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The Hausdorff Nearest Circle to a Convex Compact Set in the Plane

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Abstract. The problem of finding the nearest in the Hausdorff metric circle to a non-empty convex compact set T in the plane is considered from geometrical point of view. The consideration is based on the equivalence of this problem with the Chebyshevian best approximation of 2π-periodic functions by trigonometric polynomials of first order, whence it follows that the Hausdorff nearest circle to a convex compact set in the plane exists and is unique. It can be characterized by a geometric Chebyshevian alternance. As a consequence, in the particular case of a polygon the centre of the circle is described as an intersection of a midline between some two vertices and a bisectrix of some two sides. In the general case, geometrical algorithms corresponding to the one and the four point exchange Remez algorithms are described. They assure correspondingly linear and superlinear convergence. Following the idea, in the case of a polygon to get the exact solution in finite number of steps, a modified two-point exchange algorithm is suggested and illustrated by a numerical example. An application is given to estimate the Hausdorff distance between an arbitrary convex set and its Hausdorff nearest circle. The considered problem arises as a practical problem by measuring and pattern recognition in the production of circular machine parts.

Keywords: *Convex sets in two dimensions, geometric construction of best approximation, Hausdorff metric, approximation by circles*

AMS subject classification: 52 A 10, 52 A 27

1. Introduction

Throughout this paper we denote by:

E – the Euclidean plane

$\mathcal{T}$ – the class of the non-empty convex compact sets on E

$\mathcal{K}$ – the class of all the circles (closed Euclidean balls) on E

$B = K(0,1)$ – the closed unit circle, $K(X,r) = X + rB$

$S = \{\vec{e} \mid \|\vec{e}\| = 1\}$ – the unit circumference, where $\|\cdot\|$ is the Euclidean norm

$s_T(\vec{e}) = s_{T,O}(\vec{e}) = \max_{M\in T} \vec{e}\,\overrightarrow{OM}$ – the support function of $T \in \mathcal{T}$.

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The support function $s_T(\vec{e}) = s_{T,O}(\vec{e})$ is related to the origin O and

$$s_{T,X}(\vec{e}) = s_{T,O}(\vec{e}) - \vec{e}\,\overrightarrow{OX} \tag{1.1}$$

to another initial point X. The *Hausdorff distance* $h(T_1, T_2)$ between $T_1, T_2 \in \mathcal{T}$ is defined by

$$h(T_1, T_2) = \inf\left\{\varepsilon > 0 \,\middle|\, T_1 \subset T_2 + \varepsilon B \text{ and } T_2 \subset T_1 + \varepsilon B\right\}.$$

Let $T \in \mathcal{T}$ be fixed. We consider the problem

$$h(T, K) \longrightarrow \min \qquad (K \in \mathcal{K}), \tag{1.2}$$

that is we look for the Hausdorff nearest circle $K \in \mathcal{K}$ to a given convex compact set T. This problem was brought to the authors by an engineer who was facing the following practical situation. The engineer produces machine parts which should be (ideal) disks but due to random fluctuation he gets declining (non-ideal) disks. Provided the produced disk satisfy certain admittances requirements, then they must be sorted according to their radius. We can imagine for instance such a situation by the production of piston rings and pistons, where sorting the rings and the pistons according to their measure, we are able easily to complete later the pistons with the corresponding rings. The problem which arises is how to measure the radius of the non-ideal disk, whose shape in fact is not a circle, and how to find the position of its centre. The knowledge of this position could be important for the eventual further production operations. The problem can be extended to the case when the produced details do not satisfy the required admittance, that is they are far from ideal disks (or their measured radius is outside the interval of admittance). Such machine parts must go to trash, therefore we deal with the problem of quality control and pattern recognition.

The mathematical model of the described situation is the optimization problem (1.2), where in general h is some distance functional. We consider this problem with respect to h being the Hausdorff distance, finding it proper to treat the described practical problem.

The contemporary methods of quality control require often visualization on a screen, which is particularly desired if the initial problem admits geometrical description. It is therefore important to have algorithms being real time geometric procedures. This paper solves the posed problem. The suggested geometric procedures give the solution not only in the "nearly circle case from the practice" that motivated initially this study but also for an arbitrary convex compact body T.

A similar problem is considered in Przesławski [13]. He studies centres of convex sets in L^P metrices in the sense that he considers the problems [1)]

$$H_p(T + tB, \{y\}) \longrightarrow \min \ (y \in \mathbb{R}^n) \quad \text{and} \quad H_p(T, y + tB) \longrightarrow \min \ (y \in \mathbb{R}^n)$$

[1)] The distances H_p are defined in Gruber [5] for $1 \le p < \infty$ by $H_p(T, K) = \left(\int_S |s_T(\vec{e}) - s_K(\vec{e})|^p d\sigma(\vec{e})\right)^{1/p}$ where σ is the normalized Lebesgue measure on S.

for a real parameter $t \geq 0$. The case $H_p = h$, i.e. $p = \infty$, is relevant for our considerations. Obviously, the first problem has the Chebyshev centre as unique solution for any $t \geq 0$. The solutions of the second problem are uniquely determined [13: Theorem 4.1] and belong to T for any $t \geq 0$ and the curve $t \mapsto y_{sol}(t)$ seems to have interesting properties. In the case of a triangle $t \mapsto y_{sol}(t)$ consists of a part of a bisectrix and a part of a midline. For t sufficiently close to 0 we get the Chebyshev centre (centre of the smallest circumscribed circle) and for t sufficiently large we get the centre of the incircle. Ginchev [3] finds in a straightforward manner the solution of problem (1.2) for the case, when T is a triangle. The intersection of some bisectrix and some midline defines the centre of the ball of best approximation in this case.

Several authors investigate problem (1.2) but rather with a metric h different than the Hausdorff metric. Alt and Wagener [1] give a computational procedure to find the circle of the best approximation for a convex polygon, provided the metric h into consideration is the area of the symmetric difference between T_1 and T_2. Some works, e.g. Bani and Chalmers [2] suggest a connection with the L^2 and L^p norm. Kenderov [8] and Kenderov and Kirov [9] use the Hausdorff distance too, however they consider the approximation of a convex compact set in the plane by convex polygons.

In this paper we use the obvious fact that the considered problem is equivalent to the problem of Chebyshevian approximation of continuous or, more precisely, of sinusoidal convex functions by trigonometric polynomials of first order. This problem has a unique solution characterized by the Chebyshevian alternance property. If the support function of the convex set is known, the solution can be computed by using well-known algorithms of semi-infinite programming, Remez algorithm included [6].

However, our intention is not to repeat these known facts but to generate a geometrical procedure which gives the exact solution (in the case of a polygon) or at least an approximate solution (in the case of an arbitrary convex compact set). The procedure bases on the Remez algorithm for finding the nearest in the uniform metric trigonometric polynomial of first degree to a given continuous 2π-periodic function.

2. The alternance property

The Hausdorff distance between two convex sets $T_1, T_2 \in \mathcal{T}$ can be expressed by their support functions (see, e.g., Leichtweiß[11]):

$$h(T_1, T_2) = \|s_{T_1} - s_{T_2}\|_S := \max_{\vec{e} \in S} |s_{T_1}(\vec{e}) - s_{T_2}(\vec{e})| . \tag{2.1}$$

Using the support function $s_{K,O}(\vec{e}) = \rho + \vec{e}\,\overrightarrow{OX}$ of $K(X, \rho)$ problem (1.2) is equivalent to

$$h(T, K) = \max_{\vec{e} \in S} |s_T(\vec{e}) - (\rho + \vec{e}\,\overrightarrow{OX})| \longrightarrow \min \qquad (X \in E,\ \rho \geq 0).$$

Suppose that an orthogonal coordinate system origined at O is introduced and let $X = (a, b)$ and $\vec{e} = (\cos t, \sin t)$ be the coordinates of X and $\vec{e}$, respectively. Then we come to the problem

$$h(T, K) = \max_{t \in [0, 2\pi]} |s(t) - (\rho + a\cos t + b\sin t)| \longrightarrow \min \quad (a, b \in \mathbb{R}, \rho \geq 0) \tag{2.2}$$

where $s(t) = s_{T,O}(\cos t,\ \sin t)$ is a continuous 2π-periodic function. The function $s(t)$ is a supremum of sinusoids of the type $s_{\gamma,\phi}(t) = \gamma\cos(t-\phi)$ with $(\gamma\cos(\phi), \gamma\sin(\phi)) \in T$. Such a function is sometimes called *sinusoidal convex*. This property implies that the solution of the minimization problem (2.2) upon $a, b, \rho \in \mathbb{R}$ satisfies automatically $\rho \geq 0$. Hence, our problem is equivalent to the problem of Chebyshevian approximation of a 2π-periodic continuous functions by trigonometric polynomials of first order satisfying the Haar condition [7, 10]. From the general alternance theorem the following corollary follows.

Corollary 2.1. *If s is 2π-periodic function, then there exists a unique trigonometric polynomial Θ, $\Theta(t) = \rho + a\cos t + b\sin t$, being the nearest to s among the trigonometric polynomials of order at most 1 in the uniform on $[0, 2\pi]$ metric. The following alternance property characterizes Θ:*

The trigonometric polynomial Θ is the nearest to s if and only if there are four points $t_1 < t_2 < t_3 < t_4$, $t_4 - t_1 < 2\pi$, and a number $\varepsilon = \pm 1$, such that

$$s(t_i) - \Theta(t_i) = \varepsilon(-1)^i \|s - \Theta\| \qquad (i = 1, 2, 3, 4) \tag{2.3}$$

where we put $\|\phi\| = \max_{0 \leq t \leq 2\pi} |\phi(t)|$ for a 2π-periodic function ϕ.

Remark 1. If one considers the approximation of T by a convex n-gon with respect to the above Hausdorff distance, then the alternance property is only necessary but not sufficient for the best approximation [8, 9]. The next corollary says that the alternance property is both necessary and sufficient for the best approximation of T by a circle.

Definition 2.2. We say that the vectors $\vec{e}_i \in S$ $(i = 1, 2, \ldots, n)$ follow in a *circular order*, if there are numbers $t_1 < t_2 < \ldots < t_n$, $t_n - t_1 < 2\pi$ such that $\vec{e}_i = (\cos t_i, \sin t_i)$ $(i = 1, 2, \ldots, n)$.

The following corollary is a geometric interpretation of Corollary 2.1.

Corollary 2.3. *If T is a non-empty convex compact set on the plane, then there exists a uniquely determined circle $K = K(X, \rho)$ being the nearest to T in the Hausdorff metric. The following alternance property characterizes K:*

The circle $K = K(X, \rho)$ is the nearest to T if and only if there are four unit vectors $\vec{e}_i$ $(i = 1, 2, 3, 4)$ following in a circular order and a number $\varepsilon \in \{-1, 1\}$ such that

$$s_T(\vec{e}_i) - s_K(\vec{e}_i) = \varepsilon(-1)^i h(T, K) \qquad (i = 1, 2, 3, 4). \tag{2.4}$$

Defining the difference $\delta(\vec{e}) := s_T(\vec{e}) - s_K(\vec{e})$ as the deviation of T from $K = K(X, \rho)$ in direction $\vec{e}$, condition (2.3) can be written in the form

$$\delta(\vec{e}_i) = \varepsilon(-1)^i h(T, K) \qquad (i = 1, 2, 3, 4). \tag{2.5}$$

We show that the deviations in direction $\vec{e}_i$ $(i = 1, 2, 3, 4)$ alternate and their absolute values are the maximal possible (cf. Figure 2.1).

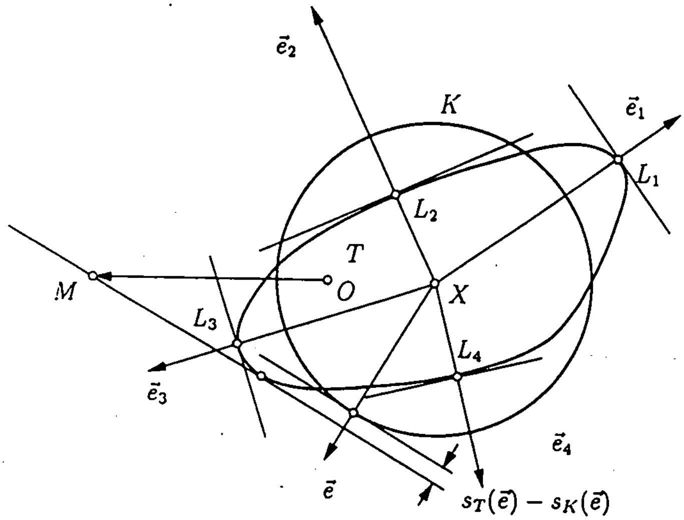

Figure 2.1: The Hausdorff nearest circle K to the plane convex compact T

We denote by $p_T(\vec{e}\,)$ the support line of T in direction $\vec{e}$, i.e. if M is the current point on $p_T(\vec{e}\,)$, then $p_T(\vec{e}\,) = \{M | \overrightarrow{OM}\ \vec{e} = s_T(\vec{e}\,)\}$. The deviation $\delta(\vec{e}\,)$ is the oriented distance between the support lines of T and K in direction $\vec{e}$ (cf. Figure 2.1). It seems that the points L_i, in which the axis in directions $\vec{e}_i$ originated in X meets the boundary of T, lie on the support lines $p_T(\vec{e}_i)$. This fact we prove in Section 3. Let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, $\vec{e}_4$ be the vectors, for which the alternance property holds. The deviation $\delta(\vec{e}\,) = s_T(\vec{e}\,) - s_K(\vec{e}\,)$ attains according to (2.4) extrema in the directions $\vec{e}_i$. We can relate in this difference the support function to different initial points, in particular to the centre X of the Hausdorff nearest circle $K = K(X, \rho)$. Then $\delta(\vec{e}\,) = s_{T,X}(\vec{e}\,) - \rho$ and therefore $s_{K,X}(\vec{e}\,)$ should attain extrema at $\vec{e}_i$ $(i = 1,2,3,4)$ where $s_{T,X}(\vec{e}_1) = s_{T,X}(\vec{e}_3)$ and $s_{T,X}(\vec{e}_2) = s_{T,X}(\vec{e}_4)$, one of these two values is a maximum, the other is a minimum. Let

$$s_{T,X}^{\min} := \min_{\vec{e}\in S} s_{T,X}(\vec{e}\,) \qquad \text{and} \qquad s_{T,X}^{\max} := \max_{\vec{e}\in S} s_{T,X}(\vec{e}\,).$$

Then formula (2.4) gives

$$h(T,K) = s_{T,X}^{\max} - \rho = \rho - s_{T,X}^{\min} \tag{2.6}$$

$$\rho = \frac{1}{2}(s_{T,X}^{\max} + s_{T,X}^{\min}) \qquad \text{and} \qquad h(T,K) = \frac{1}{2}(s_{T,X}^{\max} - s_{T,X}^{\min}) \tag{2.7}$$

which can be used for the construction of ρ.

Proposition 2.4 (see [13: Theorem 4.1]). *Let T be a convex compact set and $K = K(X,\rho)$ be the Hausdorff nearest circle to T. Then X is a point of T.*

Example 2.5. If T is a rectangular with sides $a \geq b$, then the centre X of the Hausdorff nearest circle $K = K(X, \rho)$ is the centre of the rectangular. Its radius ρ and the Hausdorff distance $h = h(T, K)$ are

$$\rho = \frac{1}{4}(\sqrt{a^2 + b^2} + b) \qquad \text{and} \qquad h = \frac{1}{4}(\sqrt{a^2 + b^2} - b).$$

3. Geometric characterization of the extremality of $\delta(\vec{e})$

The following characterizations of extremality of $\delta(\vec{e})$ are important for the geometric construction of the Hausdorff nearest circle.

Proposition 3.1. *Let T be a convex compact set and $K = K(X, \rho)$ be a circle. Let the deviation $\delta(\vec{e}) = s_T(\vec{e}) - s_K(\vec{e})$ $(\vec{e} \in S)$ attains a local extremum in direction $\vec{e}_0$ (the extremum is understood with respect to the relative topology on S). Let l be the axis containing X and $\vec{e}_0$. Then there is a non-empty segment $\overrightarrow{LL_0}$, in which l intersects T and having the direction of $\vec{e}_0$. The point L_0 lies on the support line $p_T(\vec{e}_0)$ and if the extremum is a local maximum, then $T \cap p_T(\vec{e}_0)$ consists of the single point L_0.*

Proof. The statement is trivial if $T = \{M\}$. Now let the dimension of T be larger than zero. It holds

$$\delta(\vec{e}) = s_{T,X}(\vec{e}) - s_{K,X}(\vec{e}) = s_{T,X}(\vec{e}) - \rho.$$

Therefore, any minimum (maximum) of δ is also a minimum (maximum) of $s_{T,X}$ and vice versa. Let $\{L_1\} = l \cap p_T(\vec{e}_0)$, i.e.

$$p_T(\vec{e}_0) = \{Y \,|\, \overrightarrow{XY}\, \vec{e}_0 = s_{T,X}(\vec{e}_0)\} \qquad \text{and} \qquad |s_{T,X}(\vec{e}_0)|\, \vec{e}_0 = \overrightarrow{XL_1}.$$

a) Let $s_{T,X}(\vec{e})$ have a minimum at $\vec{e}_0$. Assume $L_1 \notin T$. At first we consider the case $s_{T,X}(\vec{e}_0) \geq 0$. Using a standard separation theorem we find some $\vec{e}_1 \in S$ arbitrary close to $\vec{e}_0$ such that $\overrightarrow{XY}\, \vec{e}_1 \leq s_{T,X}(\vec{e}_1)$ for all $Y \in T$ and $\overrightarrow{XL_1}\, \vec{e}_1 > s_{T,X}(\vec{e}_1)$. Hence

$$s_{T,X}(\vec{e}_0) = s_{T,X}(\vec{e}_0)\vec{e}_0\, \vec{e}_0 \geq s_{T,X}(\vec{e}_0)\, \vec{e}_0\, \vec{e}_1 = \overrightarrow{XL_1}\, \vec{e}_1 \;\; > s_{T,X}(\vec{e}_1)$$

contradicting the assumption that $s_{T,X}(\vec{e})$ has a minimum at $\vec{e}_0$. Now let $s_{T,X}(\vec{e}_0) < 0$. Then $X \notin T$, the closed ball B around X with the radius $s_{T,X}(\vec{e}_0)$ contains L_1 and does not meet T. Hence, there is some $\vec{e}_1$ arbitrary close to $\vec{e}_0$ such that $s_{T,X}(\vec{e}_1) < s_{T,X}(\vec{e}_0)$ which again contradicts that $s_{T,X}(\vec{e})$ has a minimum at $\vec{e}_0$.

b) Let $s_{T,X}(\vec{e})$ have a maximum at $\vec{e}_0$. Assume $L_1 \notin T$. Choose M with $\overrightarrow{XM}\, \vec{e}_0 = s_{T,X}(\vec{e}_0)$. Obviously, $s_{T,X}(\vec{e}) > 0$, whenever T consists of more than one point. Since $L_1 \notin T$, we have $M \neq L_1$. We choose $\vec{e}_\lambda = (1 - \lambda)\vec{e}_0 + \lambda\overrightarrow{XM}/\|\overrightarrow{XM}\|$ where $0 < \lambda < 1$. Let P_λ be the orthogonal projection of M to the line l_λ having direction $\vec{e}_\lambda$ and passing through X. Then $\|\overrightarrow{XL_1}\| = \overrightarrow{XM}\, \vec{e}_0$ and

$$\|\overrightarrow{XP_\lambda}\| = \overrightarrow{XM}\, \frac{\vec{e}_\lambda}{\|\vec{e}_\lambda\|} = \frac{(1 - \lambda)\overrightarrow{XM}\, \vec{e}_0 + \lambda\|\overrightarrow{XM}\|}{\|\vec{e}_\lambda\|} > \frac{\|\overrightarrow{XL_1}\|}{\|\vec{e}_\lambda\|} > \|\overrightarrow{XL_1}\|.$$

Therefore,

$$s_{T,X}(\vec{e}_0) = \|\overrightarrow{XL_1}\| < \|\overrightarrow{XP_\lambda}\| = \overrightarrow{XM}\,\frac{\vec{e}_\lambda}{\|\vec{e}_\lambda\|} \le s_{T,X}\left(\frac{\vec{e}_\lambda}{\|\vec{e}_\lambda\|}\right),$$

a contradiction with the assumption that $s_{T,X}$ has a local maximum at $\vec{e}_0$. Therefore $L_1 \in T$.

We prove now that $p_T(e_0) \cap T$ contains only the single point L_0. Otherwise we can find a point $M \in p_t(\vec{e}_0) \cap T$ different than L_0 to which exactly the same reasonings as above can be applied and in the same way a contradiction can be obtained ∎

Let $X, L_0 \in T$ and $\vec{e}_0 \in S$ be such as in Proposition 3.1. Then the circle with the radius $\overline{XL_0}$ contains T (smallest circumcircle with centre X) if $\delta(\vec{e}_0)$ is a global maximum and is contained by T (largest inscribed circle with centre X) if $\delta(\vec{e}_0)$ is a global minimum. In both cases the circle is supported in L_0 by the line $p_T(\vec{e}_0)$.

If T is a convex polygon, we get some sharper characterizations of extremality of $\delta(\vec{e})$.

Proposition 3.2. *Let T be a convex polygon and $K = K(X, \rho)$ be a circle, for which $X \in T$. Let the deviation $\delta = \delta(\vec{e}) = s_T(\vec{e}) - s_K(\vec{e})$ $(\vec{e} \in S)$ attains a local extremum in direction $\vec{e}_0$. Let l be the axis containing X and having direction $\vec{e}_0$. Then there is a non-empty segment $\overrightarrow{LL_0}$, in which l intersects T and having the direction of $\vec{e}_0$. The point L_0 lies on the support line $p_T(\vec{e}_0)$.*

a) *If the extremum of $\delta(\vec{e})$ is a local minimum at $e = e_0$, then L_0 lays on the relative interior of a side of T having $\vec{e}_0$ as an outer normal (only here the assumption $X \in T$ is used).*

b) *If the extremum of $\delta(\vec{e})$ is a local maximum at $e = e_0$, then $T \cap p_T(\vec{e}_0)$ consists of the single point L_0 and is a vertex of T.*

Proof. a) Let $\delta(\vec{e})$ attains a minimum. As it is shown in Proposition 3.1, $L_0 \in T$. If L_0 is in the relative interior of a side, then since $L_0 \in p_T(\vec{e}_0)$ and $p_T(\vec{e}_0)$ is a support line of T, we see that $\vec{e}_0$ must be an outer normal of this side. If L_0 does not belong to the relative interior of a side, then it must be a vertex. Therefore, there must exist a further side of T through L_0 with an outer normal vector $\vec{e}_2 \neq \vec{e}_0$. We choose $\vec{e}_1$ between $\vec{e}_0$ and $\vec{e}_2$ arbitrary close to $\vec{e}_0$. Then $\vec{e}_1\,\vec{e}_0 < 1$ and $L_0 \in p_T(\vec{e}_0) \cap p_T(\vec{e}_1)$. Using $s_{T,X}(\vec{e}_0) \ge 0$ (because of $X \in T$) we get

$$s_{T,X}(\vec{e}_0) = s_{T,X}(\vec{e}_0)\,\vec{e}_0\,\vec{e}_0 = \overrightarrow{XL_0}\,\vec{e}_0 > \overrightarrow{XL_0}\,\vec{e}_1 = s_{T,X}(\vec{e}_1)$$

which means that $s_{T,X}(\vec{e})$ does not attain a maximum at $\vec{e}_0$ - a contradiction.

b) Let $\delta(\vec{e})$ attains a maximum at $\vec{e}_0$. Since $\{L_0\} = T \cap p_T(\vec{e}_0)$ is the single point of this intersection, L_0 must be a vertex ∎

Proposition 3.3. *Let $T \in \mathcal{T}$ and let $\mathcal{E} := \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subset S$, where $\vec{e}_k$ $(k = 1,2,3,4)$ are given in a circular order. The centre $\bar{X}$ of the circle K with*

$$\mathcal{H}(\mathcal{E},T,K) := \max_{\vec{e}\in\mathcal{E}} \{|s_T(\vec{e}) - s_K(\vec{e})|\} \to \min \qquad (K \in \mathcal{K}) \tag{3.1}$$

is the cutting point of the both bisectrices generated by the supporting lines of $p_T(\vec{e}_1)$, $p_T(\vec{e}_3)$ and $p_T(\vec{e}_2), p_T(\vec{e}_4)$. The radius $\bar{\rho}$ is given by the mean value $\frac{1}{2}(s_{T,\bar{X}}(\vec{e}_1) + s_{T,\bar{X}}(\vec{e}_2))$ of the distances of $\bar{X}$ to the lines $p_T(\vec{e}_1)$ and $p_T(\vec{e}_2)$ as well as the optimal value by $\mathcal{H}(\mathcal{E},T,\bar{K}) = \frac{1}{2}|s_{T,\bar{X}}(\vec{e}_1) - s_{T,\bar{X}}(\vec{e}_2)|$.

Proof. The solution $\bar{K}$ with the centre $\bar{X}$ and the radius $\bar{\rho}$ of (3.1) fulfils the alternance condition, $s_T(\vec{e}_k) - s_{\bar{K}}(\vec{e}_k) = s_{T,\bar{X}}(\vec{e}_k) - s_{\bar{K},\bar{X}}(\vec{e}_k)$ for any $X \in \mathbb{R}^2$ and $s_{\bar{K},\bar{X}}(\vec{e}_k) = \bar{\rho}$. Therefore, we get the bisectrices property $s_{T,\bar{X}}(\vec{e}_1) = s_{T,\bar{X}}(\vec{e}_3)$ and $s_{T,\bar{X}}(\vec{e}_2) = s_{T,\bar{X}}(\vec{e}_4)$. Since $\vec{e}_k$ $(k = 1,2,3,4)$ are given in a circular order the bisectrices cut each other. From

$$\max_{1\le i\le 4} s_{T,\bar{X}}(\vec{e}_i) = \bar{\rho} + \mathcal{H}(\mathcal{E},T,\bar{K}) \qquad \text{and} \qquad \min_{1\le i\le 4} s_{T,\bar{X}}(\vec{e}_i) = \bar{\rho} - \mathcal{H}(\mathcal{E},T,\bar{K})$$

there follows

$$2\bar{\rho} = s_{T,\bar{X}}(\vec{e}_1) + s_{T,\bar{X}}(\vec{e}_2) = s_{T,\bar{X}}(\vec{e}_3) + s_{T,\bar{X}}(\vec{e}_4)$$

and

$$\mathcal{H}(\mathcal{E},T,\bar{K}) = \frac{1}{2}\Big(\max_{1\le i\le 4} s_{T,\bar{X}}(\vec{e}_i) - \min_{1\le i\le 4} s_{T,\bar{X}}(\vec{e}_i)\Big) = \frac{1}{2}|s_{T,\bar{X}}(\vec{e}_1) - s_{T,\bar{X}}(\vec{e}_2)|$$

and the statement is proved ∎

If it happens that all $\vec{e}_i$ belong to a supporting cone of T at the same point P, then $\mathcal{H}(\mathcal{E},T,K) = \rho = 0$ since $s_{T,\bar{X}}(\vec{e}_i) = 0$ for all $i = 1,2,3,4$.

4. The Hausdorff nearest circle to a convex polygon

Now we give a characterization for the Hausdorff nearest circle to a convex polygon as intersection of some midline and besectrix of suitable vertices and sides, respectively.

Definition 4.1. Let T be a convex polygon with vertices $A_1, A_2, \dots, A_n$ and sides $a_1, a_2, \dots, a_n$. For a point X denote by $r_i(X)$ the distance from X to A_i and by $d_i(X)$ the distance from the point X to the side a_i. Let $r(X) = \max_{1\le i\le n} r_i(X)$ and $d(X) = \min_{1\le i\le n} d_i(X)$. Introduce the sets

$$P_i = \{X \in T \,|\, r(X) = r_i(X)\} \qquad \text{and} \qquad Q_i = \{X \in T \,|\, d(X) = d_i(X)\}$$

for $i = 1,2,\dots,n$,

$$\Gamma_P = \bigcup_{i=1}^{n} \partial P_i \qquad \text{and} \qquad \Gamma_Q = \bigcup_{i=1}^{n} \partial Q_i$$

where ∂P_i and ∂Q_i are the boundaries of P_i and Q_i, respectively.

Now the Hausdorff nearest circle to a convex polygon can be determined as follows.

Theorem 4.2. *Let T be a convex polygon with vertices $A_1, A_2, \ldots, A_n$ and sides $a_1, a_2, \ldots, a_n$. Then the centre X of the Hausdorff nearest circle lies at the intersection $\Gamma = \Gamma_P \cap \Gamma_Q$. It is the unique point of Γ, for which the following alternance property holds.*

There are four directions following in a circular order $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{e}_4$, such that:

(i) *There are vertices A_i and A_j, for which $r(X) = r_i(X) = r_j(X)$ and $\overrightarrow{XA_i}$ and $\overrightarrow{XA_j}$ have directions $\vec{e}_1$ and $\vec{e}_3$ (or $\vec{e}_2$ and $\vec{e}_4$).*

(ii) *There are sides a_k and a_l, for which $d(X) = d_k(X) = d_l(X)$ and the outer normals for the sides a_k and a_l have directions $\vec{e}_2$ and $\vec{e}_4$ (or $\vec{e}_1$ and $\vec{e}_3$).*

The radius of K is $\rho = \frac{1}{2}(r(X) + d(X))$ and the Hausdorff distance $h = h(T, K)$ is $h = \frac{1}{2}(r(X) - d(X))$.

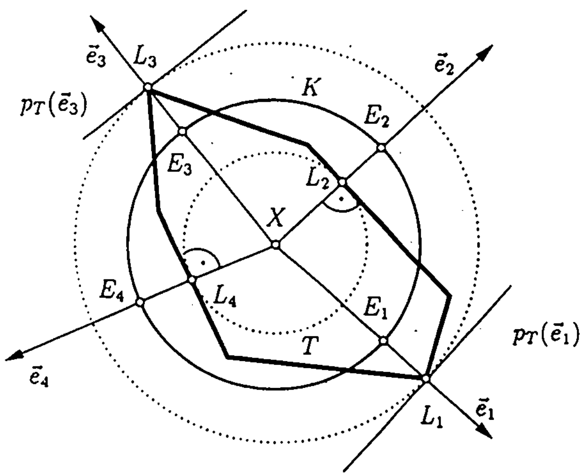

Figure 4.1: The Hausdorff nearest circle K to a polygon T

Proof (For illustration see Figure 4.1). Let $K = K(X, \rho)$ be the Hausdorff nearest circle and let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ and $\vec{e}_4$ be the vectors, for which the alternance property holds by Corollary 2.3. Suppose for determination that

$$s_{T,X}(\vec{e}_1) = s_{T,X}(\vec{e}_3) = \max_{\vec{e} \in S} s_{T,X}(\vec{e}) \quad \text{and} \quad s_{T,X}(\vec{e}_2) = s_{T,X}(\vec{e}_4) = \min_{\vec{e} \in S} s_{T,X}(\vec{e}).$$

Let l_i be the axis through X having direction $\vec{e}_i$ and $L_i = l_i \cap p_T(\vec{e}_i)$. According to Proposition 3.2 the points L_1 and L_3 are vertices, say A_i and A_j. Since the maximal property yields that for arbitrary vertex A_s it holds $r_s(X) \leq r_i(X) = r_j(X)$, we see that $X \in P_i \cap P_j \subset \partial P_i \subset \Gamma_P$.

Similarly, from Proposition 3.2 and the minimal property we see that there are sides a_k and a_l such that for arbitrary side a_s we have $d_s(X) \leq d_k(X) = d_l(X)$, therefore

$X \in Q_k \cap Q_l \subset \partial Q_k \subset \Gamma_Q$, thus $X \in \Gamma_P \cap \Gamma_Q$. The formulas for ρ and h come from (2.7).

Now, let $X \in \Gamma_P \cap \Gamma_Q$. Since X is in Γ_P, then there exist at least two regions P_i and P_j such that X is in the boundary of both of them. Let $\vec{e}_1$ and $\vec{e}_3$ be unit vectors giving the directions of $\overrightarrow{XA_i}$ and $\overrightarrow{XA_j}$ and $r_s(X) \leq r_i(X) = r_j(X)$ for arbitrary vertex A_s. Similarly, from $X \in \Gamma_Q$ there follows that X is in the boundary of some regions Q_k and Q_l. Let $\vec{e}_2$ and $\vec{e}_4$ be the outer normal vectors for the sides a_k and a_l, and $d_s(X) \geq d_k(X) = d_l(X)$ for arbitrary side a_s. We have therefore for arbitrary $\vec{e} \in S$

$$d(X) = d_k(X) = d_l(X) \leq s_{K,X}(\vec{e}) \leq r_i(X) = r_j(X) = r(X).$$

If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{e}_4$ follow in a circular order and $\rho = \frac{1}{2}(r(X)+d(X))$, $h = \frac{1}{2}(r(X)-d(X))$, then obviously the alternance property (2.4) is satisfied. Therefore $K = K(X,\rho)$ is the Hausdorff nearest circle to T ∎

Naturally, Γ_P is contained (but generally not equal to) in the union of the midlines of the segments A_iA_j, whose end points are vertices of T. Similarly, Γ_Q is contained (but generally not equal to) in the union of the bisectrices of all the angles $\langle a_i, a_j \rangle$ obtained by the sides of T. For illustration we consider the simple cases of a triangle and a quadrilateral.

Example 4.3. If T is a triangle, then the centre of its Hausdorff nearest circle is the intersection of the midline of the longest side and the bisectrix against the shortest side. If the sides of T have lenghts $a_1 \geq a_2 \geq a_3$ and its angles have measures $A_1 \geq A_2 \geq A_3$, then the radius ρ and the Hausdorff distance $h = h(T,K)$ are

$$\rho = \frac{1}{4}\,a_1\,\frac{1+\sin\frac{1}{2}A_3}{\cos\frac{1}{2}A_3} \quad \text{and} \quad h = \frac{1}{4}\,a_1\,\frac{1-\sin\frac{1}{2}A_3}{\cos\frac{1}{2}A_3}. \tag{4.1}$$

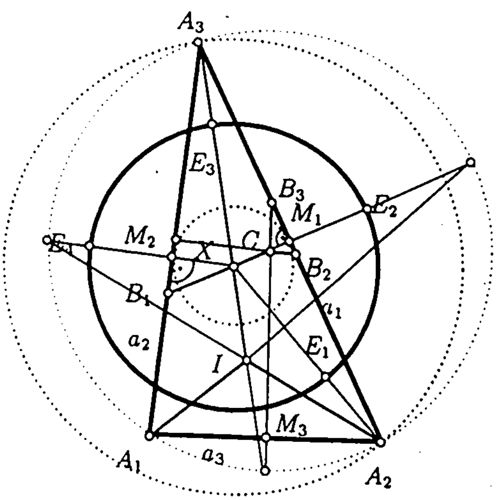

Figure 4.2: The Hausdorff nearest circle to a triangle

Indeed, let for determination $a_1 \geq a_2 \geq a_3$ (see Figure 4.2) and all the angles are acute (the reasoning does not change much if this condition is not true). Let M_1, M_2 and M_3 be the middle of the sides and let B_1, B_2 and B_3 be those points different from M_1, M_2 and M_3, in which the midlines of the sides intersect the boundary of T. Further, let C and I be the cross of the midlines and the bisectrices, respectively. Then $\Gamma_P = CB_1 \cup CB_2 \cup CB_3$ and $\Gamma_Q = IA_1 \cup IA_2 \cup IA_3$. Using the fact that a midline of a side and a bisectrix to it cross at the circumcircle, we see that that Γ_P and Γ_Q intersect at only one point X being the intersection of the midline of the longest side a_1 and the bisectrix toward the shortest side a_3. The existence of the Hausdorff nearest circle and Theorem 4.2 imply that X as the single point in the intersection $\Gamma_P \cap \Gamma_Q$, is the centre of the Hausdorff nearest circle. This can be also directly derived from the alternance property. We have

$$r(X) = r_1(X) = r_2(X) = \frac{\frac{1}{2}a_1}{\cos\frac{1}{2}A_3} \quad \text{and} \quad d(X) = d_1(X) = d_2(X) = \tfrac{1}{2}\,a_1 \tan\tfrac{1}{2}A_3,$$

whence (4.1) follows from the formulas in Theorem 4.2. On Figure 4.2 the unit vector $\vec{e}_i$ for the alternance property are the directions of $\overrightarrow{XE_i}$ $(i = 1, 2, 3, 4)$. The alternance property is better underlined by drawing the two bold-face dotted circles obtained by the Hausdorff nearest one by enlarging and diminishing its radius by the Hausdorff distance. Such circles are drawn also on Figures 4.1 and 4.3.

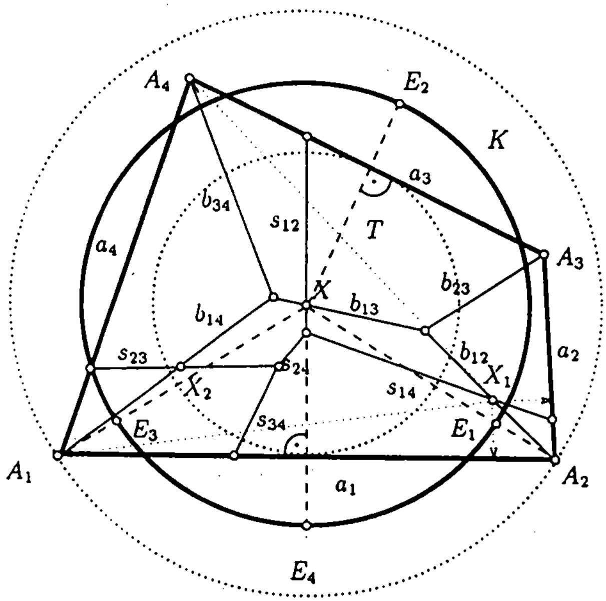

Figure 4.3: The Hausdorff nearest circle to a quadrilateral

Example 4.4. We consider the quadrilateral $A_1A_2A_3A_4$ given on Figure 4.3 (in another quadrilateral the configuration may be different). We have

$$\Gamma_P = s_{12} \cup s_{14} \cup s_{23} \cup s_{24} \cup s_{34} \qquad \text{and} \qquad \Gamma_Q = b_{12} \cup b_{13} \cup b_{14} \cup b_{23} \cup b_{34}.$$

Here s_{ij} is a part of the midline of the segment A_iA_j and b_{ij} is a part of the bisectrix of the angle $\langle a_i, a_j \rangle$. We see that

$$\Gamma_P \cap \Gamma_Q = \{X, X_1, X_2\} \qquad \text{where } X = b_{13} \cap s_{12}, X_1 = b_{12} \cap s_{23}, X_2 = b_{14} \cap s_{23}.$$

The point X is the centre of the Hausdorff nearest circle, since the unit vectors $\vec{e}_i$ $(i = 1,2,3,4)$ (those giving the directions of $\overrightarrow{XE_i}$ on the figure), in which the deviations $\delta(\vec{e}_i)$ obtain maximal absolute value and alternate in sign, follow in a circular order. The points X_1 and X_2, as seen from the figure, do not possess this property.

For an arbitrary convex polygon, similarly like in Example 4.4, the curves Γ_P and Γ_Q may intersect in more than one point. The existence and the uniqueness of the Hausdorff nearest circle implies, however, that there exist a single point $X \in \Gamma_P \cap \Gamma_Q$ and four unit vectors $\vec{e}_i$ $(i = 1,2,3,4)$ following in a circular order such that $\delta(\vec{e}_i)$, where $\delta(\vec{e}) = s_{T,X}(\vec{e}) - \rho$, $\rho = \frac{1}{2}(s_{T,X}(\vec{e}_1) + s_{T,X}(\vec{e}_2))$, obtain a maximal absolute value, alternate in signs.

5. The Remez algorithm – a geometric interpretation

5.1 Remarks to the Remez algorithm. We shortly discuss the well-known Remez algorithm for solving the problem

$$\max_{t\in[0,2\pi]} \left|s(t) - (\rho + a\cos t + b\sin t)\right| \longrightarrow \min \qquad (a, b, \rho \in \mathbb{R}) \tag{5.1}$$

where s is a 2π-periodic function (see, e.g., Laurent [10] or Karlin and Studden [7]). The basic idea is to determine four points $t_1 < t_2 < t_3 < t_4 < t_1 + 2\pi$, $\delta > 0$ and a, b, ρ such that for some $\sigma \in \{-1, 1\}$ the system

$$e(t_i, \rho, a, b) := s(t_i) - (\rho + a\cos t_i + b\sin t_i) = (-1)^i \sigma\, \delta \quad (1 \le i \le 4) \tag{5.2}$$

$$H(\rho, a, b) := \max_{t\in[0,2\pi]} \left|s(t) - (\rho + a\cos t + b\sin t)\right| = \delta \tag{5.3}$$

is satisfied. One starts with some selection $t_1 < t_2 < t_3 < t_4 < t_1 + 2\pi$ and solves system (5.2). If $\delta(\rho, a, b) = H(\rho, a, b)$, then the solution is found. Otherwise determine a value $t^* \in [0, 2\pi]$ where $H(\rho, a, b)$ is attained. Replace in the selection $t_1 < t_2 < t_3 < t_4$ one of the points (one point exchange) t_k or t_{k+1} by t^* such that $t_k < t^* < t_{k+1}$ $(k = 0, 1, ..., 4;\ t_0 = t_4 - 2\pi, t_5 = t_1 + 2\pi)$ and $\operatorname{sign}(e(t_k, \rho, a, b)) = \operatorname{sign}(e(t^*, \rho, a, b))$ where t_0 and t_5 can be identified with t_4 and t_1, respectively. The procedure now repeats with the obtained updated selection.

Remark 2. Detailed descriptions about several algorithms from the numerical point of view, the Remez algorithms included, can be found in Hettich and Zencke [6: pp. 147 ff.]. The above mentioned one point exchange is at least linearly convergent. The full exchange (four point exchange) by using the local reduction theory is superlinearly convergent.

The value of $H^j := H(\rho^j, a^j, b^j)$ strictly decreases and the solution δ^j of system (5.2) strictly increases with respect to the iteration index j in both methods. Both sequences converge to the optimal value of (5.1). They are used for error estimations and stopping rules.

Remark 3. The solution $\psi(t) := \rho + a\cos t + b\sin t$ of system (5.2) can be calculated explicitly. Using the abbreviation $k(t_1,t_2,t_3,t_4) := 2\sin\frac{1}{2}(t_4 - t_3 + t_2 - t_1)$ we get

$$\begin{aligned}
\psi(t) = &-\frac{1}{k(t_1,t_2,t_3,t_4)\sin\frac{1}{2}(t_3-t_1)} \\
&\times\left(-\cos\frac{1}{2}(t_4-t_3)\cos\frac{1}{2}(t_3-t_2)+\cos\left(t-\frac{1}{2}(t_4+t_2)\right)\right)s(t_1) \\
&+\frac{1}{k(t_1,t_2,t_3,t_4)\sin\frac{1}{2}(t_4-t_2)} \\
&\times\left(-\cos\frac{1}{2}(t_4-t_1)\cos\frac{1}{2}(t_4-t_3)+\cos\left(t-\frac{1}{2}(t_3+t_1)\right)\right)s(t_2) \\
&+\frac{1}{k(t_1,t_2,t_3,t_4)\sin\frac{1}{2}(t_3-t_1)} \\
&\times\left(-\cos\frac{1}{2}(t_4-t_1)\cos\frac{1}{2}(t_2-t_1)+\cos\left(t-\frac{1}{2}(t_4+t_2)\right)\right)s(t_3) \\
&-\frac{1}{k(t_1,t_2,t_3,t_4)\sin\frac{1}{2}(t_4-t_2)} \\
&\times\left(-\cos\frac{1}{2}(t_3-t_2)\cos\frac{1}{2}(t_2-t_1)+\cos\left(t-\frac{1}{2}(t_3+t_1)\right)\right)s(t_4).
\end{aligned}$$

Remark 4. If an approximate solution is determined with sufficient accuracy, then we can use the four point exchange. Here at each step all local maxima of (5.3) must be determined. The arguments t_i are implicitly given twice differentiable functions of the parameter a, b, ρ. The corresponding local reduction method is in this case at least superlinearly convergent (for detailed description cf. [6: Chapter 5.4]).

5.2 Geometric versions and modifications of the Remez algorithm. In this subsection we describe three algorithms for the geometrical construction of the Hausdorff nearest circle to a convex set. First we start with direct anologies to the one and four point exchange.

Algorithm 1: One point exchange (see Fig. 5.1).

Initialization: Choose $\mathcal{E}^1 = \{\vec{e}_i : i = 1,2,3,4\}$, $\vec{e}_i \in S$ $(i = 1,2,3,4)$ in a circular order and accuracy $\varepsilon > 0$.

Iteration:

Step 1: Construct with $\mathcal{E}^k$ the circle K^k with centre X^k, radius ρ^k and distance $H(\mathcal{E}^k, T, K^k)$ according to Proposition 3.3. Denote the corresponding supporting points by P_i^k $(i = 1,2,3,4)$.

Step 2: Construct the incircle and circumcircle of T using the centre X^k. Let r^k and R^k be the radius of the constructed approximately incircle and circumcircle.

If $\mathcal{H}(\mathcal{E}^k, T, K^k)+\varepsilon \geq \max(R^k-\rho^k, \rho^k-r^k)$ (i.e. each of the incircle and circumcircle supports T in at least two points with an accuracy ε)

then Stop, approximate solution with desired accuracy is found.

If $\rho^k - r^k > R^k - \rho^k$

then $\vec{e}^* \in S$ is the common outer unit normal of the incircle of T and $K(X^k, r^k)$ at a point $P^* \in T \cap K(X^k, r^k)$

else $\vec{e}^* \in S$ is the common outer unit normal of the circumcircle of T and $K(X^k, R^k)$ at a point $P^* \in T \cap K(X^k, r^k)$.

Step 3: Exchange P^* with that neighbouring P_i^k which is on the same side (inner or outer point) of K^k as P^*. This defines the new selection P_i^{k+1} $(i = 1,2,3,4)$ and the associated $\mathcal{E}^{k+1}$.

Step 4: $k := k+1$, go to **Step 1**.

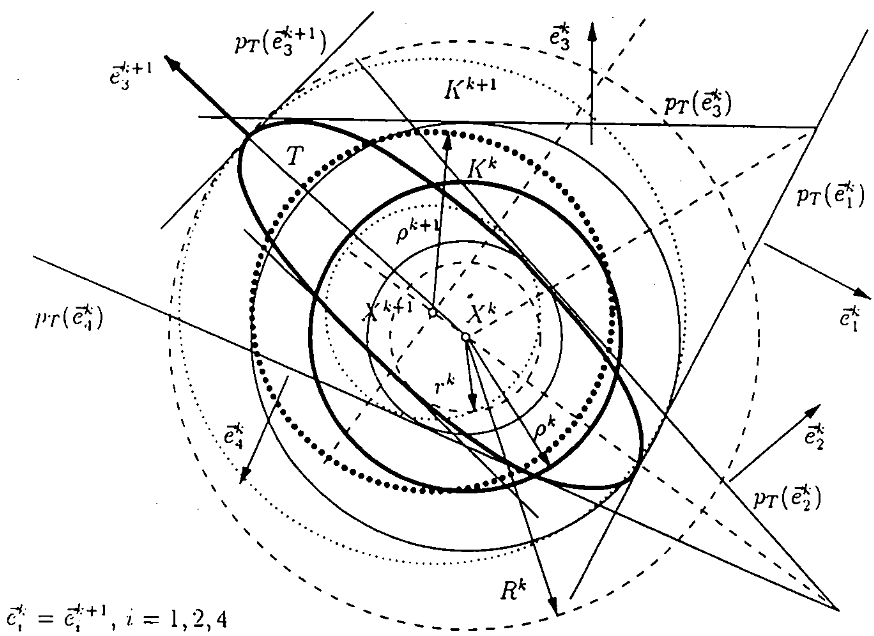

Figure 5.1: Construction of one iteration in Algorithm 1

Remark 5. This algorithm is an exact geometric interpretation of the one-point exchange Remez algorithm. Also the name "one-point exchange" is inherited from the analytical setting, but from geometrical point of view what is really exchanged by each step is one direction and not one point. Let us especially underline that in Step 1 the centre X^k of the circle K^k is the intersecting point of the bisectrix of the supporting lines through P_1^k and P_3^k and the bisectrix of the supported lines through P_2^k and P_4^k. This simple construction solves geometrically problem (5.2) replacing the rather extensive analytic solution given by the formula in Remark 3, and in some sense it is the central moment in our considerations. Let us also observe that from iteration to iteration one bisectrix remains the same. The convergence of this algorithm is at least linear [6] for arbitrary initial directions.

Algorithm 2 : Four point exchange.

Initialization: Choose X^1 in the interior of T and accuracy ε. Construct (e.g. bisection) an approximation of the two best local incircles and circumcircles of T using the centre X^1. Let $\vec{e}_1, \vec{e}_3$ be common outer unit normals of the locally incircles and T and let $\vec{e}_2, \vec{e}_4$ be common outer unit normals of the locally circumcircles and T at common supporting points. Choose $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ such that they are in a circular order. Set $\mathcal{E}^1 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$.

Iteration:

Step 1: Construct with $\mathcal{E}^k$ the circle K^k with centre X^k, radius ρ^k and distance $H(\mathcal{E}^k, T, K^k)$ according to Proposition 3.3.

Step 2: Construct (e.g. bisection) an approximation of the two best local incircles and circumcircles of T using the centre X^k. Let $\vec{e}_1, \vec{e}_3$ be common outer unit normals of the locally inscribed circles and T and let $\vec{e}_2, \vec{e}_4$ be the common outer unit normals of the locally circumcircles and T at common supporting points. Choose $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ such that they are in a circular order. Set $\mathcal{E}^{k+1} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$.

If the Hausdorff distance between the two incircles and the Hausdorff distance between the two circumcircles is smaller than ε

then Stop, approximate solution with desired accuracy is found by K^k

else $k = k + 1$, go to **Step 1**.

Remark 6. The four point exchange (Algorithm 2) is equivalent to the local reduction method (see, e.g., [6: Chapter 5.4]) which is superlinearly convergent for the Chebyshev approximation. However, the construction is only possible, whenever the centre X^k is sufficiently close to the centre of the circle of best Hausdorff approximation. As long as this construction is impossible one can start or continue with Algorithm 1 since one (locally) incircle and one (locally) circumcircle is ensured for any $X^k \in T$.

The simple characterization of the Hausdorff nearest circle for polygons obtained in Theorem 4.2 raises an interesting question: Do the considered algorithms allow in the case of a polygon the exact construction of the Hausdorff nearest circle after finite number of steps regardless of which initial directions are chosen? Unfortunately, the answer is not sure in the case of Algorithm 1. This question is a motivation to look for modifications giving affirmative answer. The next Algorithm 3 is a suggestion in this direction. It corresponds to a modified two point exchange Remez algorithm. In the case of a polygon the optimal solution is obtained after a finite number of steps. A peculiarity of this algorithm is that on each step we deal with some midline or some bisectrix, which underlines the importance of these concepts for the considered problem and justifies the special attention that we pay to the polygonal case.

Definition 5.1. We say that the points V_1, E_2, V_3 and E_4 of the boundary of a convex set follow in a *circular order* if there are supporting lines through these points whose normals follow in a circular order.

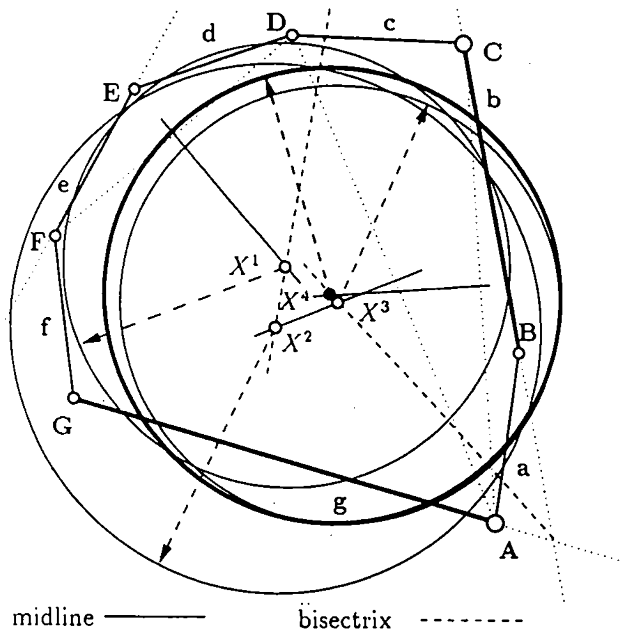

Figure 5.2: An example based on Algorithm 3

Algorithm 3 :

Initialization: Choose on the boundary of T the four points $E_1^1, V_2^1, E_3^1, V_4^1$ in a circular order and accuracy $\varepsilon > 0$, where E_1^1 and E_3^1 possess a unique supporting line l_1^1 and l_3^1 with the outer unit normals $\vec{e}_1^1$ and $\vec{e}_3^1$. If T is a polygon, then E_1^1 and E_3^1 are elements of the relative interior of edges and V_2^1, V_4^1 are vertices.

Iteration:

Step 1: Construct the centre X^k of the circle K^k as the intersection point of the besectrix b^k generated by the support lines l_1^k and l_3^k through E_1^k, E_3^k and the midline m^k with respect to $P_2^k := V_2^k$ and $P_4^k := V_4^k$. Denote the orthogonal projection of X^k to l_i^k by P_i^k $(i = 1, 3)$ and set $\rho^k := \frac{1}{2}(\|\overrightarrow{X^k P_2^k}\| + \|\overrightarrow{X^k P_1^k}\|)$ and $h^k := \frac{1}{2}(\|\overrightarrow{X^k P_2^k}\| - \|\overrightarrow{X^k P_1^k}\|)$.

Step 2: Construct the incircle and circumcircle of T using the centre X^k. Let r^k and R^k be the radius of the constructed incircle and circumcircle. Set $H^k := \max(R^k - \rho^k, \rho^k - r^k) = h(K^k, T)$. The incircle defines at least one supporting point E^* where T and the incircle has a common unique supporting line l^*. The circumcircle defines at least one supporting point (vertex in the case of a polygon) V^* where T and the circumcircle has a common supporting line l^{**}.

If $\frac{1}{2}(\|\overline{X^k P_2^k}\| - \|\overline{X^k P_1^k}\|) + \varepsilon \geq H^k$ (i.e. each of the incircle and circumcircle supports T in at least two points with an accuracy ε)

then Stop, approximate solution with desired accuracy is found.

If $\rho^k - r^k > R^k - \rho^k$

then Exchange E^* (and the associated l^*) with the nearest (with respect to positive or negative circular order) E_j^k (and the associated l_j^k) $(j = 1, 3)$ which can be reached on the circumference without meeting V_i^k $(i = 2, 4)$. The other points remain the same.

else Exchange V^* with the nearest (with respect to positive or negative circular order) V_j^k $(j = 2, 1)$ which can be reached on the circumference without meeting E_i^k $(i = 1, 3)$. The other points and supporting lines remain the same. Thus, the new collection $(E_1^{k+1}, l_1^{k+1}), V_2^{k+1}, (E_3^{k+1}, l_3^{k+1}), V_4^{k+1}$ is determined.

Step 3: $k := k + 1$, go to **Step 1**.

	A	B	C	D	E	F	G
x	3915	4095	3690	2430	1260	675	810
y	4005	2745	450	405	810	1890	3105

Table 5.1: Coordinates of vertices

Example 5.2 Following Algorithm 3 the Hausdorff nearest circle for the polygon $ABCDEFG$ (Figure 5.2) is obtained using the program xfig running under unix.

Table 5.1 gives the coordinates of the vertices in *xfig*-internal units (1 unit = 0.000875 in). As usually in computer drawing programs x coordinates increase to the right and y coordinates increase to downwards. We start with points $V_1^1 = F$, $E_2^1 \in e$, $V_3^1 = D$, $E_4^1 \in b$ (for the construction the exact E-points are not important, important are only the sides to which they belong). The exact solution is obtained after four steps. The results of the calculations are given on Table 5.2.

n	V_1	E_2	V_3	E_4	ρ	r	R	h	H	$X = (\alpha, \beta)$	
1	F	e	D	b	1645	1581	2446	77	1041	2385	2108
2	A	c	D	b	1957	924	2532	205	1033	2303	2565
3	A	g	D	b	1625	1249	2154	383	520	2767	2385
4	A	g	C	b	1687	1275	2099	412	412	2722	2325

Table 5.2: Iterations according to Algorithm 3

The first columns show how do the points V_1, E_2, V_3 and E_4 change. The ρ-column and X-column give the radius and the centre of K^k, in the last row the radius and the centre of the Hausdorff nearest circle stands. The r-column and R-column give the radii r^k and R^k. The H-column gives the Hausdorff distance $H^k = h(K^k, T)$ according to formula (5.2) and the h-column gives the above defined values h^k which are some analog to the δ^k used as lower bounds in the Remez algorithm. Turn attention that the values h^k increase and the values of H^k decrease. Their common value in the last row is the Hausdorff distance of the polygon to the nearest circle. Figure 5.2 represents the circles obtained by the successive approximations and also some of the intermediate constructions, say the needed bisectrices (dashed lines) and midlines (solid lines) on each step.

Lemma 5.3. *The sequence $(h^k)_{k\in\mathbb{N}}$ is strictly increasing. The sequence $(H^k)_{k\in\mathbb{N}}$ is strictly decreasing and $h^k < H^k$ for all $k \in \mathbb{N}$.*

Proof. We show here only the increasing property of h^k. The decrease of H^k follows similarly. From the construction in Algorithm 3 the inequality $h^k < H^k$ for all $k \in \mathbb{N}$ is obvious. We have to show the increase of h^k only in the case of exchange of a vertex. In the other case we consider the intersection $\tilde{T}$ of T with the inner halfspaces of l_j^k $(j = 1,3)$. Because of $r^k < \|\overrightarrow{XP_1^k}\| < \rho^k < \|\overrightarrow{XP_2^k}\| < R^k$ the point E^* belongs to $\tilde{T}$. Algorithm 3 for T coincides in this case with Algorithm 1 for $\tilde{T}$. Hence by Remark 2 we have the desired inequality $h^k < h^{k+1}$. In the case of exchange of a vertex the inequality follows from well-known similarity properties of circles spanning the cone generated by $l_j^k = l_j^{k+1}$. Their centers X^k and X^{k+1} lie on the bisectrix $b^k = b^{k+1}$ of l_j^k $(j = 1,3)$ defined by the cut with the midlines of V_j^k $(j = 2,4)$ and V_j^{k+1} $(j = 2,4)$. We consider the arcs A^k and A^{k+1} defined by the smallest connected subset of the intersections of the circumferences of K^k and K^{k+1}, respectively, with the cone containing T and generated by l_j^k $(j = 1,2)$. We consider the segment connecting X^{k+1} with the vertex $V_j^{k+1} = V_j^k$. The cuts of the arcs A^k and A^{k+1} with this segment are called U^k and U^{k+1}. Let W^k be the cut of A^k with the segment $[X^k, V_j^k]$. The arcs has only one cut in the quadrilateral $P_j^k P_j^{k+1} X^{k+1} X^k$. Because the segment line through X^{k+1} and V_j^k separates the segments $[X^{k+1}, X^k]$ and $[P_j^{k+1}, P_j^k]$ either for $j = 1$ or for $j = 3$, the points U^{k+1} and U^k lie on $[X^{k+1}, V_j^k]$ in the same order (linear order on the ray $[X^{k+1}, V_j^k)$) as X^{k+1} and X^k on the bisectrix (linear order on the ray $[X^{k+1}, X^k)$). The angle $X^{k+1}X^kV_j^k$ is obtuse. It follows $h^{k+1} = \|\overrightarrow{U^{k+1}V^{k+1}}\| > \|\overrightarrow{U^kV^{k+1}}\| = \|\overrightarrow{U^kV^k}\| > \|\overrightarrow{W^kV^k}\| = h^k$ ∎

The monotonicity of h^k and H^k cannot be concluded from Algorithm 1 and the property of the δ^k and H^k noticed in Remark 2 since generally it cannot be associated to the "vertices" V_j^k the direction $\overrightarrow{XV_j^k}$ as a unit outer normal of T.

Lemma 5.4. *Let* $\operatorname{int}(T) \neq \emptyset$. *Let* (E_1, V_2, E_3, V_4) *and* $(\hat{E}_1, \hat{V}_2, \hat{E}_3, \hat{V}_4)$ *according to Step 1 in Algorithm 3 be given as well as* h, H, X, ρ *and* $\hat{h}, \hat{H}, \hat{X}, \hat{\rho}$ *constructed according to Step 1 and Step 2. Then for any* $\varepsilon > 0$ *there is a constant* $\delta(\varepsilon, \hat{E}_1, \hat{V}_2, \hat{E}_3, \hat{V}_4) > 0$ *such that*

$$\|(E_1, V_2, E_3, V_4) - (\hat{E}_1, \hat{V}_2, \hat{E}_3, \hat{V}_4)\| < \delta(e) \quad \Rightarrow \quad \|(h, H, X, \rho) - (\hat{h}, \hat{H}, \hat{X}, \hat{\rho})\| < \varepsilon.$$

Proof. We have to consider disturbed linear equalities and system of two regular linear equalities as well as the projection operator. Because of the uniqueness of the l_j^k $(j = 1, 2)$, the compactness of T and $\mathrm{int}(T) \neq \emptyset$ we get by standard stability results continuous dependence of (h, H, X, ρ) from (E_1, V_2, E_3, V_4) ∎

Proposition 5.5. *The sequence $(\rho^k, X^k, E_1^k, V_2^k, E_3^k, V_4^k, h^k, H^k)_{k\in\mathbb{N}}$ generated by Algorithm 3 converges to $(\bar{\rho}, \bar{X}, \bar{E}_1, \bar{V}_2, \bar{E}_3, \bar{V}_4, \bar{h}, \bar{H})$ with $\bar{H} = \bar{h}$ where $K(\bar{X}, \bar{\rho})$ is the Hausdorff nearest circle to T. The distance $\bar{H} = \bar{h}$ is at least attained in the points $\bar{E}_1, \bar{V}_2, \bar{E}_3, \bar{V}_4 \in T$. Algorithm 3 is finite for polygons.*

Proof. Lemma 5.3 yields $\lim_{k\to\infty} h^k = \bar{h} \leq \bar{H} = \lim_{k\to\infty} H^k$. Because of the compactness of T we find a subsequence (k_n) of $\mathbb{N}$ such that

$$\lim_{n\to\infty} (\rho^{k_n}, X^{k_n}, E_1^{k_n}, V_2^{k_n}, E_3^{k_n}, V_4^{k_n}) = (\bar{\rho}, \bar{X}, \bar{E}_1, \bar{V}_2, \bar{E}_3, \bar{V}_4).$$

Assume $K(\bar{X}, \bar{\rho})$ is not the Hausdorff nearest circle. Then $\bar{h} \neq \bar{H}$ and Algorithm 3 can be started in $(\bar{\rho}, \bar{X}, \bar{E}_1, \bar{V}_2, \bar{E}_3, \bar{V}_4)$ again. Let $\bar{h}^1 \geq \bar{h} + \varepsilon$, $\varepsilon > 0$ fixed the next iteration of $\bar{h}$. Lemma 5.4 implies for some $k > n(\varepsilon, \bar{E}_1, \bar{V}_2, \bar{E}_3, \bar{V}_4)$ that

$$\|(E_1^k, V_2^k, E_3^k, V_4^k) - (\bar{E}_1, \bar{V}_2, \bar{E}_3, \bar{V}_4)\| < \delta(\tfrac{\varepsilon}{3}, \bar{E}_1, \bar{V}_2, \bar{E}_3, \bar{V}_4).$$

Hence $\|h^{k+1} - \bar{h}^1\| < \frac{\varepsilon}{3}$. From $\bar{h} > h^{k+1} > h^k$ we get the contradiction $\varepsilon \leq \bar{h}^1 - \bar{h} \leq h^{k+1} + \frac{\varepsilon}{3} - \bar{h} < \frac{\varepsilon}{3}$. Thus, $K(\bar{X}, \bar{\rho})$ is the Hausdorff nearest circle. Because of its uniqueness the above sequences cannot have more than one accumulation point. The remaining follows from the geometric optimality conditions. For polygons Algorithm 3 allows only finite constellations of points (P_1, P_2, P_3, P_4). Since h^k $(k \in \mathbb{N})$ are strictly increasing, the algorithm stops after a finite number of steps ∎

Remark 7. In the proof of finiteness the possibility of an exact construction is assumed. If we can do the geometric construction within an certain accuracy, then the algorithm can stop earlier if the polygon is roughly spoken nearly smooth.

6. Application

As a geometrical application of the Remez algorithm we derive the following estimations.

Proposition 6.1. *Let T be a convex compact set in the plane, whose diameter has length a and whose width in a direction perpendicular to the diameter is b. Then for the Hausdorff nearest circle $K = K(X, \rho)$ it holds*

$$\frac{1}{4}(a-b) \leq h(T,K) \leq \frac{1}{2}\sqrt{a^2+b^2} - \frac{1}{4}(a+b) \tag{6.1}$$

$$\frac{3}{4}a + \frac{1}{4}b - \frac{1}{2}\sqrt{a^2+b^2} \leq \rho \leq \frac{1}{2}\sqrt{a^2+b^2} - \frac{1}{4}(a-b). \tag{6.2}$$

The equality holds only if T is a segment. Estimation (6.1) for the Hausdorff distance $h(T, K)$ gives in particular

$$h(T,K) \leq \frac{1}{4}\operatorname{diam} T. \tag{6.3}$$

Proof. Choose the directions $\vec{e}_1$, $\vec{e}_3$ parallel to the diameter of T and $\vec{e}_2$, $\vec{e}_4$ perpendicular to it (see Figure 6.1).

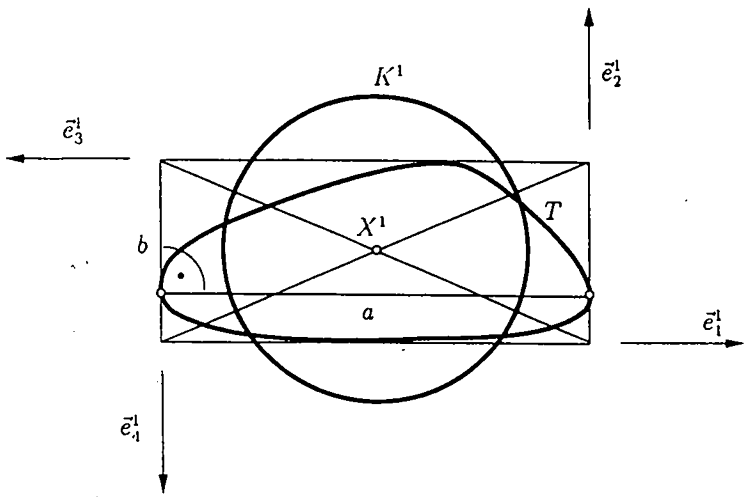

Figure 6.1: Estimation of the Hausdorff distance $H(K,T)$

The circle $K^1 = K(X^1, \rho^1)$ has a centre X^1 at the centre of the rectangular, whose sides lie at the support lines of T in directions $\vec{e}_i$ $(i = 1,2,3,4)$. The radius of this circle is $\rho^1 = \frac{1}{4}(a+b)$. We have by Remark 2

$$\frac{1}{4}(a-b) = \delta^1 \le h(T,K) \le H^1 \le \frac{1}{2}\sqrt{a^2+b^2} - \frac{1}{4}(a+b).$$

Let $h = h(T,K)$. The inclusion $K \subset T + hB$ gives $2\rho \le b + 2h$, whence

$$\rho \le \frac{1}{2}b + h \le \frac{1}{2}b + \frac{1}{2}\sqrt{a^2+b^2} - \frac{1}{4}(a+b) = \frac{1}{2}\sqrt{a^2+b^2} - \frac{1}{4}(a-b).$$

Similarly, the inclusion $T \subset K + hB$ gives $a \le 2\rho + 2h$, whence

$$\rho \ge \frac{1}{2}a - h \ge \frac{1}{2}a - \frac{1}{2}\sqrt{a^2+b^2} + \frac{1}{4}(a+b) = \frac{3}{4}a + \frac{1}{4}b - \frac{1}{2}\sqrt{a^2+b^2}.$$

Consider the function

$$\phi(a,b) = \frac{1}{2}\sqrt{a^2+b^2} - \frac{1}{4}(a+b) \qquad (a \ge 0,\, 0 \le b \le a).$$

Fix a. An easy calculation shows that $\phi(a,b)$ decreases from $\frac{1}{2}a$ to $\frac{1}{4}a(\sqrt{3}-1)$ when b varies from 0 to $\frac{1}{3}a\sqrt{3}$ and increases from $\frac{1}{4}a(\sqrt{3}-1)$ to $\frac{1}{2}a(\sqrt{2}-1)$ when b varies from $\frac{1}{3}a\sqrt{3}$ to a. Therefore

$$h(T,K) \le \max\left(\frac{1}{4}a, \frac{\sqrt{3}}{3}a\right) = \frac{1}{4}a$$

which shows (6.3) ∎

There is a result of Lessak [12] stating that for each $T \in \mathcal{T}$ there exists an ellipse K, such that $h(T, K) \leq \frac{1}{4} \operatorname{diam} T$. We see from (6.3) that this result remains true, when circles instead of ellipses are taken, in fact we obtained the more precise estimation (6.1). Since the approximation of convex bodies by circles is a particular case of the approximation by ellipses therefore our estimation is better than the cited one.

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