

On the Solvability of Linear Differential Equations with Unbounded Operators in Banach Spaces

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Abstract. This article deals with some solvability results on the Cauchy problem for linear differential equations with unbounded operators. The main result consists in the description of the set of initial data for which the corresponding solutions are represented by means of the classical exponential formula in the stationary case, and by means of the Peano matrixiant formula in the non-stationary case. In this connection a new generalization of Gelfand's lemma about analytic vectors of the generator of a strongly continuous group is proved.

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1. Introduction

Linear differential equations of the type

$$\frac{dx}{dt} = Ax \quad (1)$$

and

$$\frac{dx}{dt} = A(t)x \quad (2)$$

have been among the most important mathematical subjects of study for two centuries. Contemporary methods of studying their solvability go back to the study of two methods suggested for the construction of their solutions by Cauchy: the method of successive approximations, and the method of series expansions; essentially different approaches based on the use of integral transforms (first of all, Fourier and Laplace transforms), were suggested by Heaviside.

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For the case when equations (1) and (2) are systems of linear equations with constant or variable coefficients (in other words, A and $A(t)$ are matrices, respectively, with constant elements or elements that are continuous functions), elegant and exhaustive results were obtained relatively soon. Here one can mention the classical theorem of Picard on the convergence of successive approximations for equations (1) and (2), in the form of exponential and matrix series, and a number of others. Later, in this century, an elegant theory of linear differential equations (1) for the case of a linear continuous operator A in a certain Banach space, and of equations (2) for the case of a continuous function $A(t)$ in the space of linear continuous operators that act in a certain Banach space was formulated. This theory is a simple and natural generalization of the theory of finite linear systems of differential equations, although it also covers various important infinite systems of differential equations, integro-differential equations, and others.

The history of the above methods was different for various (first of all, parabolic and hyperbolic) types of autonomous and non-autonomous partial differential equations, which are also written in the form (1) and (2), but contain an operator A that is unbounded in a certain Banach space, and a function $A(t)$ that assumes values in the set of unbounded linear operators in a certain Banach space. The same equations can also be viewed as equations (1) and (2) with a continuous linear operator in some locally convex space and a continuous function $A(t)$ with values on the space of continuous linear operators in a locally convex linear space. Despite the importance of both cases for applications, no general results were known for a long time, with the exception of the classical theorem of Cauchy-Kovalevskaya on analytic solutions of partial differential equations which, however, was not formulated as an abstract theorem. A considerable breakthrough in the analysis of differential equations of these types occurred with the beginning of development of functional analysis; the first profound results here were obtained using the methods of integral transforms based on the theory of semigroups which goes back to W. Feller and E. Hille (theorems of Hille-Phillips-Miyadera, Solomiak-Yosida, and others, see, e.g., [13, 16]). Some time later other results were obtained using the method of successive approximations; here one should mention, first of all, the work of M. Nagumo and L. V. Ovsiannikov that, in fact, applied to equations of type (1) and (2) with operators not in Banach spaces, but in a special kind of locally convex spaces which were described as "fans" or scales of Banach spaces. Later on, in the work of F. Trèves, R. Nirenberg, T. Nishida, and others, the methods of Ovsiannikov were further developed, and the abstract theory gained its complete and modern look. More or less at the same time, numerous abstract results about the Cauchy method for the analytic representation of solutions were obtained. We also mention the results by Yu. A. Dubinskii (see, e.g., [7]) concerning spaces with operator exponent.

One should mention an essential difference between the cases of bounded and unbounded operators A and $A(t)$ in equations (1) and (2). Namely, the results for bounded operators A and $A(t)$ that are obtained using different methods (the method of Cauchy expansions, the method of successive approximations, the method of integral transforms) coincide. However, in the case of unbounded operators A and $A(t)$ the situation is quite different - the three methods lead, practically, to different types of solutions.

This article is devoted, first of all, to the analysis of the Cauchy method of analytic representations of solutions to equations (1) and (2) in the case that A is an unbounded linear operator in a Banach space \mathbb{X} , and in the case that $A(t)$ is a function whose values are unbounded linear operators in a Banach space \mathbb{X} and which are continuous in a suitable sense. (However, we need some results based on other approaches and they will be presented.) The basic results of the article describe those initial data for which the solutions to the corresponding Cauchy problems may be represented by classical series of the theory of differential equations. More precisely, this refers to the exponential series

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \tag{3}$$

in the stationary case, and to Peano's matriciant

$$U(t, \tau) = I + \sum_{n=1}^{\infty} \int_{\Delta_n(\tau, t)} A(\sigma_1) \cdots A(\sigma_n) d\sigma_n \cdots d\sigma_1 \tag{4}$$

in the non-stationary case; here $\Delta_n(\tau, t)$ is the set of points $(\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$ satisfying the condition $\tau \leq \sigma_1 \leq \dots \leq \sigma_n \leq t$ for $\tau \leq t$ and the condition $\tau \geq \sigma_1 \geq \dots \geq \sigma_n \geq t$ for $\tau \geq t$. Furthermore, the article discusses some properties of evolutionary operators (Cauchy functions), which are defined by the exponent series (3) and the matriciant (4).

It turns out that, in the stationary case, the complete description of the set of initial data for which the corresponding solutions are represented in the form (3) can be done in terms of so-called Roumieu spaces generated by the operator A and their inductive and projective limits, the Gevrey and Beurling spaces. However, in some cases these spaces can turn out to be trivial (in the "worst" cases equations (1) and (2) can not have non-trivial solutions represented in the form (3) or (4)). As a result, in the general theory some sufficient density conditions of the Roumieu, Gevrey and Beurling spaces in the original Banach space \mathbb{X} must play an important role. The authors could discover only a few such results; the first of them is Gel'fand's remarkable lemma on the density of analytic vectors of the generator of a uniformly bounded group of operators; this lemma was essentially strengthened by Yu. I. Ljubič and V. I. Matsaev. A new refinement in a different direction of Gel'fand's lemma is given in this article (Proposition 4). In order that one can catch the situation with different density results they are gathered in Section 2 (Propositions 1 - 6). The situation in the non-stationary case is more delicate; we present only the simplest result.

2. Roumieu spaces and density results

First we consider the autonomous linear equation (1) with an unbounded operator A . In this case formula (3) can not be considered as an equality in $\mathcal{L}(\mathbb{X})$; however, one can study conditions for $\xi \in \mathbb{X}$ under which the equality

$$e^{At} \xi = \sum_{n=0}^{\infty} \frac{A^n \xi t^n}{n!} \tag{5}$$

holds. In the case of a continuous operator A formula (5) defines a solution to equation (1) satisfying the initial condition

$$x(0) = \xi. \tag{6}$$

The main part of this article will be devoted to conditions under which formula (5) defines a solution to the Cauchy problem (1)/(6), and to the analysis of properties of the operator function defined by formula (5) in the case of an unbounded operator A . In order to characterize the sets of elements $\xi \in X$ under which formula (5) defines a solution to the Cauchy problem (1)/(6), it is convenient to deal with the Roumieu spaces generated by the closed operator A .

Let μ be a sequence $\mu = (M_n)$ with $M_0 = 1$, and let $0 < L < \infty$. Recall (see, e.g., [10, 11, 17]) that the *Roumieu space* $\mathcal{R}(A, \mu, L)$ is a space of elements $x \in X$ such that

$$\sup_{0 \leq n < \infty} L^{-n} M_n^{-1} \|A^n x\| < \infty$$

equipped with the norm

$$\|x\|_{\mathcal{R}(A, \mu, L)} = \sup_{0 \leq n < \infty} L^{-n} M_n^{-1} \|A^n x\|,$$

This is a Banach space which is continuously embedded in the space X . Usually it is useful to consider the closed subspace $\mathcal{R}^\circ(A, \mu, L)$ consisting of elements $x \in \mathcal{R}(A, \mu, L)$ such that

$$\lim_{n \rightarrow \infty} L^{-n} M_n^{-1} \|A^n x\| = 0.$$

Furthermore, we also need the *Beurling* and *Gevrey spaces*

$$\mathcal{B}(A, \mu) = \bigcap_{0 < L < \infty} \mathcal{R}(A, \mu, L) \quad \text{and} \quad \mathcal{G}(A, \mu) = \bigcup_{0 < L < \infty} \mathcal{R}(A, \mu, L)$$

which are locally convex spaces equipped with the topologies of inductive and projective limits, respectively.

Roumieu spaces (or special types of them) were considered by numerous authors. However, a lot of problems in their theory is still open. In particular, it is very little known about conditions under which a Roumieu (or Beurling, or Gevrey) space is dense in the original space X . The well-known example of R. Phillips [13] shows that the Roumieu space $\mathcal{R}(A, \mu, L)$ can be trivial.

It is easy to see (see Section 2 below) that initial data $\xi \in \mathcal{G}(A, \mu)$ in the case $\mu = (1)$ correspond to exponential-like entire solutions of equation (1). Similarly, initial data $\xi \in \mathcal{B}(A, \mu)$ in the case $\mu = (n!)$ correspond to entire solutions of this equation. Finally, initial data $\xi \in \mathcal{G}(A, \mu)$, $\mu = (n!)$, correspond to analytic solutions of this equation. Because of this, elements of $\mathcal{G}(A, (1))$ are usually called *exponential vectors*, elements of $\mathcal{B}(A, (n!))$ *entire vectors*; and elements of $\mathcal{G}(A, (n!))$ *analytic vectors*.

Let $\mu = (1)$. In this case the Roumieu space $\mathcal{R}(A, \mu, L)$ is the maximal A -invariant subspace of X on which the operator A is defined and has spectral radius not exceeding L (these spaces are not necessarily closed). The theory of these spaces is deeply connected

with the theory of spectral operators and their functional calculus (see, e.g., [8]). The latter allows us to formulate various conditions under which the spaces $\mathcal{R}(A, (1), L)$ are dense in the original space \mathbb{X} .

Below we formulate some basic results on the Roumieu spaces $\mathcal{R}(A, (1), L)$. These results have been obtained in an equivalent form in the articles [18, 19]. More exactly, the authors of [18, 19] proved that, under the conditions of either Proposition 1 or Proposition 2, the corresponding operator A is an S -operator. In particular, by their definition, this means that the union of all subspaces of \mathbb{X} which are invariant for A and on which A is bounded is dense in \mathbb{X} . One can see that any subspace of \mathbb{X} which is invariant for A and on which A is bounded with norm L is included in $\mathcal{R}(A, (1), L)$. As a result, the union $\mathcal{G}(A, (1))$ of $\mathcal{R}(A, (1), L)$ ($0 < L < \infty$) is also dense in \mathbb{X} .

Proposition 1. *Let \mathbb{X} be a Banach space and A a closed operator satisfying the Levinson condition*

$$\int_0^a \log \log \sup_{\operatorname{Re} \lambda \geq \theta} \|R(\lambda, A)\| d\theta < \infty \tag{7}$$

for some $a > 0$. Then the Gevrey space $\mathcal{G}(A, (1))$ is dense in \mathbb{X} .
 Conversely, if a function $M = M(\theta)$ satisfies the condition

$$\int_0^a \log \log M(\theta) d\theta = \infty \tag{8}$$

for some $a > 0$, then there exists an operator A such that

$$\sup_{\operatorname{Re} \lambda \geq \theta} \|R(\lambda, A)\| \leq M(\theta)$$

and $\mathcal{G}(A, (1)) = \{0\}$.

Proposition 2. *Let \mathbb{X} be a Banach space and A an operator acting in \mathbb{X} and being a generator of a strongly continuous group of operators $\mathbb{T}(t)$ ($-\infty < t < \infty$) which satisfy the Ostrowsky condition*

$$\int_{-\infty}^{+\infty} \frac{\log \|\mathbb{T}(t)\|}{1+t^2} dt < \infty. \tag{9}$$

Then the Gevrey space $\mathcal{G}(A, (1))$ is dense in \mathbb{X} .

Conversely, if a function $\omega = \omega(t)$ satisfies the condition

$$\int_{-\infty}^{+\infty} \frac{\log \omega(t)}{1+t^2} dt = \infty, \tag{10}$$

then there exists an operator A such that

$$\|\mathbb{T}(t)\| \leq \omega(t) \quad (t \in \mathbb{R})$$

and $\mathcal{G}(A, (1)) = \{0\}$.

Proposition 2 is an essential amplification of the classical Gel'fand result [9] on the density of analytic vectors for a generator of a uniformly bounded and strongly continuous group of operators in two directions: First, it turns out that one can assume the essentially weaker assumption (9) instead of the uniform boundedness of a group. Second, even under this weak assumption the set of exponential (not only analytic) vectors is dense in the original space.

Let us mention the articles [25 - 27], in which the space $\mathcal{G}(A, \mu), \mu = (1)$, is also considered under the restrictive assumption that A is a generator of a uniformly bounded and strongly continuous group of operators. The results obtained in these articles are special cases of Proposition 2. We also mention the article [1] in which entire vectors of strongly continuous and analytic semigroups are studied, and the article [14] in which some refinement to the "converse" part of Proposition 2 is obtained (see also [6]).

Some conditions for the density of the Roumieu spaces $\mathcal{R}(A, \mu, L)$ in the case $\mu = (n!)$ in the space \mathbb{X} are wellknown.

Proposition 3. *Let \mathbb{X} be a Banach space and A a generator of a strongly continuous semigroup which is analytic in a sector $S(A, \theta, h) = \{\lambda : \operatorname{Re} \lambda > h|\operatorname{Im} \lambda|^\theta\}$ (here $0 < \theta \leq 1, h \geq 0$). Then the Beurling space $\mathcal{B}(A, (n!))$ is dense in \mathbb{X} .*

In the case $\theta = 1$ this proposition was proved in [15] for Hilbert spaces and in [12] for Banach spaces; the general case is studied in similar way. For another proof see Corollary 8.3 of [6] and results in [1]. The assumption of Proposition 3 that A is a generator of an analytic semigroup implies the "good" solvability properties of the Cauchy problem (1)/(6) for $t > 0$. It is a rather strange but the assertion of Proposition 3 means, among other things, that there exists a dense set of $\xi \in \mathbb{X}$ for which the Cauchy problem (1)/(6) is solvable for (sufficiently small) $t < 0$.

Now we are in a position to formulate and prove a new result. This result shows that the statement of the classical Gel'fand result [9] on the density of analytic vectors holds true for arbitrary strongly continuous groups.

Proposition 4. *Let \mathbb{X} be a Banach space and A a generator of a strongly continuous group. Then the Gevrey space $\mathcal{G}(A, (n!))$ is dense in \mathbb{X} .*

Proof. Let $\mathbb{T}(t)$ ($-\infty < t < +\infty$) be a strongly continuous group of bounded linear operators in a Banach space \mathbb{X} , A its generator, and M and ω constants satisfying the condition

$$\|\mathbb{T}(t)\| \leq M \cosh \omega t \quad (-\infty < t < +\infty). \tag{11}$$

Denote by \mathcal{F} the set of functions defined and integrable with the weight $\cosh \omega t$ on \mathbb{R} together with their derivatives of all orders. Further, for each $x \in \mathbb{X}$ let

$$H(f)x = \int_{-\infty}^{+\infty} f(t) \mathbb{T}(t)x dt \quad (f \in \mathcal{F}).$$

It is easy to see that, according to (11), we have

$$\begin{aligned}
 AH(f)x &= \lim_{h \rightarrow 0} \frac{\mathbb{T}(h) - I}{h} \int_{-\infty}^{+\infty} f(t)\mathbb{T}(t)x \, dt \\
 &= \lim_{h \rightarrow 0} \int_{-\infty}^{+\infty} f(t) \frac{\mathbb{T}(t+h) - \mathbb{T}(t)}{h} x \, dt \\
 &= - \lim_{h \rightarrow 0} \int_{-\infty}^{+\infty} \frac{f(t-h) - f(t)}{h} \mathbb{T}(t)x \, dt && (f \in \mathcal{F}). \\
 &= - \int_{-\infty}^{+\infty} f'(t)\mathbb{T}(t)x \, dt \\
 &= -H(f')x
 \end{aligned}$$

Here passing to the limit under the integral sign is possible due to the integrability of the function $f'(t)$ with the weight $\cosh \omega t$.

In a similar way, the equalities

$$A^n H(f)x = (-1)^n H(f^{(n)}) \quad (n \geq 1; f \in \mathcal{F}) \tag{12}$$

are proved. These equalities imply that

$$\|A^n H(f)x\| \leq M \int_{-\infty}^{+\infty} |f^{(n)}(t)| \cosh \omega t \, dt \|x\| \quad (n \geq 1; f \in \mathcal{F}).$$

From this we get the inclusion $H(f)x \in \mathcal{D}(A^\infty)$ for any $x \in \mathbb{X}$ and any $f \in \mathcal{F}$ (here, as usual, $\mathcal{D}(A^\infty) = \bigcap_{n=1}^\infty \mathcal{D}(A^n)$). Moreover, $H(f)x \in \mathcal{G}(A, \mu)$, where $\mu = (n!)$, if f satisfies the inequalities

$$\int_{-\infty}^{+\infty} |f^{(n)}(t)| \cosh \omega t \, dt \leq c L^n n! \quad (n \geq 0) \tag{13}$$

for suitable L and c .

Denote by L_ω the space of functions $f = f(t)$ which are integrable on \mathbb{R} with weight $\cosh \omega t$, equipped with the natural norm

$$\|f\|_{L_\omega} = \int_{-\infty}^{+\infty} |f(t)| \cosh \omega t \, dt.$$

Furthermore, denote by \mathcal{L}_ω the set of functions $f = f(t)$ satisfying inequalities (13). It is evident that \mathcal{L}_ω is a subset of the space L_ω .

It is easy to see that the set

$$H(L_\omega) = \{H(f)x : f \in L_\omega \text{ and } x \in \mathbb{X}\}$$

is dense in \mathbb{X} . In fact, putting $\chi_n = 2n\chi_{[-\frac{1}{n}, \frac{1}{n}]}$ instead of f , one can easily prove that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \chi_n(t) \mathbb{T}(t)x \, dt = x$$

and thus the closure of $H(L_\omega)$ in \mathbb{X} coincides with \mathbb{X} .

Now, since $H(\mathcal{L}_\omega) \subseteq \mathcal{G}(A, \mu)$, $\mu = (n!)$, it is sufficient to show that the set \mathcal{L}_ω is dense in L_ω , and thus the set $H(\mathcal{L}_\omega) = \{H(f)x : f \in \mathcal{L}_\omega \text{ and } x \in \mathbb{X}\}$ is dense in \mathbb{X} . To this end, it suffices to remark that the functions

$$f_\epsilon(t) = \frac{1}{\gamma\epsilon} \int_{-\infty}^{+\infty} \cosh^{-1} \frac{t-s}{\epsilon} f(s) \, ds$$

where $\gamma = \int_{-\infty}^{+\infty} \cosh^{-1} s \, ds$ approximate an arbitrary function $f \in L_\omega$ in the norm of L_ω :

$$\lim_{\epsilon \rightarrow 0} \|f_\epsilon(t) - f(t)\|_{L_\omega} = 0,$$

and that all these functions f_ϵ belong to \mathcal{L}_ω .

The first of these statements is an evident consequence of the fact that the norms of the linear integral operators

$$S_\epsilon f(t) = \frac{1}{\gamma\epsilon} \int_{-\infty}^{+\infty} \cosh^{-1} \frac{t-s}{\epsilon} f(s) \, ds$$

are uniformly bounded in the space L_ω , and their values on Lipschitzian functions with compact supports (whose set is dense in L_ω) tend to the identity operator as $\epsilon \rightarrow 0$. Both these facts are elementary corollaries of the evident equalities

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\gamma\epsilon} \int_{|t-s| \geq \delta} \cosh^{-1} \frac{t-s}{\epsilon} \, ds = 1 \quad (0 < \delta < \infty)$$

$$\lim_{\epsilon \rightarrow 0} \|S_\epsilon\|_{\mathcal{L}(L_\omega)} = 1$$

where (see, e.g., [8])

$$\|S_\epsilon\|_{\mathcal{L}(L_\omega, L_\omega)} = \frac{1}{\gamma\epsilon} \sup_{-\infty < s < \infty} \cosh^{-1} \omega s \int_{-\infty}^{+\infty} \cosh \omega t \cosh^{-1} \frac{t-s}{\epsilon} \, dt.$$

In order to prove the second statement above it is necessary to obtain auxiliary estimates for all derivatives of the function $\cosh^{-1} z$.

Let

$$\Phi(z) = e^z \quad \text{and} \quad \Psi(w) = \frac{2w}{w^2 + 1} = \frac{1}{w + i} + \frac{1}{w - i}.$$

Then $\cosh^{-1} z = \Psi(\Phi(z))$. Since $\Phi^{(n)}(z) = e^z$ ($n \geq 0$) and

$$\begin{aligned} \Psi^{(n)}(w) &= \frac{(-1)^n}{(w + i)^{n+1}} + \frac{(-1)^n}{(w - i)^{n+1}} \\ &= (-1)^n \frac{(w + i)^{n+1} + (w - i)^{n+1}}{(w^2 + 1)^{n+1}} \end{aligned} \quad (n \geq 0)$$

then, due to Faà di Bruno's formula for the n -th order derivative of the superposition of two functions, we have

$$\begin{aligned} &(\cosh^{-1} z)^{(n)} \\ &= \sum_{k_1 + 2k_2 + \dots + nk_n = n} \frac{n!}{k_1! k_2! \dots k_n! (1!)^{k_1} (2!)^{k_2} \dots (n!)^{k_n}} \\ &\times (-1)^{k_1 + k_2 + \dots + k_n} \frac{(e^z + i)^{k_1 + k_2 + \dots + k_n + 1} + (e^z - i)^{k_1 + k_2 + \dots + k_n + 1}}{(e^{2z} + 1)^{k_1 + k_2 + \dots + k_n + 1}} \\ &\times e^{(k_1 + k_2 + \dots + k_n)z} \end{aligned}$$

($n \geq 1$). Therefore, the inequality

$$\begin{aligned} &|(\cosh^{-1} z)^{(n)}| \\ &\leq \sum_{k_1 + 2k_2 + \dots + nk_n = n} \frac{2n!}{k_1! k_2! \dots k_n! (1!)^{k_1} (2!)^{k_2} \dots (n!)^{k_n}} \\ &\times \frac{(e^{2z} + 1)^{\frac{k_1 + k_2 + \dots + k_n + 1}{2}} \cdot e^{(k_1 + k_2 + \dots + k_n)z}}{(e^{2z} + 1)^{k_1 + k_2 + \dots + k_n + 1}} \\ &\leq \sum_{k_1 + 2k_2 + \dots + nk_n = n} \frac{2n!}{k_1! k_2! \dots k_n! (1!)^{k_1} (2!)^{k_2} \dots (n!)^{k_n}} \\ &\times \frac{1}{(e^{2z} + 1)^{\frac{1}{2}}} \cdot \frac{e^{(k_1 + k_2 + \dots + k_n)z}}{(e^{2z} + 1)^{\frac{k_1 + k_2 + \dots + k_n}{2}}} \end{aligned}$$

($n \geq 1$) holds for $z \geq 0$. Consequently,

$$|(\cosh^{-1} z)^{(n)}| \leq \frac{1}{(e^{2z} + 1)^{\frac{1}{2}}} \sum_{k_1 + 2k_2 + \dots + nk_n = n} \frac{2n!}{k_1! k_2! \dots k_n! (1!)^{k_1} (2!)^{k_2} \dots (n!)^{k_n}}$$

($n \geq 1$). Further, it is easy to see that

$$\begin{aligned} & \sum_{k_1+2k_2+\dots+nk_n=n} \frac{2n!}{k_1!k_2! \dots k_n! (1!)^{k_1} (2!)^{k_2} \dots (n!)^{k_n}} \\ & \leq \sum_{k_1+k_2+\dots+k_n \leq n} \frac{2n!}{k_1!k_2! \dots k_n! (1!)^{k_1} (2!)^{k_2} \dots (n!)^{k_n}} \\ & \leq 2 \sum_{j=1}^n \frac{n!}{j!} \sum_{k_1+k_2+\dots+k_n=j} \frac{j!}{k_1!k_2! \dots k_n! (1!)^{k_1} (2!)^{k_2} \dots (n!)^{k_n}} \\ & \leq 2n! \sum_{j=1}^n \frac{1}{j!} \left(\frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \right)^j \\ & \leq 2e^{e-1} n! \end{aligned}$$

($n \geq 1$). Finally,

$$|(\cosh^{-1} z)^{(n)}| \leq \frac{2e^{e-1} n!}{(e^{2z} + 1)^{\frac{1}{2}}} \quad (n \geq 1; z > 0).$$

Analogously,

$$|(\cosh^{-1} z)^{(n)}| \leq \frac{2e^{e-1} n!}{(e^{-2z} + 1)^{\frac{1}{2}}} \quad (n \geq 1; z < 0).$$

Consequently, the inequality

$$|(\cosh^{-1} z)^{(n)}| \leq \frac{2e^{e-1} n!}{(e^{|2z|} + 1)^{\frac{1}{2}}} \quad (n \geq 1) \tag{14}$$

holds for all z .

Now let f be an arbitrary function from L_ω . We show that, for all functions f_ϵ ($0 < \epsilon < \infty$) inequalities (13) hold, i.e.

$$\int_{-\infty}^{+\infty} |f_\epsilon^{(n)}(t)| \cosh \omega t \, dt \leq c_\epsilon L_\epsilon^n n! \quad (n \geq 1) \tag{15}$$

for some c_ϵ and L_ϵ . In fact, the left-hand sides of these inequalities can be rewritten in the form

$$\int_{-\infty}^{+\infty} |f_\epsilon^{(n)}(t)| \cosh \omega t \, dt = \frac{1}{\gamma \epsilon^{n+1}} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} (\cosh^{-1})^{(n)} \frac{t-s}{\epsilon} f(s) \, ds \right| \cosh \omega t \, dt,$$

and therefore

$$\frac{1}{\gamma \epsilon} \int_{-\infty}^{+\infty} |f_\epsilon^{(n)}(t)| \cosh \omega t \, dt \leq \frac{1}{\gamma \epsilon^{n+1}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| (\cosh^{-1})^{(n)} \frac{t-s}{\epsilon} \right| |f(s)| \, ds \cosh \omega t \, dt.$$

Further, due to Fubini's theorem,

$$\frac{1}{\gamma \varepsilon} \int_{-\infty}^{+\infty} |f_\varepsilon^{(n)}(t)| \cosh \omega t \, dt \leq \frac{1}{\gamma \varepsilon^{n+1}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \left| (\cosh^{-1})^{(n)} \frac{t-s}{\varepsilon} \right| \cosh \omega t \, dt \right) |f(s)| \, ds.$$

Evidently, by virtue of (14),

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left| (\cosh^{-1})^{(n)} \frac{t-s}{\varepsilon} \right| \cosh \omega t \, dt \\ & \leq 2e^{\varepsilon-1} n! \int_{-\infty}^{+\infty} (e^{2e^{-1}|t-s|} + 1)^{-\frac{1}{2}} \cosh \omega t \, dt \\ & \leq 2e^{\varepsilon-1} n! \int_{-\infty}^{+\infty} e^{-\varepsilon^{-1}|t-s|} e^{\omega|t|} \, dt \\ & \leq 2e^{\varepsilon-1} n! \int_{-\infty}^{+\infty} e^{-\varepsilon^{-1}|t-s|} e^{\omega|t-s|+\omega|s|} \, dt \\ & \leq 2e^{\varepsilon-1} n! \int_{-\infty}^{+\infty} e^{(-\varepsilon^{-1}+\omega)|t-s|} \, dt e^{\omega|s|} \\ & \leq \frac{4e^{\varepsilon-1} n!}{\omega - \varepsilon^{-1}} e^{\omega|s|} \end{aligned}$$

for $\varepsilon < \omega^{-1}$, and hence

$$\frac{1}{\gamma \varepsilon} \int_{-\infty}^{+\infty} |f_\varepsilon^{(n)}(t)| \cosh \omega t \, dt \leq \frac{4e^{\varepsilon-1}}{(\omega - \varepsilon^{-1})\gamma \varepsilon} \int_{-\infty}^{+\infty} |f(s)| e^{\omega|s|} \, ds (\varepsilon^{-1})^n n!$$

for all $n \geq 0$. In other words, inequalities (15) hold for

$$c_\varepsilon = \frac{4e^{\varepsilon-1}}{(\omega - \varepsilon^{-1})\gamma \varepsilon} \int_{-\infty}^{+\infty} |f(s)| e^{\omega|s|} \, ds \quad \text{and} \quad L_\varepsilon = \varepsilon^{-1}.$$

Thus, Proposition 4 is proved ■

Proposition 4 shows that, under the conditions of Gel'fand's lemma, the assumption about uniform boundedness of the group can be omitted.

The sequence $\mu = (n!)$ seems to be a "limit case" in the problem of the density of Roumieu spaces in the original space \mathbb{X} , at least for generators of strongly continuous semigroups on $(0, \infty)$. Using methods suggested in [9] and the theory of quasianalytic function classes [20, 21, 24] one can repeat the proof of density in \mathbb{X} of the set $\mathcal{D}(A^\infty)$ and show the validity of the following "folklore" statement.

Proposition 5. *Let X be a Banach space and A a generator of a strongly continuous semigroup. Then the Gevrey space $\mathcal{G}(A, \mu)$ is dense in X if the sequence μ is not quasianalytic, i.e. if*

$$\sum_{n=0}^{\infty} H_n^{-1} < \infty \tag{16}$$

where $H_n = \inf_{k \geq n} M_k^{1/k}$ ($n \geq 0$).

We recall that the quasianalyticity condition for a Roumieu space is equivalent to the existence of non-trivial C^∞ -smooth functions with compact support in this space; numerous equivalent conditions can be found in [21]. Thus, the problem about density of the Gevrey spaces $\mathcal{G}(A, \mu)$ in the original space X has a sense only in the case if $\sum_{n=0}^{\infty} H_n^{-1} = \infty$. In the case of the classical Gevrey spaces $\mu = ((n!)^s)$ ($0 < s < \infty$), this condition is valid for $0 \leq s \leq 1$.

Below another condition about the density of Gevrey spaces is formulated (see [3 - 4]).

Proposition 6. *Let X be a Banach space. Suppose that the resolvent $R(\lambda, A)$ exists and satisfies the inequality*

$$\|R(\lambda, A)\| \leq M(1 + |\lambda|)^N$$

in a domain $\text{Re } \lambda \geq \max\{a, h|\text{Im } \lambda|^\theta\}$ for suitable $a \in \mathbb{R}, 0 < \theta < 1, h \geq 0, M > 0$ and $N \geq 0$. Then the Gevrey space $\mathcal{G}(A, \mu)$ for $\mu = ((n!)^s)$ ($s > 1$) is dense in X .

3. Stationary linear differential equations

Now we are in a position to describe the solutions which can be represented by exponents for the linear differential equation (1) with a closed unbounded linear operator A in a Banach space X . The following simple result is basic.

Theorem 1. *Let A be a closed linear operator in a Banach space $X, \mu = (n!)$, and $\xi \in X$. Then:*

a) *Equality (5) defines a solution of the Cauchy problem (1)/(6) on some interval $(-h, h)$ (on any interval $(-h, h)$) if and only if $\xi \in \mathcal{G}(A, \mu)$ ($\xi \in \mathcal{B}(A, \mu)$). More precisely, in the case $\xi \in \mathcal{R}(A, \mu, h^{-1})$ equality (5) defines the solution of the Cauchy problem (1)/(6) on the interval $(-h, h)$. Conversely, if the solution of the Cauchy problem (1)/(6) is defined on the interval $[-h, h]$, then $\xi \in \mathcal{R}^\circ(A, \mu, h^{-1})$.*

b) *For $L' < L''$ equation (3) defines, for $t \in (-h(L', L''), h(L', L''))$ with*

$$h(L', L'') = (L')^{-1} - (L'')^{-1} \tag{17}$$

a continuous linear operator e^{At} from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L'')$, such that

$$\|e^{At}\|_{\mathcal{L}(\mathcal{R}(A, \mu, L'), \mathcal{R}(A, \mu, L''))} \leq \frac{L''}{L'} \tag{18}$$

c) *The semigroup identity*

$$e^{A(t+\tau)} = e^{At} \cdot e^{A\tau} \tag{19}$$

holds for $\tau, t, t + \tau \in (-h(L', L''), h(L', L''))$ and $L' < L < L''$ where e^{At} , $e^{A\tau}$ and $e^{A(t+\tau)}$ are considered as operators from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L)$, from $\mathcal{R}(A, \mu, L)$ into $\mathcal{R}(A, \mu, L'')$, and from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L'')$, respectively.

Proof. Suppose that the series (5) with a fixed $\xi \in \mathbb{X}$ is absolutely convergent on an interval $[-h, h]$. Then

$$\lim_{n \rightarrow \infty} \frac{t^n A^n \xi}{n!} = 0 \quad (-h \leq t \leq h)$$

and, consequently,

$$\lim_{n \rightarrow \infty} \frac{h^n \|A^n \xi\|}{n!} = 0.$$

This means that $\xi \in \mathcal{R}^\circ(A, \mu, h^{-1})$ for $\mu = (n!)$. Conversely, if $\xi \in \mathcal{R}(A, \mu, h^{-1})$, then the series (5) is absolutely and uniformly convergent on every compact subset of the interval $(-h, h)$. Thus, the right-hand side of (5) defines, for each $t \in (-h, h)$, a linear operator e^{At} on the set of all $\xi \in \mathcal{R}(A, \mu, h^{-1})$ with values in the space \mathbb{X} . In general, the sum of the series (5) is defined, at least for small t , for all ξ from the space $\mathcal{G}(A, \mu)$ with $\mu = (n!)$.

Further, let $\xi \in \mathcal{G}(A, \mu)$ ($\xi \in \mathcal{B}(A, \mu)$), where $\mu = (n!)$. Then equality (5) defines a continuous function on an interval (the fixed interval) $(-h, h)$. This function is differentiable, because formal differentiation of (5) lead to the series

$$Q(t) = \sum_{n=1}^{\infty} \frac{t^{n-1} A^n \xi}{(n-1)!} = \sum_{n=0}^{\infty} \frac{t^n A^{n+1} \xi}{n!}$$

which is absolutely convergent. Due to the classical theorem on the differentiability of function series the sum $Q(t)$ of this series is the derivative of the left-hand side of (5). In addition, since the operator A is closed the identity

$$\sum_{n=0}^{\infty} \frac{t^n A^{n+1} \xi}{n!} = A \sum_{n=0}^{\infty} \frac{t^n A^n \xi}{n!}$$

holds, and, as a result, the left-hand side of (5) is a solution of the Cauchy problem (1)/(6).

Now, let $L' < L''$ and $\xi \in \mathcal{R}(A, \mu, L')$. Then for each $n \geq 0$ and $|t| < h(L', L'')$ we

have

$$\begin{aligned} & \|e^{At}\xi\|_{\mathcal{R}(A,\mu,L'')} \\ &= \sup_{0 \leq n < \infty} (L'')^{-n}(n!)^{-1} \|A^n e^{At}\xi\| \\ &\leq \sup_{0 \leq n < \infty} (L'')^{-n}(n!)^{-1} \sum_{k=0}^{\infty} \frac{|t|^k \|A^{n+k}\xi\|}{k!} \\ &\leq \|\xi\|_{\mathcal{R}(A,\mu,L')} \sup_{0 \leq n < \infty} (L'')^{-n}(n!)^{-1} (L')^n \sum_{k=0}^{\infty} \frac{|t|^k (L')^k (n+k)!}{k!} \\ &= \|\xi\|_{\mathcal{R}(A,\mu,L')} \sup_{0 \leq n < \infty} (L'')^{-n}(n!)^{-1} (L')^n \sum_{k=0}^{\infty} (|t|L')^k (k+1) \cdots (k+n), \end{aligned}$$

and after elementary calculations ($z = |t|L'$)

$$\begin{aligned} & \|e^{At}\xi\|_{\mathcal{R}(A,\mu,L'')} \\ &\leq \|\xi\|_{\mathcal{R}(A,\mu,L')} \sup_{0 \leq n < \infty} (L'')^{-n}(n!)^{-1} (L')^n \left(\frac{\partial}{\partial z}\right)^n \sum_{s=0}^{\infty} (|t|L')^s \\ &= \|\xi\|_{\mathcal{R}(A,\mu,L')} \sup_{0 \leq n < \infty} (L'')^{-n}(n!)^{-1} (L')^n \left(\frac{\partial}{\partial z}\right)^n \frac{1}{1 - L'|t|} \\ &= \|\xi\|_{\mathcal{R}(A,\mu,L')} \sup_{0 \leq n < \infty} (L'')^{-n}(n!)^{-1} (L')^n \frac{n!}{(1 - L'|t|)^{n+1}} \\ &= \frac{L''}{L'} \|\xi\|_{\mathcal{R}(A,\mu,L')}. \end{aligned}$$

Now, let $L' < L < L''$. By what has been proved above, the operator $e^{A\tau}$ is defined for $|\tau| < (L')^{-1} - L^{-1}$ and acts from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L)$, while the operator e^{At} is defined for $|t| < L^{-1} - (L'')^{-1}$ and acts from $\mathcal{R}(A, \mu, L)$ into $\mathcal{R}(A, \mu, L'')$. Therefore the operator $e^{At} \cdot e^{A\tau}$ is defined and acts from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L'')$. It remains to check the equality $e^{A(t+\tau)}\xi = e^{At} \cdot e^{A\tau}\xi$ for all $\xi \in \mathcal{R}(A, \mu, L')$, i.e. the equality

$$\sum_{n=0}^{\infty} \frac{(t+\tau)^n A^n \xi}{n!} = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j \left(\sum_{k=0}^{\infty} \frac{\tau^k A^k \xi}{k!} \right) \quad (\xi \in \mathcal{R}(A, \mu, L')).$$

Since the operator A is closed and the series $\sum_{k=0}^{\infty} \frac{\tau^k A^{k+j} \xi}{k!}$ ($j \geq 0$) are absolutely convergent for $|\tau| < (L')^{-1} - L^{-1}$, the right-hand side of the above equality can be rewritten in the form

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} A^j \left(\sum_{k=0}^{\infty} \frac{\tau^k A^k \xi}{k!} \right) = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{t^j \tau^k A^{j+k} \xi}{j!k!} \right).$$

Therefore, it is sufficient to prove the equality

$$\sum_{n=0}^{\infty} \frac{(t+\tau)^n A^n \xi}{n!} = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{t^j \tau^k A^{j+k} \xi}{j!k!} \right).$$

The right-hand side of this equality, in virtue of absolute convergence for $|\tau| < (L')^{-1} - L^{-1}$ and $|t| < L^{-1} - (L'')^{-1}$, can be rewritten in the form

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{t^j \tau^k A^{j+k} \xi}{j!k!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=n}^{\infty} \frac{n!}{j!k!} t^j \tau^k \right) A^n \xi$$

and thus the equality $e^{A(t+\tau)} = e^{At} \cdot e^{A\tau}$ holds. Theorem 1 is proved ■

Theorem 2. Let A be a closed linear operator in a Banach space \mathbb{X} , $\mu = (n!)^s$ for $0 < s < 1$. Then the operator e^{At} , defined by the right-hand side of equation (5) for all $t \in (-\infty, \infty)$, is a continuous linear operator from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L'')$ ($L' < L''$), and

$$\|e^{At}\|_{\mathcal{L}(\mathcal{R}(A, \mu, L'), \mathcal{R}(A, \mu, L''))} \leq \sum_{k=0}^{\infty} \frac{1}{(k!)^{1-s}} \left(\frac{(L'')^s L' |t|}{(L'' - L')^s} \right)^k \tag{20}$$

Moreover, the semigroup identity

$$e^{A(t+\tau)} = e^{At} \cdot e^{A\tau} \tag{21}$$

holds for $\tau, t, t + \tau \in (-\infty, \infty)$ and $L' < L < L''$ where e^{At} , $e^{A\tau}$ and $e^{A(t+\tau)}$ are considered as operators from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L)$, from $\mathcal{R}(A, \mu, L)$ into $\mathcal{R}(A, \mu, L'')$, and from $\mathcal{R}(A, \mu, L')$ into $\mathcal{R}(A, \mu, L'')$, respectively.

Proof. It is evident that, for each $\xi \in \mathcal{R}(A, \mu, L')$ and each $t \in (-\infty, \infty)$, the estimate

$$\begin{aligned} & \|e^{At} \xi\|_{\mathcal{R}(A, \mu, L'')} \\ &= \sup_{0 \leq n < \infty} (L'')^{-n} (n!)^{-s} \|A^n e^{At} \xi\| \\ &\leq \sup_{0 \leq n < \infty} (L'')^{-n} (n!)^{-s} \sum_{k=0}^{\infty} \frac{|t|^k \|A^{n+k} \xi\|}{k!} \\ &\leq \|\xi\|_{\mathcal{R}(A, \mu, L')} \sup_{0 \leq n < \infty} (L'')^{-n} (L')^n \sum_{k=0}^{\infty} \frac{|t|^k (L')^k ((n+k)!)^s}{(n!)^{-s} k!} \\ &= \|\xi\|_{\mathcal{R}(A, \mu, L')} \sup_{0 \leq n < \infty} (L'')^{-n} (L')^n \sum_{k=0}^{\infty} \frac{|t|^k (L')^k ((n+k)!)^s}{(n!)^s (k!)^s (k!)^{1-s}} \\ &= \|\xi\|_{\mathcal{R}(A, \mu, L')} \sup_{0 \leq n < \infty} (L'')^{-n} (L')^n \sum_{k=0}^{\infty} \left(\frac{(n+k)!}{n!k!} \right)^s \frac{(|t|L')^k}{(k!)^{1-s}} \end{aligned}$$

holds. Further, the elementary inequality

$$\frac{(n+k)!}{n!k!} \leq z^n \left(\frac{z}{z-1} \right)^k \quad (1 < z < \infty)$$

for $z = L''(L')^{-1}$ implies

$$\begin{aligned} \|e^{At}\xi\|_{\mathcal{R}(A,\mu,L'')} &\leq \|\xi\|_{\mathcal{R}(A,\mu,L')} \sum_{k=0}^{\infty} \left(\frac{L''}{L''-L'}\right)^{ks} \frac{(|t|L')^k}{(k!)^{1-s}} \\ &= \|\xi\|_{\mathcal{R}(A,\mu,L')} \sum_{k=0}^{\infty} \frac{1}{(k!)^{1-s}} \left(\frac{(L'')^s L' |t|}{(L''-L')^s}\right)^k. \end{aligned}$$

The latter proves the basic statement of Theorem 2. The proof of the semigroup property repeats literally the corresponding reasoning in the proof of Theorem 1 ■

The estimate (20) for $s = 1$ is rather rough in comparison with the estimate (18).

The statements of Theorems 1 and 2 can be extended to the case of a sequence $\mu = (M_n)$ which satisfies the condition

$$\sigma(\mu) = \limsup_{n \rightarrow \infty} (n!)^{-1} M_n < \infty. \tag{22}$$

In this case the operator e^{At} defined by the right-hand side of equation (5) is defined on the space $\mathcal{R}(A, \mu, L')$ and takes its values in the space $\mathcal{R}(A, \mu, L'')$ ($0 < L' < L'' < \infty$) for $t \in (-h(L', L''), h(L', L''))$, where

$$h(L', L'') = (L')^{-1} \sup \left\{ h : \Phi\left(\frac{L'}{L''}, h\right) < \infty \right\}.$$

Moreover, this operator satisfies the estimate

$$\|e^{At}\|_{\mathcal{L}(\mathcal{R}(A,\mu,L'), \mathcal{R}(A,\mu,L''))} \leq \Phi\left(\frac{L'}{L''}, L'|t|\right)$$

where

$$\Phi(u, v) = \sup_{0 \leq n < \infty} \left\{ u^n \sum_{k=0}^{\infty} \frac{M_{n+k}}{M_n k!} v^k \right\}.$$

In fact, for each $\xi \in \mathcal{R}(A, \mu, L')$ and each $t \in (-\infty, \infty)$ we have

$$\begin{aligned} \|e^{At}\xi\|_{\mathcal{R}(A,\mu,L'')} &= \sup_{0 \leq n < \infty} (L'')^{-n} M_n^{-1} \|A^n e^{At}\xi\| \\ &\leq \sup_{0 \leq n < \infty} (L'')^{-n} M_n^{-1} \sum_{k=0}^{\infty} \frac{|t|^k \|A^{n+k}\xi\|}{k!} \\ &\leq \|\xi\|_{\mathcal{R}(A,\mu,L')} \sup_{0 \leq n < \infty} (L'')^{-n} (L')^n \sum_{k=0}^{\infty} \frac{|t|^k (L')^k M_{n+k}}{M_n k!} \\ &= \|\xi\|_{\mathcal{R}(A,\mu,L')} \Phi\left(\frac{L'}{L''}, L'|t|\right). \end{aligned}$$

As a matter of fact, Theorems 1 and 2 reduce the problem of the exponential representation of solutions to the Cauchy problem for equation (5) to the analysis of the Roumieu spaces generated by the corresponding operator A for sequences $\mu = (M_n)$ with factorial growth. Propositions 1 - 6 allow us to give effective sufficient conditions under which the set of initial data of solutions which may be represented by an exponential series is sufficiently "rich" (dense in the original Banach space). In the well-known example of R. Phillips (see [13]) the statements of Theorems 1 and 2 are applicable only to the zero vector and, therefore, useless.

4. Non-stationary linear differential equations

In this section the linear equation (2) is considered, where $A(t)$ is a family of closed linear operators with a dense domains in \mathbb{X} . It is clear that in this case equation (4) can not be considered as an equation in the space $\mathcal{L}(\mathbb{X})$. However, one can study conditions for $\xi \in \mathbb{X}$ under which the equation

$$U(t, \tau)\xi = \xi + \sum_{n=1}^{\infty} \int_{\Delta_n(\tau, t)} A(\sigma_1) \cdots A(\sigma_n)\xi d\sigma_n \cdots d\sigma_1 \tag{23}$$

defines (as in the case of continuity of the operators $A(t)$) solutions of equation (2) satisfying the Cauchy initial condition

$$x(\tau) = \xi. \tag{24}$$

The analysis of the representations (23) of solutions to the Cauchy problem (2)/(24) is deeply related with the verification of rather subtle convergence conditions for the series (23), and, up to present, nobody can say that such an analysis is complete. We restrict ourselves only to a new simple result which is formulated in terms of some auxiliary operator C .

Let Γ be an infinite matrix with non-negative elements γ_{kj} ($j, k \geq 0$); in usual cases in applications the equalities

$$\gamma_{jk} = 0 \quad (j > k + l, k \geq 0)$$

are supposed for some l . The smallest l with this property can be considered as the "order" of the right-hand side of equation (2) with respect to C .

Let C be a closed linear operator in a Banach space \mathbb{X} with dense domain, μ a sequence, and \mathbb{I} an interval from \mathbb{R} . We say that an operator function $A(t)$ satisfies a Γ -condition with respect to Γ and μ on the interval \mathbb{I} , if

$$\|C^k A(t)\xi\| \leq \sum_{j=0}^{\infty} \gamma_{kj} \|C^j \xi\| \quad \left(t \in \mathbb{I}, \xi \in \bigcap_{j=0}^{\infty} D(C^j), k \geq 0 \right). \tag{25}$$

We need some notation. Let, for $n \geq 1$,

$$\theta_n(L', L'') = \inf \left\{ \theta(L_0, L_1) \cdots \theta(L_{n-1}, L_n) : L' = L_0 < L_1 < \dots < L_n = L'' \right\} \tag{26}$$

where

$$\theta(L', L'') = \sup_{1 \leq k < \infty} \left\{ (L'')^{-k} M_k^{-1} \sum_{j=0}^{\infty} \gamma_{kj} (L')^j M_j \right\} \tag{27}$$

and

$$\theta_n(L) = \lim_{\Lambda \rightarrow \infty} \theta_n(L, \Lambda) \quad (n \geq 1). \tag{28}$$

Further, let

$$w(\mu, L', L'', h) = 1 + \sum_{n=1}^{\infty} (n!)^{-1} \theta_n(L', L'') h^n \tag{29}$$

$$w(\mu, L, h) = 1 + \sum_{n=1}^{\infty} (n!)^{-1} \theta_n(L) h^n \tag{30}$$

and $h(L', L'')$, $h(L)$ be the radii of convergence of these series.

Theorem 3. *Let C be a closed linear operator in a Banach space X with dense domain, μ a sequence, and I an interval from \mathbb{R} . Suppose that each operator $A(t)$ ($t \in I$) is a closed linear operator on the Banach space X , the functions*

$$A(\sigma_1) \dots A(\sigma_n) \xi \quad (\xi \in \mathcal{D}(C^n), n \geq 1)$$

are continuous on I^n , and the operator function $A(t)$ satisfies a Γ -condition with respect to Γ and μ on the interval I . Then:

a) Equation (23) defines, for $\xi \in \mathcal{R}(C, \mu, L)$, on the interval $(\tau - h(L), \tau + h(L))$ the solution to the Cauchy problem (2)/(24) and this solution satisfies the inequality

$$\|x(t)\|_{\mathcal{R}(C, \mu, L'')} \leq w(\mu, L, |t - \tau|) \|\xi\|_{\mathcal{R}(A, \mu, L')} \quad (t, \tau \in I, |t - \tau| < h(L)). \tag{31}$$

b) For $L' < L''$ equation (23) defines, for $t, \tau \in I$ with $|t - \tau| < h(L', L'')$, a continuous linear operator $U(t, \tau)$ from $\mathcal{R}(C, \mu, L')$ into $\mathcal{R}(C, \mu, L'')$ such that

$$\|U(t, \tau)\|_{\mathcal{L}(\mathcal{R}(C, \mu, L'), \mathcal{R}(C, \mu, L''))} \leq w(\mu, L', L'', |t - \tau|). \tag{32}$$

c) The equalities

$$U'_t(t, \tau) = A(t)U(t, \tau) \quad \text{and} \quad U'_\tau(t, \tau) = -U(t, \tau)A(\tau) \tag{33}$$

hold for $t, \tau \in I$ with $|t - \tau| < h(L', L'')$, and the formula

$$U(t, s) \cdot U(s, \tau) = U(t, \tau) \tag{34}$$

holds for $t, \tau, \sigma \in I$ with $|t - \sigma| < h(L', L)$ and $|\sigma - \tau| < h(L, L'')$ were $U(t, s)$, $U(s, \tau)$ and $U(t, \tau)$ are considered as operators from $\mathcal{R}(C, \mu, L)$ into $\mathcal{R}(C, \mu, L'')$, from $\mathcal{R}(C, \mu, L')$ into $\mathcal{R}(C, \mu, L)$, and from $\mathcal{R}(C, \mu, L')$ into $\mathcal{R}(C, \mu, L'')$, respectively.

Proof. First we note that, by virtue of inequality (25), each operator $A(\sigma)$ acts from the Roumieu space $\mathcal{R}(C, \mu, L')$ into the Roumieu space $\mathcal{R}(C, \mu, L'')$ for $L' < L''$, and by (27)

$$\begin{aligned} \|A(\sigma)\xi\|_{\mathcal{R}(C, \mu, L'')} &\leq \sup_k \left\{ (L'')^{-k} M_k^{-1} \sum_{j=0}^{\infty} \gamma_{kj} (L')^j M_j \right\} \|\xi\|_{\mathcal{R}(C, \mu, L')} \\ &= \theta(L', L'') \|\xi\|_{\mathcal{R}(C, \mu, L')}. \end{aligned}$$

This inequality and (26) imply that

$$\|A(\sigma_1) \dots A(\sigma_n)\xi\|_{\mathcal{R}(C,\mu,L'')} \leq \theta_n(L', L'') \|\xi\|_{\mathcal{R}(C,\mu,L')} \quad (n \geq 1). \tag{35}$$

Further, by (28),

$$\|A(\sigma_1) \dots A(\sigma_n)\xi\| \leq \theta_n(L) \|\xi\|_{\mathcal{R}(C,\mu,L)} \quad (n \geq 1). \tag{36}$$

The last inequality implies that the series in the right-hand side of (23) for $\xi \in \mathcal{R}(C, \mu, L)$ is absolutely and uniformly convergent in the norm of the space \mathbb{X} on each interval $(\tau - h, \tau + h)$ ($h < h(L)$) where $h(L)$ is the radius of convergence of the series (30). Thus, equation (23) for $|t - \tau| < h(L)$ defines a function $x(t) = U(t, \tau)\xi$. Repeating the reasoning in the proof of Theorem 1 one can see that $x(t) = U(t, \tau)\xi$ is a solution of the Cauchy problem (2)/(24) on the interval $(\tau - h(L), \tau + h(L))$. Moreover, the estimate (31) is proved.

Let $L' < L''$ and $\xi \in \mathcal{R}(C, \mu, L')$. Then statement b) and inequality (32) follow from the chain of inequalities

$$\begin{aligned} & \|U(t, \tau)\xi\|_{\mathcal{R}(C,\mu,L'')} \\ & \leq \|\xi\|_{\mathcal{R}(C,\mu,L'')} + \sum_{n=0}^{\infty} \int_{\Delta_n(\tau,t)} \|A(\sigma_1) \dots A(\sigma_n)\xi\|_{\mathcal{R}(C,\mu,L'')} d\sigma_n \dots d\sigma_1 \\ & \leq \|\xi\|_{\mathcal{R}(C,\mu,L')} + \sum_{n=0}^{\infty} (n!)^{-1} \theta_n(L', L'') h^n \|\xi\|_{\mathcal{R}(C,\mu,L')} \\ & \leq w(\mu, L', L'', h) \|\xi\|_{\mathcal{R}(C,\mu,L')} \end{aligned}$$

for $|t - \tau| \leq h$, with $h < h(L', L'')$. In order to prove the semigroup property $U(t, s) \cdot U(s, \tau) = U(t, \tau)$ one can see that the left-hand and right-hand sides of this equality are operators which act from $\mathcal{R}(C, \mu, L')$ into $\mathcal{R}(C, \mu, L'')$ under the hypotheses of Theorem 3. Moreover, the formal composition of series

$$\begin{aligned} U(t, s) &= I + \sum_{j=0}^{\infty} \int_{\Delta_j(s,t)} A(\varphi_1) \dots A(\varphi_j) d\varphi_j \dots d\varphi_1 \\ U(s, \tau) &= I + \sum_{k=0}^{\infty} \int_{\Delta_k(\tau,s)} A(\psi_1) \dots A(\psi_k) d\psi_k \dots d\psi_1 \end{aligned}$$

can be written (after an evident substitution) in the form

$$\begin{aligned} & \sum_{j,k=0}^{\infty} \int_{\Delta_j(s,t)} \int_{\Delta_k(\tau,s)} A(\varphi_1) \dots A(\varphi_j) A(\psi_1) \dots A(\psi_k)\xi d\varphi_j \dots d\varphi_1 d\psi_k \dots d\psi_1 \\ &= \sum_{n=0}^{\infty} \int_{\Delta_n(t,\tau)} A(\sigma_1) \dots A(\sigma_n)\xi d\sigma_n \dots d\sigma_1 \\ &= U(t, \tau). \end{aligned}$$

To justify the formal composition it is sufficient to verify absolute convergence of the left-hand side of the latter equation; however, this is a consequence of the evident chain of inequalities

$$\begin{aligned} & \sum_{j,k=0}^{\infty} \int_{\Delta_j(s,t)} \int_{\Delta_k(\tau,\theta)} \|A(\varphi_1) \cdots A(\varphi_j) A(\psi_1) \cdots A(\psi_k) \xi\| d\varphi_j \cdots d\varphi_1 d\psi_k \cdots d\psi_1 \\ & \leq \left(\sum_{j,k=0}^{\infty} (j!)^{-1} (k!)^{-1} \theta_j(L', L) \theta_k(L, L'') h(L', L)^j h(L, L'')^k \right) \|\xi\|_{\mathcal{R}(C, \mu, L')} \\ & = \left(\sum_{j=0}^{\infty} (j!)^{-1} \theta_j(L', L) h(L', L)^j \right) \left(\sum_{k=0}^{\infty} (k!)^{-1} \theta^k(L, L'') h(L, L'')^k \right) \|\xi\|_{\mathcal{R}(C, \mu, L')} \\ & = w(\mu, L', L, h_1) w(\mu, L, L'', h_2) \|\xi\|_{\mathcal{R}(C, \mu, L')} \end{aligned}$$

for $|\tau - s| \leq h_1$ and $|t - s| \leq h_2$, with $h_1 < h(L', L)$ and $h_2 < h(L, L'')$. Thus, the statement of Theorem 3 is proved ■

One can see that the application of Theorem 3 can give non-trivial results only if the Roumieu spaces $\mathcal{R}(C, \mu, L)$ ($0 < L < \infty$) are sufficiently "rich", at least, dense in the original space \mathbb{X} . Thus, we need different density results for the Roumieu spaces as well as for the Gevrey and Beurling spaces. In particular, in applying Theorem 3 Propositions 1 - 6 are useful.

The conditions of Theorem 3 are rather cumbersome and tedious to verify. However, simple examples of linear partial differential equations show that they are sufficiently natural. Moreover, one can see that the calculation of the values $h(L)$, $h(L', L'')$ and $w(\mu, L', L'', h)$ is standard; in particular, one can consider the special cases from [2, 22, 23].

The case considered in [5] is more difficult. Condition (25) in this case can be written in the form

$$\|C^k A(t) \xi\| \leq \sum_{j=1}^{k+1} \frac{k!}{(j-1)!} \Lambda^{k-j} \|C^j \xi\| \quad \left(t \in \mathbb{I}, \xi \in \bigcap_{j=1}^{k+1} \mathcal{D}(C^j), k \geq 0 \right).$$

Simple calculations show that $\theta(L', L'') = \infty$ for $L'' < 1$; in the case $L'' \geq L > 1$ the inequality

$$\theta(L', L'') \leq \frac{c(L)}{(L')^{-1} - (L'')^{-1}}$$

holds. Applying Theorem 3 in this case allows us not only to get existence of solutions to the Cauchy problem on the corresponding interval, but also to define the Roumieu space in which the corresponding solutions lie.

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