathcal{E}$ (see [7, 8]). First we mention the *Leibniz rule* which states that

$$\mu(uv,w) = \tilde{u}\mu(v,w) + \tilde{v}\mu(u,w)$$

for all $u,v \in D_{\rm loc}(\mathcal{E}) \cap L^\infty_{\rm loc}(X,dm)$ and $w \in D_{\rm loc}(\mathcal{E})$. Secondly, the *chain rule* which states that, if $u \in D_{\rm loc}(\mathcal{E})$, $v \in D_{\rm loc}(\mathcal{E}) \cap L^\infty_{\rm loc}(X,dm)$ and $\phi \in C_b^1(\mathbb{R})$, it follows that $\varphi(u) \in D_{\rm loc}(\mathcal{E})$ and

$$\mu(\phi(u),v) = \phi'(\tilde{u})\mu(u,v).$$

The *truncation lemma* may be stated by saying that, if $u \in D_{\rm loc}(\mathcal{E})$ and $v \in D_{\rm loc}(\mathcal{E}) \cap L^\infty(X,dm)$ we have, for any $c \in \mathbb{R}$,

$$\mu((\tilde{u}-c)_+,v) = \mathbf{1}_{\{\tilde{u}>c\}}\mu(u,v) \quad \text{and} \quad \mu((\tilde{u}-c)_+,(\tilde{u}-c)_+) = \mathbf{1}_{\{\tilde{u}>c\}}\mu(u,u)$$

where $(\cdot)_+$ denotes the positive part. Finally, the *Schwarz inequality* says that

$$\int_X |fg|\, d|\mu(u,v)| \le \left(\int_X f^2 d\mu(u,u)\right)^{1/2} \left(\int_X g^2 d\mu(v,v)\right)^{1/2}$$

for all $u,v \in D(\mathcal{E}) \cap L^\infty(X,m)$, $f \in L^2(X,\mu(u,u))$ and $g \in L^2(X,\mu(v,v))$.

Given the strongly local Dirichlet form $\mathcal{E}$ and a positive Radon measure V on X charging no exceptional sets, we shall consider the local Dirichlet form

$$\mathcal{E}_V(u,v) = \mathcal{E}(u,v) + \int_X \tilde{u}\tilde{v}\, dV \tag{2.1}$$

for all $u,v \in D(\mathcal{E}) \cap L^2(dV)$. The first Beurling-Deny formula guarantees that each local Dirichlet form admits the decomposition (2.1) at least on a suitable core, so that we shall not loose any generality in assuming it. In the sequel, we shall also avoid mentioning explicitely that we are choosing, without exceptions, quasi-continuous versions of the functions involved when dealing with their pointwise versions, and avoid also for notational simplicity to write the superscript $\sim$.

Then we define the *intrinsic metric* $d(x,y)$ on X. It generalizes several notions of distance naturally associated to differential operators. Precisely we define, given a strongly local Dirichlet form $\mathcal{E}$ with energy measures $d\mu(u,v)$,

$$d(x,y) = \sup\left\{|v(x) - v(y)| : v \in C_c(X) \cap D(\mathcal{E}), d\mu(v,v) \le dm\right\} \tag{2.2}$$

where $d\mu(v,v) \le dm$ means that $d\mu(v,v)$ is absolutely continuous with respect to dm, and the corresponding Radon-Nikodym derivative is smaller than or equal to one dm-a.e. The definition is well posed by the strong locality of $\mathcal{E}$; the corresponding metric may be degenerate, but in our main results we shall require suitable assumptions ensuring in particular that d is a true metric.

Let us consider the intrinsic open ball $B(x,r)$ of center $x \in X$ and radius $r > 0$, and the cut-off functions introduced in [10] and defined for $x \in X$ by

$$\varrho_{r,y} = (r - d(x,y))_+ \qquad (y \in X, r > 0). \tag{2.3}$$

Clearly, the support of $\varrho_{r,y}$ is $\overline{B(y,r)}$. By a fundamental result of Sturm one has, *requiring the topology induced by the intrinsic metric to be equivalent to the original one*, $d(\cdot,y) \in D_{\text{loc}}(\mathcal{E}) \cap C(X)$ and

$$d\mu(d(\cdot,y), d(\cdot,y)) \le dm. \tag{2.4}$$

This parallels the fact the the gradient of the usual (regularized) distance function in an Euclidean domain has length not larger than one. Property (2.4) implies also, by an application of the truncation lemma, that $d\mu(\varrho_{r,y}, \varrho_{r,y}) \le dm$.

The first result of the present paper is a version of Persson's characterization of the infimum of the essential spectrum of a second-order elliptic operator in divergence form, in the context of local Dirichlet forms. First we define

$$\Sigma(\mathcal{E}) = \sup_{K \text{ compact}} \inf \left\{ \frac{\mathcal{E}(\varphi,\varphi)}{\|\varphi\|^2} \,\middle|\, \varphi \in C, \operatorname{supp}\varphi \subset X \setminus K, \varphi \neq 0 \right\}. \tag{2.5}$$

We also recall that, given a self-adjoint operator H whose spectrum we denote by $\sigma(H)$, the purely discrete spectrum $\sigma_{\mathrm{d}}(H)$ is defined to be the set of isolated eigenvalues with finite multiplicity. The essential spectrum $\sigma_{\mathrm{e}}(H)$ is then defined to be

$$\sigma_{\mathrm{e}}(H) = \sigma(H) \setminus \sigma_{\mathrm{d}}(H).$$

Our proof combines some ideas from [1] with the original Persson's arguments [9].

We need the following preliminary result.

Lemma. *The Leibniz rule for the energy measures*

$$\mu(uv,w) = u\mu(v,w) + v\mu(u,w)$$

holds also when $v \in D_{\mathrm{loc}}(\mathcal{E})$ is not necessarily bounded and $u \in D(\mathcal{E})_{\mathrm{loc}} \cap C_b(X)$ has an energy measure $d\mu(v,v)$ admitting an m-essentially bounded density with respect to dm.

Proof. First of all we show that $uv \in D(\mathcal{E})_{\mathrm{loc}}$. It clearly suffices to show that, when $u,v \in D(\mathcal{E})$, it follows that $uv \in D(\mathcal{E})$. Indeed, in this latter situation, choose functions $v_n \in C$ such that $v_n \to v$ in the $\mathcal{E}_1$-norm. Then $uv_n \in D(\mathcal{E}) \cap C_b(X)$. Next notice that, for $f,g \in D(\mathcal{E})_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(X,m)$ quasi-continuous,

$$\begin{aligned}\mu(fg,fg) &= f^2\mu(g,g) + g^2\mu(f,f) + 2fg\,\mu(f,g)\\ &\le f^2\mu(g,g) + g^2\mu(f,f) + 2|fg|\,|\mu(f,g)|\\ &\le 2f^2\mu(g,g) + 2g^2\mu(f,f)\end{aligned}$$

where we have used (cf. [2]) the fact that

$$2|kh|\,|\mu(f,g)| \le k^2\mu(f,f) + h^2\mu(g,g).$$

Hence

$$\begin{aligned}&\frac{1}{2}\mathcal{E}(u(v_n - v_m), u(v_n - v_m))\\ &\quad= \frac{1}{2}\int_X d\mu(u(v_n - v_m), u(v_n - v_m))\\ &\quad\le \int_X u^2 d\mu(v_n - v_m, v_n - v_m) + \int_X (v_n - v_m)^2 d\mu(u,u)\\ &\quad\le \|u\|_\infty^2\,\mathcal{E}(v_n - v_m, v_n - v_m) + \int_X (v_n - v_m)^2 f\, dm\\ &\quad\le \|u\|_\infty^2\,\mathcal{E}(v_n - v_m, v_n - v_m) + \|f\|_\infty\,\|v_n - v_m\|_2^2\end{aligned}$$

where by f we denote the essentially bounded, non-negative Radon-Nikodym derivative of $d\mu(u,u)$ with respect to dm. By using the fact that v_n is $\mathcal{E}_1$-Cauchy, we then have that uv_n is $\mathcal{E}$-Cauchy, hence $\mathcal{E}_1$-Cauchy, hence $\mathcal{E}_1$-convergent to a suitable $g \in D(\mathcal{E})$. By the uniqueness of the L^2-limit we have $g = uv$ so that $uv_n \to uv$ in the $\mathcal{E}_1$-norm; in particular, $uv \in D(\mathcal{E})$.

Next the Leibniz rule applied to uv_n implies that

$$\mu(uv_n, w) = u\mu(v_n, w) + v_n\mu(u, w).$$

It is known that, when a sequence $\{f_n\} \subset D(\mathcal{E})$ $\mathcal{E}$-converges to $f \in D(\mathcal{E})$, then the corresponding sequence of energy measures converges to the energy measure of f in the total variation norm. Therefore $\mu(uv_n, w) \to \mu(uv, w)$ in the total variation norm, as well as $u\mu(v_n, w) \to u\mu(v, w)$ in the same sense, since u is everywhere bounded. Finally, we can show that $v_n\mu(u, w) \to v\mu(u, w)$ by noting that, by the Schwarz rule for the energy measures,

$$\begin{aligned}
\int_X |v_n - v|\,|d\mu(u,w)| &\le \left(\int_X |v_n - v|^2 d\mu(u,u)\right)^{1/2} \left(\int_X d\mu(w,w)\right)^{1/2} \\
&= \left(\int_X |v_n - v|^2 f\, dm\right)^{1/2} \left(\int_X d\mu(w,w)\right)^{1/2} \\
&\le \text{const} \left(\int_X |v_n - v|^2 dm\right)^{1/2} \\
&\to 0 \quad \text{as } n \to +\infty
\end{aligned}$$

This completes the proof ∎

The proof of the following results is based upon a generalization, due to Biroli and Tchou [3], of a well-known result of Rellich concerning the compactness of a certain Sobolev embedding (see, e.g., [6]). We stress here that, although Theorem 1 below is stated for the sake of notational simplicity for strongly local Dirichlet forms, a similar result also holds for local Dirichlet forms of type (2.1) under suitable assumptions on the measure dV in (2.1), whose L^2-space has to be locally continuously embedded into $D(\mathcal{E})$.

Theorem 1. *Let $\mathcal{E}$ be a strongly local, irreducible, regular Dirichlet form with core $C = C_(X) \cap D(\mathcal{E})$. Assume that the topology generated by the intrinsic metric is equivalent to the original topology. Then*

$$\inf \sigma_e(H) = \Sigma(\mathcal{E}). \tag{2.6}$$

Proof. First we prove that, given $\mu < \inf \sigma_e(H)$, one has $\Sigma(\mathcal{E}) \ge \mu$ (see [1: pp. 50 - 51]). Given the spectral projection $E(\mu) := E(\mu, +\infty)$ associated with the self-adjoint generator of $\mathcal{E}$, one has the eigenfunction expansion

$$I - E(\mu) = \sum_{i=1}^{n} (\cdot, \psi_i)\, \psi_i$$

for suitable $\psi_i \in D(H) \subset D(H^{1/2}) = D(\mathcal{E})$. Given $\varepsilon > 0$, $\varphi \in C$ supported in K^c, K being a compact set, this implies that

$$\mathcal{E}([I - E(\mu)]\varphi, [I - E(\mu)]\varphi)^{\frac{1}{2}} \leq \sum_{i=1}^{n} \left(\int_{K^c} |\psi_i|^2 dm\right)^{\frac{1}{2}} \mathcal{E}(\psi_i, \psi_i)^{\frac{1}{2}} ||\varphi|| \leq \varepsilon\, ||\varphi||$$

as soon as K is suitably choosen. Indeed, recall that X is locally compact and that the intrinsic topology is equivalent to the original one. By a similar calculation one can also prove that

$$||I - E(\mu)\varphi|| \leq \varepsilon\, ||\varphi||.$$

Clearly, one also has by the spectral theorem

$$\mathcal{E}(E(\mu)\varphi, E(\mu)\varphi) \geq \mu\, ||E(\mu)\varphi||^2.$$

By using the above three inequalities one can prove that

$$\begin{aligned}\mathcal{E}(\varphi,\varphi)^{1/2} &\geq \mathcal{E}(E(\mu)\varphi, E(\mu)\varphi)^{1/2} - \mathcal{E}((I - E(\mu))\varphi, (I - E(\mu))\varphi)^{1/2} \\ &\geq \mu^{1/2}||E(\mu)\varphi|| - \varepsilon\, ||\varphi|| \\ &\geq \mu^{1/2}(||\varphi|| - ||(I - E(\mu))\varphi||) - \varepsilon\, ||\varphi|| \\ &\geq (\mu^{1/2} - \varepsilon(\mu^{1/2} + 1))||\varphi||\end{aligned}$$

for all φ as above. By definition this implies that

$$\Sigma(\mathcal{E})^{1/2} \geq \mu^{1/2} - \varepsilon[(\mu^{1/2} + 1)]^2$$

for $\varepsilon > 0$ sufficiently small. By letting $\varepsilon \to 0$ we obtain $\Sigma(\mathcal{E}) \geq \mu$ and hence

$$\Sigma(\mathcal{E}) \geq \inf \sigma_e(H).$$

To prove the reverse inequality fix $\lambda \in \sigma_e(H)$ and consider a sequence $\{u_n\}_{n\in\mathbf{N}}$ of approximate eigenfunctions for H relative to λ, so that each u_n has unit L^2-norm, $u_n \to 0$ weakly, $u_n \in D(H) \subset D(H^{1/2}) = D(\mathcal{E})$ and

$$||Hu_n - \lambda u_n|| \to 0 \qquad \text{as } n \to +\infty.$$

Then for n sufficiently large we have

$$1 \geq ||Hu_n - \lambda u_n|| \geq |\mathcal{E}(u_n, u_n) - \lambda|,$$

which also implies that $\{u_n\}_{n\in\mathbf{N}}$ is bounded in the $\mathcal{E}$-norm.

Given a compact set $K \subset X$, consider the open covering $\cup_{x\in K} B(x, \frac{r}{2})$ of K, where $r > 0$ (depending on K) is sufficiently small, so that the intrinsic balls $B(x,r)$ have compact closure for any $x \in K$. Then there exists a finite subcovering $\cup_{i=1}^{n} B(x_i, \frac{r}{2})$ for suitable $x_i \in K$ $(1 \leq i \leq n)$. Assume also, in order to use a result of [3], that

$B(x_i, 2r) \neq X$ for all $i = 1, \dots, n$ and define the compact set $\overline{K'}$ (containing K) as the closure of $K' = \cup_{i=1}^n B(x_i, r)$. Then

$$\mathcal{A} = \{u \in D_0(\mathcal{E}, K') | \, \mathcal{E}(u,u) \leq \text{const}\}$$

is precompact in $L^2(dm)$ by a result of [3] where $D_0(\mathcal{E}, K')$ is the closure in the $\mathcal{E}$-norm of $D(\mathcal{E}) \cap C_0(K')$.

Consider also a function $\varrho \in C_c(X) \cap D(\mathcal{E})$ whose energy measure has a bounded density with respect to dm and such that $\varrho = 1$ on K and $\varrho = 0$ on $X \setminus \overline{K''}$, where $K'' = \cup_{i=1}^n B(x_i, \frac{r}{2})$. Such a function exists because $C_c(X) \cap D(\mathcal{E})$ is a special standard core [8]. Then the Lemma, the L^2-normalization of each u_n, the assumption on ϱ and the Schwarz inequality imply that $\mathcal{E}(\varrho u_n, \varrho u_n)$ is well defined and bounded as a function of n. Indeed, the proof of the lemma shows that $\varrho u_n \in D(\mathcal{E})$. Moreover,

$$\begin{aligned}
&\mathcal{E}(\varrho u_n, \varrho u_n)\\
&\quad = \int_X d\mu(\varrho u_n, \varrho u_n)\\
&\quad \leq \int_X \varrho^2 d\mu(u_n, u_n) + \int_X u_n^2 d\mu(\varrho, \varrho) + 2\int_X \varrho |u_n| \, d|\mu(\varrho, u_n)|\\
&\quad \leq c_1 \mathcal{E}(u_n, u_n) + c_2 \|u_n\|^2 + 2\left(\int_X \varrho^2 d\mu(u_n, u_n)\right)^{\frac{1}{2}} \left(\int_X u_n^2 d\mu(\varrho, \varrho)\right)^{\frac{1}{2}}\\
&\quad \leq c_1 \mathcal{E}(u_n, u_n) + c_2 + 2(c_1 c_2)^{\frac{1}{2}} \mathcal{E}(u_n, u_n)^{\frac{1}{2}}\\
&\quad \leq c_3
\end{aligned}$$

independent of $n \in \mathbb{N}$. Hence the sequence $v_n = \varrho u_n$ is bounded in the $\mathcal{E}$-norm and has support contained in the compact set $\overline{K''}$. Then there exists an $L^2(K')$-convergent subsequence of $\{v_n\}$ and, by the fact that $u_n \to 0$ weakly, we can therefore conclude that $u_n \to 0$ strongly in $L^2(K)$ for all compact K.

Now take a function $\vartheta \in D_{\text{loc}}(\mathcal{E})$ such that, given a compact set $K \subset X$,

$\vartheta = 1$ on $X \setminus K$

$0 \leq \vartheta(x) \leq 1$ for all $x \in X$

ϑ has a bounded density.

By the Lemma we have

$$\begin{aligned}
\int_X d\mu(\vartheta u_n, \vartheta u_n) &= \int_X \vartheta \, d\mu(u_n, \vartheta u_n) + \int_X u_n \, d\mu(\vartheta, \vartheta u_n)\\
&= \int_X \vartheta^2 d\mu(u_n, u_n) + 2\int_X \vartheta u_n \, d\mu(\vartheta, u_n) + \int_X u_n^2 \, d\mu(\vartheta, \vartheta).
\end{aligned}$$

The third term in the right-hand side tends to zero as $n \to +\infty$ because $d\mu(\vartheta, \vartheta) = 0$ as a Radon measure on K^c, ϑ has a bounded density and $u_n \to 0$ in $L^2(K)$ as $n \to +\infty$. The second term in the right-hand side is shown to converge to zero as well by the same argument, after using the Schwarz inequality, because u_n is an $\mathcal{E}$-bounded sequence. Then

$$\mathcal{E}(\vartheta u_n, \vartheta u_n) \leq \mathcal{E}(u_n, u_n) + o(1) = \lambda||u_n||^2 + o(1)\lambda||\vartheta u_n||^2 + o(1)$$

where the second equality is a consequence of

$$|\mathcal{E}(u_n, u_n) - \lambda| = |(u_n, (H - \lambda)u_n)| \leq ||Hu_n - \lambda u_n|| \to 0$$

as $n \to +\infty$.

Finally, by the definition of $\Sigma(\mathcal{E})$ we have that, for some compact set $K_1 \subset X$, all $f \in C$ supported in $X \setminus K_1$ satisfy $\mathcal{E}(f, f) \geq (\Sigma(\mathcal{E}) - \varepsilon)||f||$ which also holds by approximation for all $f \in D(\mathcal{E})$ with the same support property. In particular, when ϑ is as above and corresponds to a compact set K' containing K,

$$(\Sigma(\mathcal{E}) - \varepsilon)||\vartheta u_n||^2 \leq \mathcal{E}(\vartheta u_n, \vartheta u_n) \leq \lambda||\vartheta u_n||^2 + o(1).$$

Therefore $\Sigma(\mathcal{E}) \leq \lambda + \varepsilon$ for all positive ε, which implies $\Sigma(\mathcal{E}) \leq \inf \sigma_e(H)$, which is the desired inequality ∎

The next corollary deals with the L^p-exponential decay of the "small eigenfunctions" of $\mathcal{E}$, that is with those (weak) eigenfunctions of H corresponding to eigenvalues lying below the bottom of the essential spectrum of H.

Corollary. *In addition to the above assumptions, suppose that (X, ϱ) is complete, where $\varrho = \varrho(\cdot, x_0)$ denotes the intrinsic distance. Let $u \in D_{\text{loc}}(\mathcal{E})$ satisfy*

$$\mathcal{E}(u, \varphi) = \alpha(u, \varphi) \qquad \text{for all } \varphi \in C, \tag{2.7}$$

for some positive $\alpha < \Sigma(\mathcal{E})$. Fix $p \in [1, +\infty)$ and assume that

$$\int_X |u|^p e^{-2\beta\varrho} dm < +\infty \tag{2.8}$$

for some positive $\beta < [\Sigma(\mathcal{E}) - \mu]^{1/2}$, where $\varrho = \varrho(\cdot, x_0)$ for some fixed point $x_0 \in X$. Then

$$\int_X |u|^p e^{2\gamma\varrho} dm < +\infty \tag{2.9}$$

for all $\gamma < [\Sigma(\mathcal{E}) - \mu]^{1/2}$. In particular, (2.9) holds whenever $u \in L^p(dm)$.

The proof of the above corollary is essentially identical to that of [1: Theorem 4.1, Corollary 4.2], given the general result of [5] concerning the L^p-exponential decay of solutions to functional equations in local Dirichlet spaces. Therefore we do not insist on details, but only mention that the compactness of intrinsic balls with small radius has to be used as well.

We stress again that the above corollary also holds for local Dirichlet forms of type (2.1) under mild assumption on dV.

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