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The 2-Dimensional Dirichlet Problem in an External Domain with Cuts

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Abstract. The Dirichlet problem for the Laplace equation in an external connected plane region with cuts is studied. The existence of a classical solution is proved by potential theory. The problem is reduced to a Fredholm equation of the second kind, which is uniquely solvable. Consequently, the solution can be computed by standard codes. The solvability of the Dirichlet problem in an internal domain with cuts is proved with the help of a conformal mapping.

Keywords: *Harmonic functions, Dirichlet problem, plane domains with cuts, potential theory, boundary integral equation method*

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1. Introduction

Boundary value problems in arbitrary plane domains with cuts were not actively studied in the theory of partial differential equations before. Problems outside cuts in the plane and problems in domains bounded by closed curves have been studied separately, because different methods for their analysis were used. The 2-dimensional Dirichlet boundary value problem for the Laplace equation in a multiply connected domain bounded by closed curves is considered, for instance, in [2, 8]. The Dirichlet problem for this equation in the exterior of cuts is studied in [8]. The present note is an attempt to join these problems together and to consider domains containing cuts. From practical stand-point such domains have great significance, because cuts model cracks, screens or wings in physical problems. Domains without cuts are a particular case of our problem. Our approach is different from [2, 8] even in this case.

The approach proposed in the present paper can be applied to other elliptic problems in domains with closed and open boundary. The Dirichlet and Neumann problems for the Helmholtz equation in plane domains with cuts have been recently investigated in [4, 6, 7]. Some nonlinear problems on fluid flow over several obstacles, including wings, were treated in [5].

The uniqueness theorem in the Dirichlet problem for the Laplace equation follows from the maximum principle, unlike the Dirichlet problem for the Helmholtz equation [4, 7], where the energy equalities are used. This enables to study the problem in the present paper under weakened smoothness conditions in comparison with [4, 7].

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2. Formulation of the problem

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [8].

In the plane $x = (x_1, x_2) \in \mathbb{R}^2$ we consider the external multiply connected domain bounded by simple open curves $\Gamma_1^1, ..., \Gamma_{N_1}^1 \in C^{2,\lambda}$ $(N_1 \geq 0)$ and simple closed curves $\Gamma_1^2, ..., \Gamma_{N_2}^2 \in C^{1,\lambda}$ $(N_2 \geq 1)$ with $\lambda \in (0, 1]$, so that the curves do not have points in common. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \qquad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \qquad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The external connected domain bounded by Γ^2 will be called $\mathcal{D}$. We assume that each curve Γ_n^k is parametrized by the arc length s:

$$\Gamma_n^k = \left\{x : x = x(s) = (x_1(s), x_2(s)),\ s \in [a_n^k, b_n^k]\right\} \qquad (n = 1, ..., N_k,\ k = 1, 2)$$

so that

$$a_1^1 < b_1^1 < ... < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < ... < a_{N_2}^2 < b_{N_2}^2$$

and the domain $\mathcal{D}$ is on the right when the parameter s increases on Γ_n^2. Therefore, points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except a_n^2, b_n^2, which correspond to the same point x for $n = 1, ..., N_2$. Below the sets of the intervals on the Os-axis

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \qquad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \qquad \bigcup_{k=1}^{2} \bigcup_{n=1}^{N_k} [a_n^k, b_n^k]$$

will be denoted by the same symbols as corresponding sets of curves, that is, by Γ^1, Γ^2 and Γ respectively.

We put

$$C^0(\Gamma_n^2) = \left\{\mathcal{F} = \mathcal{F}(s) :\ \mathcal{F} \in C^0[a_n^2, b_n^2],\ \mathcal{F}(a_n^2) = \mathcal{F}(b_n^2)\right\}$$

and

$$C^0(\Gamma^2) = \bigcap_{n=1}^{N_2} C^0(\Gamma_n^2).$$

By $\mathcal{D}_n$ we denote the internal domain bounded by the curve Γ_n^2 $(n = 1, ..., N_2)$.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s),\ \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$ and $\sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\frac{\pi}{2}$.

We consider the curves Γ^1 as a set of cuts. The side of Γ^1 which is on the left, when the parameter s increases, will be denoted by $(\Gamma^1)^+$, and the opposite side will be denoted by $(\Gamma^1)^-$.

Let us formulate the Dirichlet problem for the Laplace equation in the domain $\mathcal{D}\backslash\Gamma^1$.

Problem (U). To find a function $u = u(x)$ of class $C^0(\overline{\mathcal{D}\backslash\Gamma^1}) \cap C^2(\mathcal{D}\backslash\Gamma^1)$, so that $u(x)$ satisfies the Laplace equation

$$\frac{\partial^2}{\partial x_1^2}u(x) + \frac{\partial^2}{\partial x_2^2}u(x) = 0 \qquad (x \in \mathcal{D}\backslash\Gamma^1), \tag{1$_a$}$$

the boundary conditions

$$u(x(s))|_{(\Gamma^1)^+} = F^+(s), \qquad u(x(s))|_{(\Gamma^1)^-} = F^-(s), \qquad u(x(s))|_{\Gamma^2} = F(s) \tag{1$_b$}$$

and the condition at infinity

$$|u(x)| \leq \text{const} \qquad \left(|x| = \sqrt{x_1^2 + x_2^2} \to \infty\right). \tag{1$_c$}$$

All conditions of the problem (U) must be satisfied in the classical sense.

Remark. By $C^0(\overline{\mathcal{D}\backslash\Gamma^1})$ we denote a class of functions, which are continuously extended on cuts Γ^1 from the left and right, but their values on Γ^1 from the left and right can be different, so that the functions may have a jump on Γ^1.

If $N_1 = 0$ and cuts Γ^1 are absent, then problem (U) transforms to the classical Dirichlet problem in a domain $\mathcal{D}$ without cuts.

On the basis of the behaviour of harmonic functions in external domains [9: Subsection 26.1] and the maximum principle we can readily prove the following assertion.

Theorem 1. *Problem* (U) *has at most one solution.*

3. Integral equations at the boundary

Below we assume that the functions $F^+ = F^+(s)$, $F^- = F^-(s)$ and $F = F(s)$ in $(1)_b$ are subject to the following conditions:

$$F^+, F^- \in C^{1,\lambda}(\Gamma^1), \quad F \in C^0(\Gamma^2) \qquad (\lambda \in (0,1]) \tag{2$_a$}$$

$$F^+(a_n^1) = F^-(a_n^1), \quad F^+(b_n^1) = F^-(b_n^1) \qquad (n = 1, ..., N_1). \tag{2$_b$}$$

If $\mathcal{B}_1(\Gamma^1)$ and $\mathcal{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2, then by $\mathcal{B}_1(\Gamma^1) \cap \mathcal{B}_2(\Gamma^2)$ we denote the Banach space of functions $\mathcal{F} = \mathcal{F}(s)$, which are defined on $\Gamma = \Gamma^1 \cup \Gamma^2$ and such that $\mathcal{F}|_{\Gamma^m} \in \mathcal{B}_m(\Gamma^m)$, where $m = 1, 2$. The Banach space $\mathcal{B}_1(\Gamma^1) \cap \mathcal{B}_2(\Gamma^2)$ is endowed with the norm

$$\|\cdot\|_{\mathcal{B}_1(\Gamma^1)\cap\mathcal{B}_2(\Gamma^2)} = \|\cdot\|_{\mathcal{B}_1(\Gamma^1)} + \|\cdot\|_{\mathcal{B}_2(\Gamma^2)}.$$

An example of such a Banach space is $C^0(\Gamma) = C^0(\Gamma^1) \cap C^0(\Gamma^2)$.

We say that the function $u = u(x)$ belongs to the *smoothness class* **K** if the following conditions are fulfilled:

1) $u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}) \cap C^2(\mathcal{D}\backslash\Gamma^1)$.

2) $\nabla u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}\backslash\Gamma^2\backslash X)$, where X is a point set, consisting of the end-points of Γ^1:

$$X = \bigcup_{n=1}^{N_1} \left(x(a_n^1) \cup x(b_n^1)\right).$$

3) In the neighbourhood of any point $x(d) \in X$, for some constants $C > 0$, $\epsilon > -1$ the inequality

$$|\nabla u| \leq C|x - x(d)|^{\epsilon} \tag{3}$$

holds, where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$ $(n = 1, ..., N_1)$.

We shall construct the solution of problem (U) from the smoothness class **K** with the help of potential theory for the Laplace equation $(1)_a$.

By $\int_{\Gamma^k} ... \, d\sigma$ we mean

$$\sum_{n=1}^{N_k} \int_{a_n^k}^{b_n^k} ... \, d\sigma.$$

We consider an angular potential [1] for equation $(1)_a$:

$$v_1[\nu](x) = -\frac{1}{2\pi} \int_{\Gamma^1} \nu(\sigma) V(x, y(\sigma)) \, d\sigma. \tag{4}$$

The kernel $V(x, y(\sigma))$ is defined (up to indeterminacy $2\pi m$, $m = \pm 1, \pm 2, ...$) by the formulae

$$\cos V(x, y(\sigma)) = \frac{x_1 - y_1(\sigma)}{|x - y(\sigma)|}, \qquad \sin V(x, y(\sigma)) = \frac{x_2 - y_2(\sigma)}{|x - y(\sigma)|},$$

where

$$y = y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma^1, \quad |x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}.$$

One can see that $V(x, y(\sigma))$ is the angle between the vector $\overrightarrow{y(\sigma)x}$ and the direction of the Ox_1-axis. More precisely, $V(x, y(\sigma))$ is a many-valued harmonic function of x connected with $\ln|x - y(\sigma)|$ by the Cauchy-Riemann relations.

Below by $V(x, y(\sigma))$ we denote an arbitrary fixed branch of this function, which varies continously with σ along each curve Γ_n^1 $(n = 1, ..., N_1)$ for given fixed $x \notin \Gamma^1$.

Under this definition of $V(x, y(\sigma))$, the potential $v_1[\nu](x)$ is a many-valued function. In order that the potential $v_1[\nu](x)$ be single-valued, it is necessary to impose the following additional conditions:

$$\int_{a_n^1}^{b_n^1} \nu(\sigma) \, d\sigma = 0 \qquad (n = 1, ..., N_1). \tag{5}$$

Below we suppose that the density $\nu = \nu(s)$ belongs to $C^{0,\lambda}(\Gamma^1)$ and satisfies conditions (5). As shown in [1, 3], for such $\nu = \nu(s)$ the angular potential $v_1[\nu](x)$ belongs to the class $\mathbf{K}$. In particular, condition (3) is fulfilled for any $\epsilon \in (0,1)$. Moreover, integrating $v_1[\nu](x)$ by parts and using (5), we express the angular potential in terms of a double layer potential

$$v_1[\nu](x) = \frac{1}{2\pi}\int_{\Gamma^1} \rho(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x - y(\sigma)|\,d\sigma \tag{6}$$

with the density

$$\rho(\sigma) = \int\limits_{a_n^1}^{\sigma} \nu(\xi)\,d\xi \qquad (\sigma \in [a_n^1, b_n^1];\ n = 1,\dots,N_1). \tag{7}$$

Consequently, $v_1[\nu](x)$ satisfies both equation $(1)_a$ outside Γ^1 and the condition at infinity $(1)_c$.

Let us construct a solution of problem (U). We seek a solution of the problem in the form

$$u[\nu,\mu](x) = v_1[\nu](x) + w[\mu](x) \tag{8}$$

where $v_1[\nu](x)$ is given by (4), (6) and

$$w[\mu](x) = w_1[\mu](x) + w_2[\mu](x) + h[\mu](x). \tag{9}$$

Here

$$w_1[\mu](x) = -\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\ln|x - y(\sigma)|d\sigma$$

$$w_2[\mu](x) = -\frac{1}{2\pi}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x - y(\sigma)|d\sigma.$$

By $h[\mu](x)$ we denote the sum of point sources placed at the fixed points Y_k lying inside Γ_k^2 $(k = 1,\dots,N_2)$ and a constant:

$$h[\mu](x) = -\frac{1}{2\pi}\sum_{k=2}^{N_2}\int_{\Gamma_k^2}\mu(\sigma)\,d\sigma\ \ln|x - Y_k|$$
$$+\frac{1}{2\pi}\left[\int_{\Gamma}\mu(\sigma)\,d\sigma - \int_{\Gamma_1^2}\mu(\sigma)\,d\sigma\right]\ln|x - Y_1| + \int_{\Gamma}\mu(\sigma)\,d\sigma$$

where $Y_k \in \mathcal{D}_k$ $(k = 1,\dots,N_2)$. Clearly, $h[\mu](x)$ obeys equation $(1)_a$ and belongs to

$$C^\infty\left(\mathbb{R}^2\setminus\bigcup_{k=1}^{N_2} Y_k\right).$$

Besides, if $x(s) \in \Gamma$, then $h[\mu](x(\cdot)) \in C^{1,\lambda}(\Gamma)$.

As noted above, we will look for the density $\nu = \nu(s)$ satisfying conditions (5) and belonging to $C^{0,\lambda}(\Gamma^1)$.

We will seek $\mu = \mu(s)$ in the Banach space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$ $(\omega \in (0,1],\ q \in [0,1))$ with the norm $\|\cdot\|_{C_q^{\omega}(\Gamma^1)\cap C^0(\Gamma^2)} = \|\cdot\|_{C_q^{\omega}(\Gamma^1)} + \|\cdot\|_{C^0(\Gamma^2)}$. We say that $\mu \in C_q^{\omega}(\Gamma^1)$ if the funtion $\theta = \theta(s)$ defined by

$$\theta(s) = \mu(s) \prod_{n=1}^{N_1} |s - a_n^1|^q |s - b_n^1|^q$$

belongs to the Hölder space $C^{0,\omega}(\Gamma^1)$ with the exponent ω and

$$\|\mu\|_{C_q^{\omega}(\Gamma^1)} = \|\theta\|_{C^{0,\omega}(\Gamma^1)} .$$

It can be checked directly with the help of [3, 8] that for such $\mu = \mu(s)$ the function $w_1[\mu](x)$ obeys equation $(1)_a$ and is of class $\mathbf{K}$. In particular, inequality (3) holds with $\epsilon = -q$ if $q \in (0,1)$. The potential $w_2[\mu](x)$ satisfies equation $(1)_a$ and belongs to $C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$. Consequently, $w_2[\mu](x)$ belongs to the class $\mathbf{K}$. The function $h[\mu](x)$ is constructed in such a way that $w[\mu](x)$ meets at infinity condition $(1)_c$, because according to our assumptions $N_2 \geq 1$.

To satisfy the boundary conditions, we put (8) into $(1)_b$ and arrive at the system of integral equations for the densities $\mu(s)$, $\nu(s)$:

$$\begin{aligned} &\pm \frac{1}{2}\rho(s) - \frac{1}{2\pi}\int_{\Gamma^1} \nu(\sigma) V\left(x(s), y(\sigma)\right) d\sigma \\ &- \frac{1}{2\pi}\int_{\Gamma^1} \mu(\sigma) \ln|x(s) - y(\sigma)|\, d\sigma \qquad\qquad (s \in \Gamma^1) \qquad (10)_a \\ &- \frac{1}{2\pi}\int_{\Gamma^2} \mu(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x(s) - y(\sigma)|\, d\sigma + h[\mu](x(s)) = F^{\pm}(s) \end{aligned}$$

and

$$\begin{aligned} &- \frac{1}{2\pi}\int_{\Gamma^1} \nu(\sigma) V(x(s), y(\sigma)) d\sigma \\ &- \frac{1}{2\pi}\int_{\Gamma^1} \mu(\sigma) \ln|x(s) - y(\sigma)| d\sigma + \frac{1}{2}\mu(s) \qquad\qquad (s \in \Gamma^2) \qquad (10)_b \\ &- \frac{1}{2\pi}\int_{\Gamma^2} \mu(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x(s) - y(\sigma)| d\sigma + h[\mu](x(s)) = F(s) \end{aligned}$$

where $\rho(s)$ is defined in terms of $\nu(s)$ in (7). The kernels of the second integral term in $(10)_a$ and the third integral term in $(10)_b$ have a weak singularity as $s = \sigma$.

To derive limit formulas for the angular potential, we used its expression in the form of a double layer potential (6).

Equation $(10)_a$ is obtained as $x \to x(s) \in (\Gamma^1)^{\pm}$ and comprises two integral equations. The upper sign denotes the integral equation on $(\Gamma^1)^+$, the lower sign denotes the integral equation on $(\Gamma^1)^-$.

In addition to the integral equations (10) we have conditions (5).

Subtracting the integral equations $(10)_a$ and using (7), we find

$$\rho(s) = F^+(s) - F^-(s), \quad \nu(s) = F'^+(s) - F'^-(s), \quad F'^\pm(s) = \frac{d}{ds}F^\pm(s), \tag{11}$$

so that

$$\rho \in C^{1,\lambda}(\Gamma^1), \quad \nu \in C^{0,\lambda}(\Gamma^1).$$

Clearly, $\nu = \nu(s)$ is found completely and satisfies all required conditions, in particular, (5). Hence, the angular potential (4), (6) is found completely as well.

We introduce the function $f = f(s)$ on Γ by the formula

$$f(s) = F(s) + \frac{1}{2\pi}\int_{\Gamma^1} \left(F'^+(\sigma) - F'^-(\sigma)\right) V\left(x(s), y(\sigma)\right) d\sigma \qquad (s \in \Gamma) \tag{12}$$

where $F = F(s)$ is a function defined on Γ, so that $F(s)$ on Γ^2 is specified in $(1)_b$, while $F(s)$ on Γ^1 is specified by the relationship

$$F(s) = \frac{1}{2}\left(F^+(s) + F^-(s)\right) \qquad (s \in \Gamma^1).$$

As shown in [3], $f \in C^{1,\lambda}(\Gamma^1)$. Consequently, $f \in C^{1,\lambda}(\Gamma^1) \cap C^0(\Gamma^2)$.

Adding the integral equations $(10)_a$ and taking into account $(10)_b$, we obtain the integral equation for $\mu(s)$ on Γ

$$\begin{aligned} w[\mu](x(s))|_\Gamma &= -\frac{1}{2\pi}\int_{\Gamma^1} \mu(\sigma)\ln|x(s) - y(\sigma)|\, d\sigma + \frac{1}{2}\delta(\Gamma^2, s)\mu(s) \\ &\quad - \frac{1}{2\pi}\int_{\Gamma^2} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x(s) - y(\sigma)|\, d\sigma + h[\mu](x(s)) \qquad (s \in \Gamma) \\ &= f(s) \end{aligned} \tag{13}$$

where $f(s)$ is given in (12), and the limit values of the function (9) as $x \to x(s) \in \Gamma$ $(x \in \mathcal{D})$ are denoted by $w[\mu](x(s))|_\Gamma$. Furthermore,

$$\delta(\Gamma^2, s) = \begin{cases} 0 & \text{if } s \notin \Gamma^2 \\ 1 & \text{if } s \in \Gamma^2. \end{cases}$$

Thus, if $\mu = \mu(s)$ is a solution of equation (13) from the space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$ $(\omega \in (0,1],\ q \in [0,1))$, then the potential (8) with $\nu(s)$ from (11) satisfies all conditions of problem (U) and belongs to the class $\mathbf{K}$. The following theorem holds.

Theorem 2. *Let $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ and conditions* (2) *hold. If equation* (13) *has a solution $\mu = \mu(s)$ from the Banach space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$ for some $\omega \in (0,1]$ and $q \in [0,1)$, then a solution of problem* (U) *exists, belongs to the class* $\mathbf{K}$ *and is given by* (8), *where $\nu = \nu(s)$ is defined in* (11).

If $s \in \Gamma^2$, then (13) is an equation of the second kind. If $s \in \Gamma^1$, then (13) is an equation of the first kind, and its kernel has a logarithmic singularity. Our further treatment will be aimed to the proof of the solvability of (13) in the Banach space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$. Moreover, we reduce (13) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

By differentiating (13) on Γ^1, we reduce it to the Cauchy singular integral equation on Γ^1:

$$\begin{aligned}\frac{\partial}{\partial s} w[\mu](x(s)) &= \frac{1}{2\pi} \int_{\Gamma^1} \mu(\sigma) \frac{\sin \varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} d\sigma \\ &\quad - \frac{1}{2\pi} \int_{\Gamma^2} \mu(\sigma) \frac{\partial}{\partial s} \frac{\partial}{\partial \mathbf{n}_y} \ln |x(s) - y(\sigma)| d\sigma \qquad (s \in \Gamma^1) \qquad (14) \\ &\quad + \frac{\partial}{\partial s} h[\mu](x(s)) \\ &= f'(s)\end{aligned}$$

where $\varphi_0(x, y)$ is the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x, y)$ is taken to be positive if it is measured anticlockwise from $\mathbf{n}_x$, and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x, y)$ is continuous in $x, y \in \Gamma$ if $x \neq y$. Note, that for $x(s), y \in \Gamma$ and $x \neq y$ we have the relationships

$$\begin{aligned}\frac{\partial}{\partial s} \ln |x(s) - y| &= \frac{\partial}{\partial \tau_x} \ln |x - y| = -\frac{\partial}{\partial \mathbf{n}_x} V(x, y) \\ &= -\frac{\sin \varphi_0(x, y)}{|x - y|} = \frac{\cos (V(x(s), y) - \alpha(s))}{|x(s) - y|}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial s} V(x(s), y) &= \frac{\partial}{\partial \tau_x} V(x, y) = \frac{\partial}{\partial \mathbf{n}_x} \ln |x - y| \\ &= -\frac{\cos \varphi_0(x, y)}{|x - y|} = -\frac{\sin (V(x(s), y) - \alpha(s))}{|x(s) - y|},\end{aligned}$$

where $\alpha(s)$ is the inclination of the tangent τ_x to the Ox_1-axis, and $V(x, y)$ is the kernel of the angular potential from (4).

Equation (13) on Γ^2 we rewrite in the form

$$\mu(s) + \int_\Gamma \mu(\sigma) A_2(s, \sigma)\, d\sigma = 2f(s) \qquad (s \in \Gamma^2), \qquad (15)$$

where

$$\begin{aligned}A_2(s, \sigma) = \Big\{ &-\frac{1}{\pi} \left(1 - \delta(\Gamma^2, \sigma)\right) \ln |x(s) - y(\sigma)| - \frac{1}{\pi} \delta(\Gamma^2, \sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln |x(s) - y(\sigma)| \\ &- \frac{1}{\pi} \sum_{k=2}^{N_2} \delta(\Gamma_k^2, \sigma) \ln |x(s) - Y_k| + \frac{1}{\pi} \left(1 - \delta(\Gamma_1^2, \sigma)\right) \ln |x(s) - Y_1| + 2 \Big\}.\end{aligned}$$

The function $\delta(\Gamma^2, \sigma)$ was introduced in (13), and

$$\delta(\Gamma_k^2, \sigma) = \begin{cases} 0 & \text{if } \sigma \notin \Gamma_k^2 \\ 1 & \text{if } \sigma \in \Gamma_k^2 \end{cases} \qquad (k = 1, ..., N_2).$$

The kernel $A_2(s, \sigma)$ has a weak singularity if $s = \sigma \in \Gamma^2$. Consequently, the integral operator from (15) is a compact operator mapping $C^0(\Gamma)$ into $C^0(\Gamma^2)$.

Remark. Evidently, $f(a_n^2) = f(b_n^2)$ and $A_2(a_n^2, \sigma) = A_2(b_n^2, \sigma)$ for $\sigma \in \Gamma$, $\sigma \neq a_n^2, b_n^2$ $(n = 1, ..., N_2)$. Hence, if $\mu = \mu(s)$ is a solution of equation (15) from

$$C^0 \left(\bigcup_{n=1}^{N_2} [a_n^2, b_n^2] \right),$$

then according to the equality (15), $\mu(s)$ automatically satisfies the matching conditions $\mu(a_n^2) = \mu(b_n^2)$ for $n = 1, ..., N_2$ and, therefore, belongs to $C^0(\Gamma^2)$. This observation is true for equation (13) also and can be helpful in finding numerical solutions, since we may refuse from the matching conditions $\mu(a_n^2) = \mu(b_n^2)$ $(n = 1, ..., N_2)$, which are fulfilled automatically.

We note that equation (14) is equivalent to (13) on Γ^1 if and only if (14) is accompanied by the additional conditions

$$w[\mu](x(a_n^1)) = f(a_n^1) \qquad (n = 1, ..., N_1). \tag{16}$$

The system (14) - (16) is equivalent to the equation (13).

It can be easily proved that

$$\frac{\sin \varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - s} \in C^{0,\lambda}(\Gamma^1 \times \Gamma^1)$$

(see [3, 8] for details). Therefore, we can rewrite (14) in the form

$$2\frac{\partial}{\partial s} w[\mu](x(s)) = \frac{1}{\pi} \int_{\Gamma^1} \mu(\sigma) \frac{d\sigma}{\sigma - s} + \int_{\Gamma} \mu(\sigma) M(s, \sigma)\, d\sigma = 2f'(s) \quad (s \in \Gamma^1) \tag{17}$$

where $f'(s) = \frac{d}{ds} f(s)$ belongs to $C^{0,\lambda}(\Gamma^1)$ and

$$\begin{aligned} M(s, \sigma) = \frac{1}{\pi} \Big\{ & \left(1 - \delta(\Gamma^2, \sigma)\right) \left[\frac{\sin \varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|} - \frac{1}{\sigma - s} \right] \\ & - \delta(\Gamma^2, \sigma) \frac{\partial}{\partial s} \left[\frac{\partial}{\partial \mathbf{n}_y} \ln |x(s) - y(\sigma)| + \sum_{k=2}^{N_2} \delta(\Gamma_k^2, \sigma) \ln |x(s) - Y_k| \right] \\ & + \left(1 - \delta(\Gamma_1^2, \sigma)\right) \frac{\partial}{\partial s} \ln |x(s) - Y_1| \Big\} \in C^{0,\lambda}(\Gamma^1 \times \Gamma). \end{aligned}$$

4. The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator in (17), we arrive at the following integral equation of second kind [8]:

$$\mu(s) + \frac{1}{Q_1(s)} \int_\Gamma \mu(\sigma) A_1(s,\sigma)\, d\sigma + \frac{1}{Q_1(s)} \sum_{n=0}^{N_1-1} G_n s^n = \frac{1}{Q_1(s)} \Phi_1(s) \quad (s \in \Gamma^1) \quad (18)$$

where

$$Q_1(s) = \prod_{n=1}^{N_1} \left| \sqrt{s - a_n^1} \sqrt{b_n^1 - s} \right| \mathrm{sign}(s - a_n^1)\ ,$$

$$A_1(s,\sigma) = -\frac{1}{\pi} \int_{\Gamma^1} \frac{M(\xi,\sigma)}{\xi - s} Q_1(\xi)\, d\xi\ , \qquad \Phi_1(s) = -\frac{1}{\pi} \int_{\Gamma^1} \frac{2Q_1(\sigma) f'(\sigma)}{\sigma - s}\, d\sigma\ ,$$

and $G_0, ..., G_{N_1-1}$ are arbitrary constants. It can be shown using properties of singular integrals [8] that $\Phi_1 = \Phi_1(s)$ and $A_1 = A_1(s,\sigma)$ are Hölder functions on Γ^1 and $\Gamma^1 \times \Gamma$ respectively. Consequently, any solution of (18) belongs to $C^{\omega}_{1/2}(\Gamma^1)$ and below we look for $\mu = \mu(s)$ on Γ^1 in this space.

We put

$$Q(s) = \left(1 - \delta(\Gamma^2, s)\right) Q_1(s) + \delta(\Gamma^2, s) \qquad (s \in \Gamma).$$

Instead of $\mu \in C^{\omega}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ we introduce the new unknown function $\mu_* = \mu_*(s)$, so that $\mu_*(s) = \mu(s) Q(s)$ and $\mu_* \in C^{0,\omega}(\Gamma^1) \cap C^0(\Gamma^2)$. Now we rewrite (18), (15) in the form of one equation in terms of $\mu_*(s)$:

$$\mu_*(s) + \int_\Gamma \mu_*(\sigma) Q^{-1}(\sigma) A(s,\sigma)\, d\sigma + \left(1 - \delta(\Gamma^2, s)\right) \sum_{n=0}^{N_1-1} G_n s^n = \Phi(s) \quad (s \in \Gamma) \quad (19)$$

where

$$A(s,\sigma) = \left(1 - \delta(\Gamma^2, s)\right) A_1(s,\sigma) + \delta(\Gamma^2, s) A_2(s,\sigma)$$

$$\Phi(s) = \left(1 - \delta(\Gamma^2, s)\right) \Phi_1(s) + 2\delta(\Gamma^2, s) f(s).$$

To derive equations for $G_0, ..., G_{N_1-1}$, we substitute $\mu(s)$ from (18), (15) in the conditions (16). Then in terms of $\mu_*(s)$ we obtain

$$\int_\Gamma Q^{-1}(\xi) \mu_*(\xi) l_n(\xi)\, d\xi + \sum_{m=0}^{N_1-1} B_{nm} G_m = H_n \qquad (n = 1, ..., N_1) \qquad (20)$$

where

$$l_n(\xi) = -w\left[Q^{-1}(\cdot) A(\cdot,\xi)\right](a_n^1)\ , \qquad H_n = -w\left[Q^{-1}(\cdot)\Phi(\cdot)\right](a_n^1) + f(a_n^1)\ ,$$

$$B_{nm} = -w\left[Q^{-1}(\cdot)\left(1 - \delta(\Gamma^2, \cdot)\right)(\cdot)^m\right](a_n^1). \qquad (21)$$

By $\cdot$ we denote the variable of integration in the potential (9).

Thus, system of equations (14) - (16) for $\mu(s)$ has been reduced to system (19), (20) for the function $\mu_*(s)$ and the constants $G_0, ..., G_{N_1-1}$. It is clear from our consideration that any solution of system (19), (20) gives a solution of system (14) - (16).

As noted above, $\Phi_1 = \Phi_1(s)$ and $A_1 = A_1(s,\sigma)$ are Hölder functions on Γ^1 and $\Gamma^1 \times \Gamma$ respectively. More precisely (see [8]), $\Phi_1 \in C^{0,p}(\Gamma^1)$ $(p = \min\{\frac{1}{2}, \lambda\})$ and $A_1(\cdot,\sigma) \in C^{0,p}(\Gamma^1)$ uniformly with respect to $\sigma \in \Gamma$. We arrive at the following assertion.

Lemma. *Let* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ $(\lambda \in (0,1])$, *and* $\Phi = \Phi(s)$ *is such that* $\Phi \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, *where* $p = \min\{\lambda, \frac{1}{2}\}$. *If* $\mu_* = \mu_*(s)$ *from* $C^0(\Gamma)$ *satisfies equation* (19), *then* $\mu_* \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.

The condition $\Phi \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ holds if conditions (2) hold. Hence below we will seek $\mu_* = \mu_*(s)$ in $C^0(\Gamma)$.

It was noted above that the integral operator from (15) with the kernel $A_2(s,\sigma)$ is compact from $C^0(\Gamma)$ into $C^0(\Gamma^2)$. Since $A_1 = A_1(s,\sigma)$ belongs to $C^0(\Gamma^1 \times \Gamma)$, the integral operator from (19)

$$\mathbf{A}\mu_* = \int_\Gamma \mu_*(\sigma) Q^{-1}(\sigma) A(s,\sigma)\, d\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself. We rewrite (19) in the operator form

$$(I + \mathbf{A})\mu_* + PG = \Phi \tag{22}$$

where P is the operator multiplying the row $P = \left(1 - \delta(\Gamma^2, s)\right)\left(s^0, ..., s^{N_1-1}\right)$ by the column $G = (G_0, ..., G_{N_1-1})^T$. The operator P is finite-dimensional from E_{N_1} into $C^0(\Gamma)$ and therefore compact.

Now we rewrite equations (20) in the form

$$I_{N_1} G + L\mu_* + (B - I_{N_1})G = H \tag{23}$$

where $H = (H_1, ..., H_{N_1})^T$ is a column of N_1 elements, I_{N_1} is the identity operator in E_{N_1}, and B is an $(N_1 \times N_1)$-matrix consisting of the elements B_{nm} from (21). The operator L acts from $C^0(\Gamma)$ into E_{N_1}, so that $L\mu_* = (L_1\mu_*, ..., L_{N_1}\mu_*)^T$, where

$$L_n \mu_* = \int_\Gamma Q^{-1}(\xi)\mu_*(\xi) l_n(\xi)\, d\xi.$$

The operators $(B - I_{N_1})$ and L are finite-dimensional and therefore compact.

We consider the columns $\tilde{\mu} = \begin{pmatrix}\mu_* \\ G\end{pmatrix}$, $\tilde{\Phi} = \begin{pmatrix}\Phi \\ H\end{pmatrix}$ in the Banach space $C^0(\Gamma) \times E_{N_1}$ with the norm $\|\tilde{\mu}\|_{C^0(\Gamma)\times E_{N_1}} = \|\mu_*\|_{C^0(\Gamma)} + \|G\|_{E_{N_1}}$.

We write system (22), (23) in the form of one equation

$$(\mathbf{I} + \mathbf{R})\tilde{\mu} = \tilde{\Phi}\,, \qquad \mathbf{R} = \begin{pmatrix} \mathbf{A} & P \\ L & B - I_{N_1} \end{pmatrix} \tag{24}$$

where $\mathbf{I}$ is the identity operator in the space $C^0(\Gamma) \times E_{N_1}$.

It is clear that $\mathbf{R}$ is a compact operator mapping $C^0(\Gamma) \times E_{N_1}$ into itself. Therefore, (24) is a Fredholm equation in this space.

Let us show that the homogeneous equation (24) has only a trivial solution. Then, according to Fredholm's theorems, the inhomogeneous equation (24) has a unique solution for any right-hand side. We will prove this by a contradiction. Let $\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$ be a non-trivial solution of the homogeneous equation (24). According to the Lemma, $\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1}$ where $p = \min\{\lambda, \frac{1}{2}\}$. Therefore, the function $\mu^0(s) = \mu_*^0(s)Q^{-1}(s)$ belonging to $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ and the column G^0 convert the homogeneous equations (18), (15), (20) into identities. For instance, (15) takes the form

$$\lim_{x \to x(s) \in \Gamma^2} w[\mu^0](x) = 0 \qquad (x \in \mathcal{D}). \tag{25$_a$}$$

Using the homogeneous identities (18), (15), we check that the homogeneous identities (20) are equivalent to

$$w\left[\mu^0\right](a_n^1) = 0 \qquad (n = 1, ..., N_1). \tag{25$_b$}$$

Besides, acting on the homogeneous identity (18) with a singular operator with the kernel $(s-t)^{-1}$, we find that $\mu^0(s)$ satisfies the homogeneous equation (17):

$$\frac{\partial}{\partial s} w\left[\mu^0\right](x(s))\Big|_{\Gamma^1} = 0. \tag{25$_c$}$$

It follows from (25) that $\mu^0(s)$ satisfies the homogeneous equation (13). On the basis of Theorem 2, $u[0,\mu^0](x) = w[\mu^0](x)$ is a solution of the homogeneous problem (U). According to Theorem 1, $w[\mu^0](x) \equiv 0$ on $\mathcal{D}\backslash\Gamma^1$. Using the limit formulas for normal derivatives of a single-layer potential on Γ^1, we have

$$\lim_{x \to x(s) \in (\Gamma^1)^+} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) - \lim_{x \to x(s) \in (\Gamma^1)^-} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) = \mu^0(s) \equiv 0 \qquad (s \in \Gamma^1).$$

Hence, $w[\mu^0](x) = w_2[\mu^0](x) + h_2[\mu^0](x) \equiv 0$ for $x \in \mathcal{D}$, where

$$\begin{aligned} h_2[\mu^0](x) = &-\frac{1}{2\pi}\sum_{k=2}^{N_2}\int_{\Gamma_k^2} \mu(\sigma)\,d\sigma\ \ln|x - Y_k| \\ &+ \frac{1}{2\pi}\left[\int_{\Gamma^2} \mu(\sigma)\,d\sigma - \int_{\Gamma_1^2} \mu(\sigma)\,d\sigma\right] \ln|x - Y_1| + \int_{\Gamma^2} \mu(\sigma)\,d\sigma \end{aligned} \tag{26}$$

and $\mu^0(s)$ satisfies $(25)_a$, which can be written as

$$\frac{1}{2}\mu^0(s) - \frac{1}{2\pi}\int_{\Gamma^2} \mu^0(\sigma)\frac{\partial}{\partial \mathbf{n}_y} \ln|x(s) - y(\sigma)|\,d\sigma + h_2[\mu^0](x(s)) = 0 \quad (s \in \Gamma^2). \tag{27}$$

The Fredholm equation (27) arises when solving the Dirichlet problem for the Laplace equation $(1)_a$ in the domain $\mathcal{D}$ by the double layer potential with the sum

This inequality and (26) imply that

$$\|A(\sigma_1)\dots A(\sigma_n)\xi\|_{\mathcal{R}(C,\mu,L'')} \le \theta_n(L',L'')\ \|\xi\|_{\mathcal{R}(C,\mu,L')} \qquad (n \ge 1). \tag{35}$$

Further, by (28),

$$\|A(\sigma_1)\cdots A(\sigma_n)\xi\| \le \theta_n(L)\,\|\xi\|_{\mathcal{R}(C,\mu,L)} \qquad (n \ge 1). \tag{36}$$

The last inequality implies that the series in the right-hand side of (23) for $\xi \in \mathcal{R}(C,\mu,L)$ is absolutely and uniformly convergent in the norm of the space $\mathbb{X}$ on each interval $(\tau - h, \tau + h)$ $(h < h(L))$ where $h(L)$ is the radius of convrgence of the series (30). Thus, equation (23) for $|t-\tau| < h(L)$ defines a function $x(t) = U(t,\tau)\xi$. Repeating the reasoning in the proof of Theorem 1 one can see that $x(t) = U(t,\tau)\xi$ is a solution of the Cauchy problem (2)/(24) on the interval $\big(\tau - h(L), \tau + h(L)\big)$. Moreover, the estimate (31) is proved.

Let $L' < L''$ and $\xi \in \mathcal{R}(C,\mu,L')$. Then statement b) and inequality (32) follow from the chain of inequalities

$$\begin{aligned}
&\|U(t,\tau)\xi\|_{R(C,\mu,L'')}\\
&\quad\le \|\xi\|_{R(C,\mu,L'')} + \sum_{n=0}^{\infty} \int_{\Delta_n(\tau,t)} \|A(\sigma_1)\cdots A(\sigma_n)\xi\|_{\mathcal{R}(C,\mu,L'')}\, d\sigma_n\cdots d\sigma_1\\
&\quad\le \|\xi\|_{\mathcal{R}(C,\mu,L')} + \sum_{n=0}^{\infty}(n!)^{-1}\theta_n(L',L'')h^n\|\xi\|_{\mathcal{R}(C,\mu,L')}\\
&\quad\le w(\mu,L',L'',h)\ \|\xi\|_{\mathcal{R}(C,\mu,L')}
\end{aligned}$$

for $|t-\tau| \le h$, with $h < h(L',L'')$. In order to prove the semigroup property $U(t,s)\cdot U(s,\tau) = U(t,\tau)$ one can see that the left-hand and right-hand sides of this equality are operators which act from $\mathcal{R}(C,\mu,L')$ into $\mathcal{R}(C,\mu,L'')$ under the hypotheses of Theorem 3. Moreover, the formal composition of series

$$U(t,s) = I + \sum_{j=0}^{\infty} \int_{\Delta_j(s,t)} A(\varphi_1)\cdots A(\varphi_j)\, d\varphi_j\cdots d\varphi_1$$

$$U(s,\tau) = I + \sum_{k=0}^{\infty} \int_{\Delta_k(\tau,s)} A(\psi_1)\cdots A(\psi_k)\, d\psi_k\cdots d\psi_1$$

can be written (after an evident substitution) in the form

$$\begin{aligned}
&\sum_{j,k=0}^{\infty} \int_{\Delta_j(s,t)} \int_{\Delta_k(\tau,s)} A(\varphi_1)\cdots A(\varphi_j)\, A(\psi_1)\cdots A(\psi_k)\xi\, d\varphi_j\cdots d\varphi_1\, d\psi_k\cdots d\psi_1\\
&\quad= \sum_{n=0}^{\infty} \int_{\Delta_n(t,\tau)} A(\sigma_1)\cdots A(\sigma_n)\xi\, d\sigma_n\cdots d\sigma_1\\
&\quad= U(t,\tau).
\end{aligned}$$

To justify the formal composition it is sufficient to verify absolute convergence of the left-hand side of the latter equation; however, this is a consequence of the evident chain of inequalities

$$\begin{aligned}
&\sum_{j,k=0}^{\infty} \int_{\Delta_j(s,t)} \int_{\Delta_k(\tau,s)} \|A(\varphi_1)\cdots A(\varphi_j)\,A(\psi_1)\cdots A(\psi_k)\xi\|\,d\varphi_j\cdots d\varphi_1\,d\psi_k\cdots d\psi_1 \\
&\le \left(\sum_{j,k=0}^{\infty}(j!)^{-1}(k!)^{-1}\theta_j(L',L)\theta_k(L,L'')h(L',L)^j h(L,L'')^k\right)\|\xi\|_{\mathcal{R}(C,\mu,L')} \\
&= \left(\sum_{j=0}^{\infty}(j!)^{-1}\theta_j(L',L)h(L',L)^j\right)\left(\sum_{k=0}^{\infty}(k!)^{-1}\theta^k(L,L'')h(L,L'')^k\right)\|\xi\|_{\mathcal{R}(C,\mu,L')} \\
&= w(\mu,L',L,h_1)w(\mu,L,L'',h_2))\,\|\xi\|_{\mathcal{R}(C,\mu,L')}
\end{aligned}$$

for $|\tau - s| \le h_1$ and $|t-s| \le h_2$, with $h_1 < h(L',L)$ and $h_2 < h(L,L''))$. Thus, the statement of Theorem 3 is proved ∎

One can see that the application of Theorem 3 can give non-trivial results only if the Roumieu spaces $\mathcal{R}(C,\mu,L)$ $(0 < L < \infty)$ are sufficiently "rich", at least, dense in the original space $\mathbb{X}$. Thus, we need different density results for the Roumieu spaces as well as for the Gevrey and Beurling spaces. In particular, in applying Theorem 3 Propositions 1 - 6 are useful.

The conditions of Theorem 3 are rather cumbersome and tedious to verify. However, simple examples of linear partial differential equations show that they are sufficiently natural. Moreover, one can see that the calculation of the values $h(L)$, $h(L',L'')$ and $w(\mu,L',L'',h)$ is standard; in particular, one can consider the special cases from [2, 22, 23].

The case considered in [5] is more difficult. Condition (25) in this case can be written in the form

$$\|C^k A(t)\xi\| \le \sum_{j=1}^{k+1} \frac{k!}{(j-1)!}\Lambda^{k-j}\,\|C^j\xi\| \qquad \left(t\in\mathbb{I},\ \xi\in\bigcap_{j=1}^{k+1}\mathcal{D}(C^j),\ k\ge 0\right).$$

Simple calculations show that $\theta(L',L'') = \infty$ for $L'' < 1$; in the case $L'' \ge L > 1$ the inequality

$$\theta(L',L'') \le \frac{c(L)}{(L')^{-1} - (L'')^{-1}}$$

holds. Applying Theorem 3 in this case allows us not only to get existence of solutions to the Cauchy problem on the corresponding interval, but also to define the Roumieu space in which the corresponding solutions lie.

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